On cubic $s$-arc transitive Cayley graphs of finite simple groups

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Abstract

For a positive integer $s$, a graph $\Gamma$ is called $s$-arc transitive if its full automorphism group $\text{Aut}\Gamma$ acts transitively on the set of $s$-arcs of $\Gamma$. Given a group $G$ and a subset $S$ of $G$ with $S = S^{-1}$ and $1 \notin S$, let $\Gamma = \text{Cay}(G, S)$ be the Cayley graph of $G$ with respect to $S$ and $G_R$ the set of right translations of $G$ on $G$. Then $G_R$ forms a regular subgroup of $\text{Aut}\Gamma$. A Cayley graph $\Gamma = \text{Cay}(G, S)$ is called normal if $G_R$ is normal in $\text{Aut}\Gamma$. In this paper we investigate connected cubic $s$-arc transitive Cayley graphs $\Gamma$ of finite non-Abelian simple groups. Based on Li’s work (Ph.D. Thesis (1996)), we prove that either $\Gamma$ is normal with $s \leq 2$ or $G = A_{47}$ with $s = 5$ and $\text{Aut}\Gamma \cong A_{48}$. Further, a connected 5-arc transitive cubic Cayley graph of $A_{47}$ is constructed.

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1. Introduction

The aim of this paper is to give a complete classification of non-normal, connected, cubic, $s$-arc transitive, Cayley graphs of finite simple groups. Definitions of these concepts are given later in this section. Our main result is the following theorem.

**Theorem 1.1.** Let $G$ be a finite non-Abelian simple group, and let $\Gamma$ be a connected, cubic $s$-arc transitive, Cayley graph for $G$, where $s \geq 1$. Then
(1) either $\Gamma$ is a normal Cayley graph and $s \leq 2$, or $G = A_{47}$, $s = 5$ and $\text{Aut}(\Gamma) = A_{48}$.

Moreover,

(2) there exist connected cubic 5-arc transitive Cayley graphs for $G = A_{47}$ with automorphism group $A_{48}$, and all such graphs can be constructed as in Theorem 4.1(2).

We build on work of Li [11] who showed that the only possibilities for non-normal Cayley graphs under the hypotheses of Theorem 1.1 must arise from one of the groups $G = A_5, L_2(11), M_{11}, A_{11}, M_{23}, A_{23}, A_{47}$. In the paper we examine each of these seven groups in detail using, among other tools, a result of two of the authors with Praeger [7], showing that $G = A_{47}$ is only possible group, and that in this case $s = 5$ and $\text{Aut}(\Gamma) = A_{48}$. A general construction for graphs in this case is given in Section 4. Since Li's result depends on the finite simple group classification, our theorem also depends on that classification.

For a finite group $G$, a nonempty subset $S$ of $G$ is called a Cayley subset if $S = S^{-1} := \{s^{-1} \mid s \in S\}$ and $1 \notin S$. Given a Cayley subset $S$ of $G$, the Cayley graph $\Gamma = \text{Cay}(G, S)$ of $G$ with respect to $S$ consists of the vertex set $V \Gamma = G$ and the edge set $E \Gamma = \{(g, sg) \mid g \in G, s \in S\}$. Note that $S = S^{-1}$. Thus $\Gamma$ is undirected. It is not difficult to see that $\Gamma$ is the disjoint union of $k$ copies of $\text{Cay}(\langle S \rangle, S)$ when $k = |G : \langle S \rangle|$. Hence $\Gamma$ is connected if and only if $\langle S \rangle = G$. In this paper, we concentrate on the case where $\Gamma$ is connected. From now on, all graphs considered are undirected and connected.

It follows from the definition that the group $G_R = \{g^{-1} \mid g \in G\}$ of right translations $g \mapsto xg$ forms a transitive automorphism group of $\Gamma$, and we may identify $G$ with $G_R$. For a connected Cayley graph $\Gamma = \text{Cay}(G, S)$, by [9], we have $N_{\text{Aut}(\Gamma)}(G) = G : \text{Aut}(G, S)$, the semidirect product of $G$ and $\text{Aut}(G, S)$, where

$$\text{Aut}(G, S) = \{\sigma \in \text{Aut}(G) \mid S^\sigma = S\}.$$ 

Evidently, if $G$ is normal in $\text{Aut}(\Gamma)$, then $\text{Aut}(\Gamma) = G : \text{Aut}(G, S)$. In this case, $\Gamma$ is called a normal Cayley graph of $G$ [20]. Note that $\text{Aut}(G, S)$ fixes the vertex $\alpha = 1$, the identity of $G$, and acts faithfully on $V \Gamma$. So, if $\Gamma = \text{Cay}(G, S)$ is normal, then $|\text{Aut}(G, S)|$ is a divisor of $|S|!$.

Let $\Gamma$ be a finite simple graph with vertex set $V \Gamma$ and edge set $E \Gamma$. Given a positive integer $s$, an $s$-arc is a sequence $(v_0, v_1, \ldots, v_s)$ of $s + 1$ vertices of $V \Gamma$ such that $(v_{i-1}, v_i) \in E \Gamma$ and $v_{i-1} \neq v_{i+1}$ for all $i$. The graph $\Gamma$ is called $s$-arc transitive if $\text{Aut}(\Gamma)$ acts transitively on the set of all $s$-arcs of $\Gamma$ (a 1-arc transitive graph is also called symmetric). Clearly, for $s \geq 2$, an $s$-arc transitive graph is also $(s - 1)$-arc transitive and hence symmetric. For a subgroup $G$ of $\text{Aut}(\Gamma)$, the graph $\Gamma$ is said to be $(G, s)$-arc transitive if $G$ acts transitively on the set of $s$-arcs of $\Gamma$.

The first result which gave an upper bound $s$ for $s$-arc transitivity of graphs came from Tutte [17] who proved that there exist no cubic $s$-arc transitive graphs for $s \geq 6$. Tutte's work initiated the study of $s$-arc transitive graphs. Since then, the construction and classification of $s$-arc transitive graphs have received considerable attention (see, for example, [2, 5, 8, 10, 12, 14, 15, 18]). Clearly, a fundamental problem in determining the structure of an $s$-arc transitive Cayley graph is determining its full automorphism group $\text{Aut}(\Gamma)$, which is very difficult in general. The purpose of this paper is to investigate cubic symmetric Cayley graphs of finite simple groups. In [11], Li proved that a connected
undirected cubic symmetric Cayley graph of a finite non-Abelian simple group $G$ is normal if $G$ is not one of the groups $A_5$, $L_2(11)$, $M_{11}$, $A_{23}$, $A_47$. Later on the first and last authors of this paper proved that a connected undirected cubic symmetric Cayley graph of the alternating group $A_5$ is normal (see [21]). Further, from [6] we know that, for most finite non-Abelian simple groups $G$, the corresponding connected undirected cubic Cayley graph is normal. From these results one can see that the examples of symmetric cubic Cayley graphs of finite non-Abelian simple groups which are not normal might be very rare and it is possible to construct and classify all such graphs. Thus it is natural to ask about the existence and possible classification of non-normal connected symmetric cubic Cayley graphs for finite simple groups; answers to these questions are provided in Theorem 1.1.

For $G = A_47$, Theorem 1.1 asserts that a connected cubic 5-arc transitive Cayley graph of $G$ must be isomorphic to a graph defined in Theorem 4.1. Then it is natural to seek an explicit classification of the connected cubic 5-arc transitive Cayley graphs of $A_47$. We are very grateful to a referee who kindly made many helpful suggestions on solving this problem. We shall discuss the classification of such graphs in a separate paper.

2. Preliminaries

The terminology and notation used in this paper are standard (see for example [1, 4, 15, 20]). For a general finite connected undirected cubic $s$-arc transitive graph $\Gamma$, the first lemma describes the structure of a vertex stabiliser of $\text{Aut}\Gamma$.

**Lemma 2.1** ([19] (see also [1, 18C, p. 126])). Let $\Gamma$ be a finite undirected and connected cubic $s$-arc transitive graph with $1 \leq s \leq 5$. Let $H$ be a subgroup of $\text{Aut}\Gamma$ such that $H$ is transitive on $s$-arcs but not $(s + 1)$-arcs of $\Gamma$. For $\alpha \in V\Gamma$, let $H_\alpha$ denote the vertex stabiliser of $H$. Then the following holds:

1. $s = 1$, $H_\alpha \cong \mathbb{Z}_3$;
2. $s = 2$, $H_\alpha \cong S_3$;
3. $s = 3$, $H_\alpha \cong D_{12}$;
4. $s = 4$, $H_\alpha \cong S_4$;
5. $s = 5$, $H_\alpha \cong S_4 \times \mathbb{Z}_2$.

By Lemma 2.1, for a finite undirected and connected cubic $s$-arc transitive graph $\Gamma$, the order of the vertex stabiliser of $\text{Aut}\Gamma$ divides $48 = 2^4 \times 3$.

A permutation group $H$ on a set $\Omega$ is called *quasiprimitive* if each nontrivial normal subgroup of $G$ is transitive on $\Omega$. For a finite graph $\Gamma$ and an automorphism group $H$ of $\Gamma$, the graph is called *$H$-quasiprimitive* if $H$ acts quasiprimatively on $V\Gamma$. For a transitive automorphism group $H$ of $\Gamma$, if $H$ is not quasiprimitive on $V\Gamma$, then there is a nontrivial intransitive normal subgroup $N$ of $H$. Further, $N$ induces a *quotient graph* $\Gamma_N$, for which $V\Gamma_N$ consists of all $N$-orbits on $V\Gamma$ and $\{U, V\} \in E\Gamma_N$ if and only if there exist some $u \in U$ and $v \in V$ such that $u$ and $v$ are adjacent in $\Gamma$.

The next lemma gives a general description of the possibilities for the full automorphism group of a connected Cayley graph of a finite non-Abelian simple group.
Lemma 2.2 ([7, Theorem 1.1]). Let $G$ be a finite non-Abelian simple group and $\Gamma = \text{Cay}(G, S)$ a connected Cayley graph for $G$. Let $M$ be a subgroup of $\text{Aut}(G, S)$ containing $G$. Then either $M = G$ or one of the following holds.

1. $M$ is almost simple, $\text{soc}(M)$ contains $G$ as a proper subgroup and is transitive on $V \Gamma$; or
2. $G, \text{Inn}(G) \leq M \leq G, \text{Aut}(G, S)$ and $S$ is a self-inverse union of $G$-conjugacy classes; or
3. $M$ is not quasiprimitive on $V \Gamma$ and there is a maximal intransitive normal subgroup $K$ of $M$ such that one of the following holds:
   (a) $M/K$ is almost simple, and $\text{soc}(M/K)$ contains $GK/K \cong K$ and is transitive on $V \Gamma_K$;
   (b) $M/K = \text{AGL}_3(2)$, $G = L_2(7)$ and $\Gamma_K \cong K_8$; or
   (c) $\text{soc}(M/K) \cong T \times T$, and $GK/K \cong G$ is a diagonal subgroup of $\text{soc}(M/K)$ (see [7, Table 1] for details of $T$ and $G$).

Let $G$ be a group. A subgroup $T$ of $G$ is said to be core-free if $\bigcap_{g \in G} T^g = 1$. Suppose that $T$ is a core-free subgroup of $G$ and that an element $g \in G$ satisfies $g \not\in N_G(T), g^2 \in T$, and $\langle T, g \rangle = G$. A coset graph of $G, \Gamma^* = \Gamma(G, T, g)$ is defined by Sabidussi [16] as follows:

$$V \Gamma^* = \{Tx : x \in G\} \quad \text{and} \quad E \Gamma^* = \{(Tx, Ty) : xy^{-1} \in TgT\}. \quad (1)$$

For a finite undirected and connected $G$-symmetric graph $\Gamma$, the next lemma tells us how to reconstruct $\Gamma$ from the group $G$. Its proof can be found in [13, 16]; also see [5, Theorem 2.1] for reference.

Lemma 2.3. Let $\Gamma$ be a finite connected undirected $G$-symmetric graph of valency at least 3. Then there exists a core-free subgroup $T$ of $G$ and an element $g \in G$ having the following properties:

1. $g \not\in N_G(T), g^2 \in T$, and $\langle T, g \rangle = G$;
2. the right multiplication action of $T$ on the set of cosets $[T : T \cap T^g]$ is transitive.

Moreover, $\Gamma \cong \Gamma^* = \Gamma(G, T, g)$ defined in (1) above, and we may choose $g$ to be a 2-element. Conversely, if $G$ is a finite group with a core-free subgroup $T$ and an element $g$ satisfying (1) and (2), then $\Gamma(G, T, g)$ is a connected undirected $G$-symmetric graph, and $G$ acts faithfully on the vertices by right multiplication.

Since the graphs $\Gamma$ considered in this paper are symmetric Cayley graphs, by Lemma 2.3, there exists a subgroup $H$ of $\text{Aut}(\Gamma)$ such that $\Gamma$ is $H$-symmetric and hence we may construct the coset graph $\Gamma(H, T, g)$ with $g \in H$ and, up to isomorphism, we may identify $\Gamma = \Gamma(H, T, g)$.

The final lemma of this section gives some relations between Cayley graphs and coset graphs, which will play a very important role in proving our theorems.

Lemma 2.4. Let $\Gamma$ be a finite connected and undirected $H$-symmetric graph. If $H$ contains a regular subgroup $G$, then the following are true.
(1) there is a Cayley subset $S$ of $G$ such that $\Gamma \cong \text{Cay}(G, S)$;
(2) if the valency of $\Gamma$ is odd, then there is an involution $g \in G$ such that $\Gamma \cong \Gamma(H, H_\alpha, g)$, where $H_\alpha$ is the vertex stabiliser of $H$ for some $\alpha \in V \Gamma$.

**Proof.** Now $H = GH_\alpha = H_\alpha G$ and $G \cap H_\alpha = 1$ since $G$ acts regularly on $V \Gamma$. By Lemma 2.3, $\Gamma \cong \Gamma(H, H_\alpha, g)$ for some 2-element $g \in H$.

(1) Set $S = G \cap (H_\alpha G H_\alpha)$ and define the Cayley graph $\text{Cay}(G, S)$. Then it is easy to verify that the map $\sigma$ defined by

$$\sigma : x \mapsto H_\alpha x, \quad \text{for any} \quad x \in G$$

is a graph isomorphism from $\text{Cay}(G, S)$ to $\Gamma(H, H_\alpha, g)$ and hence $\Gamma$.

(2) Now we identify $\Gamma = \Gamma(H, H_\alpha, g) = \text{Cay}(G, S)$. If the valency of $\Gamma$ is odd, since $S = S^{-1}$, there exists at least one involution, say $s \in S$. Since $S = G \cap (H_\alpha G H_\alpha)$, $s = h_1 g h_2$ for some $h_1, h_2 \in H_\alpha$, which implies that $H_\alpha g H_\alpha = H_\alpha s H_\alpha$. It follows that $\Gamma(H, H_\alpha, g) = \Gamma(H, H_\alpha, s)$ and hence $\Gamma \cong \Gamma(H, H_\alpha, s)$. $\square$

### 3. Proof of Theorem 1.1(1)

**Lemma 3.1.** Let $\Omega = \{1, 2, \ldots, n\}$ and $L$ be a regular subgroup of $S_n$. For an involution $g \not\in L$, if $g$ normalises $L$, then the following hold.

(1) For $n = 4$ and $L \cong \mathbb{Z}_2^n$, either $g$ centralises $L$ or $g$ is an odd permutation of $S_4$.

(2) For $n = 8$ and $L \cong D_8$, where $L = \langle a, b \rangle$, $a = (1234)(5678)$, $b = (15)(28)(37)(46)$ and $a^b = a^{-1}$, if $b^g = ba$, then $g$ is an odd permutation in $S_8$.

**Proof.** For $n = 4$, the conclusion holds immediately from the fact that $L$ contains all involutions of $A_4$.

For $n = 8$, write $\text{Fix}(g) = \{i \in \Omega \mid i^g = i\}$. Consider first that $\text{Fix}(g) \neq \emptyset$ and assume that $1 \in \text{Fix}(g)$. Now


Since $1^g = 1$ and $b^g = ba$, $5^g = 6$ and $6^g = 5$, which implies that 2$^g$ = 4 and 4$^g$ = 2. It follows that 7$^g$ = 8, 8$^g$ = 7 and 3$^g$ = 3. Thus $g = (56)(24)(78)$ is an odd permutation of $S_8$. Suppose then that $\text{Fix}(g) = \emptyset$. In this case, for a 2-cycle factor $(ij)$ of $b$, $\{i, j\} \cap \{i^g, j^g\} = \emptyset$ since $b$ and $b^g = ba$ have no common 2-cycle factor. Thus $(15)^g$ is either (38) or (47). In the former case we have $8^g \in \{1, 5\}$, and hence $(28)^g = (16)$. So $(37)^g = (16)$ and consequently $(46)^g = (47)$, which implies that $4^g = 4$, contradicting $\text{Fix}(g) = \emptyset$. In the latter case, a similar argument also yields a contradiction. $\square$

For $\Gamma = \text{Cay}(G, S)$ given in Theorem 1.1, write $\text{Aut}\Gamma = A$ and set

$$\Sigma = \{L_2(11), M_{11}, A_{11}, M_{23}, A_{23}, A_{47}\}.$$

By [11, Theorem 7.1.3] and the results of [21], it is sufficient to consider only the case $G \in \Sigma$. The next lemma examines the case where $\text{Aut}\Gamma$ contains a simple subgroup which acts transitively on the arcs of $\Gamma$. 
Lemma 3.2. For $G \in \Sigma$ and $\Gamma = \text{Cay}(G, S)$ given in Theorem 1.1, if there is a simple subgroup $H$ of $\text{Aut}\Gamma$ containing $G$ properly, then $(H, G) = (A_{48}, A_{47})$.

**Proof.** By [11, Theorem 7.1.3] the only possibilities for the pair of simple groups $(H, G)$ are $(A_{48}, A_{47})$, $(M_{11}, L_2(11))$, $(M_{12}, M_{11})$, $(A_{12}, A_{11})$, $(M_{24}, M_{23})$ and $(A_{24}, A_{23})$.

Now we show that, except for $(A_{48}, A_{47})$, all cases do not occur. Note that $\Gamma$ is $H$-symmetric. So we can identify $\Gamma = \Gamma(H, T, g)$, where $T = H_n$ and $g$ satisfies the conditions given by Lemma 2.3. Now $H = \text{Aut} G$. Let $\Omega = \{H : G\} = \{Gh : h \in H\}$.

Then $T$ acts regularly on $\Omega$ and $G$ acts transitively on $\Omega \setminus \{G\}$ by right multiplication. Write $\Omega = \{\delta_1, \delta_2, \ldots, \delta_m\}$ with $\delta_i \cong G$ and $m = |H : G|$. Since $G$ is simple, $G$ acts faithfully on $\Omega \setminus \{\delta_i\}$ and is therefore isomorphic to a subgroup of $S_m$. Note that $G$ has no subgroups of index $2$. Thus $G$ is isomorphic to a subgroup of $A_m$. From now on $H$ and $G$ are considered as subgroups of $A_m$. Write $L = T \cap T^g$. Since $T$ is regular on $\Omega$, $L$ acts semiregularly on $\Omega$. Further $L$ has exactly three $L$-orbits on $\Omega$, namely $\Omega_i = \{\delta_{ij} : j = 1, \ldots, n\}$, where $n = m/3$ for $i = 1, 2, 3$. For a subgroup $B$ of $T$ and an element $u \in S_m$ fixing $\Omega_i$, write $B_i = B^{\Omega_i}$ and $u_i = u^\Omega_i$, the components of $B$ and $u$ on $\Omega_i$, for $i = 1, 2, 3$.

Let $K$ denote the kernel of $H_n$ acting on $\Gamma(\alpha)$, the set of vertices adjacent to $\alpha$ in $\Gamma$. We treat the following two cases separately.

**Case 1.** $G \in \{L_2(11), M_{11}, A_{11}\}$.

Now $m = 12$, $n = 4$ and $T = H_n = \langle x, y \rangle \cong D_{12}$, where $x^6 = y^2 = 1$ and $x^y = x^{-1}$. Since $K \cong \mathbb{Z}_2$, $K$ is the center of $T$ and hence $K = \langle x^3 \rangle \trianglelefteq T$. Write $a = x^3$ and $b = a^8$. Since $T$ is core-free, $K^a \neq K$, which implies that $a \neq b$. It follows that $L = T \cap T^g = \langle a, b \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, which acts regularly on each $\Omega_i$. Clearly, $L \cong L_2$. Note that $g$ is an involution and $L = T \cap T^g$. Thus $L^g = L$. Since there are three $L$-orbits $\Omega_i$ on $\Omega$, $g$ induces a permutation on these three $L$-orbits. It follows that either $g$ fixes some $L$-orbit, say $\Omega_1$, and interchanges $\Omega_2$ and $\Omega_3$, or $g$ fixes each $\Omega_i$ for $i = 1, 2, 3$. In the former case we have $g = g_1g'$, where $g' = g^{\Omega_1 \cup \Omega_2 \cup \Omega_3}$. Note that $|\Omega_1| = 4$ and $g'$ interchanges $\Omega_2$ and $\Omega_3$. So $g'$ is an even permutation on $\Omega_2 \cup \Omega_3$. Now $L_{g_1} = L_1$ with $g_1 \in S_{\Omega_1} = S_4$. If $g_1$ centralises $L_1$, then $a^{g_1} = b^{g_1}$ since $a^g = b$. It follows that $a^g$ fixes $\Omega_1$ pointwise, which contradicts $L$ being semiregular on $\Omega$. So $g_1$ acts nontrivially on $L_1$ by conjugation. Then by Lemma 3.1(1), $g_1$ is an odd permutation of $S_{\Omega_1}$. Thus $g$ is an odd permutation of $A_{12}$, which means that there is no involution $g$ of $A_{12}$ satisfying $\Gamma = \Gamma(H, H_n, g)$. In the latter case, $g$ fixes each $\Omega_i$ setwise. So we have $g = g_1g_2g_3$ and $L_i^{g_1} = L_i$ with $g_i$ an involution of $S_{\Omega_i} = S_4$. A similar argument shows that $g_i$ does not centralise $L_i$ for each $i = 1, 2, 3$. Then by Lemma 3.1(1), $g_i$ is an odd permutation of $S_{\Omega_i}$, for each $i = 1, 2, 3$. Thus $g$ is an odd permutation of $A_{12}$, and so there exists no involution $g$ of $A_{12}$ such that $\Gamma = \Gamma(H, H_n, g)$. This shows that Case 1 does not occur.

**Case 2.** $G \in \{M_{23}, A_{23}\}$.

By Lemma 2.1, $\Gamma$ is $(H, 4)$-arc transitive and $T = H_n \cong S_4$. Now $m = 24$ and $n = 8$. Since $T/K \cong S_3$, $K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (because $S_4$ has a unique normal subgroup of order 4). Write $L = \langle a, b \rangle$ with $a^4 = b^2 = 1$ and $a^b = a^{-1}$. Since $K \trianglelefteq L$ and $Z(L) = \langle a^2 \rangle < K$, we may assume further that $K = \langle a^2, b \rangle$. Note that $L \cong D_8$ contains exactly two subgroups isomorphic to $\mathbb{Z}_2^2$, namely $K = \langle a^2, b \rangle$ and $M = \langle a^2, ba \rangle$. 
Since \( L^g = L \) and \( K^g \neq K, K^g = M \). Note that \( (a^2)^g = a^2 \) since \( (a^2) = Z(L) \). Thus \( b^g = ba \) or \( ba^{-1} \). Without loss of generality we may assume \( b^g = ba \). Now, for each \( i \in \{1, 2, 3\} \), it is easy to see that \( L \cong L_i = \langle a_i, b_i \rangle \) acts regularly on \( \Omega_i \). Further, we may write
\[
a_i = (\delta_1 \delta_2 \delta_3 \delta_4)(\delta_5 \delta_6 \delta_7 \delta_8), \quad b_i = (\delta_2 \delta_3 \delta_4 \delta_5)(\delta_6 \delta_7 \delta_8)(\delta_9 \delta_{10}).
\]
As in Case 1, since \( L^g = L \), we know that either \( g \) fixes some \( L \)-orbit, say \( \Omega_i \), and interchanges \( \Omega_2 \) and \( \Omega_3 \), or \( g \) fixes each of \( \Omega_i \) setwise. By Lemma 3.1(2), a similar argument to that in Case 1 shows that there is no involution \( g \) of \( A_{24} \) such that \( \text{Cay}(G, S) = \Gamma(H, T, g) \). So Case 2 does not occur, which completes the proof of the lemma. □

For \( \Gamma = \text{Cay}(G, S) \) given by Theorem 1.1, write \( A = \text{Aut} \Gamma \). If \( G \) is normal in \( A \), then \( A = G \cdot \text{Aut}(G, S) \). Since \( \text{Aut}(G, S) \) acts faithfully on \( S \) and \( |S| = 3 \), \( s \leq 2 \) and \( A \leq G \cdot S \), So we assume that \( G \) is not normal in \( A \).

To complete the proof of Theorem 1.1(1), we discuss the following two cases separately.\[\]
Case 1. \( A \) is quasiprimitive on \( V \Gamma \).

In this case, by Lemma 2.2, either \( A \) is an almost simple group and \( \text{soc}(A) \) contains \( G \) properly or \( G \cdot \text{Inn}(G) \leq A \) and \( S \) is a self-inverse union of \( G \)-conjugacy classes. Note that \( |S| = 3 \). Thus the latter case does not occur. In the former case, by Lemma 3.2, \( G = A_{47} \) and \( \text{soc}(A) = A_{48} \). Hence, by Lemma 2.1, \( s = 5 \) and \( A = A_{47}(S_3 \times Z_2) = A_{48} \).

Case 2. \( A \) is not quasiprimitive on \( V \Gamma \).

Now \( A \) contains a maximal intransitive normal subgroup \( M \). Since the valency of \( \Gamma \) is 3, again by Lemma 2.2, we conclude that \( A/M \) is almost simple and \( \text{soc}(A/M) \) contains \( GM/M \cong G \) which is transitive on \( V \Gamma_M \).

We claim first that \( M \) is centralised by \( G \) and hence \( |M| \) divides 6. Note that \( G \cap M = 1 \). Thus \( |M| \) divides \( 2^4 \times 3 \). If \( G \) does not centralise \( M \), then \( G \) is isomorphic to a subgroup of \( \text{Aut}(M) \). On the other hand, for \( G \in \Sigma' \), it is easy to see that this is impossible.

Now \( GM = G \times M \) and \( \text{soc}(A/M) \) is a finite simple group containing \( GM/M \). If \( \text{soc}(A/M) \cong G \), we must have \( G \leq A \) since \( M \) centralises \( G \), contradicting our assumption. Note that \( |\text{soc}(A/M) : GM/M| \) divides \( 2^4 \times 3 \). Therefore, if \( \text{soc}(A/M) \not\cong G \), by checking [3] we know that the pair of simple groups \( (A/M, G) \) must be one of the following:
\[
(M_{11}, L_2(11)), (M_{12}, M_{11}), (A_{12}, A_{11}), (M_{24}, M_{23}), (A_{24}, A_{23}).
\]
We have \( M \cong \mathbb{Z}_2 \) since \( 3^2 \) does not divide \( |A_{a} \rangle \). Then, by [11, Proposition 7.1.1], \( A \) contains a finite simple group \( H \) such that the pair of simple groups \( (H, G) \) is one of those listed in (2) above. Thus \( \Gamma = \text{Cay}(G, S) \) is also an \( H \)-symmetric graph for a finite non-Abelian simple group \( H \), contradicting Lemma 3.2. This completes the proof of Theorem 1.1(1). □

4. Proof of Theorem 1.1(2)

Let \( A_{48} \) act naturally on \( \Omega = \{1, 2, \ldots, 48\} \) and let \( A_{47} \) be the stabiliser of the point 48. Choose a regular subgroup \( T \cong S_4 \times \mathbb{Z}_2 \) of \( A_{48} \). Then \( A_{48} = A_{47}T \). Set
\[
\Omega(A_{47}, T) = \{ g \in A_{47} \mid |g| = 2, T \cap T^g \cong D_8 \times \mathbb{Z}_2, \langle T, g \rangle = A_{48} \}.
\]
We state and prove the following theorem which leads directly to Theorem 1.1(2).

**Theorem 4.1.** (1) Let $\Gamma = \text{Cay}(A_{47}, S)$ be a connected cubic 5-arc transitive graph with $\text{Aut}\Gamma = A_{48}$. Then there exists a subgroup $T$ of $A_{48}$ and an involution $g \in \Omega(A_{47}, T)$ such that $\Gamma \cong \Gamma(A_{48}, T, g)$. (2) For $A_{48}$, there exists a regular subgroup $T$ of $A_{48}$ with $T \cong S_4 \times \mathbb{Z}_2$ such that $\Omega(A_{47}, T) \neq \emptyset$. Furthermore, for each $g \in \Omega(A_{47}, T)$, define the coset graph $\Gamma = \Gamma(A_{48}, T, g)$. Then $\Gamma$ is a connected cubic 5-arc transitive Cayley graph of $A_{47}$.

**Remarks on Theorem 4.1.** (1) Theorem 4.1 gives the first example of a non-normal cubic symmetric Cayley graph of a finite non-Abelian simple group. Moreover, in this example, the full automorphism group of the graph is $A_{48}$ which contains $A_{47}$ as a maximal subgroup. Hence $A_{47}$ is self-normalised in $\text{Aut}\Gamma$. So it also answers the question raised in [6], that of “finding an example of non-normal cubic $s$-arc transitive Cayley graph of finite non-Abelian simple groups satisfying that $\text{Aut}(G, S) = 1$” (see [6, p. 70]).

(2) It is worth mentioning that some computation by computer gives us some clue as to how to construct examples of connected cubic 5-arc transitive Cayley graphs of $A_{47}$. For a regular subgroup $T$ of $A_{48}$ given in our proof, we choose many different involutions of $A_{47}$ and find that there are several involutions $g \in \Omega(A_{47}, T)$. Thus, in principle, there may exist several nonisomorphic connected cubic 5-arc transitive Cayley graphs of $A_{47}$.

**Proof of Theorem 4.1.** The first part of the theorem follows immediately from Lemmas 2.1 and 2.4. To prove part (2) we need the following lemma.

**Lemma 4.2.** In the natural action of $A_{48}$ on $\Omega = \{1, 2, \ldots, 48\}$, let $u$ be a 3-cycle of $A_{48}$ and $p$ a 47-cycle. If $\langle u, p \rangle$ is transitive on $\Omega$, then $A_{48} = \langle u, p \rangle$.

**Proof.** Clearly $\langle p \rangle$ is primitive on the support of $p$. So $\langle u, p \rangle$ is primitive on $\Omega$. Since $\langle u, p \rangle$ contains a 3-cycle, by Jordan’s theorem, $\langle u, p \rangle = A_{48}$. $\square$

Now we prove part (2) of Theorem 4.1. It is sufficient to find a regular subgroup $T$ of $A_{48}$ such that $\Omega(A_{47}, T) \neq \emptyset$. Write $H = A_{48}$ and $G = A_{47}$. Let $T = \langle a, b, c, t \rangle$, where

- $a = (1, 2, 3, 4)(5, 6, 7, 8)(9, 10, 11, 12)(13, 14, 15, 16)(17, 18, 19, 20)(21, 22, 23, 24)(25, 26, 27, 28)(29, 30, 31, 32)(33, 34, 35, 36)(37, 38, 39, 40)(41, 42, 43, 44)(45, 46, 47, 48),
Direct computation shows that
\[ a^4 = b^2 = c^2 = 1, \quad a^b = a^{-1}, \quad a^c = a, \quad b^c = b, \quad a^{b'} = a^{-1}, \quad b' = a^2b, \quad c' = c, \]
and \( T \cong S_4 \times \mathbb{Z}_2 \). Moreover, it is not hard to verify that \( T \) is a regular subgroup of \( H \) and hence \( H = GT \) with \( G \cap T = 1 \). Clearly \( \langle a, b \rangle \cong D_8 \) and \( c \) centralises \( \langle a, b \rangle \). It follows that \( N := \langle a, b, c \rangle \cong D_8 \times \mathbb{Z}_2 \), and hence \( N \) is a Sylow 2-subgroup of \( T \). Take an involution \( g \in G \) such that
\[
g = (1, 25)(2, 28)(3, 27)(4, 26)(5, 30)(6, 29)(7, 32)(8, 31)
(9, 19)(10, 18)(11, 17)(12, 20)(13, 24)(14, 23)(15, 22)
\]
Then it is easy to check that \( N^g = N \) and \( T^g \neq T \). Thus \( T \cap T^g = N \) and \( |T : T \cap T^g| = 3 \). Set \( L = \langle T, g \rangle \). We shall show that \( L = H \), which is equivalent to showing that \( L \) is transitive on \( \Omega \) and contains a 3-cycle and a 47-cycle by Lemma 4.2.
Write \( v_1 = gt \) and \( v_2 = gt^2 \). Then
\[
v_1 = (1, 10, 48)(2, 33, 32, 47, 22, 39, 27, 23, 28, 44, 6, 38, 9, 7, 13, 5, 12, 41, 17,
31, 21, 29, 20, 37, 30, 42, 3, 15, 4, 40, 16, 46, 8, 35, 25, 18, 36, 11)
(14, 43, 24, 34)(19, 26, 45),
\]
\[
v_2 = (1, 36, 31, 46, 21, 38, 26, 22, 27, 43, 5, 37, 12, 6, 16, 8, 11, 44, 20, 30, 24,
32, 19, 40, 29, 41, 2, 14, 3, 39, 15, 45, 7, 34, 28, 17, 35, 10)(4, 9, 47)
(13, 42, 23, 33)(18, 25, 48).
\]
Then it is easy to see that \( |v_1| = |v_2| = 228 = 2^2 \times 3 \times 19 \). Now
\[
v_1^3 = (2, 47, 27, 44, 9, 5, 17, 29, 30, 15, 16, 35, 36, 33, 22, 23, 6, 7, 12, 31, 20,
42, 4, 46, 25, 11, 32, 29, 38, 38, 13, 41, 21, 37, 3, 40, 8, 18)(14, 34, 24, 43),
\]
\[
v_1^4 = (1, 10, 48)(2, 11, 28, 9, 12, 21, 30, 4, 8, 36, 32, 27, 6, 13, 17, 20, 3, 16, 25)
(5, 31, 17, 46, 18, 33, 39, 44, 7, 41, 29, 42, 20, 35, 11, 47, 23, 38)
(19, 26, 45),
\]
\[
v_2^3 = (1, 31, 21, 26, 27, 5, 12, 16, 11, 20, 24, 19, 29, 2, 3, 15, 7, 28, 35)(4, 47, 9)
(6, 8, 44, 30, 22, 39, 41, 14, 39, 45, 34, 17, 10, 36, 46, 38, 22, 43, 37)
(13, 23)(18, 48, 25)(33, 42),
\]
\[
v_2^5 = (1, 38, 5, 8, 24, 41, 15, 17, 31, 22, 12, 44, 19, 14, 7, 10, 21, 43, 16, 30, 29,
39, 28, 36, 27, 11, 32, 2, 45, 35, 46, 27, 6, 20, 40, 3, 34)(4, 47, 9)
(13, 42, 23, 33)(18, 48, 25).
\]
Since \( |v_1| = |v_2| = 2^2 \times 3 \times 19 \), we have \( |v_1^3| = 2^2 \times 19 \), \( |v_1^4| = 3 \times 19 \), \( |v_2^3| = 2 \times 3 \times 19 \) and \( |v_2^5| = 2^2 \times 3 \times 19 \). Now
\[
v_1 v_2 = (2, 13, 37, 24, 28, 20, 12)(3, 45, 40, 8, 10, 18, 31, 38, 47, 27, 33, 19,
22, 15, 9, 34)(4, 29, 30, 23, 17, 46, 11, 14, 5, 6, 26, 7, 42, 39, 43, 32)
(16, 21, 41, 35, 48, 36, 44),
\]
$v_2v_1 = (1, 11, 6, 46, 29, 17, 25)(2, 43, 12, 38, 45, 13, 3, 27, 24, 47, 40, 20, 42, 28, 31, 8)(4, 7, 14, 15, 19, 16, 35, 48, 36, 21, 9, 22, 23, 32, 26, 39)(5, 30, 34, 44, 37, 41, 33),$

$(v_1v_2)^2 = (2, 37, 28, 12, 13, 24, 20)(3, 40, 10, 31, 47, 33, 22, 9)(4, 30, 17, 11, 5, 26, 42, 43)(6, 7, 39, 32, 29, 23, 46, 14)(8, 18, 38, 27, 19, 15, 34, 45)(16, 41, 48, 44, 21, 35, 36).$

Let $p = v_1^4v_2^2(v_1v_2)^2$. Then

$p = (1, 16, 38, 13, 31, 7, 6, 46, 44, 12, 42, 48, 47, 24, 15, 27, 18, 43, 28, 30, 33, 8, 14, 32, 26, 45, 23, 9, 41, 37, 39, 17, 20, 34, 11, 3, 5, 35, 2, 4, 21, 29, 22, 36, 10, 25, 40).$

Hence $|p| = 47$ and $L$ contains a 47-cycle of $H$ fixing 19. Next show that $L$ contains a 3-cycle. Let $q = v_1^3v_2v_1^5v_1v_2$. Then

$q = (1, 4, 43, 46, 47, 35, 32, 27, 14, 22, 13, 37, 26, 20, 44, 2, 45, 39, 15, 11, 24, 16, 25, 12, 28, 48, 7, 6, 42, 18, 21, 9, 30, 5, 31, 17, 38, 3, 8, 36, 10, 41, 29)(19, 23, 33).$

Clearly, $|q| = 43 \cdot 3$ and $u = q^{43} = (19, 23, 33)$. So $L$ contains a 3-cycle of $H$. Finally, since $(p, u)$ is transitive on $\Omega$, by Lemma 4.2, $(u, P) = A_{48}$. Since $u, p \in L$, we have $L = H$, that is, $(T, g) = H$.

Let $\Gamma = \Gamma(H, T, g)$ be the coset graph. Recall that $|T : T^8 \cap T| = 3$ and that $|H : H_2| = 48$. So $\Gamma$ is a cubic $(H, 5)$-arc transitive graph by Lemmas 2.3 and 2.1. Moreover, since $G$ acts regularly on $V\Gamma$, by Lemma 2.4, there exists some Cayley subset $S$ of $G$ such that $\Gamma \cong \text{Cay}(G, S)$.

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\section*{References}