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Existence theorems for the one-dimensional singular p-Laplacian equation with a nonlinear boundary condition ☆

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Abstract

In this paper, general existence theorems are presented for the singular equation

$$-(\varphi_p(y'))' = q(t)f(t,y), \quad 0<t<1,$$
$$y(0)=0, \quad \Psi(y(1)) + y'(1) = 0.$$

Throughout, our nonlinearity is allowed to change sign. The singularity may occur at $y=0$, $t=0$ and $t=1$. In addition, Ψ may be nonlinear.

Keywords: One-dimensional p-Laplacian; Positive solution; Upper and lower solution

1. Introduction

In this paper, we study the singular boundary value problem

$$-(\varphi_p(y'))' = q(t)f(t,y), \quad 0<t<1,$$
$$y(0)=0, \quad \Psi(y(1)) + y'(1) = 0, \tag{1.1}$$

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where $\varphi_p(s)=|s|^{p-2}s,\ \ p>1$. The singularity may occur at $y=0$, $t=0$ and $t=1$, and the function f is allowed to change sign. In addition, Ψ may be nonlinear. Note f may not be a Carathéodory function because of the singular behaviour of the y variable. For $p=2$, problem (1.1) was motivated by a singular problem arising in the theory of membrane response of a spherical cap [3,11], namely

$$-y''=\frac{t^2}{32y^2}-\frac{\lambda^2}{8},\quad 0<t<1,$$
$$y(0)=0,\quad 2y'(1)-(1+v)y(1)=0,\ \ 0<v<1 \text{ and } \lambda>0. \tag{1.2}$$

In [12], D. O'Regan has proved (1.2) has a solution $y\in C[0,1]\cap C^1(0,1]\cap C^2(0,1)$ with $y(t)>0$ for $t\in(0,1)$. For $p\neq 2$, equations of the form (1.1) occur in the study of the p-Laplace equation, non-Newtonial fluid theory, and the turbulent flow of a gas in a porous medium [10]. Several results on the existence of positive solutions for the one dimensional p-Laplacian boundary value problems have been established in the literature (see [1,2,5–16]). The key condition used is that the nonlinearity is nonnegative so the solution u is concave down; if the nonlinearity f is negative somewhere, then the solution u need no longer be concave down. As a result it is difficult to find positive solutions of the p-Laplacian equation when f changes sign. In [1,5,16], the Dirichlet problem has been studied when the nonlinearity is allowed to change sign. Motivated by [1,12], where the function f is allowed to change sign, we consider the p-Laplacian equation (1.1).

To date no paper has appeared in the literature which discusses the p-Laplacian singular boundary value problem when the boundary condition at one is nonlinear (or even of the form $ay'(1)+by(1)=0$, with $a\geqslant 0$, $b\leqslant 0$, $a^2+b^2>0$) and when the nonlinearity in the differential equation may change sign. This paper attempts to fill this gap in the literature and we present a very general upper and lower solution theory in Section 2 for this type of problem. Moreover in Section 3 we present easily verifiable (and new) criteria so that upper and lower solutions can be constructed. An example is given in Section 4 to show how easily the theory in Section 3 can be applied in practice. We finally note that very little is known concerning the computation of the solution to (1.1). However if one constructs "good" upper and lower solutions (as described in Section 2 and 3) the shooting method in [3, Section 5] to numerically compute the solution may be used for certain boundary value problems of the form (1.1).

2. General existence theorem

We assume throughout that $\Psi(x)\leqslant 0$ for $x\geqslant 0$. Our theory involves approximating (1.1) by a sequence of nonsingular problems (each of which has a lower solution ρ_n and an upper solution β_n). Using the Schauder fixed point theorem we establish the existence of a solution (which lies between the lower and upper solution) for each approximating problem. The Arzela–Ascoli theorem will then complete the proof.

Theorem 2.1. *Let $n_0\in\{3,4,\ldots\}$ be fixed and suppose the following conditions are satisfied*:

$$f:[0,1]\times(0,\infty)\to R \text{ is continuous}, \tag{2.1}$$

$$q\in C(0,1) \text{ with } q>0 \text{ on } (0,1) \text{ and } q\in L^1[0,1], \tag{2.2}$$

$$\Psi: R\to R \text{ is continuous and } \Psi(x)\leqslant 0 \text{ for } x\geqslant 0, \tag{2.3}$$

$$
\begin{aligned}
&|f(t,y)| \leqslant g(y)+h(y) \textit{ on } [0,1]\times(0,\infty) \textit{ with}\\
&g>0 \textit{ continuous and nonincreasing on } (0,\infty),\\
&h\geqslant 0 \textit{ continuous on } [0,\infty), \textit{ and } \frac{h}{g}\\
&\textit{nondecreasing on } (0,\infty),
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
&\textit{let } n\in\{n_0,n_0+1,\ldots\}\equiv N_0 \textit{ and associated with each } n\in N_0\\
&\textit{we have a constant } \rho_n \textit{ such that } \{\rho_n\} \textit{ is a nonincreasing}\\
&\textit{sequence with } \lim_{n\to\infty}\rho_n=0 \textit{ and such that for}\\
&\frac{1}{n}\leqslant t\leqslant 1 \textit{ we have } q(t)f(t,\rho_n)\geqslant 0,
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
&\exists\alpha\in C[0,1]\cap C^1(0,1],\ \varphi_p(\alpha')\in C^1(0,1),\\
&\alpha(0)=0,\ \alpha'(1)+\Psi(\alpha(1))<0,\\
&\alpha>0 \textit{ on } (0,1] \textit{ such that for each } n\in N_0\\
&\textit{we have } (\varphi_p(\alpha'))'+q(t)f(t,y)>0 \textit{ for}\\
&(t,y)\in\left[\frac{1}{n},1\right)\times\{y\in(0,\infty): y<\alpha(t)\}\\
&\textit{and } (\varphi_p(\alpha'))'+q(t)f\left(\frac{1}{n},y\right)>0 \textit{ for}\\
&(t,y)\in\left(0,\frac{1}{n}\right)\times\{y\in(0,\infty): y<\alpha(t)\},
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
&\textit{for any } R>0,\ \frac{1}{g} \textit{ is differentiable on } (0,R] \textit{ with } g'<0\\
&\textit{a.e. on } (0,R],\ \frac{|g'|^{1/p}}{g^{2/p}}\in L^1[0,R] \textit{ and } \int_0^\infty \frac{|g'(t)|^{1/p}}{(g(t))^{2/p}}\,\mathrm{d}t=\infty,
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
&\textit{for each } n\in N_0,\ \exists\beta_n\in C[0,1]\cap C^1(0,1],\ \varphi_p(\beta_n')\in C^1(0,1)\\
&\textit{with } \beta_n(t)\geqslant\rho_n \textit{ for } t\in[0,1],\ \beta_n'(1)+\Psi(\beta_n(1))>0,\\
&\beta_n(1)\geqslant\alpha(1) \textit{ and we have } (\varphi_p(\beta_n'))'+q(t)f(t,\beta_n(t))\leqslant 0\\
&\textit{for } t\in\left[\frac{1}{n},1\right),\ \textit{and } (\varphi_p(\beta_n'))'+q(t)f\left(\frac{1}{n},\beta_n(t)\right)\leqslant 0\\
&\textit{for } t\in\left(0,\frac{1}{n}\right)
\end{aligned}
\tag{2.8}
$$

and

$$\max\left\{\sup_{t\in[0,1]} \beta_n(t) : n \in N_0\right\} < \infty. \tag{2.9}$$

Then (1.1) *has a solution* $y \in C[0,1] \cap C^1(0,1]$, $\varphi_p(y') \in C^1(0,1)$ *with* $y(t) \geqslant \alpha(t)$ *for* $t \in [0,1]$.

Proof. Fix $n \in N_0$. Consider the boundary value problem

$$\begin{aligned} &-(\varphi_p(y'))' = q(t) f^*(t,y), \quad 0<t<1,\\ &y(0)=\rho_n, \quad y'(1)+\Psi^*(y(1))=0, \end{aligned} \tag{2.10^n}$$

where

$$f^*(t,y)=\begin{cases} f\left(\frac{1}{n},\beta_n(t)\right)+r(\beta_n(t)-y), & y\geqslant\beta_n(t) \text{ and } 0\leqslant t\leqslant\frac{1}{n}\\ f(t,\beta_n(t))+r(\beta_n(t)-y), & y\geqslant\beta_n(t) \text{ and } \frac{1}{n}\leqslant t\leqslant 1\\ f\left(\frac{1}{n},y\right), & \rho_n\leqslant y\leqslant\beta_n(t) \text{ and } 0\leqslant t\leqslant\frac{1}{n}\\ f(t,y), & \rho_n\leqslant y\leqslant\beta_n(t) \text{ and } \frac{1}{n}\leqslant t\leqslant 1\\ f(t,\rho_n)+r(\rho_n-y), & y<\rho_n \text{ and } \frac{1}{n}\leqslant t\leqslant 1\\ f\left(\frac{1}{n},\rho_n\right)+r(\rho_n-y), & y<\rho_n \text{ and } 0\leqslant t\leqslant\frac{1}{n} \end{cases}$$

with

$$\Psi^*(z)=\begin{cases} \psi(\beta_n(1)), & z>\beta_n(1),\\ \Psi(z), & \alpha(1)\leqslant z\leqslant\beta_n(1),\\ \Psi(\alpha(1)), & z<\alpha(1) \end{cases}$$

and $r : R \to [-1,1]$ is the radial retraction defined by

$$r(u)=\begin{cases} u, & |u|\leqslant 1,\\ \dfrac{u}{|u|}, & |u|>1. \end{cases} \qquad \square$$

Remark 2.1. Note $\Psi^*(z)\leqslant 0$ for $z \in R$.

Let

$$C_0[0,1]=\{u\in C[0,1] : u(0)=0\}$$

and

$$C^1_{\rho_n}[0,1]=\{u\in C^1 : u(0)=\rho_n\}.$$

Define the mappings $L_p, F : C^1_{\rho_n}[0,1] \to C_0[0,1]\times R$ by

$$L_p y(t) = (\varphi_p(y'(t)) - \varphi_p(y'(0)), -y'(1))$$

and

$$Fy(t) = \left(-\int_0^t q(x) f^*(x, y(x)\,\mathrm{d}x, \Psi^*(y(1))) \right).$$

Now F is continuous and compact (by the Arzela–Ascoli theorem). Also if $L_p y = (u, \gamma)$, with $u \in C_0[0,1]$ and $\gamma \in R$, then

$$y(t) = \rho_n + \int_0^t \varphi_p^{-1}(u(x) - \varphi_p(\gamma) - u(1))\,\mathrm{d}x,$$

so L_p^{-1} exists and is continuous. Solving $(2.10)^n$ is equivalent to finding a fixed point of $y = L_p^{-1} F y \equiv Ny$ where $N = L_p^{-1} F : C^1_{\rho_n}[0,1] \to C^1_{\rho_n}[0,1]$ is continuous and compact. Schauder's fixed point theorem guarantees that $(2.10)^n$ has a solution $y_n \in C^1[0,1]$ and $\varphi_p(y_n') \in C^1(0,1)$. First we show

$$y_n(t) \geqslant \rho_n \quad \text{for } t \in [0,1]. \tag{2.11}$$

If (2.11) is not true then $y_n - \rho_n$ has a negative absolute minimum at say $t_0 \in (0,1]$. If $t_0 \in (0,1)$, then $y_n'(t_0) = 0$ and $(\varphi_p(y_n'))'(t_0) \geqslant 0$. However

$$(\varphi_p(y_n'))'(t_0) = \begin{cases} -q(t_0)[f(t_0, \rho_n) + r(\rho_n - y_n(t_0))], & \text{if } \dfrac{1}{n} \leqslant t_0 < 1, \\ -q(t_0)\left[f\left(\dfrac{1}{n}, \rho_n\right) + r(\rho_n - y_n(t_0))\right], & \text{if } 0 \leqslant t_0 \leqslant \dfrac{1}{n}, \end{cases}$$

i.e., $(\varphi_p(y_n'))'(t_0) < 0$, a contradiction. It remains to discuss the case $t_0 = 1$. If $t_0 = 1$ there exists δ, $0 \leqslant \delta < 1$ with $\rho_n - y_n(t) > 0$ for $t \in (\delta, 1]$ and $\rho_n - y_n(\delta) = 0$. In addition for $t \in (\delta, 1)$ we have

$$(\varphi_p((\rho_n - y_n)'))' = q(t) f^*(t, y_n(t)) > 0,$$

so $\varphi_p((\rho_n - y_n)')$ is increasing. Since $\varphi_p : R \to R$ is increasing, we have $(\rho_n - y_n)'$ is increasing and so $\rho_n - y_n$ is convex on $(\delta, 1)$. Now [4, pp. 134] guarantees that

$$-y_n'(1) \geqslant \frac{[\rho_n - y_n(1)] - [\rho_n - y_n(\delta)]}{1 - \delta} \geqslant \rho_n - y_n(1)$$

and this together with Remark 2.1 yields

$$0 < \rho_n - y_n(1) \leqslant -y_n'(1) = \Psi^*(y_n(1)) \leqslant 0,$$

a contradiction. Thus (2.11) holds. Next we show

$$y_n(t) \leqslant \beta_n(t) \quad \text{for } t \in [0,1]. \tag{2.12}$$

If (2.12) is not true then $y_n - \beta_n$ would have a positive absolute maximum at say $t_0 \in (0,1]$. Suppose first that $t_0 \in (0,1)$. Then $y_n'(t_0) = \beta_n'(t_0)$.

We first prove that

$$(\varphi_p(y_n'))'(t_0) - (\varphi_p(\beta_n'))'(t_0) \leqslant 0. \tag{2.13}$$

Since $y_n - \beta_n$ have a positive absolute maximum at $t_0 \in (0,1)$, with $y_n'(t_0) - \beta_n'(t_0) = 0$ and there exists $\delta \in (0,1)$ with $y_n'(t) - \beta_n'(t) \geqslant 0$ for $t \in (t_0-\delta, t_0)$, namely $\varphi_p(y_n')(t) - \varphi_p(\beta_n')(t) \geqslant 0$ for $t \in (t_0-\delta, t_0)$. Then we have

$$\frac{(\varphi_p(y_n'(t)) - \varphi_p(\beta_n'(t))) - (\varphi_p(y_n'(t_0)) - \varphi_p(\beta_n'(t_0)))}{t-t_0} \leqslant 0 \quad \text{for } t \in (t_0-\delta, t_0),$$

so

$$\frac{\varphi_p(y_n'(t)) - \varphi_p(y_n'(t_0))}{t-t_0} \leqslant \frac{\varphi_p(\beta_n'(t)) - \varphi_p(\beta_n'(t_0))}{t-t_0} \quad \text{for } t \in (t_0-\delta, t_0).$$

Consequently,

$$\begin{aligned}(\varphi_p(y_n'))'(t_0) &= \lim_{t(\in(t_0-\delta,t_0))\to t_0^-} \frac{\varphi_p(y_n'(t)) - \varphi_p(y_n'(t_0))}{t-t_0} \\ &\leqslant \lim_{t(\in(t_0-\delta,t_0))\to t_0^-} \frac{\varphi_p(\beta_n'(t)) - \varphi_p(y_n'(t_0))}{t-t_0} \\ &= (\varphi_p(\beta_n'))'(t_0),\end{aligned}$$

i.e.,

$$(\varphi_p(y_n'))'(t_0) - (\varphi_p(\beta_n'))'(t_0) \leqslant 0.$$

We now consider two cases, namely $t_0 \in \left[\frac{1}{n}, 1\right)$ and $t_0 \in \left(0, \frac{1}{n}\right)$.

Case i: $t_0 \in \left[\frac{1}{n}, 1\right)$.

Then since $y_n(t_0) > \beta_n(t_0)$ we have, using (2.8), that

$$\begin{aligned}(\varphi_p(y_n'))'(t_0) - (\varphi_p(\beta_n'))'(t_0) &= -q(t_0)[f(t_0,\beta_n(t_0)) + r(\beta_n(t_0) - y_n(t_0))] - (\varphi_p(\beta_n'))'(t_0) \\ &= -[(\varphi_p(\beta_n'))'(t_0) + q(t_0)f(t_0,\beta_n(t_0))] - q(t_0)r(\beta_n(t_0) \\ &\quad - y_n(t_0)) > 0,\end{aligned}$$

a contradiction.

Case ii: $(t_0 \in \left(0, \frac{1}{n}\right)$.

Then (2.8) gives

$$\begin{aligned}(\varphi_p(y_n'))'(t_0) - (\varphi_p(\beta_n'))'(t_0) &= -q(t_0)\left[f\left(\frac{1}{n},\beta_n(t_0)\right) + r(\beta_n(t_0) - y_n(t_0))\right] - (\varphi_p(\beta_n'))'(t_0) \\ &= -\left[(\varphi_p(\beta_n'))'(t_0) + q(t_0)f\left(\frac{1}{n},\beta_n(t_0)\right)\right] - q(t_0)r(\beta_n(t_0) \\ &\quad - y_n(t_0)) > 0,\end{aligned}$$

a contradiction.

It remains to discuss the case $t_0 = 1$. If $t_0 = 1$ there exists δ, $0 \leqslant \delta < 1$ with $y_n(t) - \beta_n(t) > 0$ for $t \in (\delta, 1]$ and $y_n(\delta) - \beta_n(\delta) = 0$. Then $y_n'(\delta) \geqslant \beta_n'(\delta)$, and in addition, $(\varphi_p(y_n'))'(t) - (\varphi_p(\beta_n'))'(t) = -q(t)f^*(t, y_n(t)) - (\varphi_p(\beta_n'))'(t)$ for $t \in (\delta, 1)$. We claim

$$(\varphi_p(y_n'))'(t) \geqslant (\varphi_p(\beta_n'))'(t) \quad \text{for } t \in (\delta, 1). \tag{2.14}$$

If $\delta \geqslant \frac{1}{n}$ then (2.8) guarantees that (2.14) true. If $\delta < \frac{1}{n}$ then $(\varphi_p(y_n'))'(t) > (\varphi_p(\beta_n'))'(t)$, for $t \in \left[\frac{1}{n}, 1\right)$ by (2.8) and for $t \in \left(\delta, \frac{1}{n}\right)$ we have from (2.8) that

$$\begin{aligned}(\varphi_p(y_n'))'(t) - (\varphi_p(\beta_n'))'(t) = &-q(t)\left[f\left(\frac{1}{n}, \beta_n(t)\right) + r(\beta_n(t) - y_n(t))\right] - (\varphi_p(\beta_n'))'(t)\\ = &-\left[(\varphi_p(\beta_n'))'(t) + q(t)f\left(\frac{1}{n}, \beta_n(t)\right)\right] - q(t)r(\beta_n(t)\\ &- y_n(t)) > 0.\end{aligned}$$

Thus in all cases (2.14) holds. Integrate (2.14) from δ to 1 to obtain

$$\varphi_p(y_n'(1)) - \varphi_p(y_n'(\delta)) \geqslant \varphi_p(\beta_n'(1)) - \varphi_p(\beta_n'(\delta))$$

and so

$$\varphi_p(y_n'(1)) - \varphi_p(\beta_n'(1)) \geqslant \varphi_p(y_n'(\delta)) - \varphi_p(\beta_n'(\delta)) \geqslant 0.$$

This together with $\beta_n'(1) > -\Psi(\beta_n(1))$ gives

$$\begin{aligned}0 \leqslant y_n'(1) - \beta_n'(1) = &-\Psi^*(y_n(1)) - \beta_n'(1) < -\Psi^*(y_n(1)) + \Psi(\beta_n(1))\\ = &-\Psi(\beta_n(1)) + \Psi(\beta_n(1)) = 0,\end{aligned}$$

a contradiction. Thus (2.12) holds. In particular

$$y_n(t) \leqslant a_0 = \max\left\{\sup_{t\in[0,1]} \beta_n(t) : n \in N_0\right\} \quad \text{for } t \in [0,1].$$

Next we obtain a sharper lower bound on y_n, namely we will show

$$y_n(t) \geqslant \alpha(t) \quad \text{for } t \in [0,1]. \tag{2.15}$$

Suppose (2.15) is not true. Then $y_n - \alpha$ has a negative absolute minimum at say $t_1 \in (0,1]$. Suppose first that $t_1 \in (0,1)$. Then $y_n'(t_1) = \alpha'(t_1)$. As the proof of (2.13), we can prove that

$$(\varphi_p(y_n'))'(t_1) - (\varphi_p(\alpha'))'(t_1) \geqslant 0.$$

We now consider two cases, namely $t_1 \in \left[\frac{1}{n}, 1\right)$ and $t_1 \in \left(0, \frac{1}{n}\right)$.

Case i: $t_1 \in \left[\frac{1}{n}, 1\right)$.

Now $0 < y_n(t_1) < \alpha(t_1)$, $\rho_n \leqslant y_n(t_1) \leqslant \beta_n(t_1)$, and (2.6) implies

$$(\varphi_p(y_n'))'(t_1) - (\varphi_p(\alpha'))'(t_1) = -[q(t_1)f(t_1, y_n(t_1) + (\varphi_p(\alpha'))'(t_1))] < 0,$$

a contradiction.

Case ii: $t_1 \in \left(0, \frac{1}{n}\right)$.

In this case (2.6) also implies

$$(\varphi_p(y_n'))'(t_1) - (\varphi_p(\alpha'))'(t_1) = -\left[q(t_1)f\left(\frac{1}{n}, y_n(t_1)\right) + (\varphi_p(\alpha'))'(t_1)\right] < 0,$$

a contradiction.

It remains to discuss the case $t_1 = 1$. If $t_1 = 1$ there exists δ, $0 \leqslant \delta < 1$ with $\alpha(t) - y_n(t) > 0$ for $t \in (\delta, 1]$, $\alpha(\delta) = y_n(\delta) = 0$, and $\alpha'(\delta) \geqslant y_n'(\delta)$. We claim

$$(\varphi_p(\alpha'))'(t) > (\varphi_p(y_n'))'(t) \quad \text{for } t \in (\delta, 1). \tag{2.16}$$

If $\delta \geqslant \frac{1}{n}$ then for $t \in (\delta, 1)$ we have

$$\begin{aligned}(\varphi_p(\alpha'))'(t) - (\varphi_p(y_n'))'(t) &= q(t) f^*(t, y_n(t)) + (\varphi_p(\alpha'))'(t) \\ &= q(t) f(t, y_n(t)) + (\varphi_p(\alpha'))'(t) > 0,\end{aligned}$$

so (2.16) is true in this case. If $\delta < \frac{1}{n}$ then for $t \in \left[\frac{1}{n}, 1\right)$ we have

$$(\varphi_p(\alpha'))'(t) - (\varphi_p(y_n'))'(t) = q(t) f(t, y_n(t)) + (\varphi_p(\alpha'))'(t) > 0,$$

whereas for $t \in \left(0, \frac{1}{n}\right)$ we have

$$\begin{aligned}(\varphi_p(\alpha'))'(t) - (\varphi_p(y_n'))'(t) &= q(t) f^*(t, y_n(t)) + (\varphi_p(\alpha'))'(t) \\ &= q(t) f\left(\frac{1}{n}, y_n(t)\right) + (\varphi_p(\alpha'))'(t) > 0.\end{aligned}$$

Thus (2.16) is also true in this case. Consequently (2.16) holds. Integrate (2.16) from δ to 1 to obtain

$$\varphi_p(\alpha'(1)) - \varphi_p(\alpha'(\delta)) \geqslant \varphi_p(y_n'(1)) - \varphi_p(y_n'(\delta)).$$

Thus

$$\varphi_p(\alpha'(1)) - \varphi_p(y_n'(1)) \geqslant \varphi_p(\alpha'(\delta)) - \varphi_p(y_n'(\delta)) \geqslant 0.$$

This together with $\alpha'(1) < -\Psi(\alpha(1))$ gives

$$\begin{aligned}0 &\leqslant \alpha'(1) - y_n'(1) < -\Psi(\alpha(1)) + \Psi^*(\alpha(1)) \\ &\leqslant -\Psi(\alpha(1)) + \Psi(\alpha(1)) = 0,\end{aligned}$$

a contradiction. Thus (2.15) holds.

Remark 2.2. It is possible to check

$$\alpha(t) \leqslant \beta_n(t) \quad \text{for } t \in [0, 1]. \tag{2.17}$$

If this is not true then $\alpha - \beta_n$ would have a positive absolute maximum at say $t_1 \in (0, 1)$ (note $\beta_n(1) \geqslant \alpha(1)$). Then $\alpha'(t_1) = \beta_n'(t_1)$ and there exist $\tau_0, \tau_1 \in [0, 1]$ with $t_1 \in (\tau_0, \tau_1)$ and

$$\alpha(\tau_0) - \beta_n(\tau_0) = \alpha(\tau_1) - \beta_n(\tau_1) = 0 \quad \text{and} \quad \beta_n(t) - \alpha(t) < 0, \; t \in (\tau_0, \tau_1).$$

We can claim

$$(\varphi_p(\beta_n'(t)))' - (\varphi_p(\alpha'(t)))' \leqslant 0 \quad \text{for a.e. } t \in (\tau_0, \tau_1). \tag{2.18}$$

We first show that if (2.18) is true then (2.17) will follow. Let

$$w(t) = \beta_n(t) - \alpha(t) < 0, \quad \text{for } t \in (\tau_0, \tau_1).$$

Then

$$\int_{\tau_0}^{\tau_1} ((\varphi_p(\beta_n'(t)))' - (\varphi_p(\alpha'(t)))')w(t)\,\mathrm{d}t \geqslant 0.$$

On the other hand, using the inequality

$$(\varphi_p(b) - \varphi_p(a))(b-a) \geqslant 0, \quad \text{for } a, b \in R$$

and the fact that there exists $\tau^* \in (\tau_0, \tau_1)$ with $\alpha'(\tau^*) \neq \beta_n'(\tau^*)$ we have

$$\int_{\tau_0}^{\tau_1} ((\varphi_p(\beta_n'(t)))' - (\varphi_p(\alpha'(t)))')w(t)\,\mathrm{d}t = -\int_{\tau_0}^{\tau_1} (\varphi_p(\beta_n'(t)) - \varphi_p(\alpha'(t)))(\beta_n'(t) - \alpha'(t))\,\mathrm{d}t$$
$$< 0,$$

a contradiction. It remains to show (2.18) is true. Now $\alpha(t) > \beta_n(t)$, for $t \in (\tau_0, \tau_1)$ and (2.6) gives

$$q(t)f(t, \beta_n(t)) + (\varphi_p(\alpha'))'(t) > 0 \quad \text{if } t \in (\tau_0, \tau_1) \cap \left[\frac{1}{n}, 1\right)$$

and

$$q(t)f\left(\frac{1}{n}, \beta_n(t)\right) + (\varphi_p(\alpha'))'(t) > 0 \quad \text{if } t_1 \in (\tau_0, \tau_1) \cap \left(0, \frac{1}{n}\right).$$

Now if $t \in (\tau_0, \tau_1) \cap \left[\frac{1}{n}, 1\right)$ then (2.8) implies

$$(\varphi_p(\alpha'))'(t) - (\varphi_p(\beta_n'))'(t) \geqslant (\varphi_p(\alpha'))'(t) + q(t)f(t, \beta_n(t)) > 0.$$

Next if $t \in (\tau_0, \tau_1) \cap \left(0, \frac{1}{n}\right)$ then (2.8) implies

$$(\varphi_p(\alpha'))'(t) - (\varphi_p(\beta_n'))'(t) \geqslant (\varphi_p(\alpha'))'(t) + q(t)f\left(\frac{1}{n}, \beta_n(t)\right) > 0.$$

Thus (2.18) holds.

We now show

$$\{y_n\}_{n \in N_0} \text{ is a bounded, equicontinuous family on } [0, 1]. \tag{2.19}$$

To see (2.19) first notice from (2.4), (2.11) and (2.12) that

$$|f^*(t, y_n(t))| \leqslant g(y_n(t)) + h(y_n(t))$$
$$\leqslant g(y_n(t))\left\{1 + \frac{h(a_0)}{g(a_0)}\right\} \quad \text{for } t \in (0, 1)$$

and so

$$-(\varphi_p(y_n'))' \leqslant q(t)g(y_n(t))\left\{1 + \frac{h(a_0)}{g(a_0)}\right\} \quad \text{for } t \in (0, 1). \tag{2.20}$$

Also $y_n'(1) = -\Psi^*(y_n(1))$ gives

$$|y_n'(1)| \leqslant \sup_{s \in [\alpha(1), a_0]} |\Psi(z)| \equiv K_0.$$

Divide (2.20) by $g(y_n(t))$ and integrate from 0 to 1 to obtain

$$\frac{-\varphi_p(y_n'(1))}{g(y_n(1))} + \frac{\varphi_p(y_n'(0))}{g(\rho_n)} + \int_0^1 \left\{ \frac{-g'(y_n(x))}{g^2(y_n(x))} \right\} |y_n'(x)|^p \, dx \leqslant \left\{ 1 + \frac{h(a_0)}{g(a_0)} \right\} \int_0^1 q(s) \, ds.$$

Then since $y_n'(0) \geqslant 0$ (note $y_n(0) = \rho_n$ and $y_n(t) \geqslant \rho_n$ for $t \in [0, 1]$) we have

$$\int_0^1 \left\{ \frac{-g'(y_n(x))}{g^2(y_n(x))} \right\} |y_n'(x)|^p \, dx \leqslant K_1,$$

where

$$K_1 = \frac{\varphi_p(K_0)}{g(a_0)} + \left\{ 1 + \frac{h(a_0)}{g(a_0)} \right\} \int_0^1 q(s) \, ds.$$

Now consider

$$I(z) = \int_0^z \frac{[-g'(u)]^{1/p}}{[g(u)]^{2/p}} \, du.$$

For $t, s \in [0, 1]$ we have

$$\begin{aligned} |I(y_n(t)) - I(y_n(s))| &= \left| \int_s^t \frac{[-g'(y_n(x))]^{1/p}}{[g(y_n(x))]^{2/p}} y_n'(x) \, dx \right| \\ &\leqslant |t-s|^{1/q} \left(\int_s^t \left\{ \frac{-g'(y_n(x))}{g^2(y_n(x))} \right\} |y_n'(x)|^p \, dx \right)^{1/p} \\ &\leqslant |t-s|^{1/q} K_1^{1/p}, \end{aligned}$$

where $p + q = pq$. It follows from the above inequality, the uniform continuity of I^{-1} of $[0, I(a_0)]$ and

$$|y_n(t) - y_n(s)| = |I^{-1}(I(y_n(t))) - I^{-1}(I(y_n(s)))|$$

that $\{y_n\}_{n \in N_0}$ is equicontinuous on $[0, 1]$. Thus (2.19) holds. The Arzela–Ascoli Theorem guarantees the existence of a subsequence N_1 of N_0 and a function $y \in C[0, 1]$ with y_n converging uniformly on $[0, 1]$ to y as $n \to \infty$ through N_1. Also $y(0) = 0$ and $\alpha(t) \leqslant y(t) \leqslant a_0$ for $t \in [0, 1]$. Fix $t \in (0, 1)$ and $m \in \{n_0, n_0 + 1, \ldots\}$ be such that $\frac{1}{m} < t < 1$. Define the operator $L : C\left[\frac{1}{m}, 1\right] \to C\left[\frac{1}{m}, 1\right]$ by

$$(Lu)(t) = u(1) - \int_t^1 \varphi_p^{-1} \left(-\varphi_p(\Psi^*(u(1))) + \int_x^1 q(s) f(s, u(s)) \, ds \right) dx.$$

As in the proof of Theorem 2.2 [10], $L : C\left[\frac{1}{m}, 1\right] \to C\left[\frac{1}{m}, 1\right]$ is continuous. Let $N_1^* = \{n \in N_1 : n \geqslant m\}$. Now y_n, $n \in N_1^*$, satisfies

$$\begin{aligned} y_n(t) &= y_n(1) - \int_t^1 \varphi_p^{-1} \left(-\varphi_p(\Psi^*(y_n(1))) + \int_x^1 q(s) f(s, y_n(s)) \, ds \right) dx \\ &= y_n(1) - \int_t^1 \varphi_p^{-1} \left(-\varphi_p(\Psi(y_n(1))) + \int_x^1 q(s) f(s, y_n(s)) \, ds \right) dx \end{aligned}$$

since $\alpha(t) \leqslant y_n(t) \leqslant \beta_n(t)$ and $t > \frac{1}{m}$. Let $n \to \infty$ through N_1^* to obtain

$$y(t) = y(1) - \int_t^1 \varphi_p^{-1}\left(-\varphi_p(\Psi(y(1))) + \int_x^1 q(s)f(s, y(s))\,\mathrm{d}s\right)\mathrm{d}x.$$

We can do this argument for each $t \in (0,1)$, so $-(\varphi_p(y'))' = q(t)f(t,y)$ for $t \in (0,1)$, $y \in C^1(0,1]$ and $\varphi_p(y') \in C^1(0,1)$ and $y'(1) = -\Psi(y(1))$.

Remark 2.3. Condition (2.7) can be removed in the statement in Theorem 2.1 if β_n satisfies the following condition:

$$\text{for each } t \in [0,1] \text{ we have that } \{\beta_n(t)\}_{n \in N_0} \text{ is a nonincreasing sequence and } \lim_{n\to\infty} \beta_n(0) = 0. \tag{2.21}$$

To see this fix $n \in N_0$. We obtain as in Theorem 2.1 a solution y_n to $(2.10)^n$ with (2.11), (2.12) and (2.15) holding. Look at the interval $\left[\frac{1}{n_0}, 1\right]$. Let (recall $\alpha > 0$ on $(0,1]$)

$$R_{n_0} = \sup\left\{|f(x,y)| : x \in \left[\frac{1}{n_0}, 1\right] \text{ and } \alpha(x) \leqslant y \leqslant a_0\right\}.$$

Now since $y_n'(1) = -\Psi^*(y_n(1))$ we have $|y_n'(1)| \leqslant \sup_{z\in[\alpha(1),a_0]}|\Psi(z)| = K_0$ and so

$$\begin{aligned}|y'(t)| &\leqslant \varphi_p^{-1}\left(\varphi_p(K_0) + \left|\int_t^1 (\varphi_p(y_n'(x)))'\,\mathrm{d}x\right|\right)\\ &\leqslant \varphi_p^{-1}\left(\varphi_p(K_0) + R_{n_0}\int_0^1 q(x)\,\mathrm{d}x\right) \quad \text{for } t \in \left[\frac{1}{n_0}, 1\right].\end{aligned}$$

As a result

$$\{y_n\}_{n\in N_0} \text{ is a bounded, equicontinuous family on } \left[\frac{1}{n_0}, 1\right].$$

The Arzela–Ascoli theorem guarantees the existence of a subsequence N_{n_0} of N_0 and a function $z_{n_0} \in C\left[\frac{1}{n_0}, 1\right]$ with y_n converging uniformly on $\left[\frac{1}{n_0}, 1\right]$ to z_{n_0} as $n \to \infty$ through N_{n_0}. Proceed inductively to obtain subsequences of integers

$$N_{n_0} \supseteq N_{n_0+1} \supseteq \cdots \supseteq N_k \supseteq \cdots$$

and functions $z_k \in C\left[\frac{1}{k}, 1\right]$ with

$$y_n \text{ converging uniformly on } \left[\frac{1}{k}, 1\right] \text{ to } z_k \text{ as } n \to \infty \text{ through } N_k$$

and $z_{k+1} = z_k$ on $\left[\frac{1}{k}, 1\right]$. Define a function $y : [0,1] \to [0,\infty)$ by $y(x) = z_k(x)$ on $\left[\frac{1}{k}, 1\right]$ and $y(0) = 0$. Notice y is well defined and $\alpha(t) \leqslant y(t) \leqslant a_0$ for $t \in (0,1]$. Next fix $t \in (0,1)$ and let $m \in \{n_0, n_0+1, \ldots\}$

be such that $\frac{1}{m} < t < 1$. Let $N_m^+ = \{n \in N_m : n \geqslant m\}$. Now y_n, $n \in N_m^*$, satisfies

$$y_n(t) = y_n(1) - \int_t^1 \varphi_p^{-1}\left(-\varphi_p(\Psi(y_n(1))) + \int_x^1 q(s) f(s, y_n(s))\,\mathrm{d}s\right)\mathrm{d}x.$$

Let $n \to \infty$ through N_m^* to obtain (note $z_m(s) = y(s)$ for $s \in [t, 1]$),

$$y(t) = y(1) - \int_t^1 \varphi_p^{-1}\left(-\varphi_p(\Psi(y(1))) + \int_x^1 q(s) f(s, y(s))\,\mathrm{d}s\right)\mathrm{d}x.$$

We can do this argument for each $t \in (0, 1)$, so $-(\varphi_p(y'))' = q(t) f(t, y(t))$ for $t \in (0, 1)$ and $y'(1) = -\Psi(y(1))$. It remains to show y is continuous at 0. Let $\varepsilon > 0$ be given. Now since $\lim_{n\to\infty} \beta_n(0) = 0$ there exists $n_1 \in N_0$ with $\beta_{n_1}(0) < \frac{\varepsilon}{2}$. Next since $\beta_{n_1} \in C[0, 1]$ there exists $\delta_{n_1} > 0$ with $\beta_{n_1}(t) < \frac{\varepsilon}{2}$ for $t \in [0, \delta_{n_1}]$. Now for $n \geqslant n_1$ we have, since $\{\beta_n(t)\}_{n\in N_0}$ is nonincreasing for each $t \in [0, 1]$,

$$\beta_n(t) \leqslant \beta_{n_1}(t) < \frac{\varepsilon}{2} \quad \text{for } t \in [0, \delta_{n_1}].$$

This together with $\alpha(t) \leqslant y_n(t) \leqslant \beta_n(t)$ for $t \in (0, 1)$ implies that for $n \geqslant n_1$ that $\alpha(t) \leqslant y_n(t) < \frac{\varepsilon}{2}$ for $t \in [0, \delta_{n_1}]$. Consequently $0 \leqslant \alpha(t) \leqslant y(t) \leqslant \frac{\varepsilon}{2} < \varepsilon$ for $t \in (0, \delta_{n_1}]$, so y is continuous at 0.

Remark 2.4. If in (2.5) we replace $\frac{1}{n} \leqslant t < 1$ with $0 \leqslant t \leqslant 1 - \frac{1}{n}$ then one would replace (2.6), (2.8) with

$$\begin{aligned}
&\exists \alpha \in C[0, 1] \cap C^1(0, 1], \varphi_p(\alpha') \in C^1(0, 1),\\
&\alpha(0) = 0, \alpha'(1) + \Psi(\alpha(1)) < 0,\\
&\alpha > 0 \text{ on } (0, 1] \text{ such that for each } n \in N_0\\
&\text{ we have } (\varphi_p(\alpha'))' + q(t) f(t, y) > 0 \text{ for}\\
&(t, y) \in \left(0, 1 - \frac{1}{n}\right] \times \{y \in (0, \infty) : y < \alpha(t)\}\\
&\text{ and } (\varphi_p(\alpha'))' + q(t) f\left(1 - \frac{1}{n}, y\right) > 0 \text{ for}\\
&(t, y) \in \left(1 - \frac{1}{n}, 1\right) \times \{y \in (0, \infty) : y < \alpha(t)\},
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
&\text{for each } n \in N_0, \exists \beta_n \in C[0, 1] \cap C^1(0, 1), \varphi_p(\beta_n') \in C^1(0, 1)\\
&\text{with } \beta_n(t) \geqslant \rho_n \text{ for } t \in [0, 1], \beta_n'(1) + \Psi(\beta_n(1)) > 0,\\
&\beta_n(1) \geqslant \alpha(1) \text{ and we have } (\varphi_p(\beta_n'))' + q(t) f(t, \beta_n(t)) \leqslant 0\\
&\text{for } t \in \left(0, 1 - \frac{1}{n}\right] \text{ and } (\varphi_p(\beta_n'))' + q(t) f\left(1 - \frac{1}{n}, \beta_n(t)\right) \leqslant 0\\
&\text{for } t \in \left(1 - \frac{1}{n}, 1\right).
\end{aligned} \tag{2.23}$$

If in (2.5) we replace $\frac{1}{n} \leqslant t \leqslant 1$ with $\frac{1}{n} \leqslant t \leqslant 1 - \frac{1}{n}$ then essentially the same reasoning as in Theorem 2.1 establishes the following result.

Theorem 2.2. *Let $n_0 \in \{3, 4, \ldots\}$ be fixed and suppose* (2.1)–(2.4) *and* (2.7) *hold. In addition assume the following conditions are satisfied*:

$$\begin{aligned}
&\textit{let } n \in \{n_0, n_0+1, \ldots\} \equiv N_0 \textit{ and associated with each } n \in N_0\\
&\textit{we have a constant } \rho_n \textit{ such that } \{\rho_n\} \textit{ is a nonincreasing}\\
&\textit{sequence with } \lim_{n\to\infty} \rho_n = 0 \textit{ and such that for}\\
&\frac{1}{n} \leqslant t \leqslant 1 - \frac{1}{n} \textit{ we have } q(t)f(t, \rho_n) \geqslant 0,
\end{aligned} \tag{2.24}$$

$$\begin{aligned}
&\exists \alpha \in C[0,1] \cap C^1(0,1],\ \varphi_p(\alpha') \in C^1(0,1),\\
&\alpha(0) = 0,\ \alpha'(1) + \Psi(\alpha(1)) < 0,\\
&\alpha > 0 \textit{ on } (0,1] \textit{ such that for each } n \in N_0\\
&\textit{we have } (\varphi_p(\alpha'))' + q(t)f(t,y) > 0 \textit{ for}\\
&(t,y) \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \times \{y \in (0,\infty) : y < \alpha(t)\}\\
&\textit{and } (\varphi_p(\alpha'))' + q(t)f\left(\frac{1}{n}, y\right) > 0 \textit{ for}\\
&(t,y) \in \left(0, \frac{1}{n}\right) \times \{y \in (0,\infty) : y < \alpha(t)\}\\
&\textit{and } (\varphi_p(\alpha'))' + q(t)f\left(1 - \frac{1}{n}, y\right) > 0 \textit{ for}\\
&(t,y) \in \left(1 - \frac{1}{n}, 1\right) \times \{y \in (0,\infty) : y < \alpha(t)\},
\end{aligned} \tag{2.25}$$

$$\begin{aligned}
&\textit{for each } n \in N_0, \exists \beta_n \in C[0,1] \cap C^1(0,1],\ \varphi_p(\beta_n') \in C^1(0,1)\\
&\textit{with } \beta_n(t) \geqslant \rho_n \textit{ for } t \in [0,1],\ \beta_n'(1) + \Psi(\beta_n(1)) > 0,\\
&\beta_n(1) \geqslant \alpha(1) \textit{ and we have } (\varphi_p(\beta_n'))' + q(t)f(t, \beta_n(t)) \leqslant 0\\
&\textit{for } t \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \textit{ and } (\varphi_p(\beta_n'))' + q(t)f\left(\frac{1}{n}, \beta_n(t)\right) \leqslant 0\\
&\textit{for } t \in \left(0, \frac{1}{n}\right) \textit{ and } (\varphi_p(\beta_n'))' + q(t)f\left(1 - \frac{1}{n}, \beta_n(t)\right) \leqslant 0\\
&\textit{for } t \in \left(1 - \frac{1}{n}, 1\right)
\end{aligned} \tag{2.26}$$

and

$$\max\left\{\sup_{t\in[0,1]} \beta_n(t) : n \in N_0\right\} < \infty. \tag{2.27}$$

Then (1.1) *has a solution* $y \in C[0,1] \cap C^1(0,1]$, $\varphi_p(y') \in C^1(0,1)$ *with* $y(t) \geqslant \alpha(t)$ *for* $t \in [0,1]$.

Remark 2.5. If in (2.5), $\frac{1}{n} \leqslant t \leqslant 1$ is replaced by $0 \leqslant t \leqslant 1$ then it is easy to see that (2.9) is not needed in the statement in Theorem 2.1 provided in (2.8) we assume $(\varphi_p(\beta_n'))' + q(t)f(t,\beta_n(t)) \leqslant 0$ for $t \in (0,1)$.

3. Construction of α and β_n

In this section, we discuss how to construct the lower solution α in (2.6) or (2.22) or (2.25) and the upper solution β_n in (2.8) or (2.23) or (2.26).

Lemma 3.1. *Let* $e_n = \left[\frac{1}{n+1}, 1\right]$ $(n \geqslant 1)$, $e_0 = \emptyset$. *If there exist a sequence* $\{\varepsilon_n\} \downarrow 0$ *and* $\varepsilon_n > 0$ *for* $n \geqslant 1$, *then there exist a function* $\lambda \in C^1[0,1]$ *such that*

(1) $\varphi_p(\lambda') \in C^1[0,1]$ *and* $\max_{0\leqslant t \leqslant 1} |(\varphi_p(\lambda'(t)))'| > 0$;
(2) $\lambda(0) = 0$, $\lambda'(1) + \Psi(\lambda(1)) < 0$ *and* $0 < \lambda(t) \leqslant \varepsilon_n$, $t \in e_n \backslash e_{n-1}$, $n \geqslant 1$.

Proof. Let $r : [0,1] \to [0,+\infty)$ such that $r(0)=0$, $r(t)=\varepsilon_n^{p-1}$, $t \in e_n\backslash e_{n-1}$, $n \geqslant 1$. Let $u(t)=\int_0^t r(s)\,ds$, $v(t) = \left[\int_0^t u(s)\,ds\right]^{1/(p-1)}$, $w(t) = \int_0^t v(s)\,ds$. Then $u, v, w : \left[0, \frac{3}{4}\right] \to [0,+\infty)$ are continuous and increasing functions with $w\left(\frac{3}{4}\right) < \varepsilon_1$.

Let

$$a(t) = \left[c_0\left(\frac{7}{8} - t\right)^2 + c_1\left(\frac{7}{8} - t\right)\right]^{1/(p-1)},$$

where

$$c_0 = -8u(3/4) - 64(v(3/4))^{p-1}$$

and

$$c_1 = u(3/4) + 16(v(3/4))^{p-1}.$$

Let

$$b(t) = \int_{\frac{3}{4}}^t a(s)\,ds + w\left(\frac{3}{4}\right) \quad \text{for } t \in \left[\frac{3}{4}, \frac{7}{8}\right]$$

and

$$P(t) = \begin{cases} b(t) & \text{for } t \in \left[\frac{3}{4}, \frac{7}{8}\right], \\ b\left(\frac{7}{4} - t\right) & \text{for } t \in \left[\frac{7}{8}, 1\right]. \end{cases}$$

Define $\lambda : [0,1] \to [0,+\infty)$ with

$$\lambda(t) = \begin{cases} w(t), & 0 \leqslant t \leqslant \frac{3}{4}, \\ P(t), & \frac{3}{4} \leqslant t \leqslant 1. \end{cases}$$

We prove that λ satisfies the condition.

We easily prove that $w\left(\frac{3}{4}\right) = P\left(\frac{3}{4}\right)$, $w'\left(\frac{3}{4}\right) = P'\left(\frac{3}{4}\right)$, $(\varphi_p(w'))'\left(\frac{3}{4}\right) = (\varphi_p(P'))'\left(\frac{3}{4}\right)$ and $w \in C^1\left[0, \frac{3}{4}\right]$, $P \in C^1\left[\frac{3}{4}, 1\right]$, $\varphi_p(w') \in C^1\left[0, \frac{3}{4}\right]$, $\varphi_p(P') \in C^1\left[\frac{3}{4}, 1\right]$. As a result we have $\lambda \in C^1[0,1]$, $\varphi_p(\lambda') \in C^1[0,1]$ and $\max_{0 \leqslant t \leqslant 1} |(\varphi_p(\lambda'(t)))'| > 0$.Since $w(t) > 0$ for $t \in \left(0, \frac{3}{4}\right]$ and $P(t) > 0$ for $t \in \left[\frac{3}{4}, 1\right]$, we have $0 < \lambda(t)$ for $t \in (0,1]$. On the other hand,

$$\begin{aligned} \lambda\left(\frac{7}{8}\right) &= \int_{\frac{3}{4}}^{t} a(s)\,\mathrm{d}s + w\left(\frac{3}{4}\right) \leqslant \frac{1}{8} \max_{t \in \left[\frac{3}{4}, \frac{7}{8}\right]} \left[c_0\left(\frac{7}{8} - t\right)^2 + c_1\left(\frac{7}{8} - t\right)\right]^{1/(p-1)} + w\left(\frac{3}{4}\right) \\ &\leqslant \frac{1}{8}\left[\frac{c_0}{64} + \frac{c_1}{8}\right]^{1/(p-1)} + w\left(\frac{3}{4}\right) \\ &= \frac{1}{8}\left[\frac{-8u(3/4) - 64(v(3/4))^{p-1}}{64} + \frac{u(3/4) + 16(v(3/4))^{p-1}}{8}\right]^{1/(p-1)} + w\left(\frac{3}{4}\right) \\ &\leqslant \frac{v(3/4)}{8} + w\left(\frac{3}{4}\right) \end{aligned}$$

and

$$u\left(\frac{3}{4}\right) = \int_0^{\frac{3}{4}} r(s)\,\mathrm{d}s \leqslant \frac{3\varepsilon_1^{p-1}}{4},$$

$$\begin{aligned} v\left(\frac{3}{4}\right) &= \left[\int_0^{\frac{3}{4}} u(s)\,\mathrm{d}s\right]^{1/(p-1)} \\ &\leqslant \left[\frac{3\varepsilon_1^{p-1}}{4} \times \frac{3}{4}\right]^{1/(p-1)} = \left(\frac{9}{16}\right)^{1/(p-1)} \varepsilon_1, \end{aligned}$$

$$w\left(\frac{3}{4}\right) \leqslant \left(\frac{9}{16}\right)^{1/(p-1)} \varepsilon_1 \times \frac{3}{4}.$$

Thus

$$\begin{aligned} \lambda\left(\frac{7}{8}\right) &\leqslant \frac{v(3/4)}{8} + w\left(\frac{3}{4}\right) \\ &\leqslant \frac{1}{8} \times \left(\frac{9}{16}\right)^{1/(p-1)} \varepsilon_1 + \left(\frac{9}{16}\right)^{1/(p-1)} \varepsilon_1 \times \frac{3}{4} \\ &= \frac{7}{8} \times \left(\frac{9}{16}\right)^{1/(p-1)} \varepsilon_1 < \varepsilon_1, \end{aligned}$$

so,

$$\lambda(t) \leqslant \varepsilon_n, \quad t \in e_n \backslash e_{n-1}, \quad n \geqslant 1.$$

Now using

$$\begin{aligned}\lambda'(1) &= -P'\left(\frac{3}{4}\right) \\ &= -\left[\frac{-8u(3/4) - 64(v(3/4))^{p-1}}{64} + \frac{u(3/4) + 16(v(3/4))^{p-1}}{8}\right]^{1/(p-1)} \\ &= -v(3/4) < 0,\end{aligned}$$

we have

$$\lambda(0) = 0, \quad \lambda'(1) + \Psi(\lambda(1)) < 0.$$

Thus λ satisfies all the conditions of Lemma 3.1.

Next we discuss how to construct the lower solution α in (2.6) or (2.22) or (2.25). We begin our discussion with (2.6). Suppose the following conditions are satisfied:

$$\begin{aligned}&\text{let } n \in \{n_0, n_0+1, \ldots\} \text{ and associated with each } n \text{ we} \\ &\text{have a constant } \rho_n \text{ such that } \{\rho_n\} \text{ is a decreasing} \\ &\text{sequence with } \lim_{n\to\infty} \rho_n = 0 \text{ and there exists a} \\ &\text{constant } k_0 > 0 \text{ such that for } \frac{1}{n} \leqslant t \leqslant 1, \\ &\text{and } 0 < y \leqslant \rho_n \text{ we have } q(t) f(t, y) \geqslant k_0,\end{aligned} \tag{3.1}$$

$$f(\cdot, y) \text{ is nondecreasing on } \left(0, \frac{1}{3}\right) \text{ for each fixed } y \in (0, \infty), \tag{3.2}$$

$$\begin{aligned}&\text{for each } n \in N_0, \exists \beta_n \in C[0,1] \cap C^1(0,1], \varphi_p(\beta_n') \in C^1(0,1), \\ &\text{with } \beta_n(t) \geqslant \rho_n \text{ for } t \in [0,1], \beta_n'(1) + \Psi(\beta_n(1)) > 0 \text{ and } \beta_n(1) \geqslant \rho_1 \\ &\text{and we have } (\varphi_p(\beta_n'))' + q(t) f(t, \beta_n(t)) \leqslant 0 \\ &\text{for } t \in \left[\frac{1}{n}, 1\right), \text{ and } (\varphi_p(\beta_n'))' + q(t) f\left(\frac{1}{n}, \beta_n(t)\right) \leqslant 0 \\ &\text{for } t \in \left(0, \frac{1}{n}\right). \qquad \square\end{aligned} \tag{3.3}$$

Theorem 3.1. *Let* $n_0 \in \{3, 4, \ldots\}$ *be fixed and suppose* (2.1)–(2.4), (2.7), (3.1)–(3.3) *and* (2.9) *hold. Then* (1.1) *has a solution* $y \in C[0,1] \cap C^1(0,1]$ *with* $\varphi_p(y') \in C^1(0,1)$ *with* $y(t) > 0$, $t \in (0,1]$.

Proof. The result follows from Theorem 2.1, once we prove (2.5), (2.6) and (2.8) hold. Now (3.1) guarantees that (2.5) holds. From Lemma 3.1, there exist a function $\lambda \in C^1[0,1]$ such that

(1) $\varphi_p(\lambda') \in C^1[0,1]$ and $R_1 = \max_{0\leqslant t\leqslant 1}|(\varphi_p(\lambda'(t)))'| > 0$;
(2) $\lambda(0)=0$, $\lambda'(1)+\Psi(\lambda(1))<0$ and $0<\lambda(t)\leqslant \rho_n$, $t\in e_n\backslash e_{n-1}$, $n\geqslant 1$.

Let $m = \min\left\{1, \left(\frac{k_0}{2R}\right)^{1/(p-1)}, \frac{\rho_1}{|\lambda|_\infty}\right\}$, here k_0 is as in (3.1) and ρ_1 is as in (3.1). Let

$$\alpha(t) = m\lambda(t) \quad \text{for } t\in[0,1].$$

Then $\alpha \in C[0,1]\cap C^1(0,1]$, $\varphi_p(\alpha')\in C^1(0,1)$ $\alpha(0)=0$, $\alpha'(1)+\Psi(\alpha(1))<0$ and $0<\alpha(t)\leqslant\lambda(t)$ for $t\in(0,1]$. For each $n\in N_0$ and $(t,y)\in\left[\frac{1}{n},1\right)\times\{y\in(0,\infty): y<\alpha(t)\}$ we have

$$\begin{aligned} q(t)f(t,y) + (\varphi_p(\alpha'(t)))' &\geqslant k_0 + (\varphi_p(m\lambda'(t)))' \\ &= k_0 + m^{p-1}(\varphi_p(\lambda'(t)))' \\ &\geqslant k_0 - m^{p-1}|(\varphi_p(\lambda'(t)))'| \\ &\geqslant k_0 - \left(\frac{k_0}{2R_1}\right)\cdot|(\varphi_p(\lambda'(t)))'| \\ &\geqslant k_0 - \left(\frac{k_0}{2R_1}\right)\cdot \max_{0\leqslant t\leqslant 1}|(\varphi_p(\lambda'(t)))'| \\ &= \frac{k_0}{2} > 0. \end{aligned}$$

For each $n\in N_0$ and $(t,y)\in\left(0,\frac{1}{n}\right)\times\{y\in(0,\infty): y<\alpha(t)\}$ we have

$$\begin{aligned} q(t)f\left(\frac{1}{n},y\right) + (\varphi_p(\alpha'(t)))' \geqslant q(t)f(t,y) + (\varphi_p(\alpha'(t)))' &\geqslant k_0 + (\varphi_p(m\lambda'(t)))' \\ &\geqslant \frac{k_0}{2} > 0. \end{aligned}$$

Thus (2.6) is satisfied. On the other hand, $\alpha(1)\leqslant|\alpha|_\infty = m|\lambda|_\infty\leqslant\rho_1$. So for each $n\in N_0$,

$$\beta_n(1)\geqslant\rho_1\geqslant\alpha(1).$$

Then (2.8) is satisfied. From Theorem 2.1, (1.1) has a solution $y\in C[0,1]\cap C^1(0,1]$ with $\varphi_p(y')\in C^1(0,1)$ with $y(t)>0$ for $t\in(0,1]$. □

Remark 3.1. (a). Note the α constructed in Theorem 3.1 satisfies $|\alpha|_\infty\leqslant\rho_1$.

(b). One could replace (3.2) with the more general condition: there exists $\delta\in\left(0,\frac{1}{3}\right)$ with $f(\cdot,y)$ nondecreasing on $(0,\delta)$ for each fixed $y\in(0,\infty)$.

Remark 3.2. If we replace $1/n\leqslant t\leqslant 1$ with $0\leqslant t\leqslant 1-(1/n)$ in (3.1) and (3.3) we can easily obtain (see Remark 2.4) the analogue of Theorem 3.1 in this situation.

Next suppose the following conditions are satisfied:

$$\begin{aligned}&\text{let } n\in\{n_0,n_0+1,\ldots\} \text{ and associated with each } n \text{ we}\\ &\text{have a constant } \rho_n \text{ such that } \{\rho_n\} \text{ is a decreasing}\\ &\text{sequence with } \lim_{n\to\infty}\rho_n=0 \text{ and there exists a}\\ &\text{constant } k_0>0 \text{ such that for } \frac{1}{n}\leqslant t\leqslant 1-\frac{1}{n},\\ &\text{and } 0<y\leqslant\rho_n \text{ we have } q(t)f(t,y)\geqslant k_0,\end{aligned} \tag{3.4}$$

$$f(\cdot,y) \text{ is nondecreasing on } \left(0,\frac{1}{3}\right) \text{ for each fixed } y\in(0,\infty), \tag{3.5}$$

$$f(\cdot,y) \text{ is nonincreasing on } \left(\frac{2}{3},1\right) \text{ for each fixed } y\in(0,\infty), \tag{3.6}$$

$$\begin{aligned}&\text{for each } n\in N_0, \exists\beta_n\in C[0,1]\cap C^1(0,1], \varphi_p(\beta_n')\in C^1(0,1),\\ &\text{with } \beta_n(t)\geqslant\rho_n \text{ for } t\in[0,1], \beta_n'(1)+\Psi(\beta_n(1))>0 \text{ and } \beta_n(1)\geqslant\rho_1\\ &\text{and we have } (\varphi_p(\beta_n'))'+q(t)f(t,\beta_n(t))\leqslant 0\\ &\text{for } t\in\left(0,1-\frac{1}{n}\right] \text{ and} (\varphi_p(\beta_n'))'+q(t)f\left(1-\frac{1}{n},\beta_n(t)\right)\leqslant 0\\ &\text{for } t\in\left(1-\frac{1}{n},1\right),\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\text{for each } n\in N_0, \exists\beta_n\in C[0,1]\cap C^1(0,1], \varphi_p(\beta_n')\in C^1(0,1),\\ &\text{with } \beta_n(t)\geqslant\rho_n \text{ for } t\in[0,1], \beta_n'(1)+\Psi(\beta_n(1))>0 \text{ and } \beta_n(1)\geqslant\rho_1\\ &\text{and we have } (\varphi_p(\beta_n'))'+q(t)f(t,\beta_n(t))\leqslant 0\\ &\text{for } t\in\left[\frac{1}{n},1\right), \text{ and } (\varphi_p(\beta_n'))'+q(t)f\left(\frac{1}{n},\beta_n(t)\right)\leqslant 0\\ &\text{for } t\in\left(0,\frac{1}{n}\right).\end{aligned} \tag{3.8}$$

Then (2.24)–(2.26) hold. The proof is same as the proof of Theorem 3.1. Combining this with Theorem 2.2 gives the following result. □

Theorem 3.2. *Let* $n_0\in\{3,4,\ldots\}$ *be fixed and suppose* (2.1)–(2.4), (2.7), (3.4)–(3.8) *and* (2.27) *hold. Then* (1.1) *has a solution* $y\in C[0,1]\cap C^1(0,1]$ *with* $\varphi_p(y')\in C^1(0,1)$ *with* $y(t)>0$, $t\in(0,1]$.

Looking at Theorems 3.1 and 3.2 one sees that the main difficulty when discussing examples is constructing the β_n in (2.8) or (2.26). As a result we present a theorem which removes (2.8) or (2.26) and replaces it with an easy verifiable condition. As in Theorems 2.1 and 2.2 we first present the results in their full generality.

Theorem 3.3. *Let* $n_0 \in \{3, 4, \ldots\}$ *be fixed and suppose* (2.1)–(2.7) *hold. In addition assume there exist*

$$M_1 > 0 \quad \text{and} \quad M_2 > \max\left\{\sup_{t\in[0,1]} \alpha(t), \rho_1\right\} \tag{3.9}$$

with

$$\text{for each } n \in N_0, q(t)f(t, M_1t + M_2) < 0 \text{ for } t \in \left[\frac{1}{n}, 1\right),$$

$$q(t)f\left(\frac{1}{n}, M_1t + M_2\right) < 0 \text{ for } t \in \left(0, \frac{1}{n}\right) \tag{3.10}$$

and

$$M_1 + \Psi(M_1 + M_2) > 0 \tag{3.11}$$

holding. Then (1.1) *has a solution* $y \in C[0, 1] \cap C^1(0, 1]$ *with* $\varphi_p(y') \in C^1(0, 1)$ *with* $y(t) > 0, t \in (0, 1]$.

Proof. Define $\beta_n(t) = M_1t + M_2$ for $t \in [0, 1]$ and $n \in N_0$. Then

$$\text{for each } n \in N_0, \beta_n \in C[0, 1] \cap C^1(0, 1], \ \varphi_p(\beta_n') \in C^1(0, 1),$$

$$\text{with } \beta_n(t) \geqslant \rho_n \ \text{ for } t \in [0, 1], \beta_n'(1) + \Psi(\beta_n(1)) = \rho_1 > 0,$$

$$\beta_n(1) \geqslant \alpha(1) \text{ and we have } (\varphi_p(\beta_n'))' + q(t)f(t, \beta_n(t)) = 0$$

$$\text{for } t \in \left[\frac{1}{n}, 1\right), \text{ and } (\varphi_p(\beta_n'))' + q(t)f\left(\frac{1}{n}, \beta_n(t)\right) = 0$$

$$\text{for } t \in \left(0, \frac{1}{n}\right)$$

and

$$\max\left\{\sup_{t\in[0,1]} \beta_n(t) : n \in N_0\right\} < \infty.$$

Then (2.8) and (2.9) hold. The result now follows Theorem 2.1 i.e., (1.1) has a solution $y \in C[0, 1] \cap C^1(0, 1]$ with $\varphi_p(y') \in C^1(0, 1)$ with $y(t) > 0, t \in (0, 1]$. □

Combining Theorem 2.2 and the above proof gives the following existence result.

Theorem 3.4. *Let* $n_0 \in \{3, 4, \ldots\}$ *be fixed and suppose* (2.1)–(2.4), (2.7), (2.24), (2.25), (3.9) *and* (3.11) *hold. Suppose the following condition also holds*:

$$\begin{aligned}
&\text{for each } n \in N_0,\ q(t)f(t, M_1t + M_2) < 0 \text{ for } t \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right],\\
&q(t)f\left(\frac{1}{n}, M_1t + M_2\right) < 0 \text{ for } t \in \left(0, \frac{1}{n}\right)\\
&q(t)f\left(1 - \frac{1}{n}, M_1t + M_2\right) < 0 \text{ for } t \in \left(1 - \frac{1}{n}, 1\right).
\end{aligned} \tag{3.12}$$

Then (1.1) *has a solution* $y \in C[0, 1] \cap C^1(0, 1]$ *with* $\varphi_p(y') \in C^1(0, 1)$ *with* $y(t) > 0$, $t \in (0, 1]$.

Combining Theorems 3.1 and 3.3 yields the following theorem.

Theorem 3.5. *Let* $n_0 \in \{3, 4, \ldots\}$ *be fixed and suppose* (2.1)–(2.4), (2.7), (3.1), (3.2). *In addition, suppose that there exist*

$$M_1, M_2 > 0 \tag{3.13}$$

with (3.10) *and* (3.11) *hold. Then* (1.1) *has a solution* $y \in C[0, 1] \cap C^1(0, 1]$ *with* $\varphi_p(y') \in C^1(0, 1)$ *with* $y(t) > 0$, $t \in (0, 1]$.

Proof. By (3.1), there exist $n_0 > 0$ such that $\rho_{n_0} < M_2$. Without loss of generality, we suppose that

$$M_2 > \rho_1 > \cdots > \rho_n > \rho_{n+1} > \cdots \text{ and } \lim_{n\to\infty} \rho_n = 0. \tag{3.14}$$

As the proof of the Theorem 3.1, there exists α with

$$\begin{aligned}
&\alpha \in C[0, 1] \cap C^1(0, 1],\ \varphi_p(\alpha') \in C^1(0, 1),\\
&\alpha(0) = 0,\ \alpha'(1) + \Psi(\alpha(1)) < 0\\
&\alpha > 0 \text{ on } (0, 1] \text{ such that for each } n \in N_0\\
&\text{ we have } (\varphi_p(\alpha'))' + q(t)f(t, y) > 0 \text{ for}\\
&(t, y) \in \left[\frac{1}{n}, 1\right) \times \{y \in (0, \infty) : y < \alpha(t)\}\\
&\text{ and } (\varphi_p(\alpha'))' + q(t)f\left(\frac{1}{n}, y\right) > 0 \text{ for}\\
&(t, y) \in \left(0, \frac{1}{n}\right) \times \{y \in (0, \infty) : y < \alpha(t)\}
\end{aligned} \tag{3.15}$$

and

$$\alpha(1) \leqslant |\alpha|_\infty \leqslant \rho_1. \tag{3.16}$$

By (3.14) and (3.16) we have

$$M_2 > \max\left\{\sup_{t\in[0,1]} \alpha(t), \rho_1\right\}.$$

Thus all the conditions of the Theorem 3.3 are satisfied. Then (1.1) has a solution $y \in C[0,1] \cap C^1(0,1]$ with $\varphi_p(y') \in C^1(0,1)$ with $y(t) > 0$, $t \in (0,1]$.

Combining Theorems 3.2 and 3.4 yields the following theorem. The proof is similar to the proof of Theorem 3.5. We omit it. □

Theorem 3.6. *Let* $n_0 \in \{3,4,\ldots\}$ *be fixed and suppose* (2.1)–(2.4), (2.7), (3.4), (3.5), (3.6), (3.11), (3.12) *and* (3.13) *hold. Then* (1.1) *has a solution* $y \in C[0,1] \cap C^1(0,1]$ *with* $\varphi_p(y') \in C^1(0,1)$ *with* $y(t) > 0$, $t \in (0,1]$.

4. Example

Example. Consider the boundary value problem

$$-(|y'|^{p-2}y')' = \mu(At^{\gamma}y^{-a} - \delta^2),\ 0 < t < 1,$$
$$y(0) = 0, \quad y'(1) - \frac{(1+v)}{2}y(1) = 0 \tag{4.1}$$

with $1 < p < \infty$, $0 < v < 1$, $a > 0$, $A > 0$, $\gamma \geqslant 0$, $\mu > 0$, and $\delta > 0$. Then (4.1) has a solution $y \in C[0,1] \cap C^1(0,1]$ with $\varphi_p(y') \in C^1(0,1)$ with $y(t) > 0$, for $t \in (0,1)$.

To see this we apply Theorem 3.5. Let

$$q(t) = \mu, \quad f(t,y) = \frac{At^{\gamma}}{y^a} - \delta^2, \quad \Psi(z) = -\frac{(1+v)}{2}z \quad \text{and}$$
$$g(y) = Ay^{-a}, \quad h(y) = \delta^2. \tag{4.2}$$

Clearly (2.1)–(2.4) and (3.2) are satisfied. We next prove (2.7) holds.

We have

$$g'(y) = \frac{-Aa}{y^{a+1}} < 0 \quad \text{for } y > 0$$

and

$$\frac{|g'|^{1/p}}{g^{2/p}} = \frac{\left(\frac{Aa}{y^{a+1}}\right)^{1/p}}{(Ay^{-a})^{2/p}} = \left(\frac{a}{A}\right)^{1/p} y^{(a-1)/p}.$$

Thus for any $R > 0$,

$$\int_0^R \frac{|g'|^{1/p}}{g^{2/p}}\,\mathrm{d}y = \int_0^R \left(\frac{a}{A}\right)^{1/p} y^{(a-1)/p}\,\mathrm{d}y < \infty$$

and

$$\int_0^\infty \frac{|g'|^{1/p}}{g^{2/p}}\,\mathrm{d}y = \left(\frac{a}{A}\right)^{1/p}\int_0^\infty y^{(a-1)/p}\,\mathrm{d}y$$
$$= \left(\frac{a}{A}\right)^{1/p}\frac{p}{p+a-1}y^{(p+a-1)/p}\Big|_0^\infty$$
$$=\infty,$$

so (2.7) holds.

Next let

$$\rho_n = \left(\frac{A}{n^\gamma(\delta^2+1)}\right)^{1/a} \quad \text{and} \quad k_0=\mu \tag{4.3}$$

and notice for $n\in\{3,4,\ldots\}$, $\frac{1}{n}\leqslant t\leqslant 1$ and $0<y\leqslant\rho_n$ that we have

$$q(t)f(t,y)\geqslant\mu\left(A\left(\frac{1}{n}\right)^\gamma[\rho_n]^{-a}-\delta^2\right)=\mu([\delta^2+1]-\delta^2)=\mu.$$

Thus (3.1) holds. Finally let $M_2=\left(\frac{2A}{\delta^2}\right)^{1/a}>0$ and $M_1=\frac{2(1+v)}{1-v}M_2>0$. Then (3.10) holds since

$$f(t,M_1t+M_2)<f(t,M_2)=\frac{At^\gamma}{M_2^a}-\delta^2$$
$$=\frac{At^\gamma}{2A/\delta^2}-\delta^2=\left(\frac{t^\gamma}{2}-1\right)\delta^2<0 \text{ for } t\in\left[\frac{1}{n},1\right)$$

and

$$f\left(\frac{1}{n},M_1t+M_2\right)<f\left(\frac{1}{n},M_2\right)<0 \text{ for } t\in\left(0,\frac{1}{n}\right).$$

On the other hand, (3.11) is true since

$$M_1+\left(-\frac{1+v}{2}\right)(M_1+M_2)=M_1-\frac{1+v}{2}M_1-\frac{1+v}{2}M_2$$
$$=\frac{1-v}{2}M_1-\frac{1+v}{2}M_2$$
$$=\frac{1-v}{2}\cdot\frac{2(1+v)}{1-v}M_2-\frac{1+v}{2}M_2$$
$$=\frac{1+v}{2}M_2>0.$$

Thus all the conditions of the Theorem 3.5 are satisfied. Then (4.1) has a solution $y\in C[0,1]\cap C^1(0,1]$ with $\varphi_p(y')\in C^1(0,1)$ with $y(t)>0$, for $t\in(0,1)$.

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