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On Algebras with Two Multiplications, Including Hopf Algebras and Bose–Mesner Algebras

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INTRODUCTION

Let A be the Bose–Mesner algebra of a symmetric association scheme [1, 2, 4]. Then A is a space of $n \times n$-matrices over $\mathbb{C}$. We wish to draw attention to the following properties of A. It is equipped with two multiplications: the usual matrix product and the Hadamard (or componentwise) product $\circ$. The identity element of either algebra structure spans a one-dimensional ideal of the other. Either algebra possesses a $\mathbb{C}$-valued algebra map that defines a non-degenerate associative bilinear form for the other structure, making the latter into a Frobenius algebra. Up to a scalar factor, the two maps are the trace map tr and the map $M \mapsto \Sigma(M)$, where $\Sigma(M)$ denotes the sum of the elements of the matrix M. Matrix transposition is an algebra (anti) automorphism for both structures. Finally, the important identity

$$\mathrm{tr}(MN) = \sum (M^T \circ N) \qquad \text{for all } M, N \in A \tag{1}$$

holds.

Now, let H be a finite-dimensional Hopf algebra over a field k [13]. Then H and $H^* = \mathrm{Hom}_k(H, k)$ are associative algebras, and one can use a certain canonical linear isomorphism $H \cong H^*$ to bring the multiplication of H^* to H. Thus, we have two algebra structures on H. This system has properties very similar to those of A described above; for example, an analogy of (1) holds. For more details see Example 2.1.

The purpose of this paper is to study such properties, common to A and H, in the framework of a general axiom system. Section 1 presents axioms

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for what we call a *double Frobenius algebra*. The definition was designed to cover both of the mentioned cases (as closely as was easily attained) and to be self-dual in a certain sense.

Section 2 consists of a list of examples. In Section 3–7 we develop some general structure theory, and we view the results frequently in our two main examples of Hopf and Bose–Mesner algebras (or more generally, the adjacency algebras of association schemes, possibly non-commutative). In Section 8 we point out one further similarity between the two examples: cocleftness and imprimitivity are nearly analogous conditions when considered in our general framework. In Section 9 we look at this situation in each case more closely. We discover a common phenomenon that seems hard, perhaps impossible, to deduce from the axioms. This suggests that the axiom system could be enlarged, still keeping the two main examples. We hope to be able to return to this question in another paper.

1. DOUBLE FROBENIUS ALGEBRAS

Throughout the paper k is a fixed field. All vector spaces, tensor products, etc., are over k. For a vector space V we write $V^* = \mathrm{Hom}_k(V, k)$ and usually we denote the natural map $V^* \times V \to k$ by $\langle -,- \rangle$.

DEFINITION 1.1. Let A be a finite-dimensional k-space, equipped with two binary operations, $\cdot$ and $\star$, and with two maps ϵ and $\omega : A \to k$. Assume that

(A1) $(A, \cdot\,)$ is an associative k-algebra with identity element 1;

(A2) $(A, \star)$ is an associative k-algebra with identity element ι;

(A3) ϵ is a k-algebra map $(A, \cdot\,) \to k$;

(A4) ω is a k-algebra map $(A, \star) \to k$;

(A5) the bilinear form $A \times A \to k$, $(a, b) \mapsto \langle \epsilon, a \star b \rangle$, is non-degenerate;

(A6) the bilinear form $A \times A \to k$, $(a, b) \mapsto \langle \omega, a \cdot b \rangle$, is non-degenerate.

By (A5), (A6) there is a unique bijective map $\sigma : A \to A$ (denoted $a \mapsto a^\sigma$) that satisfies

(A7) $\langle \epsilon, a^\sigma \star b \rangle = \langle \omega, a \cdot b \rangle$ for all a, $b \in A$.

Let $\sigma^{-1} = \tau$. Finally, assume that for some $s, t \in A$ we have

(A8) $(a \star b)^\sigma = b^\sigma \star s \star a^\sigma$ for all $a, b \in A$;

(A9) $(a \cdot b)^\tau = b^\tau \cdot t \cdot a^\tau$ for all $a, b \in A$.

Then we call $(A, \cdot, \bigstar, \epsilon, \omega)$ a *double Frobenius algebra*, or briefly a *dF-algebra*.

The axioms (A5) and (A6) imply that $(A, \bigstar)$ and $(A, \cdot)$ are both Frobenius algebras.

In a dF-algebra the elements s and t are unique, since s is the inverse of ι^σ in $(A, \bigstar)$ by (A8) and t is the inverse of 1^τ in $(A, \cdot)$ by (A9). Hence, given A and $(\cdot, \bigstar, \epsilon, \omega)$ that satisfy the axioms, the rest $(1, \iota, \sigma, \tau, s, t)$ of the structure is determined uniquely. Instead of $(A, \cdot, \bigstar, \epsilon, \omega)$, we may also use notations $(A, \cdot, \bigstar, \epsilon, \omega, \sigma, \tau, s, t)$, or $(A, \cdot, \bigstar)$, or just A, and so on.

From the definition it is clear that if $(A, \cdot, \bigstar, \epsilon, \omega, \sigma, \tau, s, t)$ is a dF-algebra, then so is $(A, \bigstar, \cdot, \omega, \epsilon, \tau, \sigma, t, s)$. We call these dF-algebras *duals* of each other. Every statement about a general dF-algebra has a dual counterpart, and a proof of one of them also proves the other.

The two multiplications are in completely symmetric positions in the definition. However, to simplify notation, the following convention is used: the product $\cdot$ is denoted by juxtaposition, and in expressions it binds more strongly than $\bigstar$. For example, then $(a \cdot b) \bigstar (c \cdot (d \bigstar e))$ is written as $ab \bigstar c(d \bigstar e)$.

2. EXAMPLES

In this section we list some examples of dF-algebras. Only the first two will be used later. The examples are presented very briefly, omitting any detailed (and sometimes lengthy) calculations.

EXAMPLE 2.1 (Hopf algebra). We use freely the basic facts and standard notation on coalgebras and Hopf algebras. The reader may consult [13].

Let A be a finite-dimensional Hopf algebra over the field k. The counit is denoted by ϵ_A and the antipode by S. Fix left integrals $\lambda \in A$, $\lambda^* \in A^*$ with $\langle \lambda^*, \lambda \rangle = 1$. The map $\Phi\colon A \to A^*$, $\Phi(a) = \lambda^* \leftharpoonup S^{-1}(a)$ for $a \in A$, has an inverse Φ^{-1} given by $\Phi^{-1}(a^*) = a^* \rightharpoonup \lambda$ for $a^* \in A^*$. We use Φ to bring the multiplication of A^* to A: we define

$$a \bigstar b = \Phi^{-1}(\Phi(a)\Phi(b)) \qquad \text{for } a, b \in A.$$

Then $(A, \bigstar)$ is an associative algebra with identity element λ. Explicitly, the product is

$$a \bigstar b = \sum \langle \lambda^*, S^{-1}(a_{(1)})b \rangle a_{(2)} = \sum \langle \lambda^*, S^{-1}(a) b_{(2)} \rangle b_{(1)}.$$

One can show that $(A, \cdot, \bigstar, \epsilon = \epsilon_A, \omega = \lambda^*, \sigma = S, \tau = S^{-1}, s = \alpha^{-1} \rightharpoonup \lambda, t = 1)$ is a dF-algebra, where $\alpha \in A^*$ is determined by $\lambda a = \langle \alpha, a\rangle\lambda$ for all $a \in A$. Note, in particular, that s is not ι, in general. This example is the reason for introducing the elements s and t in Definition 1.1.

When one checks that A indeed is a dF-algebra, the following three identities are likely to be useful: $\alpha \rightharpoonup \lambda = S(\lambda)$, $\langle \lambda^*, S^{-1}(\lambda)\rangle = 1$, $\langle \lambda^*, ab\rangle = \langle \lambda^*, b(\alpha^{-1} \rightharpoonup S^2(a))\rangle$. The first is proved in [12, Proposition 5] and the second is an easy consequence. The third can be verified by starting with the identity $S(a)1 = 1S(a)$, writing the 1's as $1 = \sum\lambda_{(1)}\langle \lambda^*, \lambda_{(2)}\rangle$, using the properties of λ to move a inside $\langle \lambda^*, -\rangle$, and finally applying non-degeneracy of the integrals.

EXAMPLE 2.2 (Homogeneous coherent algebra, Bose–Mesner algebra). Let A be the adjacency algebra of a homogeneous coherent configuration [6, 7] or, in other words, the adjacency algebra of a (non-commutative) association scheme [1]. We call such an algebra in this paper a *homogeneous coherent algebra*. By the definition, $k = \mathbb{C}$, A is a subalgebra of the algebra of $n \times n$-matrices with the usual matrix product $\cdot$, and A has a basis $\{a_0, a_1, \dots, a_d\}$ of $(0, 1)$-matrices with the following properties: $a_0 = I$ (the identity matrix); $a_0 + \cdots + a_d = J$ (the matrix with all entries 1); for each i the transpose a_i^T of a_i is one of the a_j's.

More specially, one can consider the Bose–Mesner algebra of a symmetric association scheme [1, 2, 4]. Then $a_i^T = a_i$ for each i.

Since the a_i's are $(0, 1)$-matrices with sum J, we have $a_i \circ a_j = \delta_{ij}a_i$, where $\circ$ is the Hadamard product or the componentwise product. It follows that A is an associative algebra under the Hadamard product.

One can show that a_i has the same number n_i of 1's on each row, hence $a_iJ = n_iJ$. Let $\Sigma(M)$ denote the sum of all entries of a matrix M. Then $a_iJ = (1/n)\Sigma(a_i)J$, hence $aJ = (1/n)\Sigma(a)J$ for all $a \in A$. So, $\langle \epsilon, a\rangle = (1/n)\Sigma(a)$ defines an algebra map $\epsilon: (A, \cdot) \to k$. Similarly, since $a_i \circ I = \delta_{0i}I = (1/n)\mathrm{tr}(a_i)I$, the formula $\langle \omega, a\rangle = (1/n)\mathrm{tr}(a)$ defines an algebra map $\omega: (A, \circ) \to k$. For any matrices M, N we have $\Sigma(M^T \circ N) = \mathrm{tr}(MN)$, hence $\langle \epsilon, a^T \circ b\rangle = \langle \omega, ab\rangle$ for $a, b \in A$. Finally, since $\langle \epsilon, a_i \circ a_j\rangle = n_i\delta_{ij}$, the bilinear form $\langle \epsilon, (-)\circ(-)\rangle$ is non-degenerate. It is now clear that $(A, \cdot, \circ, \epsilon, \omega, \sigma = T, \ \tau = T, \ s = J, \ t = I)$ is a dF-algebra with identity elements $1 = I$, $\iota = J$.

If A is commutative, this example is a special case of the following.

EXAMPLE 2.3 (C-algebra). Let A be a C-algebra (character algebra) [1, 2.5]. Thus, A is a commutative algebra over $k = \mathbb{C}$ with a basis $\{x_0, \dots, x_d\}$ satisfying the following properties: $x_0 = 1$; there is an involution $i \mapsto i'$ of $\{0, \dots, d\}$ such that $x_i \mapsto x_{i'}$ defines an algebra automorphism of A; the structure constants with respect to the basis are real; the coefficient of x_0

in the basis expression of $x_i x_j$ is $\delta_{ij'}\xi_i$ where ξ_i is positive; $x_i \mapsto \xi_i$ gives an algebra map $\epsilon\colon A \to k$.

Define a bilinear binary operation $\star$ and linear maps $\sigma\colon A \to A$, $\omega\colon A \to k$ by $x_i \star x_j = \delta_{ij} x_i$, $x_i^\sigma = x_{i'}$, and $\langle \omega, x_i \rangle = \delta_{i0}$. We have $\langle \epsilon, x_i^\sigma \star x_j \rangle = \delta_{i'j}\xi_j = \langle \omega, x_i x_j \rangle$. Then A is a dF-algebra with $1 = x_0$, $\iota = x_0 + \cdots + x_d$, $s = \iota$, and $t = 1$ (cf. the discussion following Theorem 5.10 in [1]).

EXAMPLE 2.4 (Dual of a group algebra). It is interesting to note the following special case of Example 2.1. Let G be a finite group with identity element 1, and let $A = (kG)^*$, the dual Hopf algebra of the group algebra kG. Then a (left) integral $\lambda \in A$ is given by $\langle \lambda, x \rangle = \delta_{1x}$ for $x \in G$, and $\lambda^* = \sum_{x \in G} x$ is a (left) integral in $A^* \cong kG$. Now the two multiplications on A are the following: for $f, g \in A$, $x \in G$,

$$\langle f \cdot g, x \rangle = \langle f, x \rangle \langle g, x \rangle$$
$$\langle f \star g, x \rangle = \sum_{zy = x} \langle f, y \rangle \langle g, z \rangle.$$

The opposite product of the latter is often called convolution.

EXAMPLE 2.5 (Truncated polynomial algebra). Let $A = k[x]/(x^{n+1})$ for fixed $n \geq 0$, and let $\cdot$ be the usual product on A. Then $\langle \epsilon, x^i \rangle = \delta_{i0}$ gives an algebra map $\epsilon\colon (A, \cdot) \to k$. We define another product $\star$ on A by requiring that $x^i \mapsto x^{n-i}$ be an isomorphism between $(A, \cdot)$ and $(A, \star)$. Then $x^i \star x^j = x^{i+j-n}$ if $i + j \geq n$ and $x^i \star x^j = 0$ otherwise. The isomorphism transforms ϵ to $\omega\colon A \to k$, $\langle \omega, x^i \rangle = \delta_{in}$. Now $\langle \epsilon, x^i \star x^j \rangle = \delta_{i+j,n} = \langle \omega, x^i \cdot x^j \rangle$. So, A becomes a dF-algebra with $\sigma = \mathrm{id}$, $s = \iota = x_n$, $t = 1 = x_0$.

EXAMPLE 2.6 (Double Frobenius algebra attached to a Boolean algebra). Let P be a finite Boolean algebra, that is, a finite distributive lattice $(P, \leq)$ with binary meet $\wedge$ and join $\vee$, the smallest and largest elements $\hat{0}$ and $\hat{1}$, and a unique complement $\bar{p}$ for each $p \in P$. Let A be a k-space with a set $\{x_p \mid p \in P\}$ as a basis. Define bilinear binary operations on A by

$$x_p \cdot x_q = \begin{cases} x_{p \vee q} & \text{if } p \wedge q = \hat{0}, \\ 0 & \text{otherwise,} \end{cases}$$
$$x_p \star x_q = \begin{cases} x_{p \wedge q} & \text{if } p \vee q = \hat{1}, \\ 0 & \text{otherwise,} \end{cases}$$

and linear maps $\epsilon, \omega\colon A \to k$ by $\langle \epsilon, x_p \rangle = \delta_{p\hat{0}}$, $\langle \omega, x_p \rangle = \delta_{p\hat{1}}$. Then $\langle \epsilon, x_p \star x_q \rangle = \delta_{p\bar{q}} = \langle \omega, x_p \cdot x_q \rangle$. Now A is a dF-algebra with $1 = x_{\hat{0}}$, $\iota = x_{\hat{1}}$, $\sigma = \tau = \mathrm{id}$, $s = \iota$, $t = 1$.

3. ONE-DIMENSIONAL IDEALS

In the rest of the paper A is a double Frobenius algebra.

Recall that ι is the identity element in $(A, \star)$. In this section we prove that $k\iota$ and $k\iota^\tau$ are ideals in $(A, \cdot\,)$ and that, dually, $k1$ and $k1^\sigma$ are ideals in $(A, \star)$. This shows that one of the properties mentioned in the introduction follows from the others. As for other subspaces $k\iota^{\tau^n}$ (and $k1^{\sigma^n}$) for $n \in \mathbb{Z}$, in Section 6 we shall see that $k\iota^{\sigma^2} = k\iota$, hence $k\iota^{\tau^n}$ is always $k\iota$ or $k\iota^\tau$.

PROPOSITION 3.1. *We have $\langle \epsilon, 1 \rangle = \langle \epsilon, 1^\sigma \rangle = 1$ and $\langle \omega, \iota \rangle = \langle \omega, \iota^\tau \rangle = 1$.*

Proof. First, $\langle \epsilon, 1 \rangle = 1 = \langle \omega, \iota \rangle$ by (A3), (A4). Then (A7) gives $\langle \epsilon, 1^\sigma \rangle = \langle \epsilon, 1^\sigma \star \iota \rangle = \langle \omega, 1\iota \rangle = \langle \omega, \iota \rangle = 1$. By duality $\langle \omega, \iota^\tau \rangle = 1$. ∎

In general $\langle \epsilon, 1^\tau \rangle$ and $\langle \omega, \iota^\sigma \rangle$ need not be 1. In the Hopf algebra case (Example 2.1), $\langle \omega, \iota^\sigma \rangle = \langle \lambda^*, S(\lambda) \rangle$ which is not always 1.

The following is well known.

LEMMA 3.2. *Let B be a Frobenius algebra over k and $\beta\colon B \to k$ an algebra map. Then there is $x \in B$ such that $bx = \langle \beta, b \rangle x$ for all $b \in B$, and x is unique up to scalar multiple. The space kx is a two-sided ideal.*

PROPOSITION 3.3. *We have $a\iota = \langle \epsilon, a^\sigma \rangle \iota$ for all $a \in A$, and $k\iota$ is a two-sided ideal in $(A, \cdot\,)$. Dually, $a \star 1 = \langle \omega, a^\tau \rangle 1$ for all $a \in A$, and $k1$ is a two-sided ideal in $(A, \star)$.*

Proof. By (A9) $(a^\sigma b^\sigma)^\tau = bta$ for all a, b, hence $a^\sigma b^\sigma = (bta)^\sigma$. Since t is the inverse of 1^τ in $(A, \cdot\,)$, we have

$$(ba)^\sigma = a^\sigma (b1^\tau)^\sigma \qquad \text{for all } a, b \in A.$$

Then, by (A7), (A2), and (A3),

$$\begin{aligned}\langle \omega, ba\iota \rangle &= \langle \epsilon, (ba)^\sigma \star \iota \rangle = \langle \epsilon, (ba)^\sigma \rangle \\ &= \langle \epsilon, a^\sigma (b1^\tau)^\sigma \rangle = \langle \epsilon, a^\sigma \rangle \langle \epsilon, (b1^\tau)^\sigma \rangle.\end{aligned}$$

For $a = 1$ this gives $\langle \omega, b\iota \rangle = \langle \epsilon, (b1^\tau)^\sigma \rangle$ since $\langle \epsilon, 1^\sigma \rangle = 1$ by 3.1. Hence,

$$\langle \omega, ba\iota \rangle = \langle \epsilon, a^\sigma \rangle \langle \omega, b\iota \rangle \qquad \text{for all } a, b \in A.$$

Now $a\iota = \langle \epsilon, a^\sigma \rangle \iota$ follows from (A6). Consequently, $a \mapsto \langle \epsilon, a^\sigma \rangle$ is an algebra map $(A, \cdot) \to k$. By 3.2, $k\iota$ is a two-sided ideal ∎

Therefore, $(A, \cdot)$ acts in $k\iota$ as scalars from the right too. The algebra map $\gamma: (A, \cdot) \to k$, defined by $\iota a = \langle \gamma, a \rangle \iota$ for $a \in A$, can be expressed as follows:

$$\langle \gamma, a \rangle = \langle \gamma, a \rangle 1 = \langle \gamma, a \rangle \langle \omega, \iota \rangle = \langle \omega, \iota a \rangle = \langle \epsilon, \iota^\sigma \bigstar a \rangle.$$

Dually, $1 \bigstar a = \langle \epsilon, 1 \bigstar a \rangle 1 = \langle \omega, 1^\tau a \rangle 1$ for $a \in A$.

PROPOSITION 3.4. *We have $\iota^\tau a = \langle \epsilon, a \rangle \iota^\tau$ for all $a \in A$, and $k\iota^\tau$ is a two-sided ideal in $(A, \cdot)$. Dually, $1^\sigma \bigstar a = \langle \omega, a \rangle 1^\sigma$ for all $a \in A$, and $k1^\sigma$ is a two-sided ideal in $(A, \bigstar)$.*

Proof. By (A7), (A2), and (A3) we have for all $a, b \in A$

$$\langle \omega, \iota^\tau ab \rangle = \langle \epsilon, \iota \bigstar ab \rangle = \langle \epsilon, ab \rangle = \langle \epsilon, a \rangle \langle \epsilon, b \rangle.$$

For $a = 1$ this becomes $\langle \omega, \iota^\tau b \rangle = \langle \epsilon, b \rangle$, hence $\langle \omega, \iota^\tau ab \rangle = \langle \epsilon, a \rangle \langle \omega, \iota^\tau b \rangle$. By (A6) then $\iota^\tau a = \langle \epsilon, a \rangle \iota^\tau$. From 3.2 follows that $k\iota^\tau$ is a two-sided ideal. ∎

4. THE COALGEBRA (A, Δ) AND THE ELEMENT $\Delta(\iota)$

In this and the next section we define two coalgebra structures on the double Frobenius algebra A, as follows: The two algebra structures on A give two coalgebra structures on A^* (the dual coalgebras [13]). The non-degenerate bilinear forms (A5), (A6) define linear isomorphisms $A \cong A^*$, and using them we bring the coalgebra structures to A.

In this section we take the dual coalgebra of $(A, \bigstar)$ and use the isomorphism $A \cong A^*$, $a \mapsto \langle \epsilon, a \bigstar (-) \rangle$. As usually [13], we denote the resulting diagonalization map by $\Delta: A \to A \otimes A$ and we write $\Delta(a) = \sum a_{(1)} \otimes a_{(2)} = \sum_{(a)} a_{(1)} \otimes a_{(2)}$. If $f \in A^*$, then $\Delta_{A^*}(f)$ in the dual coalgebra of $(A, \bigstar)$ is defined by

$$\langle f, x \bigstar y \rangle = \sum \langle f_{(1)}, x \rangle \langle f_{(2)}, y \rangle \qquad \text{for all } x, y \in A,$$

hence $\Delta(a)$ for $a \in A$ is defined by

$$\langle \epsilon, a \bigstar x \bigstar y \rangle = \sum \langle \epsilon, a_{(1)} \bigstar x \rangle \langle \epsilon, a_{(2)} \bigstar y \rangle \qquad \text{for all } x, y \in A,$$

or

$$a \bigstar x = \sum \langle \epsilon, a_{(1)} \bigstar x \rangle a_{(2)} \qquad \text{for all } x \in A. \tag{4.1}$$

In 4.4 we prove the following mirror version of (4.1) that can then also be used to compute $\Delta(a)$:

$$x \star a = \sum \langle \epsilon, x \star a_{(2)} \rangle a_{(1)} \qquad \text{for all } x \in A. \tag{4.1$'$}$$

The counit is $\langle \epsilon, (-) \star \iota \rangle = \epsilon$, since the counit of A^* is $f \mapsto \langle f, \iota \rangle$.

We use the following notational convention: $(\sigma \otimes 1)\Delta(a) = \sum (a_{(1)})^\sigma \otimes a_{(2)}$ is written as $\sum a^\sigma_{(1)} \otimes a_{(2)}$, and so on.

EXAMPLE 4.1 (Hopf algebra). If we denote by Δ_A the original diagonalization of the Hopf algebra A (Example 2.1) and write $\Delta_A(a) = \sum_i a_i \otimes a'_i$, then

$$a \star x = \sum_i \langle \lambda^*, S^{-1}(a_i)x \rangle a'_i = \sum_i \langle \omega, a_i^\tau x \rangle a'_i = \sum_i \langle \epsilon, a_i \star x \rangle a'_i.$$

Thus, $\Delta_A = \Delta$, and the process gives back the original coalgebra structure of A. It is an easy exercise to show that in this dF-algebra we have

$$c(a \star b) = \sum c_{(2)} a \star c_{(1)} b$$

for all $a, b, c \in A$, and one can prove also some other identities that allow one to change the order of the two products. (Such identities do not hold in a general dF-algebra.)

EXAMPLE 4.2 (Homogeneous coherent algebra). In Example 2.2, $a_i \star a_j = a_i \circ a_j = \delta_{ij} a_i$ and $\epsilon(a_i) = n_i$. Using (4.1) one shows easily that $\Delta(a_i) = (1/n_i) a_i \otimes a_i$.

PROPOSITION 4.3. *We have* $\Delta(a \star b) = \sum a_{(1)} \star b \otimes a_{(2)} = \sum b_{(1)} \otimes a \star b_{(2)}$ *for all* $a, b \in A$.

Proof. The first expression for $\Delta(a \star b)$ follows when we write $a \star b \star x$ as

$$a \star b \star x = a \star (b \star x) = \sum \langle \epsilon, a_{(1)} \star b \star x \rangle a_{(2)},$$

and the second follows from

$$\begin{aligned} a \star b \star x = a \star (b \star x) &= a \star \Big(\sum \langle \epsilon, b_{(1)} \star x \rangle b_{(2)} \Big) \\ &= \sum \langle \epsilon, b_{(1)} \star x \rangle a \star b_{(2)}. \quad \blacksquare \end{aligned}$$

COROLLARY 4.4. *We have* $a \star b = \sum \langle \epsilon, a \star b_{(2)} \rangle b_{(1)}$ *for all* $a, b \in A$.

Proof. Use in 4.3 the counit property: $x = (1 \otimes \epsilon)\Delta(x)$. $\blacksquare$

The element $\Delta(\iota)$ of $A \otimes A$ has some nice properties. The first ones are derived by applying 4.3 and 4.4 to $a = \iota \star a = a \star \iota$:

COROLLARY 4.5. *We have* $\Delta(a) = \sum \iota_{(1)} \star a \otimes \iota_{(2)} = \sum \iota_{(1)} \otimes a \star \iota_{(2)}$ *for all* $a \in A$.

COROLLARY 4.6. *We have* $a = \sum \langle \epsilon, \iota_{(1)} \star a \rangle \iota_{(2)} = \sum \langle \epsilon, a \star \iota_{(2)} \rangle \iota_{(1)}$ *for all* $a \in A$.

COROLLARY 4.7. *The element* ι *generates* A *as a left or as a right coideal. In other words,*

$$A = \sum \langle A^*, \iota_{(1)} \rangle \iota_{(2)} = \sum \langle A^*, \iota_{(2)} \rangle \iota_{(1)}.$$

In the Hopf algebra case the bialgebra properties hold:

$$\Delta(ab) = \sum a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)}, \quad \Delta(1) = 1 \otimes 1,$$
$$\langle \epsilon, ab \rangle = \langle \epsilon, a \rangle \langle \epsilon, b \rangle, \quad \epsilon(1) = 1.$$

In a general dF-algebra the first need not be true, but the other three always are by (A3) and the following:

PROPOSITION 4.8. *We have* $\Delta(1) = 1 \otimes 1$ *and* $\Delta(1^\sigma) = 1^\sigma \otimes 1^\sigma$.

Proof. By Proposition 3.3, $x \star 1 = \langle \omega, x^\tau \rangle 1 = \langle \epsilon, \langle \omega, x^\tau \rangle 1 \rangle 1 = \langle \epsilon, x \star 1 \rangle 1$, hence by (4.1′), $\Delta(1) = 1 \otimes 1$. The other claim is proved similarly using 3.4, 3.1, and (4.1). ∎

5. THE COALGEBRA (A, ∇)

We apply now the results of the previous section to the dual dF-algebra $(A, \star, \cdot)$. This gives another coalgebra structure on A. We denote the new diagonalization by ∇ and write $\nabla(a) = \sum a^{(1)} \otimes a^{(2)}$. By (the duals of) (4.1) and (4.1′), $\nabla(a)$ is determined by either of the following:

$$ax = \sum \langle \omega, a^{(1)} x \rangle a^{(2)} \qquad \text{for all } x \in A, \tag{5.1}$$

$$xa = \sum \langle \omega, x a^{(2)} \rangle a^{(1)} \qquad \text{for all } x \in A. \tag{5.1′}$$

The counit is ω. Also, $\nabla(\iota) = \iota \otimes \iota$ and $\nabla(\iota^\tau) = \iota^\tau \otimes \iota^\tau$ by 4.8. The element $\Delta(\iota)$ is almost the same as its dual counterpart $\nabla(1)$:

PROPOSITION 5.1. *We have* $\nabla(1) = (\tau \otimes 1)\Delta(\iota)$, *that is,* $\sum 1^{(1)} \otimes 1^{(2)} = \sum \iota_{(1)}^\tau \otimes \iota_{(2)}$.

Proof. Let $x \in A$. Then $1x = x = \iota \star x = \sum\langle \epsilon, \iota_{(1)} \star x\rangle\iota_{(2)} = \sum\langle \omega, \iota_{(1)}^{\tau}x\rangle\iota_{(2)}$. Now use (5.1). ∎

COROLLARY 5.2. *We have* $\nabla(a) = \sum\iota_{(1)}^{\tau}a \otimes \iota_{(2)} = \sum\iota_{(1)}^{\tau} \otimes a\iota_{(2)}$ *for all* $a \in A$.

Proof. The dual of Corollary 4.5 says $\nabla(a) = \sum 1^{(1)}a \otimes 1^{(2)} = \sum 1^{(1)} \otimes a1^{(2)}$. Insert here $\nabla(1) = (\tau \otimes 1)\Delta(\iota)$. ∎

EXAMPLE 5.3 (Homogeneous coherent algebra). Let A be as in Examples 2.2 and 4.2, and assume that A is commutative. Since $(A, \cdot)$ is semisimple [6, 3.1], it has a basis $\{e_0, \dots, e_d\}$ of orthogonal idempotents with sum $1 = I$. Let $f_i = \langle \omega, e_i\rangle$, i.e., $I \circ e_i = f_i I$; see 3.4. Then $f_i \neq 0$ by (A6) since $\langle \omega, e_i A\rangle = k\langle \omega, e_i\rangle = kf_i$. In Example 4.2 we saw that $\Delta(a_i) = (1/n_i)a_i \otimes a_i$. Similarly, $e_i e_j = \delta_{ij}e_i$ implies $\nabla(e_i) = (1/f_i)e_i \otimes e_i$. We have $\Delta(\iota) = \Delta(J) = \Delta(\sum_i a_i) = \sum_i(1/n_i)a_i \otimes a_i$ and $\nabla(1) = \nabla(I) = \nabla(\sum_i e_i) = \sum_i(1/f_i)e_i \otimes e_i$. Hence, in this case 5.1 says that $\sum_i(1/f_i)e_i \otimes e_i = \sum_i(1/n_i)a_i^T \otimes a_i$. This shows that 5.1 generalizes the formula [4, 6.1].

Remark 5.4. Let $\{a_1, \dots, a_n\}$ be a basis of A. By 4.7 we can write $\sum\iota_{(1)}^{\tau} \otimes \iota_{(2)} = \sum_i a_i \otimes b_i$ where $\{b_1, \dots, b_n\}$ is another basis. Using 5.2 and the counit property of ω,

$$b_j = (\omega \otimes 1)\nabla(b_j) = \sum_{(\iota)}\langle \omega, \iota_{(1)}^{\tau}b_j\rangle\iota_{(2)} = \sum_i\langle \omega, a_i b_j\rangle b_i.$$

Then $\langle \omega, a_i b_j\rangle = \delta_{ij}$, i.e., the two bases are dual bases with respect to $\langle \omega, (-)(-)\rangle$ [3, (62.7)]. This fact can be used when we apply the theory of Frobenius algebras to dF-algebras. For example, a theorem of Higman [5; 3, 71.6] implies that $(A, \cdot)$ is separable if and only if $\sum\iota_{(2)}z\iota_{(1)}^{\tau} = 1$ for some $z \in A$.

6. PROPERTIES OF σ^2

In this section we show that σ^2 (or τ^2), multiplied by suitable scalars, gives automorphisms of the various structures of A.

Recall that t is the inverse of 1^{τ} in $(A, \cdot)$ and s is the inverse of ι^{σ} in $(A, \star)$. Hence, $\langle \epsilon, t\rangle = \langle \epsilon, 1^{\tau}\rangle^{-1}$ and $\langle \omega, s\rangle = \langle \omega, \iota^{\sigma}\rangle^{-1}$.

PROPOSITION 6.1. *We have*

$$\iota^{\sigma^2} = \langle \omega, \iota^{\sigma}\rangle\iota,$$

$$(a \star b)^{\sigma^2} = \langle \omega, s\rangle a^{\sigma^2} \star b^{\sigma^2} \qquad \textit{for } a, b \in A.$$

In other words, $\langle \omega, s\rangle\sigma^2$ *is an algebra automorphism of* $(A, \star)$ (*and so is its inverse* $\langle \omega, \iota^\sigma\rangle\tau^2$). *Dually,* $\langle \epsilon, t\rangle\tau^2$ *is an algebra automorphism of* $(A, \cdot)$ (*and so is* $\langle \epsilon, 1^\tau\rangle\sigma^2$).

Proof. We have

$$(ab)^\sigma = b^\sigma(a1^\tau)^\sigma = (1^\tau b)^\sigma a^\sigma \qquad \text{for all } a, b \in A.$$

The first equality was shown in the proof of 3.3 and almost the same argument gives the second. Now, $(a\iota)^\sigma = \langle \epsilon, a^\sigma\rangle\iota^\sigma$ by 3.3, and then by the above, $\iota^\sigma(a1^\tau)^\sigma = \langle \epsilon, a^\sigma\rangle\iota^\sigma$. Since $t1^\tau = 1$, inserting $a^\tau t$ in place of a, we obtain

$$\iota^\sigma a = \langle \epsilon, (a^\tau t)^\sigma\rangle\iota^\sigma \qquad \text{for all } a \in A.$$

By the above, $(a^\tau t)^\sigma = (1^\tau t)^\sigma a = 1^\sigma a$, hence $\langle \epsilon, (a^\tau t)^\sigma\rangle = \langle \epsilon, 1^\sigma a\rangle = \langle \epsilon, 1^\sigma\rangle\langle \epsilon, a\rangle = \langle \epsilon, a\rangle$ by 3.1. It follows that $\iota^\sigma a = \langle \epsilon, a\rangle\iota^\sigma$ for all $a \in A$. By 3.2 and 3.4 we have $\iota^\sigma \in k\iota^\tau$. If we let $\iota^\sigma = \xi\iota^\tau$ with $\xi \in k$, then by 3.1, $\langle \omega, \iota^\sigma\rangle = \langle \omega, \xi\iota^\tau\rangle = \xi$. So, $\iota^\sigma = \langle \omega, \iota^\sigma\rangle\iota^\tau$, or $\iota^{\sigma^2} = \langle \omega, \iota^\sigma\rangle\iota$.

Since $s \star \iota^\sigma = \iota^\sigma \star s = \iota$, using (A8) we have $\iota^{\sigma^2} \star s \star s^\sigma \star s = (s \star \iota^\sigma)^\sigma \star s = \iota^\sigma \star s = \iota$, and by the above this gives $s \star s^\sigma \star s = \langle \omega, s\rangle\iota$. From (A8) follows $(a \star b \star c)^\sigma = c^\sigma \star s \star b^\sigma \star s \star a^\sigma$ for $a, b, c \in A$, hence

$$(a \star b)^{\sigma^2} = (b^\sigma \star s \star a^\sigma)^\sigma = a^{\sigma^2} \star s \star s^\sigma \star s \star b^{\sigma^2} = \langle\omega, s\rangle a^{\sigma^2} \star b^{\sigma^2}.$$

∎

PROPOSITION 6.2. *For all $a \in A$ we have*

$$\langle \epsilon, a^{\sigma^2}\rangle = \langle \epsilon, t\rangle\langle \epsilon, a\rangle,$$
$$\Delta(a^{\sigma^2}) = \langle \epsilon, 1^\tau\rangle(\sigma^2 \otimes \sigma^2)\Delta(a).$$

In other words, $\langle \epsilon, 1^\tau\rangle\sigma^2$ *is a coalgebra automorphism of* (A, Δ). *Dually,* $\langle \omega, \iota^\sigma\rangle\tau^2$ *is a coalgebra automorphism of* (A, ∇).

Proof. By (A9) and 3.3, $\iota^\tau t a^\tau = (a\iota)^\tau = \langle \epsilon, a^\sigma\rangle\iota^\tau$. On the other hand, by 3.4 and (A3), $\iota^\tau t a^\tau = \langle \epsilon, t a^\tau\rangle\iota^\tau = \langle \epsilon, t\rangle\langle \epsilon, a^\tau\rangle\iota^\tau$, hence $\langle \epsilon, a^\sigma\rangle = \langle \epsilon, t\rangle\langle \epsilon, a^\tau\rangle$, which implies the first claim. Using this, (4.1), and 6.1,

$$\begin{aligned}(a \star b)^{\sigma^2} &= \sum \langle \epsilon, a_{(1)} \star b\rangle a_{(2)}^{\sigma^2} \\ &= \sum \langle \epsilon, 1^\tau\rangle\langle \epsilon, (a_{(1)} \star b)^{\sigma^2}\rangle a_{(2)}^{\sigma^2} \\ &= \sum \langle \epsilon, 1^\tau\rangle\langle \omega, s\rangle\langle \epsilon, a_{(1)}^{\sigma^2} \star b^{\sigma^2}\rangle a_{(2)}^{\sigma^2}.\end{aligned}$$

On the other hand, $(a \star b)^{\sigma^2} = \langle \omega, s \rangle a^{\sigma^2} \star b^{\sigma^2}$, hence for all a, x,

$$a^{\sigma^2} \star x = \sum \langle \epsilon, 1^\tau \rangle \langle \epsilon, a_{(1)}^{\sigma^2} \star x \rangle a_{(2)}^{\sigma^2}.$$

Now (4.1) implies $\Delta(a^{\sigma^2}) = \sum \langle \epsilon, 1^\tau \rangle a_{(1)}^{\sigma^2} \otimes a_{(2)}^{\sigma^2}$. ∎

7. $\Delta(\iota)$ AND ACTIONS OF A ON $A \otimes A$

We show now that two "centralizing relations" determine $\Delta(\iota)$ uniquely up to scalar multiple.

PROPOSITION 7.1. *An element $\sum_i x_i \otimes y_i$ of $A \otimes A$ satisfies*

$$\sum_i x_i^\tau a \otimes y_i = \sum_i x_i^\tau \otimes a y_i, \tag{7.1}$$

$$\sum_i x_i \star a \otimes y_i = \sum_i x_i \otimes a \star y_i \tag{7.2}$$

if and only if it is a scalar multiple of $\Delta(\iota)$.

Proof. First, $\Delta(\iota)$ satisfies (7.1) and (7.2) by 4.5 and 5.2. Conversely, assume that $\sum_i x_i \otimes y_i$ satisfies the two relations. Applying $\omega \otimes 1$ to (7.1) and $\epsilon \otimes 1$ to (7.2) and using the fact $\langle \omega, x_i^\tau a \rangle = \langle \epsilon, x_i \star a \rangle$, we find that

$$a \cdot \left(\sum_i \langle \omega, x_i^\tau \rangle y_i \right) = a \star \left(\sum_i \langle \epsilon, x_i \rangle y_i \right).$$

When $a = \iota$ this gives $\sum_i \langle \epsilon, x_i \rangle y_i \in k\iota$ by 3.3. Now, using Corollary 4.6 and (7.2),

$$\begin{aligned} \sum_i x_i \otimes y_i &= \sum_i \sum_{(\iota)} \langle \epsilon, x_i \star \iota_{(2)} \rangle \iota_{(1)} \otimes y_i \\ &= \sum_i \sum_{(\iota)} \langle \epsilon, x_i \rangle \iota_{(1)} \otimes \iota_{(2)} \star y_i \\ &= \sum_{(\iota)} \iota_{(1)} \otimes \iota_{(2)} \star \left(\sum_i \langle \epsilon, x_i \rangle y_i \right). \end{aligned}$$

This belongs to $k\Delta(\iota)$ since $\iota_{(2)} \star \iota = \iota_{(2)}$. ∎

If $\sum_i x_i \otimes y_i = \xi \Delta(\iota)$ with $\xi \in k$, then the scalar ξ may be computed, for example, from $\xi = \sum_i \langle \epsilon, x_i \rangle \langle \omega, y_i \rangle$. Namely, by the counit property of ϵ, $(\epsilon \otimes \omega)\Delta(\iota) = \langle \omega, \iota \rangle = 1$.

8. COCLEFTNESS AND IMPRIMITIVITY

Let A be a dF-algebra. Consider the following conditions for an element $x \in A$:

(B1) $(a \star x)(a' \star x) = (a \star x)a' \star x$ for $a, a' \in A$,

(B2) $1 \in A \star x$.

In this section we study these conditions in our two main examples. We show that (B1), (B2) are closely related to imprimitivity in the case of a Bose−Mesner algebra, and to A being cocleft over a right coideal subalgebra in the case of a Hopf algebra.

The two conditions imply that $B = A \star x$ is a subalgebra of $(A, \cdot\,)$. Note also that $B = A \star x$ is a left ideal of $(A, \star)$, and by (4.1′) and (A5) this is equivalent to $\Delta(B) \subseteq B \otimes A$, i.e., to B being a right coideal of (A, Δ).

First we derive some general consequences of (B1) and (B2). The following two propositions show that, in some respects, x has in B a role similar to that of ι in A; compare with 5.2 and 3.3.

PROPOSITION 8.1. *Let $x \in A$ and $B = A \star x$. Then* (B1) *is equivalent to*

(B1′) $\sum bx_{(1)} \otimes x_{(2)} = \sum x_{(1)} \otimes tb^{\tau}x_{(2)}$ *for all $b \in B$.*

Proof. Let $b = a \star x \in B$, $a \in A$. The left-hand side of (B1) can be written, using (4.1′), as

$$b(a' \star x) = \sum \langle \epsilon, a' \star x_{(2)} \rangle bx_{(1)},$$

and the right-hand side, using (4.1′), (A7), and (A9), can be written as

$$\begin{aligned} ba' \star x &= \sum \langle \epsilon, ba' \star x_{(2)} \rangle x_{(1)} \\ &= \sum \langle \omega, (ba')^{\tau} x_{(2)} \rangle x_{(1)} \\ &= \sum \langle \omega, a'^{\tau} tb^{\tau} x_{(2)} \rangle x_{(1)} \\ &= \sum \langle \epsilon, a' \star tb^{\tau} x_{(2)} \rangle x_{(1)}. \end{aligned}$$

The claim follows by comparing these and using (A5). ∎

PROPOSITION 8.2. *If $x \in A$ satisfies* (B1) *then $bx = \langle \epsilon, b^{\sigma} \rangle x$, $b^{\tau}x = \langle \epsilon, b \rangle 1^{\tau}x$, and $b \star x = \langle \epsilon, 1 \star x \rangle b$ for all $b \in B$.*

Notice that b^{τ} or 1^{τ} need not be in B.

Proof. Let $b \in B$. When we apply $1 \otimes \epsilon$ to (B1′) and use the facts that ϵ is a counit for Δ and an algebra map $(A, \cdot\,) \to k$, we get $bx = \langle \epsilon, tb^{\tau} \rangle x$. By the proof of 6.2 we have $\langle \epsilon, tb^{\tau} \rangle = \langle \epsilon, b^{\sigma} \rangle$, hence $bx = \langle \epsilon, b^{\sigma} \rangle x$.

Similarly, applying $\epsilon \otimes 1$ to (B1′) we obtain $tb^\tau x = \langle \epsilon, b \rangle x$, i.e., $b^\tau x = \langle \epsilon, b \rangle 1^\tau x$. Finally, letting $b = a \bigstar x$ and $a' = 1$ in (B1), we have $b \bigstar x = (a \bigstar x)1 \bigstar x = (a \bigstar x)(1 \bigstar x) = (a \bigstar x)\langle \epsilon, 1 \bigstar x \rangle 1 = \langle \epsilon, 1 \bigstar x \rangle b$; see the remark following the proof of 3.3. ∎

PROPOSITION 8.3. *Assume that* $x \in A$ *satisfies* (B1) *and* (B2), *and let* $B = A \bigstar x$. *If also* $B = x \bigstar A$, *then* $B = tB^\tau$ *and* $\Delta(B) \subseteq B \otimes B$.

Proof. Assume first $B = A \bigstar x = x \bigstar A$. Then $A \bigstar B \subseteq B$ and $B \bigstar A \subseteq B$, hence $\Delta(B) \subseteq (B \otimes A) \cap (A \otimes B) = B \otimes B$. Further, $1 = x \bigstar a$ for some $a \in A$. Set $f = \langle \epsilon, (-) \bigstar a \rangle \in A^*$. Then by (4.1′), $1 = x \bigstar a = \sum \langle f, x_{(1)} \rangle x_{(2)}$. Applying $f \otimes 1$ to (B1′), we obtain $\sum \langle f, bx_{(1)} \rangle x_{(2)} = tb^\tau$. This belongs to B since $\Delta(x) \in \Delta(B) \subseteq B \otimes B$. So, $tB^\tau \subseteq B$, and by dimensions $tB^\tau = B$. ∎

EXAMPLE 8.4 (Hopf algebra). Let A be a finite-dimensional Hopf algebra, viewed as a dF-algebra as in Example 2.1. Consider a right coideal subalgebra B (i.e., a subalgebra with $\Delta(B) \subseteq B \otimes A$). An element $x \in B$ is a *left integral* for B if $bx = \langle \epsilon, b \rangle x$ for $b \in B$ and a *right integral* if $xb = \langle \epsilon, b \rangle x$ for $b \in B$. Recall that A is *right* B*-cocleft* if one of certain equivalent, nice conditions holds; see [8, 10, 11, 9]. In Example 9.1 we say a little more about this situation, but at this point we need only to know that the cocleftness is equivalent to the condition

$(*)$ B is generated as a right coideal by its left integral and B contains a non-zero right integral.

The equivalence is seen from 2.2(d), 2.2(e), and 2.3 in [9].

Now let $B \subseteq A$ be any subspace. We show that the following are equivalent:

$$B \text{ is a right coideal subalgebra of } A \text{ and } A \text{ is right } B\text{-cocleft}; \tag{8.1}$$

$$B = A \bigstar x \text{ for some } x \text{ satisfying (B1) and (B2) and } B \text{ contains a non-zero right integral}. \tag{8.2}$$

That B is generated by x as a right coideal means precisely that $B = A \bigstar x$. Since (B1) and (B2) imply that $A \bigstar x$ is a subalgebra and that x is its left integral (by 8.2), the implication (8.2) ⇒ (8.1) is clear by $(*)$. For the converse, let B be as in (8.1). By $(*)$ $B = A \bigstar x$ where x is a left integral of B. We only have to show that x satisfies (B1). But for $b \in B$, $a' \in A$ we have

$$b(a' \bigstar x) = \sum b_{(2)} a' \bigstar b_{(1)} x = \sum b_{(2)} a' \bigstar \langle \epsilon, b_{(1)} \rangle x = ba' \bigstar x;$$

see Example 4.1 and note that $\Delta(b) \in B \otimes A$.

(Condition (8.2) would look better without the assumption about the right integral. The assumption comes from $(*)$ and whether it can be dropped is unknown.)

EXAMPLE 8.5 (Homogeneous coherent algebra). Let A be a homogeneous coherent algebra (see Example 2.2). Recall that it consists of $n \times n$-matrices over $\mathbb{C}$. Given a matrix a, we denote the (p,q)-entry by a_{pq}. The index set $\{1,\dots,n\}$ can be identified with an underlying coherent configuration X [1, 6]. The configuration X is *imprimitive* if it admits an equivalence relation $\sim$ such that A contains the matrix x where $x_{pq} = 1$ for $p \sim q$ and $x_{pq} = 0$ for $p \nsim q$ and if $x \neq I, J$ [1, p. 165]. Such x is idempotent in $(A, \bigstar)$, hence it is the sum of some of the elements a_i; recall that the a_i's are orthogonal idempotents and a basis of $(A, \bigstar)$.

We show that the following are equivalent:

$$\text{the underlying configuration is imprimitive;} \tag{8.3}$$

$$\text{there is } x \in A \text{ satisfying (B1) and (B2), and } A \bigstar x \neq kI, A. \tag{8.4}$$

(One should compare the treatment here with Section 2.9 in [1] and Section 2.4 in [2].)

Assume first that (8.4) holds. Write $B = A \bigstar x$. By 8.2 we have $x \bigstar x = \langle \epsilon, 1 \bigstar x\rangle x = \langle \omega, 1^{\tau} x\rangle x = \langle \omega, x\rangle x$. Since $\bigstar$ is the Hadamard product and $x \neq 0$, we cannot have $x \bigstar x = 0$. Hence $\langle \omega, x\rangle \neq 0$, and by multiplying x with $\langle \omega, x\rangle^{-1}$ we may assume that $x \bigstar x = x$. But then x is idempotent in $(A, \bigstar)$, hence it is a $(0,1)$-matrix and some sum of a_i's. Those a_i's that occur in the sum form a basis of B. By (B2), a_0 is one of them. By re-numbering the other a_i's we may assume that $x = a_0 + a_1 + \cdots + a_s$. Since B is not ka_0 or A, we have $0 < s < d$, that is, $x \neq I, J$. The product $\bigstar$ is commutative; hence 8.3 implies $B = tB^{\tau}$, i.e., $B = B^T$. Since $\{a_0, \dots, a_s\}$ is a basis of B and in general a_i^T is some a_j, transposition permutes $a_0, \dots, a_s$ (and fixes a_0). Thus, $x^T = x$. By 8.2, $x^2 = \langle \epsilon, x^{\sigma}\rangle x = \langle \epsilon, x\rangle x$, where $\langle \epsilon, x\rangle = (1/n)\Sigma(x) > 0$ (see Example 2.2). If we define a relation $\sim$ in X by $p \sim q \Leftrightarrow x_{pq} = 1$, then from the above properties of x it follows that $\sim$ is an equivalence relation; hence we have (8.3).

Conversely, assume imprimitivity. Let x and $\sim$ be as in its definition. The relation $\sim$ is reflexive; hence the diagonal elements of x are all 1. Therefore, $1 = I = I \bigstar x \in A \bigstar x$, which shows (B2). As for (B1), a much stronger fact holds: for any $n \times n$-matrices a, a', a'' we have

$$(a \circ x)(a' \circ xa'') = (a \circ x)a' \circ xa'',$$

where $\circ$ is the Hadamard product (that extends $\star$). Namely, a direct calculation shows that the (p,q)-entry of each side is

$$\sum_{i\sim p}\sum_{j\sim p} a_{pi}a'_{iq}a''_{jq}.$$

9. MORE ON COCLEFTNESS AND IMPRIMITIVITY

Let A be a dF-algebra. Consider the following conditions, stronger than (B1) and (B2):

(C1) $(a\star x)(a'\star xa'')(a\star x)a'\star xa''$ for $a, a', a'' \in A$,

(C2) for some $z \in A$ we have $1 = z\star x$ and $\iota = xz$.

The first is suggested by Example 8.5, if nothing else. In fact, we show that these conditions hold in the two examples studied in the previous section. It would be interesting to know how generally (B1), (B2) imply (C1), (C2) in a dF-algebra.

If (C1) and (C2) hold then $B = A\star x$ is a subalgebra of $(A,\cdot)$ and $C = xA$ is a subalgebra of $(A,\star)$. (Look at (C1) with $a'' = 1$ and with $a = \iota$.) We can write (C1) nicely as

$$b(a\star c) = ba\star c \qquad \text{for all } a\in A,\, b\in B,\, c\in C;$$

that is, the natural module actions $B\otimes A\to A$ and $A\otimes C\to A$ commute.

Example 9.1 (Hopf algebra). Let A be a finite-dimensional Hopf algebra, and assume that $x\in A$ satisfies (B1). Then x satisfies (C1). Namely, writing $a\star x = b\in B$ and using example 4.1,

$$(a\star x)(a'\star xa'') = b(a'\star xa'') = \sum b_{(2)}a'\star b_{(1)}xa''.$$

Since $\Delta(B)\subseteq B\otimes A$, we may insert $b_{(1)}x = \langle \epsilon, b^{\sigma}_{(1)}\rangle x = \langle \epsilon, b_{(1)}\rangle x$ by 8.2, hence

$$(a\star x)(a'\star xa'') = ba'\star xa'' = (a\star x)a'\star xa''.$$

Assume now that A is right cocleft over $B = A\star x$, that is, (8.1) and (8.2) are in force. We recall first some facts concerning this situation [8, 10, 11, 9]. As an algebra B is Frobenius. There is a left B-linear $\zeta\colon A\to B$ that has an inverse ζ^- in the convolution algebra $\mathrm{Hom}(A, B^{\mathrm{op}})$. Replacing ζ by $\zeta^-(1)\zeta$ we may assume that $\zeta(1) = \zeta^-(1) = 1$. Define $q\colon A\to A$ by $q(a) = \sum\zeta^-(a_{(2)})a_{(1)}$. Then the multiplication $\cdot$ induces an isomorphism

$B \otimes \mathrm{Im}\, q \to A$, the inverse sending $a \in A$ to $\sum \zeta(a_{(2)}) \otimes q(a_{(1)})$. Letting $V = \mathrm{Im}\, q$, we have $B \otimes V \cong BV = A$ and $1 = q(1) \in V$.

By 8.2, x is a left integral in B, and since B is Frobenius, x is unique such up to scalar multiple. It follows that if we write the left integral $\lambda = \iota$ as $\iota = \sum_i b_i z_i$ where $b_i \in B$, $z_i \in V$, then this decomposition is just $\iota = xz$ for a unique $z \in V$. This gives one half of (C2), and next we show $1 = z \bigstar x$. For $a, a' \in A$,

$$q(a) \bigstar xa' = \sum \zeta^-(a_{(2)}) a_{(1)} \bigstar xa' = \sum \zeta^-(a_{(2)})(a_{(1)} \bigstar xa')$$

by (C1) since $\zeta^-(A) \subseteq B$. But $\sum a_{(1)} \bigstar xa' \otimes a_{(2)} = \Delta(a \bigstar xa')$ by 4.3, hence

$$q(a) \bigstar xa' = \sum \zeta^-((a \bigstar xa')_{(2)})(a \bigstar xa')_{(1)} = q(a \bigstar xa').$$

Thus, $V \bigstar xA \subseteq V$. In particular, $z \bigstar xA \subseteq V$. Consider the maps $xA \to V$, $xa \mapsto z \bigstar xa$, and $V \to xV$, $v \mapsto xv$. The composite $xA \to V \to xV \subseteq xA$ is identity, since by (C1)

$$x(z \bigstar xa) = xz \bigstar xa = \iota \bigstar xa = xa.$$

Further, $V \to xV$ is injective since $BV \cong B \otimes V$. It follows that $xV = xA$ and the maps $xA \to V$ and $V \to xV = xA$ are inverses of each other. Then $z \bigstar xv = v$ for all $v \in V$. The case $v = 1 \in V$ yields $z \bigstar x = 1$.

Now we have proved (C1) and (C2), and as mentioned previously, then xA is a subalgebra of $(A, \bigstar)$. This is a known fact in disguise: By [10, 9], if B is a right coideal subalgebra such that A is right B-cocleft, then $(A/B^+A)^*$ is such a subalgebra for A^*, where $B^+ = B \cap \mathrm{Ker}\, \epsilon$. One can check that simply $xA = S^{-2}\Phi^{-1}((A/B^+A)^*)$ where Φ is the isomorphism $A \cong A^*$ in Example 2.1.

Finally, we make the following observation. Let $C = xA$. From $C = xA \cong V$ and $B \otimes V \cong BV = A$ we obtain an isomorphism $B \otimes C \cong A$ that sends $b \otimes c$ to $b(z \bigstar c) = bz \bigstar c$. We show in Example 9.3 that the corresponding claim is not always true in a Bose–Mesner algebra.

EXAMPLE 9.2. (Homogeneous coherent algebra). We continue with Example 8.5. Let x satisfy (B1) and (B2). By Example 8.5, (C1) holds and we may assume that $x \bigstar x = x$. Since $x \in A$, we have $xJ = x\iota = n'J$ where $n' = \langle \epsilon, x^\sigma \rangle = \langle \epsilon, x \rangle$. Then $x^2 = n'x$ by 8.2. It is easy to see that $z = I + (1/n')(J - x)$ satisfies (C2).

Without going into any details, we mention that the subalgebras $A \bigstar x$ and Ax are (in the case of a symmetric association scheme) closely related to the Bose–Mesner algebras of a subscheme and a quotient scheme, respectively [2, 2.4] (see also [1, p. 166]).

EXAMPLE 9.3 (Hamming scheme). Let A be the Bose–Mesner algebra of the Hamming scheme $H(2,2)$ [1, 3.2]. We can describe A by saying that it consists of 4×4-matrices over $\mathbb{C}$ and has a basis $\{a_0, a_1, a_2\}$ where a_0 is the identity matrix, a_2 is the matrix with 1's on the skew diagonal and 0's elsewhere, and $a_0 + a_1 + a_2 = J$, the matrix with all entries 1.

It is easy to see that $x = a_0 + a_2$ satisfies (C1) and that $z = I + \frac{1}{2}(J - x) = I + \frac{1}{2}a_1$ is the only element of A for which (C2) holds. Now, $B = A \star x$ has basis $\{a_0, a_2\}$, and $C = xA$ has basis $\{x, a_1\}$. By dimensions, we cannot have $B \otimes C \cong A$. In fact, the map $B \otimes C \to A$, $b \otimes c \mapsto b(z \star c)$ (cf. 9.1) is surjective but has kernel $k(a_0 - a_2) \otimes a_1$.

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