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$X = M$ for symmetric powers

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Abstract

The $X = M$ conjecture of Hatayama et al. asserts the equality between the one-dimensional configuration sum X expressed as the generating function of crystal paths with energy statistics and the fermionic formula M for all affine Kac–Moody algebras. In this paper we prove the $X = M$ conjecture for tensor products of Kirillov–Reshetikhin crystals $B^{1,s}$ associated to symmetric powers for all nonexceptional affine algebras.

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1. Introduction

In two extraordinary papers, Hatayama et al. [6,7] recently conjectured the equality between the one-dimensional configuration sum X and the fermionic formula M for all affine Kac–Moody algebras. The one-dimensional configuration sum X originates from the corner-transfer-matrix method [1] used to solve exactly solvable lattice models in statistical mechanics. It is the generating function of highest weight crystal paths graded by the energy statistic. The fermionic formula M comes from the Bethe Ansatz [2] and exhibits the

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quasiparticle structure of the underlying model. In combinatorial terms, it can be written as the generating function of rigged configurations.

The one-dimensional configuration sum depends on the underlying tensor product of crystals. In [6,7], the $X = M$ conjecture was formulated for tensor products of Kirillov–Reshetikhin (KR) crystals $B^{r,s}$. Kirillov–Reshetikhin crystals are crystals for finite-dimensional irreducible modules over quantum affine algebras. The irreducible finite-dimensional $U_q'(\mathfrak{g})$-modules were classified by Chari and Pressley [3,4] in terms of Drinfeld polynomials. The Kirillov–Reshetikhin modules $W^{r,s}$, labeled by a Dynkin node r of the underlying classical algebra and a positive integer s, form a special class of these finite-dimensional modules. They naturally correspond to the weight $s\Lambda_r$, where Λ_r is the rth fundamental weight of $\mathfrak{g}$. It was conjectured in [6,7], that there exists a crystal $B^{r,s}$ for each $W^{r,s}$. In general, the existence of $B^{r,s}$ is still an open question. For type $A_n^{(1)}$ the crystal $B^{r,s}$ is known to exist [12] and its combinatorial structure has been studied [22]. The crystals $B^{1,s}$ for nonexceptional types, which are relevant for this paper, are also known to exist and their combinatorics has been worked out [10,12].

The purpose of this paper is to establish the $X = M$ conjecture for tensor products of KR crystals of the form $B^{1,s}$ for nonexceptional affine algebras. This extends [18], where $X = M$ is proved for tensor powers of $B^{1,1}$, and [14,15], where $X = M$ is proved for type $A_n^{(1)}$.

Our method to prove $X = M$ for symmetric powers combines various previous results and techniques. $X = M$ is first proved for $\mathfrak{g}$ such that $\bar{\mathfrak{g}}$ is simply-laced (see Corollary 8.9). This is accomplished by exhibiting a grade-preserving bijection from $U_q'(\bar{\mathfrak{g}})$-highest weight vectors (paths) to rigged configurations (RCs). This was already proved for the root system $A_n^{(1)}$ [15]. For type $D_n^{(1)}$ we exhibit such a path–RC bijection. The proof essentially reduces to the previously known $s = 1$ case [18] using the "splitting" maps $B^{1,s} \to B^{1,s-1} \otimes B^{1,1}$ which are $U_q(\bar{\mathfrak{g}})$-equivariant grade-preserving embeddings.

To prove that the bijection preserves the grading, we consider an involution denoted $*$ on crystal graphs that combines contragredient duality with the action of the longest element w_0 of the Weyl group of $\bar{\mathfrak{g}}$. This duality on the crystal graph, corresponds under the path–RC bijection to the involution on RCs given by complementing the quantum numbers with respect to the vacancy numbers.

We then reduce to the case that $\bar{\mathfrak{g}}$ is simply-laced. This is achieved using the embedding of an affine algebra $\mathfrak{g}$ into one (call it $\mathfrak{g}_Y$) whose canonical simple Lie subalgebra is simply-laced. On the X side we use the virtual crystal construction developed in [16,17]. It is shown in [17] that the KR $U_q'(\mathfrak{g})$-crystals $B^{1,s}$ embed into tensor products of KR $U_q'(\mathfrak{g}_Y)$-crystals such that the grading is respected. One may define the VX ("virtual X") formula in terms of the image of this embedding and show that $X = VX$ (see Section 3.10). This is proved for tensor products of crystals $B^{1,s}$ in [17]. On the M side, it is observed in [17] that the RCs giving the fermionic formula M for type $\mathfrak{g}$, embed into the set of RCs giving a fermionic formula for type $\mathfrak{g}_Y$. Let us denote by VM ("virtual M") the generating function over the image of this embedding of fermionic formulas. It is shown in [17] that $M = VM$. It then suffices to prove $VX = VM$. That is, one must show that the path-to-RC bijection that has already been established for the simply-laced cases, restricts to a bijection between the subsets of objects in the formulas *VX* and *VM*. This is shown in Theorem 10.1 and as a corollary proves $X = M$ for nonsimply-laced algebras, as stated in Corollary 10.2.

In Section 2 we review the crystal theory, the definition of the one-dimensional configuration sum X, contragredient duality and the $*$ involution. Virtual crystals are reviewed in Section 3. Right and left splitting of crystals are discussed in Sections 4 and 5, respectively. Rigged configurations and the analogs of the splitting maps are subject of Section 6. The fermionic formulas M and their virtual counterparts *VM* are stated in Section 7. The $X = M$ conjecture for types $A_n^{(1)}$ and $D_n^{(1)}$ is proven in Section 8 by establishing a statistic preserving bijection. Finally, in Section 10 the equality $X = VX = VM = M$ is established for nonsimply-laced types.

2. Formula X

2.1. Affine algebras

Let $\mathfrak{g} \supset \mathfrak{g}' \supset \bar{\mathfrak{g}}$ be a nonexceptional affine Kac–Moody algebra, its derived subalgebra and canonical simple Lie subalgebra [8]. Denote the corresponding quantized universal enveloping algebras by $U_q(\mathfrak{g}) \supset U_q'(\mathfrak{g}) \supset U_q(\bar{\mathfrak{g}})$ [9]. Let $I = \bar{I} \cup \{0\}$ (respectively $\bar{I}$) be the vertex set of the Dynkin diagram of $\mathfrak{g}$ (respectively $\bar{\mathfrak{g}}$). For $i \in I$, let α_i, h_i, Λ_i be the simple roots, simple coroots, and fundamental weights of $\mathfrak{g}$. Let $\{\overline{\Lambda}_i \mid i \in \bar{I}\}$ be the fundamental weights of $\bar{\mathfrak{g}}$. Let $(a_0, a_1, \ldots, a_n)$ be the smallest tuple of positive integers giving a dependency relation on the columns of the Cartan matrix of $\mathfrak{g}$. Write $a_i^\vee$ for the corresponding integers for the Langlands dual Lie algebra, the one whose Cartan matrix is the transpose of that of $\mathfrak{g}$. Let $c = \sum_{i \in I} a_i^\vee h_i$ be the canonical central element and $\delta = \sum_{i \in I} a_i \alpha_i$ the generator of null roots. Let Q, $Q^\vee$, P be the root, coroot, and weight lattices of $\mathfrak{g}$. Let $\langle \cdot, \cdot \rangle : Q^\vee \otimes P \to \mathbb{Z}$ be the pairing such that $\langle h_i, \Lambda_j \rangle = \delta_{ij}$. Let $P \to P' \to \overline{P}$ be the natural surjections of weight lattices of $\mathfrak{g} \supset \mathfrak{g}' \supset \bar{\mathfrak{g}}$. Let $\overline{P}^+ \subset \overline{P}$ be the dominant weights for $\bar{\mathfrak{g}}$. Let W and $\overline{W}$ be the Weyl groups of $\mathfrak{g}$ and $\bar{\mathfrak{g}}$, respectively.

2.2. Crystal graphs

Let M be a finite-dimensional $U_q'(\mathfrak{g})$-module. Such modules are not highest weight modules (except for the zero module) and therefore need not have a crystal base. Suppose M has a crystal base B. This is a special basis of M; it possesses the structure of a colored directed graph called the *crystal graph*. By abuse of notation the vertex set of the crystal graph is also denoted B. The edges of the crystal graph are colored by the set I. It has the following properties (that of a regular $\overline{P}$-weighted I-crystal):

(1) Fix an $i \in I$. If all edges are removed except those colored i, the connected components are finite directed linear paths called the *i-strings* of B. Given $b \in B$, define $f_i(b)$ (respectively $e_i(b)$) to be the vertex following (respectively preceding) b in its i-string; if there is no such vertex, declare the result to be the special symbol $\emptyset$. Define $\varphi_i(b)$ (respectively $\varepsilon_i(b)$) to be the number of arrows from b to the end (respectively beginning) of its i-string.

(2) There is a function $\mathrm{wt}: B \to \overline{P}$ such that

$$\mathrm{wt}\big(f_i(b)\big) = \mathrm{wt}(b) - \alpha_i,$$
$$\varphi_i(b) - \varepsilon_i(b) = \big\langle h_i, \mathrm{wt}(b)\big\rangle.$$

A *morphism* $g : B \to B'$ of $\overline{P}$-weighted I-crystals is a map $g : B \cup \{\emptyset\} \to B' \cup \{\emptyset\}$ such that $g(\emptyset) = \emptyset$ and for any $b \in B$ and $i \in I$, $g(f_i(b)) = f_i(g(b))$ and $g(e_i(b)) = e_i(g(b))$. An isomorphism of crystals is a morphism of crystals which is a bijection whose inverse bijection is also a morphism of crystals.

If B_i is the crystal base of the $U_q'(\mathfrak{g})$-module M_i for $i = 1, 2$ then the *tensor product* $M_2 \otimes M_1$ is a $U_q'(\mathfrak{g})$-module with crystal base denoted $B_2 \otimes B_1$. Its vertex set is just the cartesian product $B_2 \times B_1$. Its edges are given in terms of those of B_1 and B_2 as follows.

Remark 2.1. We use the opposite of Kashiwara's tensor product convention.

One has $\mathrm{wt}(b_2 \otimes b_1) = \mathrm{wt}(b_2) + \mathrm{wt}(b_1)$ and

$$f_i(b_2 \otimes b_1) = \begin{cases} f_i(b_2) \otimes b_1, & \text{if } \varepsilon_i(b_2) \geqslant \varphi_i(b_1), \\ b_2 \otimes f_i(b_1), & \text{otherwise}, \end{cases}$$

$$e_i(b_2 \otimes b_1) = \begin{cases} e_i(b_2) \otimes b_1, & \text{if } \varepsilon_i(b_2) > \varphi_i(b_1), \\ b_2 \otimes e_i(b_1), & \text{otherwise}, \end{cases}$$

where the result is declared to be $\emptyset$ if either of its tensor factors are.

The tensor product construction is associative up to isomorphism.

Define $\varphi, \varepsilon : B \to P'$ by

$$\varphi(b) = \sum_{i \in I} \varphi_i(b) \Lambda_i, \qquad \varepsilon(b) = \sum_{i \in I} \varepsilon_i(b) \Lambda_i.$$

Every irreducible integrable finite-dimensional $U_q(\bar{\mathfrak{g}})$-module is a highest weight module with some highest weight $\lambda \in \overline{P}^+$; denote its crystal graph by $B(\lambda)$. It is a $\overline{P}$-weighted $\bar{I}$-crystal with a unique classical highest weight vector.

A *classical component* of the crystal graph B of a $U_q'(\mathfrak{g})$-module is a connected component of the graph obtained by removing all 0-arrows from B. The vertex $b \in B$ is a *classical highest weight vector* if $\varepsilon_i(b) = 0$ for all $i \in \bar{I}$. Each classical component of a $U_q'(\mathfrak{g})$-module has a unique classical highest weight vector.

2.3. Finite crystals

Let $\mathcal{C}^{\mathrm{fin}}$ be the category of *finite* crystals as defined in [5]. Every $B \in \mathcal{C}^{\mathrm{fin}}$ has the following properties.

(1) B is the crystal base of an irreducible $U_q'(\mathfrak{g})$-module and is therefore connected.

(2) There is a weight $\lambda \in \overline{P}^+$ such that there is a unique $u(B) \in B$ with $\mathrm{wt}(u(B)) = \lambda$ and for all $b \in B$, $\mathrm{wt}(b)$ is in the convex hull of $\overline{W}\lambda$.

$\mathcal{C}^{\mathrm{fin}}$ is a tensor category [5]. If $B, B' \in \mathcal{C}^{\mathrm{fin}}$ then $B \otimes B' \in \mathcal{C}^{\mathrm{fin}}$ is connected and $u(B \otimes B') = u(B) \otimes u(B')$. Due to the existence of the universal R-matrix for $U_q'(\bar{\mathfrak{g}})$ it follows from [11] that:

(1) there is a unique $U_q'(\mathfrak{g})$-crystal isomorphism $R_{B,B'} : B \otimes B' \to B' \otimes B$ called the *combinatorial R-matrix*;
(2) there is a unique function (the *local coenergy*) $H = H_{B,B'} : B \otimes B' \to \mathbb{Z}_{\geqslant 0}$ that is constant on classical components, zero on $u(B \otimes B')$, and is such that if $R_{B,B'}(b \otimes b') = c' \otimes c$ then

$$H\big(e_0(b \otimes b')\big) = H(b \otimes b') + \begin{cases} 1, & \text{if } e_0(b \otimes b') = e_0(b) \otimes b' \text{ and} \\ & \quad e_0(c' \otimes c) = e_0(c') \otimes c, \\ -1, & \text{if } e_0(b \otimes b') = b \otimes e_0(b') \text{ and} \\ & \quad e_0(c' \otimes c) = c' \otimes e_0(c), \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

The combinatorial R-matrices satisfy

$$R_{B,B} = 1_{B \otimes B},$$
$$R_{B_1,B_2} \circ R_{B_2,B_1} = 1_{B_2 \otimes B_1}$$

and the Yang–Baxter equation, the equality of isomorphisms $B_3 \otimes B_2 \otimes B_1 \to B_1 \otimes B_2 \otimes B_3$ given by

$$\begin{aligned} &(1_{B_1} \otimes R_{B_3,B_2}) \circ (R_{B_3,B_1} \otimes 1_{B_2}) \circ (1_{B_3} \otimes R_{B_2,B_1}) \\ &\quad = (R_{B_2,B_1} \otimes 1_{B_3}) \circ (1_{B_2} \otimes R_{B_3,B_1}) \circ (R_{B_3,B_2} \otimes 1_{B_1}). \end{aligned} \tag{2.2}$$

We shall abuse notation and write R_j (respectively H_j) to denote the application of an appropriate combinatorial R-matrix (respectively local coenergy function) on the $(j+1)$th and jth tensor factors *from the right*. Then (2.2) reads $R_1 R_2 R_1 = R_2 R_1 R_2$. One has the following identities on a three-fold tensor product:

$$H_2 + H_1 R_2 = H_2 R_1 + H_1 R_2 R_1,$$
$$H_1 + H_2 R_1 = H_1 R_2 + H_2 R_1 R_2.$$

Proposition 2.2 [16]. *Let $B = B_L \otimes \cdots \otimes B_1$ and $B' = B'_M \otimes \cdots \otimes B'_1$.*

(1) *$R_{B,B'}$ is equal to any composition of R-matrices of the form R_{B_i,B'_j} which shuffle the B_i to the right, past the B'_j.*

(2) *For $b \otimes b' \in B \otimes B'$, the value of $H_{B,B'}$ is the sum of the values $H_{B_i \otimes B'_j}$ evaluated at the pairs of elements in $B_i \otimes B'_j$ that must be switched by an R-matrix R_{B_i,B'_j} in the computation of $R_{B,B'}(b \otimes b')$.*

2.4. Categories $\mathcal{C}$ and $\mathcal{C}^A$ of KR crystals

We work with two categories of crystals. Let $\mathfrak{g}$ be of nonexceptional affine type. The KR modules $W_s^{(1)}$ and their crystal bases $B^s := B^{1,s}$ were constructed in [10]. See also [17] for an explicit description of B^s. Let $\mathcal{C}$ be the category of tensor products of KR crystals of the form B^s. One has that $\mathcal{C} \subset \mathcal{C}^{\text{fin}}$.

With the labeling of the Dynkin nodes as in [16,17], the crystal B^s has the $U_q(\bar{\mathfrak{g}})$-decomposition

$$B^s \cong \begin{cases} B(s\overline{\Lambda}_1), & \text{for } A_n^{(1)}, B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, \\ \bigoplus_{r=0}^{s} B((s-r)\overline{\Lambda}_1), & \text{for } A_{2n}^{(2)}, D_{n+1}^{(2)}, \\ \bigoplus_{r=0}^{\lfloor \frac{s}{2} \rfloor} B((s-2r)\overline{\Lambda}_1), & \text{for } C_n^{(1)}, A_{2n}^{(2)\dagger}. \end{cases} \tag{2.3}$$

In particular $u(B^s)$ is the unique vector of weight $s\overline{\Lambda}_1$ in B^s.

Let $\mathcal{C}^A$ be the category of all tensor products of KR crystals $B^{r,s}$ in type $A_n^{(1)}$. Here $B^{r,s} \cong B(s\overline{\Lambda}_r)$. So $u(B^{r,s})$ is the unique vector in $B^{r,s}$ of weight $s\overline{\Lambda}_r$. $B^{r,s}$ consists of the semistandard Young tableaux of shape given by an $r \times s$ rectangle, with entries in the set $\{1, 2, \ldots, n+1\}$ [13]. The structure of $B^{r,s}$ as an affine crystal was given explicitly in [22].

We fix some notation for $B \in \mathcal{C}$ or $B \in \mathcal{C}^A$. Let $\mathcal{H} = \bar{I} \times \mathbb{Z}_{>0}$ where recall that $\bar{I} = \{1, 2, \ldots, n\}$ is the set of Dynkin nodes for $\bar{\mathfrak{g}}$. The multiplicity array of B is the array $L = (L_i^{(a)} \mid (a,i) \in \mathcal{H})$ such that $L_i^{(a)}$ is the number of times $B^{a,i}$ occurs as a tensor factor in B for all $(a,i) \in \mathcal{H}$. Up to reordering of tensor factors $B = \bigotimes_{(a,i)\in\mathcal{H}} (B^{a,i})^{\otimes L_i^{(a)}}$.

2.5. Intrinsic coenergy

For $B \in \mathcal{C}^{\text{fin}}$, say that $D : B \to \mathbb{Z}$ is an *intrinsic coenergy* function for B if $D(u(B)) = 0$, D is constant on $U_q(\bar{\mathfrak{g}})$-components, and

$$D\bigl(e_0(b)\bigr) - D(b) \leqslant 1 \quad \text{for all } b \in B.$$

A *graded crystal* is a pair (B, D) where $B \in \mathcal{C}^{\text{fin}}$ and D is an intrinsic coenergy function on B.

We shall give each $B \in \mathcal{C}$ a particular graded crystal structure.

For $B \in \mathcal{C}^{\text{fin}}$ define

$$\operatorname{level}(B) = \min\bigl\{\bigl\langle c, \varphi(b)\bigr\rangle \mid b \in B\bigr\}.$$

One may verify that there is a unique element $b^\natural \in B^s$ such that

$$\varphi(b^\natural) = \operatorname{level}\bigl(B^s\bigr)\Lambda_0.$$

Define the intrinsic coenergy function $D_{B^s} : B^s \to \mathbb{Z}$ by

$$D_{B^s}(b) = H_{B^s,B^s}\big(b \otimes b^\natural\big) - H_{B^s,B^s}\big(u\big(B^s\big) \otimes b^\natural\big).$$

Example 2.3. D_{B^s} has value r on the rth summand in (2.3).

Proposition 2.4 [16]. *Graded crystals form a tensor category as follows. If (B_j, D_j) is a graded crystal for $1 \leqslant j \leqslant L$, then their tensor product $B = B_L \otimes \cdots \otimes B_1$ is a graded crystal with*

$$D_B = \sum_{1 \leqslant i < j \leqslant L} H_i R_{i+1} R_{i+2} \dots R_{j-1} + \sum_{j=1}^{L} D_{B_j} R_1 R_2 \dots R_{j-1} \tag{2.4}$$

where D_{B_j} acts on the rightmost tensor factor.

2.6. X formula

Let (B, D) be a graded crystal. For $\lambda \in \overline{P}^+$ let $P(B, \lambda)$ be the set of classical highest weight vectors in B of weight λ. Define the one-dimensional sum

$$X_{B,\lambda}(q) = \sum_{b \in P(B,\lambda)} q^{D_B(b)/a_0}. \tag{2.5}$$

Recall that $a_0 = 1$ unless $\mathfrak{g} = A_{2n}^{(2)}$ in which case $a_0 = 2$.

2.7. Contragredient duality

Given a $U_q'(\mathfrak{g})$-module M with crystal base B, the contragredient dual module $M^\vee$ has a crystal base $B^\vee = \{b^\vee \mid b \in B\}$ such that

$$\mathrm{wt}(b^\vee) = -\mathrm{wt}(b),$$
$$f_i(b^\vee) = e_i(b)^\vee$$

for $i \in I$ and $b \in B$ such that $e_i(b) \neq \emptyset$.

Proposition 2.5.

$$(B_2 \otimes B_1)^\vee \cong B_1^\vee \otimes B_2^\vee.$$

Example 2.6. Assume type $A_n^{(1)}$. We have

$$B^{r,s\vee} \cong B^{n+1-r,s}. \tag{2.6}$$

The composite map

$$B^{r,s} \xrightarrow{\vee} B^{r,s\vee} \cong B^{n+1-r,s}$$

is given explicitly as follows. Let $b \in B^{r,s}$. Replace each column of b, viewed as a subset of $\{1, 2, \dots, n+1\}$ of size r, by the column of size $n+1-r$ given by its complement. Then reverse the order of the columns. For $n = 5$, $r = 2$, and $s = 3$, a tableau $b \in B^{r,s}$ and its image in $B^{n+1-r,s}$ are given below:

$$\begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 3&4&6\\ \hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline 1&2&2\\ \hline 3&3&4\\ \hline 4&5&5\\ \hline 5&6&6\\ \hline\end{array}.$$

Example 2.7. By definition $B^{1,1\vee}$ is defined by replacing each element of $b \in B^{1,1}$ by an element $b^\vee$ and reversing arrows. $B^{1,s\vee}$ can be realized by the weakly increasing words of length s in the alphabet $\{(n+1)^\vee < \cdots < 2^\vee < 1^\vee\}$. The arrow-reversing map from B^s to $B^{s\vee}$ is given by taking a word of length s, replacing each symbol i with $i^\vee$, and reversing.

2.8. Dynkin automorphisms

Let σ be an automorphism of the Dynkin diagram of $\mathfrak{g}$. Then this induces isometries $\sigma : P \to P$ and $\sigma : \overline{P} \to \overline{P}$ given by $\sigma(\Lambda_i) = \Lambda_{\sigma(i)}$ for $i \in I$, $\sigma(\delta) = \delta$, and $\sigma(\overline{\Lambda}_i) = \overline{\Lambda}_{\sigma(i)}$ for $i \in \overline{I}$.

If M is a $U'_q(\mathfrak{g})$-module with crystal base B, then by carrying out the construction of M but with i replaced everywhere by $\sigma(i)$, there is a $U'_q(\mathfrak{g})$-module M^σ with crystal base B^σ and a bijection $\sigma : B \to B^\sigma$ such that

$$\mathrm{wt}\big(\sigma(b)\big) = \sigma\big(\mathrm{wt}(b)\big),$$
$$\sigma\big(e_i(b)\big) = e_{\sigma(i)}(b),$$
$$\sigma\big(f_i(b)\big) = f_{\sigma(i)}(b)$$

for all $b \in B$ and $i \in I$.

In particular, if the appropriate KR modules have been constructed then

$$\big(B^{r,s}\big)^\sigma = B^{\sigma(r),s}.$$

2.9. The Dynkin involution τ

We fix a canonical Dynkin automorphism τ of the affine Dynkin diagram in the following manner. There is a length-preserving involution on $\overline{W}$ given by conjugation by the longest element $w_0 \in \overline{W}$. Restricting this involution to elements of length one, one obtains an involution τ on the set of simple reflections $\{s_i \mid i \in \bar{I}\}$ of $\overline{W}$. For simplicity of notation this can be written as an involution on the index set $\bar{I}$. This gives an automorphism of the Dynkin diagram of τ. Call the resulting Dynkin automorphism τ.

Explicitly, τ is the identity except when $\bar{\mathfrak{g}} = A_{n-1}$ where τ exchanges i and $n-i$, and $\bar{\mathfrak{g}} = D_n$ with n odd, where τ exchanges $n-1$ and n and fixes all other Dynkin nodes. τ may be extended to the Dynkin diagram of $\mathfrak{g}$ by fixing the 0 node. It satisfies $w_0 s_i w_0 = s_{\tau(i)}$ for all $i \in I$.

The automorphism τ induces the following action on the weight lattice P:

$$\tau(\Lambda_i) = \Lambda_{\tau(i)} \quad \text{for } i \in I.$$

One may show that this is equivalent to

$$\tau(\Lambda) = -w_0\Lambda \quad \text{for } \Lambda \in P.$$

In particular

$$\tau(\alpha_i) = \alpha_{\tau(i)} = -w_0\alpha_i. \tag{2.7}$$

2.10. The $*$-involution

Let M be a $U'_q(\mathfrak{g})$-module with crystal base B. With τ as above, define the module

$$M^* = M^{\tau\vee}.$$

It has crystal base B^*, with elements b^* for $b \in B$ such that

$$\mathrm{wt}(b^*) = w_0\mathrm{wt}(b) \tag{2.8}$$

and

$$\begin{aligned} e_i(b^*) &= f_{\tau(i)}(b)^*, \\ f_i(b^*) &= e_{\tau(i)}(b)^* \end{aligned} \tag{2.9}$$

for all $i \in I$.

Remark 2.8. By (2.9) for $i \in \bar{I}$ it follows that the map $*$ sends classical components of B to classical components of B^*, which by (2.8) must have the same classical highest weight.

Proposition 2.9. $(B_1 \otimes B_2)^* \cong B_2^* \otimes B_1^*$ *with* $(b_1 \otimes b_2)^* \mapsto b_2^* \otimes b_1^*$.

Conjecture 2.10. *Let* $B \in \mathcal{C}^{\mathrm{fin}}$. *Then there is a unique involution* $* : B \to B$ *such that* (2.8) *and* (2.9) *hold.*

Uniqueness follows from the connectedness of B and the fact that $u(B)$ is the unique vector in B of its weight.

Remark 2.11. The crystals satisfying Conjecture 2.10 form a tensor category. Given involutions $*$ on B_1 and B_2 satisfying Conjecture 2.10, define $*$ on $B_1 \otimes B_2$ by $(b_1 \otimes b_2)^* = R(b_2^* \otimes b_1^*)$.

Remark 2.12. For $\lambda \in \overline{P}^+$ define the involution $*$ on $B(\lambda)$ to be the unique map that sends the highest weight vector u_λ to the lowest weight vector (the unique vector of weight $w_0(\lambda)$) and satisfies (2.9) for $i \in \bar{I}$. By (2.7) it follows that $\mathrm{wt}(b^*) = w_0 \mathrm{wt}(b)$ for all $b \in B(\lambda)$.

Explicitly, the involution $*$ on the $U_q(\bar{\mathfrak{g}})$-crystal $B(\overline{\Lambda}_1)$ is given by

$$i \leftrightarrow \bar{i},$$
$$\circ \leftrightarrow \circ$$

except for

$$\bar{\mathfrak{g}} = A_{n-1}: \quad i \leftrightarrow n+1-i,$$
$$\bar{\mathfrak{g}} = D_n,\ n \text{ odd}: \quad n \leftrightarrow n, \quad \bar{n} \leftrightarrow \bar{n}.$$

Here we use that the crystal of $B(\overline{\Lambda}_1)$ has underlying set [13]:

$$\{1 < 2 < \cdots < n\}, \quad \text{for } A_{n-1},$$
$$\{1 < 2 < \cdots < n < \circ < \bar{n} < \cdots < \bar{2} < \bar{1}\}, \quad \text{for } B_n,$$
$$\{1 < 2 < \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1}\}, \quad \text{for } C_n,$$
$$\{1 < 2 < \cdots < \begin{matrix} n \\ \bar{n} \end{matrix} < \cdots < \bar{2} < \bar{1}\}, \quad \text{for } D_n.$$

2.11. Explicit formula for $*$

We wish to determine the map $*$ of Conjecture 2.10 explicitly for $B^s \in \mathcal{C}$ and for $B^{r,s} \in \mathcal{C}^A$. The map $*: B^s \to B^s$ must stabilize classical components by Remark 2.8 and the multiplicity-freeness of B^s as a classical crystal. On each classical component $B(s'\overline{\Lambda}_1)$ of B^s, $*$ is uniquely defined by Remark 2.12. Using the $U_q(\bar{\mathfrak{g}})$-embedding $B(s'\overline{\Lambda}_1) \to B(\overline{\Lambda}_1)^{\otimes s'}$ and Proposition 2.9, we have $(b_1 b_2 \dots b_{s'})^* = b_{s'}^* \dots b_2^* b_1^*$. For $B^{r,s} \in \mathcal{C}^A$ and $b \in B^{r,s}$, b^* is the tableau obtained by replacing every entry c of b by c^* and then rotating by 180 degrees. The resulting tableau is sometimes called the *antitableau* of b.

Example 2.13. For type $D_5^{(1)}$ we have

$$\boxed{1}\boxed{1}\boxed{3}\boxed{\bar{5}}^{\,*} = \boxed{\bar{5}}\boxed{\bar{3}}\boxed{\bar{1}}\boxed{\bar{1}}.$$

For type $A_4^{(1)}$:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline\end{array}^{\,*} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 5 & 5 \\ \hline\end{array}.$$

Proposition 2.14. *With $*$ defined as above, Conjecture* 2.10 *holds for $B^s \in \mathcal{C}$ and for $B^{r,s} \in \mathcal{C}^A$.*

Remark 2.15. From now on the notation $*$ will only be used in the following way. Let $B = B_L \otimes \cdots \otimes B_1$ be a tensor product of factors $B_j = B^{s_j}$. Since $*$ may be regarded as an involution on B^s, by Proposition 2.9 we may write $B^* = B_1^* \otimes \cdots \otimes B_L^* = B_1 \otimes \cdots \otimes B_L$ for the *reversed* tensor product. Then $*: B \to B^*$ is defined by $(b_L \otimes \cdots \otimes b_1)^* \mapsto b_1^* \otimes \cdots \otimes b_L^*$.

Proposition 2.16. *Let R_j be the R-matrix acting at the jth and $(j+1)$st tensor positions from the right. On an L-fold tensor product of crystals of the form B^s,*

$$R_j \circ * = * \circ R_{L-j} \tag{2.10}$$

for $1 \leqslant j \leqslant L-1$.

Proof. One may reduce to the case $L = 2$. Since $B_2 \otimes B_1$ is connected, R is an isomorphism, and since (2.9) holds, it suffices to check (2.10) on $u(B_2 \otimes B_1)$. But this holds by weight considerations. $\square$

3. Virtual crystals

We review the virtual crystal construction [16,17]. This allows one to reduce the study of affine crystal graphs to those of simply-laced type.

3.1. Embeddings of affine algebras

Any affine algebra $\mathfrak{g}$ of type X can be embedded into a simply-laced affine algebra $\mathfrak{g}_Y$ of type Y [6]. For $\mathfrak{g}$ nonexceptional the embeddings are listed below. The notation $A_{2n}^{(2)}$ and $A_{2n}^{(2)\dagger}$ is used for two different vertex labelings of the same Dynkin diagram, in which α_0 is respectively the extra short and extra long root.

$$\begin{aligned} C_n^{(1)}, A_{2n}^{(2)}, A_{2n}^{(2)\dagger}, D_{n+1}^{(2)} &\hookrightarrow A_{2n-1}^{(1)}, \\ B_n^{(1)}, A_{2n-1}^{(2)} &\hookrightarrow D_{n+1}^{(1)}. \end{aligned} \tag{3.1}$$

3.2. Folding automorphism

Let σ be the following automorphism of the Dynkin diagram of Y. For $A_{2n-1}^{(1)}$, $\sigma(i) = 2n - i \pmod{2n}$. For type $D_{n+1}^{(1)}$, σ exchanges the nodes n and $n+1$ and fixes all others. Let I^X and I^Y be the vertex sets of the diagrams X and Y, respectively, I^Y/σ the set of orbits of the action of σ on I^Y, and $\iota : I^X \to I^Y/\sigma$ a bijection which preserves edges and sends 0 to 0.

Example 3.1. If $Y = A_{2n-1}^{(1)}$, then $\iota(0) = \{0\}$, $\iota(i) = \{i, 2n-i\}$ for $0 < i < n$ and $\iota(n) = \{n\}$. If $Y = D_{n+1}^{(1)}$, then $\iota(i) = i$ for $i < n$ and $\iota(n) = \{n, n+1\}$.

3.3. Embedding of weight lattices

For $i \in I^X$ define γ_i as follows.

(1) Let $Y = D_{n+1}^{(1)}$.
   (a) Suppose the arrow points towards the component of 0. Then $\gamma_i = 1$ for all $i \in I^X$.
   (b) Suppose the arrow points away from the component of 0. Then γ_i is the order of σ for i in the component of 0 and is 1, otherwise.

(2) Let $Y = A_{2n-1}^{(1)}$. Then $\gamma_i = 1$ for $1 \leqslant i \leqslant n-1$. For $i \in \{0, n\}$, $\gamma_i = 2$ (which is the order of σ) if the arrow incident to i points away from it and is 1, otherwise.

Example 3.2. For $X = B_n^{(1)}$ and $Y = D_{n+1}^{(1)}$ we have $\gamma_i = 2$ if $0 \leqslant i \leqslant n-1$ and $\gamma_n = 1$. For $X = A_{2n-1}^{(2)}$ and $Y = D_{n+1}^{(1)}$ we have $\gamma_i = 1$ for all i.

The embedding $\Psi : P^X \to P^Y$ of weight lattices is defined by

$$\Psi\left(\Lambda_i^X\right) = \gamma_i \sum_{j \in \iota(i)} \Lambda_j^Y.$$

As a consequence we have

$$\Psi\left(\alpha_i^X\right) = \gamma_i \sum_{j \in \iota(i)} \alpha_j^Y, \qquad \Psi\left(\delta^X\right) = a_0^X \gamma_0 \delta^Y. \tag{3.2}$$

3.4. Virtual crystals

Fix an embedding $\mathfrak{g}_X \hookrightarrow \mathfrak{g}_Y$ in (3.1). Let $\hat{V}$ be a Y-crystal. For $i \in I^X$ define the virtual crystal operators $\hat{e}_i$, $\hat{f}_i$ on $\hat{V}$, as the composites of Y-crystal operators e_j, f_j given by

$$\hat{e}_i = \prod_{j \in \iota(i)} e_j^{\gamma_i}, \qquad \hat{f}_i = \prod_{j \in \iota(i)} f_j^{\gamma_i}.$$

A *virtual crystal* (aligned in the sense of [16,17]) is an injection $\Psi : B \to \hat{V}$ from an X-crystal B to a Y-crystal $\hat{V}$ such that:

(1) for all $b \in B$, $i \in I^X$, and $j \in \iota(i) \subset I^Y$, $\varphi_j(\Psi(b)) = \gamma_i \varphi_i(b)$ and $\varepsilon_j(\Psi(b)) = \gamma_i \varepsilon_i(b)$;
(2) $\Psi \circ e_i = \hat{e}_i$ and $\Psi \circ f_i = \hat{f}_i$ for all $i \in I^X$.

A virtual crystal realizes the X-crystal B as the subset of the Y-crystal $\hat{V}$ given by its image under Ψ, equipped with the virtual Kashiwara operators $\hat{e}_i$ and $\hat{f}_i$.

A morphism g of virtual crystals $\Psi : B \to \hat{V}$ and $\Psi' : B' \to \hat{V}'$ consists of a morphism $g_X : B \to B'$ of X-crystals and a morphism $g_Y : \hat{V} \to \hat{V}'$ of Y-crystals, such that the diagram commutes:

$$\begin{tikzcd} B \arrow[r, "\Psi"] \arrow[d, "g_X"'] & \hat{V} \arrow[d, "g_Y"] \\ B' \arrow[r, "\Psi'"'] & \hat{V}'. \end{tikzcd}$$

An isomorphism g of virtual crystals is a morphism (g_X, g_Y) such that g_X (respectively g_Y) is an isomorphism of X- (respectively Y-) crystals.

3.5. Tensor product of virtual crystals

Let $\Psi : B \to \hat{V}$ and $\Psi' : B' \to \hat{V}'$ be virtual crystals. It is straightforward to verify that $\Psi \otimes \Psi' : B \otimes B' \to \hat{V} \otimes \hat{V}'$ is a virtual crystal. Virtual crystals form a tensor category [16].

3.6. Virtual B^s

We recall from [17] the virtual crystal construction of $B^s = B^{1,s}$ for $\mathfrak{g}$ of nonexceptional affine type. Let $\hat{V}^s$ be given by

$$\hat{V}^s = \begin{cases} B_Y^{s\vee} \otimes B_Y^s, & \text{if } \mathfrak{g}_Y = A_{2n-1}^{(1)}, \\ B_Y^s, & \text{if } \mathfrak{g}_Y = D_{n+1}^{(1)} \text{ and } \mathfrak{g} = A_{2n-1}^{(2)}, \\ B_Y^{2s}, & \text{if } \mathfrak{g}_Y = D_{n+1}^{(1)} \text{ and } \mathfrak{g} = B_n^{(1)}. \end{cases}$$

Theorem 3.3 [17]. *There is a unique virtual crystal $\Psi : B^s \to \hat{V}^s$ such that $\Psi(u(B^s)) = u(\hat{V}^s)$.*

Example 3.4. Let $X = B_3^{(1)}$ and $Y = D_4^{(1)}$. Then $\hat{V}^s = B_Y^{2s}$. Let $b = \boxed{1}\boxed{\circ}\boxed{\bar{2}} \in B_X^3$. Then

$$\Psi(b) = \boxed{1}\boxed{1}\boxed{3}\boxed{\bar{3}}\boxed{\bar{2}}\boxed{\bar{2}} \quad \text{and} \quad f_3(b) = \boxed{1}\boxed{\bar{3}}\boxed{\bar{2}}.$$

Furthermore,

$$\hat{f}_3\big(\Psi(b)\big) = f_3 \circ f_4\big(\Psi(b)\big) = \boxed{1}\boxed{1}\boxed{\bar{3}}\boxed{\bar{3}}\boxed{\bar{2}}\boxed{\bar{2}}.$$

3.7. Virtual R-matrix

Proposition 3.5 [17]. *Let $\hat{R} : \hat{V}^t \otimes \hat{V}^s \to \hat{V}^s \otimes \hat{V}^t$ be the composition of combinatorial R-matrices of type Y. Then the diagram commutes*:

$$\begin{tikzcd}
B^t \otimes B^s \arrow[r, "\Psi"] \arrow[d, "R"'] & \hat{V}^t \otimes \hat{V}^s \arrow[d, "\hat{R}"] \\
B^s \otimes B^t \arrow[r, "\Psi"'] & \hat{V}^s \otimes \hat{V}^t.
\end{tikzcd}$$

That is, the pair $(R, \hat{R})$ is an isomorphism of virtual crystals $\Psi : B^t \otimes B^s \to \hat{V}^t \otimes \hat{V}^s$ and $\Psi : B^s \otimes B^t \to \hat{V}^s \otimes \hat{V}^t$.

3.8. Virtual local coenergy

Proposition 3.6 [17]. *Let $\Psi : B \to \hat{V}$ and $\Psi' : B' \to \hat{V}'$ be virtual crystals where $B, B' \in \mathcal{C}$ both of type X. Then*

$$H^X_{B,B'} = \frac{1}{\gamma_0} \cdot H^Y_{\hat{V},\hat{V}'} \circ \big(\Psi \otimes \Psi'\big).$$

3.9. Virtual graded crystal

Proposition 3.7 [17]. *Let $B \in \mathcal{C}$ be a crystal of type X and $\Psi : B \to \hat{V}$ the corresponding virtual crystal. Then*

$$D^X = \frac{1}{\gamma_0} \cdot D^Y \circ \Psi.$$

3.10. Virtual X formula

Let $B \in \mathcal{C}$ be a crystal of type X. Let $\Psi : B \to \hat{V}$ be the corresponding virtual crystal. For $\lambda \in \overline{P}^+$ let $P^v(B,\lambda)$ be image under Ψ of the set $P(B,\lambda)$. Define the virtual X formula by

$$VX_{B,\lambda}(q) = \sum_{b \in P^v(B,\lambda)} q^{D(\Psi(b))/\gamma_0}.$$

Theorem 3.8 ($X = VX$ [17]). *For $\mathfrak{g}$ of nonexceptional affine type X and $B \in \mathcal{C}$ a crystal of type X, one has $X_{B,\lambda}(q) = VX_{B,\lambda}(q)$.*

3.11. Virtual crystals and $*$-duality

We believe that the following is true for any virtual crystal, namely, that up to R-matrices, Ψ takes the $*$ involution of type X to the $*$ involution of type Y.

Proposition 3.9. *Let $B^s \in \mathcal{C}$ and let $\Psi : B^s \to \hat{V}^s$ be a virtual crystal. Then the following diagram commutes, where ι is either a composition of R-matrices or the identity*:

$$\begin{tikzcd} B^s \arrow[r, "\Psi"] \arrow[d, "*"'] & \hat{V}^s \arrow[d, "\iota\circ *"] \\ B^s \arrow[r, "\Psi"'] & \hat{V}^s. \end{tikzcd}$$

Proof. Note that for the nonsimply-laced types X the Dynkin involution τ_X is the identity. The virtual raising and lowering operators are invariant under τ_Y. It is therefore sufficient to check the above commutation on $v \in P(B^s)$, where $P(B^s)$ is the set of classical highest weight vectors in B^s. But B^s and $\hat{V}^s$ are multiplicity-free as a classical crystals and $*$ stabilizes classical components and modifies the weight of a crystal element by applying w_0. The following are equivalent:

(1) $\Psi(v)^* \in \hat{V}^s$ is a classical lowest weight vector and $\mathrm{wt}(\Psi(v)^*) = w_0^Y(\Psi(\lambda))$.
(2) $\Psi(v) \in P(\hat{V}^s)$ and $\mathrm{wt}(\Psi(v)) = \Psi(\lambda)$.
(3) $v \in P(B^s)$ and $\mathrm{wt}(v) = \lambda$.
(4) $v^* \in B^s$ is a classical lowest weight vector and $\mathrm{wt}(v^*) = w_0^X\lambda$.
(5) $\Psi(v^*)$ is a classical lowest weight vector and $\mathrm{wt}(\Psi(v^*)) = \Psi(w_0^X\lambda)$.

One may verify that $w_0^Y(\Psi(\lambda)) = \Psi(w_0^X(\lambda))$ using linearity, to reduce to the case $\lambda = \overline{\Lambda}_i^X$ for $i \in \bar{I}$. It follows that $\Psi(v)^*$ and $\Psi(v^*)$ are classical lowest weight vectors in $\hat{V}^s$ of the same weight. But then they must be equal. $\square$

4. Right splitting

Let $\mathfrak{g}$ be of nonexceptional affine type. We define a family of $U_q(\bar{\mathfrak{g}})$-crystal embeddings which is well behaved with respect to intrinsic coenergy. They are denoted $\mathrm{rs} := \mathrm{rs}_{r;a,b}$ which stands for "right-split," because when $b = 0$, the map splits off the rightmost column of an element in $B^{r,s}$.

Conjecture 4.1. *Let $a - 2 \geqslant b \geqslant 0$. Suppose $\mathcal{C}'$ is a set of KR crystals whose modules have been constructed, which contains $B^{r,s}$ for a particular $r \in \bar{I}$ and all $s \in \mathbb{Z}_{>0}$. Then there is an injective $U_q(\bar{\mathfrak{g}})$-crystal morphism*

$$\mathrm{rs}_{r;a,b} : B^{r,a} \otimes B^{r,b} \to B^{r,a-1} \otimes B^{r,b+1}$$

such that for any crystal B which is the tensor product of crystals in $\mathcal{C}'$, the map

$$1_B \otimes \mathrm{rs}_{r;a,b} : B \otimes B^{r,a} \otimes B^{r,b} \to B \otimes B^{r,a-1} \otimes B^{r,b+1} \tag{4.1}$$

is an injective $U_q(\bar{\mathfrak{g}})$-crystal morphism which preserves intrinsic coenergy.

Theorem 4.2. *Conjecture* 4.1 *holds for $\mathfrak{g} = A_n^{(1)}$ for all $r \in \bar{I}$ and the set $\mathcal{C}'$ of all KR crystals.*

Proof. This follows from [20–22]. □

Theorem 4.3. *Conjecture* 4.1 *holds for any nonexceptional affine algebra $\mathfrak{g}$ for $r = 1$ and $\mathcal{C}'$ the set of KR crystals of the form $B^{1,s}$.*

The proof of Theorem 4.3 occupies the remainder of this section.

4.1. Explicit definition of splitting

This paper only requires the case $b = 0$ for the map rs. Except for type $A_n^{(1)}$ only the case $r = 1$ is needed. For $s \geqslant 2$ define the map $\mathrm{rs} := \mathrm{rs}_{1;s,0}$ as follows. For types $A_n^{(1)}$, $D_n^{(1)}$, $B_n^{(1)}$, and $A_{2n-1}^{(2)}$, define $\mathrm{rs} : B^s \to B^{s-1} \otimes B^1$ by $\mathrm{rs}(wx) = w \otimes x$ for $x \in B^1$ and $w \in B^{s-1}$ such that $wx \in B^s$. For the other types, in addition to the above rules we have $\mathrm{rs}(x) = \emptyset \otimes x$ for $x \in B(\overline{\Lambda}_1) \subseteq B^s$, and $\mathrm{rs}(\emptyset) = \bar{1} \otimes 1$. For $B^{r,s} \in \mathcal{C}^A$ and $b \in B^{r,s}$, let $\mathrm{rs}(b) = b_2 \otimes b_1$, where b_1 is the rightmost column of the rectangular tableau b and b_2 is the rest of b.

Remark 4.4. Suppose $s \geqslant 2$. Here $r = 1$ for $\mathcal{C}$. For $B^{r,s} \in \mathcal{C}$ (or $\mathcal{C}^A$) we write rs for the map $1_B \otimes \mathrm{rs}$ on $B \otimes B^{r,s}$ and write $\mathrm{rs}(B \otimes B^{r,s}) := B \otimes B^{r,s-1} \otimes B^{r,1}$.

4.2. Simply-laced $\mathfrak{g}$

The case $\mathfrak{g} = A_n^{(1)}$ is covered by Theorem 4.2. The other simply-laced nonexceptional family is $\mathfrak{g} = D_n^{(1)}$.

It is straightforward to check directly using the explicit description of B^s in [17] that rs is an injective $U_q(\bar{\mathfrak{g}})$-crystal morphism. Let B be the tensor product of crystals in $\mathcal{C}'$. To check that $1_B \otimes \mathrm{rs}$ preserves intrinsic coenergy, by (2.4) it suffices to check this property for B the trivial crystal and for $B = B^t$. Since $1_B \otimes \mathrm{rs}$ is a $U_q(\bar{\mathfrak{g}})$-crystal morphism, it is sufficient to prove that intrinsic coenergy is preserved for classical highest weight vectors. Suppose B is trivial. By (2.3) B^s has a single classical highest weight vector, namely, $u(B^s) = 1^s$. By Example 2.3 $D_{B^s} = 0$. On the other hand $\mathrm{rs}(1^s) = 1^{s-1} \otimes 1 = u(B^{s-1} \otimes B^1)$ so its intrinsic energy is also zero. For $B = B^t$ we require the following lemma, which is easily verified directly.

Lemma 4.5. *For $\mathfrak{g} = D_n^{(1)}$ and $s, t \geqslant 1$, $P(B^t \otimes B^s)$ consists of the elements*

$$v_{p,q}^{t,s} = 1^{t-p-q}\, 2^p\, \bar{1}^q \otimes 1^s$$

where $p + q \leqslant \min(s, t)$. In particular, $B^t \otimes B^s$ is multiplicity-free as a $U_q(D_n)$-crystal.

Recall that D_{B^t} and D_{B^s} are identically zero by Example 2.3. By (2.4) and explicit calculation,

$$D_{B^t \otimes B^s}\big(v_{p,q}^{t,s}\big) = H_{B^t,B^s}\big(v_{p,q}^{t,s}\big) = p + 2q. \tag{4.2}$$

Since R is a $U_q(D_n^{(1)})$-crystal isomorphism, Lemma 4.5 implies that

$$R_{B^t,B^s}\big(v_{p,q}^{t,s}\big) = v_{p,q}^{s,t}.$$

We compute $H_{B^t,B^{s-1}\otimes B^1}$ on $\mathrm{rs}(v_{p,q}^{t,s}) = (1^{t-p-q}2^p\bar{1}^q \otimes 1^{s-1} \otimes 1)$ using Proposition 2.2. We have

$$R_{B^t,B^{s-1}}\big(1^{t-p-q}2^p\bar{1}^q \otimes 1^{s-1}\big) = \begin{cases} 1^{s-1-p-q}2^p\bar{1}^q \otimes 1^t, & \text{if } p+q<s, \\ \bar{1}^{q-1} \otimes 1^{t-1}\bar{1}, & \text{if } p+q=s,\ q=s, \\ 2^{p-1}\bar{1}^q \otimes 1^{t-1}2, & \text{if } p+q=s,\ q<s. \end{cases}$$

By (4.2) we have

$$H_{B^t,B^{s-1}}\big(1^{t-p-q}2^p\bar{1}^q \otimes 1^{s-1}\big) = \begin{cases} p+2q, & \text{if } p+q<s, \\ p+2q-2, & \text{if } p+q=s,\ q=s, \\ p+2q-1, & \text{if } p+q=s,\ q<s. \end{cases}$$

By (4.2) we have $H(1^t \otimes 1) = 0$, $H(1^{t-1}\bar{1} \otimes 1) = 2$, and $H(1^{t-1}2 \otimes 1) = 1$. It follows in any case that $1_{B^t} \otimes \mathrm{rs}$ preserves intrinsic coenergy.

4.3. *Nonsimply-laced* $\mathfrak{g}$

Suppose $\mathfrak{g}$ is not simply-laced. Let $\mathfrak{g} \hookrightarrow \mathfrak{g}_Y$ be as in (3.1). It is not hard to show that rs_X is a $U_q(\bar{\mathfrak{g}})$-crystal injection. To show that the map (4.1) preserves intrinsic coenergy (and thereby complete the proof of Theorem 4.3), by Proposition 3.7 the following result suffices.

Proposition 4.6. *There is an injective $U_q(\mathfrak{g}_Y)$-crystal map $\widehat{\mathrm{rs}} : \hat{V}^s \to \hat{V}^{s-1} \otimes \hat{V}^1$ such that*:

(1) *the following diagram commutes*:

$$\begin{array}{ccc} B_X^s & \xrightarrow{\Psi} & \hat{V}^s \\ \downarrow{\scriptstyle \mathrm{rs}_X} & & \downarrow{\scriptstyle \widehat{\mathrm{rs}}} \\ B_X^{s-1} \otimes B_X^1 & \xrightarrow[\Psi\otimes\Psi]{} & \hat{V}^{s-1} \otimes \hat{V}^1; \end{array} \tag{4.3}$$

(2) *for any $B \in \mathcal{C}$, let $\Psi : B \to \hat{V}$ be its virtual crystal embedding. Then $1_{\hat{V}} \otimes \widehat{\mathrm{rs}}$ preserves intrinsic coenergy*;

(3) *if $v \in \hat{V}^s$ and $\widehat{\mathrm{rs}}(v) \in \mathrm{Im}(\Psi \otimes \Psi)$ then $v \in \mathrm{Im}(\Psi)$.*

Proof. Suppose first that $Y = A^{(1)}_{2n-1}$. Then $\hat{V}^s = B^{s\vee}_Y \otimes B^s_Y$. Define the map $\widehat{\mathrm{rs}}: \hat{V}^s \to \hat{V}^{s-1} \otimes \hat{V}^1$ by the composition

$$\begin{aligned} B^{s\vee}_Y \otimes B^s_Y &\xrightarrow{1\otimes \mathrm{rs}_Y} B^{s\vee}_Y \otimes B^{s-1}_Y \otimes B^1_Y \xrightarrow{R} B^{s-1}_Y \otimes B^1_Y \otimes B^{s\vee}_Y \\ &\xrightarrow{1\otimes 1\otimes \mathrm{rs}^\vee_Y} B^{s-1}_Y \otimes B^1_Y \otimes B^{s-1\vee}_Y \otimes B^{1\vee}_Y \xrightarrow{R} B^{s-1\vee}_Y \otimes B^{s-1}_Y \otimes B^{1\vee}_Y \otimes B^1_Y. \end{aligned} \tag{4.4}$$

Here $\mathrm{rs}^\vee_Y(wx) = w \otimes x$ where $wx \in B^{s\vee}_Y$, $w \in B^{s-1\vee}_Y$ and $x \in B^{1\vee}_Y$. Note that (4.4) is a composition of combinatorial R-matrices and rs maps for type A. Point (2) holds by Theorem 4.2.

One need only verify (1) on classical highest weight vectors, by the definition of Ψ and the fact that rs_Y and $\mathrm{rs}^\vee_Y$ (respectively rs_X) are morphisms of $U_q(\overline{Y})$- (respectively $U_q(\overline{X})$-) crystals.

Let $N = 2n$. The classical highest weight vectors in B^s_X have the form 1^{s-p} for $0 \leqslant p \leqslant s$; if $\mathfrak{g}$ is $C^{(1)}_n$ or $A^{(2)\dagger}_{2n}$ then p must also be even. For $p < s$ the element $1^{s-p} \in B((s-p)\overline{\Lambda}_1) \subset B^s_X$ is sent to the following elements under the maps in (4.3):

$$\begin{array}{cc} 1^{s-p} & 1^{s-1-p} \otimes 1 \\ N^{\vee(s-p)} 1^{\vee p} \otimes 1^s & N^{\vee s-p-1} 1^{\vee p} \otimes 1^{s-1} \otimes N^\vee \otimes 1 \end{array}$$

where the intermediate results under the maps in (4.4) are given by $N^{\vee(s-p)} 1^{\vee p} \otimes 1^{s-1} \otimes 1$, $1^{s-p-1} N^p \otimes 1 \otimes N^{\vee s}$, $1^{s-p-1} N^p \otimes 1 \otimes N^{\vee(s-1)} \otimes N^\vee$, and $N^{\vee s-p-1} 1^{\vee p} \otimes 1^{s-1} \otimes N^\vee \otimes 1$.

Under the maps in (4.3), the element $\emptyset$ is sent to

$$\begin{array}{cc} \emptyset & \bar{1} \otimes 1 \\ N^s \otimes N^{\vee s} & 1^{\vee s-1} \otimes 1^{s-2} N \otimes N^\vee \otimes 1 \end{array}$$

with intermediate values in (4.4) given by $N^s \otimes N^{\vee s-1} \otimes N^\vee$, $1^{\vee s-1} \otimes 1^\vee \otimes 1^s$, $1^{\vee s-1} \otimes 1^\vee \otimes 1^{s-1} \otimes 1$, and $1^{\vee s-1} \otimes 1^{s-2} N \otimes N^\vee \otimes 1$.

Since these are all the possible classical highest weight vectors, point (1) follows.

For point (3), let $v \in \hat{V}^s$ and $\widehat{\mathrm{rs}}(v) \in \mathrm{Im}(\Psi \otimes \Psi)$. Without loss of generality we may assume that $v \in P(\hat{V}^s)$ since $\widehat{\mathrm{rs}}$ is a $U_q(\overline{Y})$-morphism. Now v must have the form $v_{s,p} := N^{\vee(s-p)} 1^{\vee p} \otimes 1^s$ for $0 \leqslant p \leqslant s$. By computations similar to those above, $\widehat{\mathrm{rs}}(v) = v_{s-1,p} \otimes \Psi(1)$ if $p < s$ and $\widehat{\mathrm{rs}}(v) = 1^{\vee(s-1)} \otimes 1^{s-2} N \otimes \Psi(1)$ if $p = s$. But $\widehat{\mathrm{rs}}(v) \in \mathrm{Im}(\Psi \otimes \Psi)$ means that $v_{s-1,p} \in \mathrm{Im}(\Psi)$ if $p < s$ and $1^{\vee(s-1)} \otimes 1^{s-2} N \in \mathrm{Im}(\Psi)$ if $p = s$. The parity condition for this to occur implies the parity condition that guarantees that $v_{s,p} \in \mathrm{Im}(\Psi)$.

Suppose next that $Y = D^{(1)}_{n+1}$ and $X = A^{(2)}_{2n-1}$. Then $\hat{V}^s = B^s_Y$. Define $\widehat{\mathrm{rs}} = \mathrm{rs}_Y : B^s_Y \to B^{s-1}_Y \otimes B^1_Y$. Point (2) follows by the simply-laced $D^{(1)}_n$ case. Point (3) is trivial. For point (1) it is enough to consider elements of $P(B^s) = \{1^s\}$. Under the maps in (4.3), 1^s goes to

$$\begin{array}{cc} 1^s & 1^{s-1} \otimes 1 \\ 1^s & 1^{s-1} \otimes 1 \end{array}$$

and (4.3) commutes.

Suppose that $Y = D_{n+1}^{(1)}$ and $X = B_n^{(1)}$. Then $\hat{V}^s = B_Y^{2s}$. Define $\widehat{\mathrm{rs}} : B_Y^{2s} \to B_Y^{2s-2} \otimes B_Y^2$ by $wv \mapsto w \otimes v$ where $wv \in B_Y^{2s}$, $w \in B_Y^{2s-2}$, and $v \in B_Y^2$. This map is clearly injective and $U_q(\overline{Y})$-equivariant. Point (3) is obvious. For point (1) it is enough to consider the unique element $1^s \in P(B_X^s)$. Under (4.3) 1^s goes to

$$\begin{array}{cc} 1^s & 1^{s-1} \otimes 1 \\ 1^{2s} & 1^{2s-2} \otimes 1^2 \end{array}$$

so that (4.3) commutes. For point (2) define $\widehat{\mathrm{rs}}' : B_Y^{2s} \to B_Y^{2s-2} \otimes B_Y^1 \otimes B_Y^1$ by the composite map

$$\begin{aligned} B_Y^{2s} &\xrightarrow{\mathrm{rs}_Y} B_Y^{2s-1} \otimes B_Y^1 \xrightarrow{R} B_Y^1 \otimes B_Y^{2s-1} \\ &\xrightarrow{1 \otimes \mathrm{rs}_Y} B_Y^1 \otimes B_Y^{2s-2} \otimes B_Y^1 \xrightarrow{R} B_Y^{2s-2} \otimes B_Y^1 \otimes B_Y^1. \end{aligned}$$

Since $\widehat{\mathrm{rs}}'$ is the composition of rs_Y maps and R-matrices, it preserves intrinsic coenergy by the simply-laced case. It suffices to show that

$$\begin{tikzcd} B_Y^{2s} \arrow[d, "\widehat{\mathrm{rs}}"'] \arrow[dr, "\widehat{\mathrm{rs}}'"] & \\ B_Y^{2s-2} \otimes B_Y^2 \arrow[r, "1 \otimes \mathrm{rs}_Y"'] & B_Y^{2s-2} \otimes B_Y^1 \otimes B_Y^1 \end{tikzcd}$$

commutes since $\widehat{\mathrm{rs}}'$ and $1 \otimes \mathrm{rs}_Y$ both preserve intrinsic coenergy. It suffices to check this for the lone classical highest weight vector $1^{2s} \in P(B_Y^{2s})$. Clearly $\widehat{\mathrm{rs}}'(1^{2s}) = 1^{2s-2} \otimes 1 \otimes 1$, while $\widehat{\mathrm{rs}}(1^{2s}) = 1^{2s-2} \otimes 1^2$ and this is sent by $1 \otimes \mathrm{rs}_Y$ to $1^{2s-2} \otimes 1 \otimes 1$, as desired. □

5. Left splitting and duality

We define dual analogues of the intrinsic coenergy D and right splitting.

5.1. *Tail coenergy*

For $B^s \in \mathcal{C}$ define $\overleftarrow{D}_{B^s} = D_{B^s}$. For $B^{r,s} \in \mathcal{C}^A$, define $\overleftarrow{D}_{B^{r,s}} = D_{B^{r,s}} = 0$. If $B_1, B_2, \ldots, B_L \in \mathcal{C}$ (or $\mathcal{C}^A$) and $B = B_L \otimes \cdots \otimes B_1$ are such that $\overleftarrow{D}_{B_j} : B_j \to \mathbb{Z}_{\geqslant 0}$ are given, then define

$$\overleftarrow{D}_B = \sum_{1 \leqslant i < j \leqslant L} H_{j-1} R_{j-2} \ldots R_{i+1} R_i + \sum_{j=1}^{L} \overleftarrow{D}_{B_j} R_{L-1} R_{L-2} \ldots R_j \tag{5.1}$$

with $\overleftarrow{D}_{B_j}$ acting on the leftmost tensor position. This is a different associative tensor product on graded crystals than the one given in Section 2.5.

Recall the notation B^* of Remark 2.15.

Proposition 5.1. *Let $B \in \mathcal{C}$ (or $B \in \mathcal{C}^A$) and $b \in B$. Then $\overleftarrow{D}_B(b) = D_{B^*}(b^*)$.*

Proof. For B a single KR crystal, the result follows from the fact that the involution $*$ on B stabilizes classical components. By Proposition 2.16 and comparing (5.1) with (2.4) it suffices to show that

$$H_{B_1,B_2}(b^*) = H_{B_2,B_1}(b) \tag{5.2}$$

for B_1, B_2 KR crystals. Since $B_2 \otimes B_1$ is connected, the proof may proceed by induction on the number of steps (either of the form e_i or f_i) in $B_2 \otimes B_1$ from $u(B_2 \otimes B_1)$ to b. Suppose first that $b = u(B_2 \otimes B_1)$. By the definition of $u(B)$ in Section 2.3, $B_2 \otimes B_1$ (and therefore $B_1 \otimes B_2$) contain a unique classical component isomorphic to $B(\lambda)$ where $\lambda = \mathrm{wt}(b)$. And $B(\lambda)$ contains a unique vector of the extremal weight $w_0\lambda$. Since $\mathrm{wt}(b^*) = w_0\mathrm{wt}(b)$ it follows that b^* and $u(B_1 \otimes B_2)$ are in the same classical component, so that $H_{B_1,B_2}(b^*) = H_{B_1,B_2}(b) = 0$ by the definition of H.

Now suppose $b = f_i(c)$ where c is closer to $u(B_2 \otimes B_1)$ than b is. If $i \neq 0$ then we are done since both sides of (5.2) do not change under passing from c to b, by the definition of H and (2.9). So assume $i = 0$. By (2.9) $b^* = e_0(c^*)$. But then one may conclude the validity of (5.2) for b from that of c using rules for the Kashiwara operators on the tensor product and (2.1). □

Define $\overleftarrow{X}$ just like the one-dimensional sum X but use $\overleftarrow{D}_B$ instead of D_B. Proposition 5.1 has this corollary.

Corollary 5.2. $\overleftarrow{X}(B, \lambda) = X(B, \lambda)$.

5.2. *Left splitting*

Whenever the right-splitting map $\mathrm{rs}: B^{r,s} \to B^{r,s-1} \otimes B^{r,1}$ is defined, we may define the left-splitting map $\mathrm{ls}: B^{r,s} \to B^{r,1} \otimes B^{r,s-1}$ by the commutation of the diagram

$$\begin{tikzcd} B^{r,s} \arrow[r, "\mathrm{ls}"] \arrow[d, "*"'] & B^{r,1} \otimes B^{r,s-1} \arrow[d, "*"] \\ B^{r,s} \arrow[r, "\mathrm{rs}"'] & B^{r,s-1} \otimes B^{r,1}. \end{tikzcd} \tag{5.3}$$

In particular, it is defined for $B^s \in \mathcal{C}$ and $B^{r,s} \in \mathcal{C}^A$.

Corollary 5.3. *Here $r = 1$ for the category $\mathcal{C}$.* ls *is a $U_q(\bar{\mathfrak{g}})$-crystal embedding such that, for any $B^{r,s} \in \mathcal{C}$ (or $\mathcal{C}^A$) and for any $B \in \mathcal{C}$ (or $\mathcal{C}^A$), the map*

$$\mathrm{ls} \otimes 1_B : B^{r,s} \otimes B \to B^{r,1} \otimes B^{r,s-1} \otimes B$$

is injective and preserves $\overleftarrow{D}$.

Proof. ls is a $U_q(\bar{\mathfrak{g}})$-crystal embedding since rs is, by Theorem 4.3, the definition of $*$ and (5.3). For the preservation of $\overleftarrow{D}$, let $b_1 \otimes b_2 \in B^{r,s} \otimes B$. We have

$$\overleftarrow{D}\big(\mathrm{ls}(b_1) \otimes b_2\big) = D\big(b_2^* \otimes \mathrm{ls}(b_1)^*\big) = D\big(b_2^* \otimes \mathrm{rs}\big(b_1^*\big)\big) = D\big(b_2^* \otimes b_1^*\big) = \overleftarrow{D}(b_1 \otimes b_2)$$

by Proposition 5.1 and (5.3). □

Remark 5.4. Suppose $s \geqslant 2$. Here $r = 1$ for $\mathcal{C}$ as usual. For $B^{r,s} \in \mathcal{C}$ (or $\mathcal{C}^A$) we write ls for the map $\mathrm{ls} \otimes 1_B$ on $B^{r,s} \otimes B$. Also we write $\mathrm{ls}(B^{r,s} \otimes B) := B^{r,1} \otimes B^{r,s-1} \otimes B$.

5.3. *Explicit left-splitting*

Lemma 5.5. *For $B^s \in \mathcal{C}$ the map $\mathrm{ls} : B^s \to B^1 \otimes B^{s-1}$ is given explicitly by $\mathrm{ls}(xw) = x \otimes w$ for $x \in B^1$ and $w \in B^{s-1}$ such that $xw \in B^s$, $\mathrm{ls}(x) = x \otimes \emptyset$ for $x \in B(\overline{\Lambda}_1) \subseteq B^s$, $\mathrm{ls}(\emptyset) = \bar{1} \otimes 1$. For $B^{r,s} \in \mathcal{C}^A$ and $b \in B^{r,s}$, $\mathrm{ls}(b) = b_2 \otimes b_1$ where b_2 is the leftmost column in the $r \times s$ semistandard tableau b and b_1 is the rest of b.*

5.4. *Box-splitting*

Let $B^{r,1} \in \mathcal{C}^A$ with $r \geqslant 2$. There is a $U_q(\bar{\mathfrak{g}})$-crystal embedding $\mathrm{lb} : B^{r,1} \to B^{1,1} \otimes B^{r-1,1}$ given by $b \mapsto b_2 \otimes b_1$ where b_2 is the bottommost entry in the column tableau b of height r, and b_1 is the remainder of b. There is a $U_q(\bar{\mathfrak{g}})$-crystal embedding $\mathrm{rb} : B^{r,1} \to B^{r-1,1} \otimes B^{1,1}$ given by $b \mapsto b_2 \otimes b_1$ where b_1 is the topmost entry in the column b and b_2 is the rest of b.

The map lb is only used to define the path–RC bijection for $B \in \mathcal{C}^A$ in Section 8.

In general, morphism rb does not preserve intrinsic coenergy, but another grading called intrinsic energy. It was proved in [15] that the path–RC bijection preserves the grading for $\mathcal{C}^A$ using a different method, namely, the rank-level duality for type $A^{(1)}$.

5.5. *Projections and commutations*

Define the ("left-hat") map $\mathrm{lh} : B_2 \otimes B_1 \to B_1$ by $b_2 \otimes b_1 \mapsto b_1$. It just removes the left tensor factor. Define the "right-hat" map $\mathrm{rh} : B_2 \otimes B_1 \to B_2$ by $b_2 \otimes b_1 \mapsto b_2$.

It is immediate that the following diagram commutes:

$$\begin{tikzcd} B_2 \otimes B_1 \arrow[r, "\mathrm{lh}"] \arrow[d, "*"'] & B_1 \arrow[d, "*"] \\ B_1 \otimes B_2 \arrow[r, "\mathrm{rh}"'] & B_1. \end{tikzcd} \tag{5.4}$$

Let $P(B)$ be the set of classical highest weight vectors in B, or equivalently, the set of classical components of B.

Lemma 5.6. *The maps* $\mathrm{lh}: B_2 \otimes B_1 \to B_1$ *and* $\mathrm{rh}: B_2 \otimes B_1 \to B_2$ *induce maps* $\mathrm{lh}: P(B_2 \otimes B_1) \to P(B_1)$ *and* $\mathrm{rh}: P(B_2 \otimes B_1) \to P(B_2)$.

Proof. If $b_2 \otimes b_1$ is a classical highest weight vector of $B_2 \otimes B_1$ then by the definitions, b_1 is a classical highest weight vector of B_1. Thus lh is well defined on components.

For rh we work with classical components. By (2.9) the map $*$ takes classical components to classical components. But then rh is well defined on components since lh is, by (5.4). □

Example 5.7. Let $b = \boxed{3} \otimes \boxed{2\,|\,\bar{2}} \otimes \boxed{1\,|\,2} \otimes \boxed{1} \in P(B^1 \otimes B^2 \otimes B^2 \otimes B^1)$ of type $D_4^{(1)}$. Then

$$\mathrm{lh}(b) = \boxed{2\,|\,\bar{2}} \otimes \boxed{1\,|\,2} \otimes \boxed{1}$$

and

$$\mathrm{rh}(b) = \boxed{3} \otimes \boxed{2\,|\,\bar{2}} \otimes \boxed{1\,|\,2}.$$

The induced map on highest weight vectors yields $\mathrm{rh}(b) = \boxed{3} \otimes \boxed{2\,|\,2} \otimes \boxed{1\,|\,1}$.

One has the commutation of induced maps on classical highest weight vectors:

$$\begin{tikzcd}
P(B_2 \otimes B_1) \arrow[r, "\mathrm{lh}"] \arrow[d, "*"'] & P(B_1) \arrow[d, "*"] \\
P(B_1 \otimes B_2) \arrow[r, "\mathrm{rh}"'] & P(B_1).
\end{tikzcd}$$

Remark 5.8. From now on, unless explicitly indicated otherwise, we only consider the map lh (respectively rh) on tensor products whose left (respectively right) factor is B^1. In these cases, we use the notation $\mathrm{lh}(B^1 \otimes B) = B$ and $\mathrm{rh}(B \otimes B^1) = B$.

For $\lambda \in \overline{P}^+$ let

$$\lambda^- = \{\mu \in \overline{P}^+ \mid B(\lambda) \text{ occurs in } B^1 \otimes B(\mu)\}$$

where B^1 is regarded as a $U_q(\bar{\mathfrak{g}})$-crystal by restriction.

By Lemma 5.6 there are well-defined bijections

$$\mathrm{lh}: P(B, \lambda) \to \bigcup_{\mu \in \lambda^-} P(\mathrm{lh}(B), \mu), \tag{5.5}$$

$$\mathrm{rh}\colon P(B,\lambda) \to \bigcup_{\mu\in\lambda^-} P\big(\mathrm{rh}(B),\mu\big) \tag{5.6}$$

except in the case $\mathfrak{g} = D_{n+1}^{(2)}$. Note that B^1 has at most one vector of each weight except when $\mathfrak{g} = D_{n+1}^{(2)}$, which has two vectors 0 and $\emptyset$ of weight 0. If $\mu = \lambda$, then there can be elements $b \in P(\mathrm{lh}(B),\lambda)$ such that both $0 \otimes b$ and $\emptyset \otimes b$ are in $P(B,\lambda)$. If so, then the right-hand side of (5.5) must be modified to include two copies of b, one coming from $\emptyset \otimes b$ and the other from $0 \otimes b$. There is no analogous problem for rh since $0 \notin P(B^1)$.

Proposition 5.9. *Let $r = r' = 1$ for $\mathcal{C}$.*

(1) $[\mathrm{lh},\mathrm{rh}] = 0$ *on* $B^1 \otimes B \otimes B^1$.
(2) $[\mathrm{lh},\mathrm{rs}] = 0$ *on* $B^1 \otimes B \otimes B^{r,s}$ *for* $s \geqslant 2$.
(3) $[\mathrm{rh},\mathrm{ls}] = 0$ *on* $B^{r,s} \otimes B \otimes B^1$ *for* $s \geqslant 2$.
(4) $[\mathrm{ls},\mathrm{rs}] = 0$ *on* $B^{r,s} \otimes B \otimes B^{r',s'}$ *for* $s, s' \geqslant 2$.
(5) $* \circ \mathrm{lh} = \mathrm{rh} \circ *$ *on* $B^1 \otimes B$.

Moreover, these commutations also hold for the induced maps on sets of classical highest weight vectors.

Proof. The operators on the entire crystals commute more or less by definition. We now prove that these identities hold for the induced maps between sets of classical highest weight vectors.

The proof is again trivial except for cases involving rh. Point (1) follows from Lemma 5.6. Point (3) follows from Lemma 5.6 and the $U_q(\bar{\mathfrak{g}})$-equivariance of ls given in Corollary 5.3. Finally, point (5) follows from Lemma 5.6 and the fact (2.9) that the map $*$ respects classical raising and lowering operators. □

5.6. Right hat and classical highest weight vectors

We need to know precisely how the highest weights change when passing from an element of $P(B^1 \otimes B \otimes B^1)$ to $P(B)$ via either $\mathrm{rh} \circ \mathrm{lh}$ or $\mathrm{lh} \circ \mathrm{rh}$. In this section we assume type D_n. The answer is given by van Leeuwen [23]. We translate his answer into the language of partitions.

Let P be the set of dominant weights that can occur in a tensor product of crystals of the form $B(\overline{\Lambda}_1)$. A dominant weight $\sum_{i=1}^n a_i \overline{\Lambda}_i$ is in P if and only if a_{n-1} and a_n have the same parity. We put a graph structure on P by declaring that weights λ and μ are adjacent if there is an element $x \in B(\overline{\Lambda}_1)$ such that $\lambda - \mu = \mathrm{wt}(x)$.

We realize P as a subset of $\mathbb{Z}^n$ by letting $\overline{\Lambda}_i = (1^i, 0^{n-i})$ for $1 \leqslant i \leqslant n-2$, $\overline{\Lambda}_{n-1} = \frac{1}{2}(1^n)$ and $\overline{\Lambda}_n = \frac{1}{2}(1^{n-1}, -1)$. As such P is given by the tuples $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{n-1} \geqslant |\lambda_n|$.

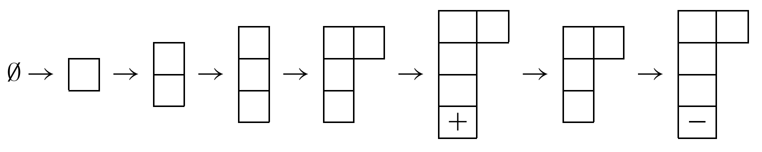
We modify this notation slightly in order to use partitions. Let Y be the lattice of partitions $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n) \in \mathbb{Z}^n_{\geqslant 0}$ with at most n parts. A graph structure on Y is given by declaring that two partitions are connected with an edge if their partition diagrams differ by one cell. Define the graph G by glueing two copies Y_+ and Y_- of Y together such that, if $\lambda \in Y$ is such that $\lambda_n = 0$, then $\lambda \in Y_+$ and $\lambda \in Y_-$ are identified.

Then $P \cong G$ where the weight $(\mu_1, \ldots, \mu_n)$ is identified with the partition $(\mu_1, \mu_2, \ldots, \mu_{n-1}, 0)$ if $\mu_n = 0$, with the "positive" partition $(\mu_1, \ldots, \mu_{n-1}, \mu_n) \in Y_+$ if $\mu_n > 0$, and with the "negative" partition $(\mu_1, \ldots, \mu_{n-1}, -\mu_n) \in Y_-$ if $\mu_n < 0$.

Let μ and λ be adjacent in P, and $x \in B(\overline{\Lambda}_1)$ such that $\lambda - \mu = \mathrm{wt}(x)$. We think of this as walking from μ to λ by the step x. In terms of partitions, if $x = i$ for $1 \leqslant i \leqslant n-1$ then a cell is added to the ith row. If $x = \bar{i}$ for $1 \leqslant i \leqslant n-1$ then a cell is removed from the ith row. If $i = n$ then the above rules hold provided that $\lambda, \mu \in Y_+$. If $\lambda, \mu \in Y_-$ then the roles of n and $\bar{n}$ are reversed.

Let $B = B(\overline{\Lambda}_1)^{\otimes L}$. Let $b = b_L \ldots b_1 \in P(B)$ with $b_j \in B(\overline{\Lambda}_1)$. In the usual way, b can be regarded as a path in the set of dominant weights: the ith weight is given by the weight of $b_i \ldots b_1$. Alternatively b describes a walk in G from the empty partition to the element of G corresponding to the weight of b.

Example 5.10. Let $n = 4$. Consider $b = \bar{4}\bar{4}41321 \in P(B)$ where $B = B(\overline{\Lambda}_1)^{\otimes 7}$. The element b corresponds to the walk in G given by



where the $+$ and $-$ markings on a partition indicate membership in Y_+ and Y_-, respectively.

In the following proposition, for weights $\lambda, \mu \in P$, we write $\mu \subset \lambda$ if the corresponding elements of G are both in Y_+ or both in Y_- and the diagram of the partition associated with μ is contained in that of λ.

Proposition 5.11. *Suppose $b \in P(B^1 \otimes B \otimes B^1, \lambda)$, $\mathrm{rh}(b) \in P(B^1 \otimes B, \alpha)$, $\mathrm{lh}(b) \in P(B \otimes B^1, \beta)$ and $\mathrm{rh}(\mathrm{lh}(b)) = \mathrm{lh}(\mathrm{rh}(b)) \in P(B, \gamma)$. Then α is uniquely determined by λ, β, and by γ. More precisely,*

(1) *If $|\lambda| = |\gamma| + 2$:*
   (a) *if the cells λ/β and β/γ are in different rows and different columns, then $\alpha = \lambda - \{\beta/\gamma\}$;*
   (b) *if λ/β and β/γ are in the same row or in the same column, then $\alpha = \beta$.*

(2) *If $|\lambda| = |\gamma| - 2$:*
   (a) *if the cells β/λ and γ/β are in different rows and different columns, then $\alpha = \lambda \cup \{\gamma/\beta\}$;*
   (b) *if β/λ and γ/β are in the same row or the same column, then $\alpha = \beta$.*

(3) *If $|\lambda| = |\gamma|$ and $\lambda \neq \gamma$:*
   (a) *if $\lambda \supset \beta$ then $\alpha = \lambda \cup \{\gamma/\beta\}$;*

   (b) *if* $\lambda \subset \beta$ *then* $\alpha = \lambda - \{\beta/\gamma\}$.
(4) *If* $\lambda = \gamma$:
   (a) *if* $\lambda \subset \beta$:
      (i) *if* β/λ *is in the first column of* λ:
         (A) *if* β/λ *is in the* n*th row, for* $\beta \in Y_{\pm}$ *let* $\alpha \in Y_{\mp}$ *be the corresponding partition*;
         (B) *otherwise let* $\alpha = \beta$;
      (ii) *else* $\alpha \subset \lambda$ *and* α *is obtained from* λ *by removing the corner cell in the column to the left of* β/λ;
   (b) *if* $\lambda \supset \beta$ *then* α *is obtained from* λ *by adjoining a cell to the column to the right of* λ/β.

Proof. The rule for the weight α is given by van Leeuwen [23, Rule 4.1.1]: α is the unique dominant element in the Weyl group orbit of the weight $\lambda + \gamma - \beta$. Using this rule the proof is straightforward. $\square$

Remark 5.12. The two operations $\mathrm{rh} \circ \mathrm{lh}$ and $\mathrm{lh} \circ \mathrm{rh}$ define a pair of two-step walks in the graph G from λ to γ, whose intermediate vertices are β and α, respectively. If there is only one such walk then $\alpha = \beta$; this occurs in cases (1)(b) and (2)(b). If there are exactly two such walks then α is always chosen to be the intermediate vertex not equal to β; this occurs in cases (1)(a), (2)(a), (3)(a), and (3)(b). In the case that $\lambda = \gamma$ there may be many such walks; the proper choice of α given β is described in the proposition.

Example 5.13. Let b be as in Example 5.10. Then $\mathrm{lh}(b) = \bar{4}41321$, $\mathrm{rh}(b) = 2\bar{4}3121$, and $\mathrm{rh}(\mathrm{lh}(b)) = \mathrm{lh}(\mathrm{rh}(b)) = \bar{4}3121$. Therefore λ is the weight $(2, 1, 1, -1)$ or the partition $(2, 1, 1, 1) \in Y_-$, β is the weight and partition $(2, 1, 1, 0)$, α is the weight $(2, 2, 1, -1)$ and the partition $(2, 2, 1, 1) \in Y_-$, and γ is the weight $(2, 1, 1, -1)$ and the partition $(2, 1, 1, 1) \in Y_-$. Since $\lambda = \gamma$ (as elements of P or G) and $\beta \subset \lambda$ as partitions, case (4)(b) applies. The cell λ/β is in the first column; therefore α should be obtained from γ by adjoining a cell at the end of the second column, which agrees with the example.

Example 5.14. In $D_4^{(1)}$ let $b = 4\bar{4}321 \in (B^{1,1})^{\otimes 5}$. Then $\mathrm{lh}(b) = \bar{4}321$, $\mathrm{rh}(b) = 4321$, $\mathrm{rh}(\mathrm{lh}(b)) = \mathrm{lh}(\mathrm{rh}(b)) = 321$. Therefore $\lambda = (1, 1, 1, 0)$, β is the weight $(1, 1, 1, -1)$ or the partition $(1, 1, 1, 1) \in Y_-$, $\gamma = (1, 1, 1, 0)$, and α is the weight $(1, 1, 1, 1)$ or the partition $(1, 1, 1, 1) \in Y_+$. This is case (4aiA).

6. Rigged configurations

In this section it is assumed that $\mathfrak{g}$ is nonexceptional and simply-laced, that is, $\mathfrak{g} = A_n^{(1)}$ or $\mathfrak{g} = D_n^{(1)}$.

6.1. Definition

Let $B \in \mathcal{C}$ for type $D_n^{(1)}$ and $B \in \mathcal{C}^A$ for type $A_n^{(1)}$. Recall the notation in Section 2.4, where $L = (L_i^{(a)} \mid (a, i) \in \mathcal{H})$ is the multiplicity array of B. The sequence of partitions

$\nu = \{\nu^{(a)} \mid a \in \bar{I}\}$ is a (L, λ)-*configuration* if

$$\sum_{(a,i)\in\mathcal{H}} i m_i^{(a)} \alpha_a = \sum_{(a,i)\in\mathcal{H}} i L_i^{(a)} \overline{\Lambda}_a - \lambda,$$

where $m_i^{(a)}$ is the number of parts of length i in partition $\nu^{(a)}$. A (L, λ)-configuration is *admissible* if $p_i^{(a)} \geqslant 0$ for all $(a, i) \in \mathcal{H}$, where $p_i^{(a)}$ is the *vacancy number*

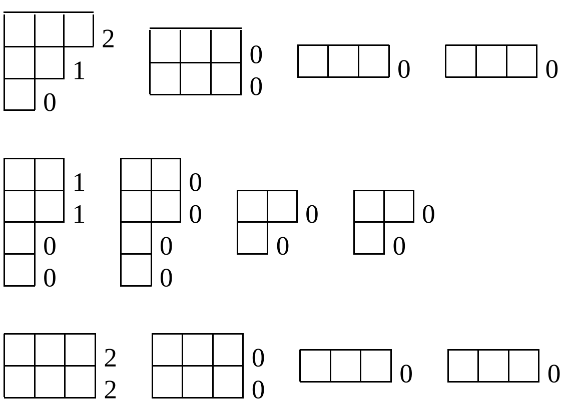
$$p_i^{(a)} = \sum_{j\geqslant 1} \min(i, j) L_j^{(a)} - \sum_{b\in\bar{I}} (\alpha_a|\alpha_b) \sum_{j\geqslant 1} \min(i, j) m_j^{(b)}.$$

Here $(\cdot|\cdot)$ is the normalized invariant form on P such that $(\alpha_i|\alpha_j)$ is the Cartan matrix. Let $\mathrm{C}(L, \lambda)$ be the set of admissible (L, λ)-configurations. A *rigged configuration* (ν, J) consists of a configuration $\nu \in \mathrm{C}(L, \lambda)$ together with a double sequence of partitions $J = \{J^{(a,i)} \mid (a, i) \in \mathcal{H}\}$ such that the partition $J^{(a,i)}$ is contained in a $m_i^{(a)} \times p_i^{(a)}$ rectangle. The set of rigged configurations is denoted by $\mathrm{RC}(L, \lambda)$.

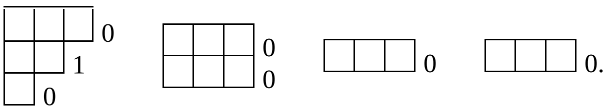
The partition $J^{(a,i)}$ is called *singular* if it has a part of size $p_i^{(a)}$. The partition $J^{(a,i)}$ is called *cosingular* if it has a part of size zero, or equivalently, its complement in the rectangle of size $m_i^{(a)} \times p_i^{(a)}$ has a part of size $p_i^{(a)}$.

It is often useful to view a rigged configuration (ν, J) as a sequence of partitions ν where the parts of size i in $\nu^{(a)}$ are labeled by the parts of $J^{(a,i)}$. The pair (i, x) where i is a part of $\nu^{(a)}$ and x is a part of $J^{(a,i)}$ is called a *string* of the ath rigged partition $(\nu, J)^{(a)}$. The label x is called a *rigging* or *quantum number*. The corresponding *coquantum number* is $p_i^{(a)} - x$.

Example 6.1. Let $\mathfrak{g} = D_4^{(1)}$, $B = B^1 \otimes B^2 \otimes B^2 \otimes B^3$ and $\lambda = 2\overline{\Lambda}_1$. Then the following three sequences of partitions are admissible (L, λ)-configurations:



where the corresponding vacancy numbers are written next to each part. Hence, writing the parts of $J^{(a,i)}$ next to the parts of size i of partition $\nu^{(a)}$ the following would be a particular rigged configuration:

$(\nu, J) =$ 

6.2. Quantum number complementation

Let $\theta = \theta_L : \mathrm{RC}(L,\lambda) \to \mathrm{RC}(L,\lambda)$ be the involution that preserves configurations and complements riggings with respect to the vacancy numbers. More precisely, each partition $J^{(a,i)}$ is replaced by the partition that is complementary to it within the $m_i^{(a)} \times p_i^{(a)}$ rectangle.

6.3. The RC reduction steps $\bar\delta$ and $\tilde\delta$

Suppose $L_1^{(1)} > 0$. Let $\mathrm{lh}(L)$ and $\mathrm{rh}(L)$ be obtained from L by removing one tensor factor B^1. In particular, if B has B^1 as its left (respectively right) tensor factor, then $\mathrm{lh}(L)$ (respectively $\mathrm{rh}(L)$) is the multiplicity array for $\mathrm{lh}(B)$ (respectively $\mathrm{rh}(B)$). In [17] a quantum number bijection $\overline{\phi} : P(B) \to \mathrm{RC}(L)$ was defined when B is a tensor power of B^1. The key step in the definition of $\overline{\phi}$ is an algorithm that defines a map $\bar\delta : \mathrm{RC}(L) \to \mathrm{RC}(\mathrm{lh}(L))$. The same algorithm defines such a map for the current case.

For $(\nu, J) \in \mathrm{RC}(L,\lambda)$, the algorithm produces a smaller rigged configuration $\bar\delta(\nu, J) \in \mathrm{RC}(\mathrm{lh}(L), \mu)$ for some $\mu \in \lambda^-$ and an element $\mathrm{rk}(\nu, J) \in B^1$ such that

$$\mu + \mathrm{wt}\big(\mathrm{rk}(\nu, J)\big) = \lambda. \tag{6.1}$$

We recall the algorithm for $\bar\delta$ explicitly for type $A_n^{(1)}$ and $D_n^{(1)}$. Although we do not use them here, the explicit algorithms exists for the other nonexceptional affine types and can be found in [18].

String selection for type $A_n^{(1)}$. Set $\ell^{(0)} = 1$ and repeat the following process for $a = 1, 2, \ldots, n$ or until stopped. Find the smallest index $i \geqslant \ell^{(a-1)}$ such that $J^{(a,i)}$ is singular. If no such i exists, set $\mathrm{rk}(\nu, J) = a$ and stop. Otherwise set $\ell^{(a)} = i$ and continue with $a + 1$.

String selection for type $D_n^{(1)}$. Set $\ell^{(0)} = 1$ and repeat the following process for $a = 1, 2, \ldots, n-2$ or until stopped. Find the smallest index $i \geqslant \ell^{(a-1)}$ such that $J^{(a,i)}$ is singular. If no such i exists, set $\mathrm{rk}(\nu, J) = a$ and stop. Otherwise set $\ell^{(a)} = i$ and continue with $a + 1$. Set all yet undefined $\ell^{(a)}$ to ∞.

If the process has not stopped at $a = n-2$, find the minimal indices $i, j \geqslant \ell^{(n-2)}$ such that $J^{(n-1,i)}$ and $J^{(n,j)}$ are singular. If neither i nor j exist, set $\mathrm{rk}(\nu, J) = n-1$ and stop. If i exists, but not j, set $\ell^{(n-1)} = i$, $\mathrm{rk}(\nu, J) = n$ and stop. If j exists, but not i, set $\ell^{(n)} = j$, $\mathrm{rk}(\nu, J) = \bar n$ and stop. If both i and j exist, set $\ell^{(n-1)} = i$, $\ell^{(n)} = j$ and continue with $a = n-2$.

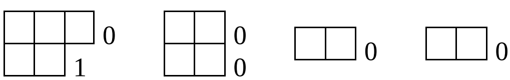
Now continue for $a = n-2, n-3, \ldots, 1$ or until stopped. Find the minimal index $i \geqslant \bar{\ell}^{(a+1)}$ where $\bar{\ell}^{(n-1)} = \max(\ell^{(n-1)}, \ell^{(n)})$ such that $J^{(a,i)}$ is singular (if $i = \ell^{(a)}$ then there need to be two parts of size $p_i^{(a)}$ in $J^{(a,i)}$). If no such i exists, set $\mathrm{rk}(\nu, J) = \overline{a+1}$ and stop. If the process did not stop, set $\mathrm{rk}(\nu, J) = \bar{1}$. Set all yet undefined $\ell^{(a)}$ and $\bar{\ell}^{(a)}$ to ∞.

The new rigged configuration. The rigged configuration $(\tilde{\nu}, \tilde{J}) = \bar{\delta}(\nu, J)$ is obtained by removing a box from the selected strings and making the new strings singular again. Explicitly (ignoring the statements about $\bar{\ell}^{(a)}$ for type $A_n^{(1)}$)

$$m_i^{(a)}(\tilde{\nu}) = m_i^{(a)}(\nu) + \begin{cases} 1, & \text{if } i = \ell^{(a)} - 1, \\ -1, & \text{if } i = \ell^{(a)}, \\ 1, & \text{if } i = \bar{\ell}^{(a)} - 1 \text{ and } 1 \leqslant a \leqslant n-2, \\ -1, & \text{if } i = \bar{\ell}^{(a)} \text{ and } 1 \leqslant a \leqslant n-2, \\ 0, & \text{otherwise.} \end{cases}$$

The partition $\tilde{J}^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $p_i^{(a)}(\nu)$ for $i = \ell^{(a)}$ and $i = \bar{\ell}^{(a)}$, adding a part of size $p_i^{(a)}(\tilde{\nu})$ for $i = \ell^{(a)} - 1$ and $i = \bar{\ell}^{(a)} - 1$, and leaving it unchanged otherwise.

Example 6.2. For the rigged configuration (ν, J) of Example 6.1, we have

$\bar{\delta}(\nu, J) =$ 

with $\mathrm{rk}(\nu, J) = \bar{2}$.

The next proposition was proved in [15,18].

Proposition 6.3. *The map* $\bar{\delta} : \mathrm{RC}(L, \lambda) \to \bigcup_{\mu \in \lambda^-} \mathrm{RC}(\mathrm{lh}(L), \mu)$ *is injective.*

Note that for simply-laced type, knowing λ and μ uniquely determines $\mathrm{rk}(\nu, J)$ by (6.1). We may define the inverse of $\bar{\delta}$. To this end, let

$$\lambda^+ = \bigl\{\mu \in \overline{P}^+ \bigm| B(\mu) \text{ occurs in } B^1 \times B(\lambda)\bigr\}.$$

Denote by $\widetilde{\mathrm{RC}}(L, \lambda)$ the subset of $\mathrm{RC}(L, \lambda) \times B^1$ given by $((\nu, J), b)$ such that $\lambda + \mathrm{wt}(b) \in \overline{P}^+$. By abuse of notation define

$$\bar{\delta}^{-1} : \widetilde{\mathrm{RC}}(L, \lambda) \to \bigcup_{\beta \in \lambda^+} \mathrm{RC}\bigl(\mathrm{lh}^{-1}(L), \beta\bigr)$$

by the following algorithm, where $\mathrm{lh}^{-1}(L)$ is obtained from L by replacing $L_1^{(1)}$ by $L_1^{(1)} + 1$.

String selection for type $A_n^{(1)}$. In this case $\mathrm{wt}(b)=\epsilon_r$ for some $1\leqslant r\leqslant n+1$, where ϵ_r is the rth canonical unit vector in $\mathbb{Z}^{n+1}$. Set $s^{(r)}=\infty$ and repeat the following process for $a=r-1,r-2,\dots,1$. Find the largest index $i\leqslant s^{(a+1)}$ such that $J^{(a,i)}$ is singular and set $s^{(a)}=i$; if no such i exists set $s^{(a)}=0$. Set all undefined $s^{(a)}$ to infinity.

String selection for type $D_n^{(1)}$. In this case $\mathrm{wt}(b)=\epsilon_r$ or $\mathrm{wt}(b)=-\epsilon_r$ for $1\leqslant r\leqslant n$, where ϵ_r is the rth canonical unit vector in $\mathbb{Z}^n$. In the first case proceed exactly as for type $A_n^{(1)}$. Throughout the whole algorithm, if an index i does not exist, set $i=0$.

If $\mathrm{wt}(b)=-\epsilon_n$, find the largest index i such that $J^{(n,i)}$ is singular and set $s^{(n)}=i$. Find the largest index $i\leqslant s^{(n)}$ such that $J^{(n-2,i)}$ is singular and set $s^{(n-2)}=i$. Then proceed as in type $A_n^{(1)}$.

If $\mathrm{wt}(b)=-\epsilon_{n-1}$, find the largest indices i and j such that $J^{(n-1,i)}$ and $J^{(n,j)}$ are singular and set $s^{(n-1)}=i$ and $s^{(n)}=j$. Then find the largest index $i\leqslant\min\{s^{(n-1)},s^{(n)}\}$ such that $J^{(n-2,i)}$ is singular and set $s^{(n-2)}=i$. After this proceed as in type $A_n^{(1)}$.

Finally, if $\mathrm{wt}(b)=-\epsilon_r$ for $1\leqslant r\leqslant n-2$, set $\bar{s}^{(r-1)}=\infty$ and proceed for $a=r,r+1,\dots,n-2$ as follows. Find the largest index $i\leqslant\bar{s}^{(a-1)}$ such that $J^{(a,i)}$ is singular and set $\bar{s}^{(a)}=i$. Then find the largest indices $i\leqslant\bar{s}^{(n-2)}$ and $j\leqslant\bar{s}^{(n-2)}$ such that $J^{(n-1,i)}$ and $J^{(n,j)}$ are singular and set $s^{(n-1)}=i$ and $s^{(n)}=j$. After this proceed as for the case $\mathrm{wt}(b)=-\epsilon_{n-1}$.

Set all yet undefined $s^{(a)}$ and $\bar{s}^{(a)}$ to ∞.

The new rigged configuration. The rigged configuration $(\tilde{\nu},\tilde{J})=\bar{\delta}^{-1}(\nu,J)$ is obtained by adding a box to the selected strings and making the new strings singular again.

Define $\tilde{\delta}:\mathrm{RC}(L)\to\mathrm{RC}(\mathrm{lh}(L))$ by $\theta_{\mathrm{lh}(L)}\circ\bar{\delta}\circ\theta_L$. Alternatively, $\tilde{\delta}$ is defined by a coquantum number version of the map $\bar{\delta}$. Instead of selecting singular strings it selects cosingular strings and keeps coquantum numbers constant for unselected strings. It also produces an element $\widetilde{\mathrm{rk}}(\nu,J)\in B^1$. If $(\nu,J)\in\mathrm{RC}(L,\lambda)$ and $\tilde{\delta}(\nu,J)\in\mathrm{RC}(\mathrm{lh}(L),\mu)$ then

$$\mu+\mathrm{wt}\big(\widetilde{\mathrm{rk}}(\nu,J)\big)=\lambda.$$

6.4. Splitting on RCs

Let $s\geqslant 2$. Suppose B contains a distinguished tensor factor $B^{r,s}$, which is the case when we consider the maps ls and rs. Let L be the multiplicity array of B and $\mathrm{ls}(L)$ that which is obtained from L by replacing $B^{r,s}$ by $B^{r,1}$ and $B^{r,s-1}$.

Proposition 6.4. *Let L be such that $L_s^{(r)}\geqslant 1$ for a particular $(r,s)\in\mathcal{H}$ with $s\geqslant 2$ and let $\mathrm{ls}(L)$ be defined with respect to (r,s). Then $\mathrm{C}(L,\lambda)\subset\mathrm{C}(\mathrm{ls}(L),\lambda)$. Under this inclusion map, the vacancy number $p_i^{(a)}$ for ν increases by $\delta_{a,r}\chi(i<s)$ where $\chi(P)=1$ if P is true and $\chi(P)=0$ otherwise. Hence there are well-defined injective maps $\bar{j},\tilde{j}:\mathrm{RC}(L)\to\mathrm{RC}(\mathrm{ls}(L))$ given by*:

(1) $\bar{j}(\nu,J)=(\nu,J)$;

(2) *$\tilde{j}(\nu, J) = (\nu, J')$ where J' is obtained from J by adding 1 to the rigging of each string in $(\nu, J)^{(r)}$ of length strictly less than s.*

In particular, $\bar{j}$ preserves quantum numbers, $\tilde{j}$ preserves coquantum numbers, and

$$\tilde{j} = \theta_{\mathrm{ls}(L)} \circ \bar{j} \circ \theta_L. \tag{6.2}$$

Proof. Immediate from the definitions. □

6.5. Box-splitting for RCs

Suppose $r \geqslant 2$ and $B \in \mathcal{C}^A$ has a distinguished tensor factor $B^{r,1}$. Let L be the multiplicity array for B and $\mathrm{lb}(L)$ that for the crystal obtained from B by replacing $B^{r,1}$ by $B^{1,1}$ and $B^{r-1,1}$.

Proposition 6.5. *Let L be such that $L_1^{(r)} \geqslant 1$ for some $r \geqslant 2$. Let $\mathrm{lb}(L)$ be defined with respect to r. Then there are injections $\bar{i}, \tilde{i} : \mathrm{RC}(L, \lambda) \to \mathrm{RC}(\mathrm{lb}(L), \lambda)$ defined by adding singular (respectively cosingular) strings of length 1 to $(\nu, J)^{(a)}$ for $1 \leqslant a < r$. Moreover, the vacancy numbers stay the same.*

7. Fermionic formula *M*

In this section we state the fermionic formula M associated with rigged configurations for simply-laced algebras as introduced in [7] and virtual fermionic formulas for nonsimply-laced algebras (see [16,17]).

7.1. Fermionic formula M

Let $(q)_m = (1-q)(1-q^2)\cdots(1-q^m)$ and let us define the q-binomial coefficient for $m, p \in \mathbb{Z}_{\geqslant 0}$ as

$$\begin{bmatrix} m+p \\ m \end{bmatrix} = \frac{(q)_{m+p}}{(q)_m (q)_p}.$$

The fermionic formula for types $A_n^{(1)}$ and $D_n^{(1)}$ is given by [7]:

$$M_{L,\lambda}(q) = \sum_{\nu \in \mathrm{C}(L,\lambda)} q^{cc(\nu)} \prod_{(a,i) \in \mathcal{H}} \begin{bmatrix} m_i^{(a)} + p_i^{(a)} \\ m_i^{(a)} \end{bmatrix} \tag{7.1}$$

with $m_i^{(a)}$, $p_i^{(a)}$ and $\mathrm{C}(L, \lambda)$ as in Section 6.1 and

$$cc(\nu) = \frac{1}{2} \sum_{a,b \in \bar{I}} \sum_{j,k \geqslant 1} (\alpha_a | \alpha_b) \min(j,k) m_j^{(a)} m_k^{(b)}.$$

Fermionic formula (7.1) can be restated solely in terms of rigged configurations. To this end recall that the q-binomial coefficient $\left[\begin{smallmatrix} m+p \\ m \end{smallmatrix}\right]$ is the generating function of partitions in a box of width p and height m. Hence

$$M_{L,\lambda}(q) = \sum_{(\nu,J)\in \mathrm{RC}(L,\lambda)} q^{cc(\nu,J)}, \tag{7.2}$$

where $cc(\nu, J) = cc(\nu) + \sum_{(a,i)\in\mathcal{H}} |J^{(a,i)}|$.

7.2. Virtual fermionic formula

Fermionic formulae for nonsimply-laced algebras were defined in [6, Section 4]. For $A_{2n}^{(2)\dagger}$ it was defined in [16]. Here we recall virtual rigged configurations in analogy to virtual crystals as defined in [17].

Definition 7.1. Let X and Y be as in Section 3.1, and λ, B and L as in Section 6.1 for type X. Let $\Psi : B \to \hat{V}$ be the corresponding virtual Y-crystal and $\hat{L}$ the multiplicity array corresponding to $\hat{V}$. For $X \notin \{A_{2n}^{(2)}, A_{2n}^{(2)\dagger}\}$, $\mathrm{RC}^v(L,\lambda)$ is the set of elements $(\hat{\nu}, \hat{J}) \in \mathrm{RC}(\hat{L}, \Psi(\lambda))$ such that:

(1) for all $i \in \mathbb{Z}_{>0}$, $\hat{m}_i^{(a)} = \hat{m}_i^{(b)}$ and $\hat{J}^{(a,i)} = \hat{J}^{(b,i)}$ if a and b are in the same σ-orbit in I^Y;
(2) for all $i \in \mathbb{Z}_{>0}$, $a \in \bar{I}^X$, and $b \in \iota(a) \subset \bar{I}^Y$, we have $\hat{m}_j^{(b)} = 0$ if $j \notin \gamma_a\mathbb{Z}$ and the parts of $\hat{J}^{(b,i)}$ are multiples of γ_a.

For $X = A_{2n}^{(2)}$ the following changes must be made:

(A2) $\hat{m}_j^{(n)}$ may be positive for any $j \geqslant 1$.

For $X = A_{2n}^{(2)\dagger}$ one makes the exception (A2) and the additional condition that:

(A2D) the parts of $\hat{J}^{(n,i)}$ must have the same parity as i.

Theorem 7.2 [17, Theorem 4.2]. *There is a bijection $\Psi : \mathrm{RC}(L,\lambda) \to \mathrm{RC}^v(L,\lambda)$ sending $(\nu, J) \mapsto (\hat{\nu}, \hat{J})$ given as follows. For all $a \in \bar{I}^X$, $b \in \iota(a) \subset \bar{I}^Y$, and $i \in \mathbb{Z}_{>0}$,*

$$\hat{m}_{\gamma_a i}^{(b)} = m_i^{(a)},$$
$$\hat{J}^{(b,\gamma_a i)} = \gamma_a J^{(a,i)},$$

except when $X = A_{2n}^{(2)}$ or $X = A_{2n}^{(2)\dagger}$ and $a = n$, in which case

$$\hat{m}_i^{(n)} = m_i^{(n)},$$
$$\hat{J}^{(n,i)} = 2J^{(n,i)}.$$

The cocharge changes by $cc(\hat{\nu}, \hat{J}) = \gamma_0\, cc(\nu, J)$.

Defining the virtual fermionic formula as

$$VM_{L,\lambda}(q) = \sum_{(\hat{\nu},\hat{J})\in \mathrm{RC}^v(L,\lambda)} q^{cc(\hat{\nu},\hat{J})/\gamma_0}$$

we obtain as a corollary:

Corollary 7.3 ($M = VM$). *$M_{L,\lambda}(q) = VM_{L,\lambda}(q)$.*

8. Bijection

8.1. Quantum number bijection

The following result defines the bijection from paths to rigged configurations. It is valid for both $B \in \mathcal{C}$ and $B \in \mathcal{C}^A$.

Proposition 8.1. *There exists a unique family of bijections $\bar{\phi} : P(B,\lambda) \to \mathrm{RC}(L,\lambda)$ such that the empty path maps to the empty rigged configuration, and*:

(1) *Suppose $B = B^1 \otimes B'$. Let $\mathrm{lh}(B) = B'$ with multiplicity array $\mathrm{lh}(L)$. Then the diagram*

$$\begin{tikzcd} P(B,\lambda) \arrow[r, "\bar{\phi}"] \arrow[d, "\mathrm{lh}"'] & \mathrm{RC}(L,\lambda) \arrow[d, "\bar{\delta}"] \\ \bigcup_{\mu\in\lambda^-} P(\mathrm{lh}(B),\mu) \arrow[r, "\bar{\phi}"'] & \bigcup_{\mu\in\lambda^-} \mathrm{RC}(\mathrm{lh}(L),\mu) \end{tikzcd} \tag{8.1}$$

commutes.

(2) *Suppose $B = B^{r,s} \otimes B'$ with $s \geqslant 2$ (and $r = 1$ for $\mathcal{C}$). Let $\mathrm{ls}(B) = B^{r,1} \otimes B^{r,s-1} \otimes B'$ with multiplicity array $\mathrm{ls}(L)$. Then the diagram*

$$\begin{tikzcd} P(B,\lambda) \arrow[r, "\bar{\phi}"] \arrow[d, "\mathrm{ls}"'] & \mathrm{RC}(L,\lambda) \arrow[d, "\bar{j}"] \\ P(\mathrm{ls}(B),\lambda) \arrow[r, "\bar{\phi}"'] & \mathrm{RC}(\mathrm{ls}(L),\lambda) \end{tikzcd} \tag{8.2}$$

commutes.

(3) *For $\mathcal{C}^A$, suppose $B = B^{r,1} \otimes B'$ with $r \geqslant 2$. Let* $\mathrm{lb}(B) = B^{1,1} \otimes B^{r-1,1} \otimes B'$ *and* $\mathrm{lb}(L)$ *its multiplicity array. Then the following diagram commutes*:

$$\begin{tikzcd} P(B,\lambda) \arrow[r, "\bar\phi"] \arrow[d, "\mathrm{lb}"'] & \mathrm{RC}(L,\lambda) \arrow[d, "\bar\imath"] \\ P(\mathrm{lb}(B),\lambda) \arrow[r, "\bar\phi"'] & \mathrm{RC}(\mathrm{lb}(L),\lambda). \end{tikzcd} \tag{8.3}$$

For type $A_n^{(1)}$ the existence of $\overline{\phi}$ was proven in [15]. The proof in case (1) for other nonexceptional types is essentially done in [18]. It remains to prove case (2) for type $D_n^{(1)}$.

Lemma 8.2. *Let $B = B^s \otimes B'$ with $s \geqslant 2$. For type $D_n^{(1)}$, the map $\overline{\phi} : P(\mathrm{ls}(B), \lambda) \to \mathrm{RC}(\mathrm{ls}(L), \lambda)$ restricts to a bijection $\overline{\phi} : \mathrm{ls}(P(B, \lambda)) \to \overline{j}(\mathrm{RC}(L, \lambda))$.*

Proof. Let $b = x \otimes b_2 \otimes b' \in B^1 \otimes B^{s-1} \otimes B'$ and $\mathrm{ls}(b_2) = y \otimes b_3 \in B^1 \otimes B^{s-2}$. Then $b \in \mathrm{Im}(\mathrm{ls})$ if and only if $x \leqslant y$. (Note that this implies in particular that n and $\bar{n}$ cannot appear in the same one-row crystal element.)

By Proposition 6.4, $(\nu, J) \in \mathrm{RC}(\mathrm{ls}(L), \lambda)$ is in the image of $\overline{j}$ if and only if $(\nu, J)^{(1)}$ has no singular strings of length smaller than s.

Let us first show that if $b \in \mathrm{Im}(\mathrm{ls})$ then $\overline{\phi}(b) \in \mathrm{Im}(\overline{j})$. Hence assume that $b = x \otimes b_2 \otimes b'$ with $x \leqslant y$ with y as defined above. By induction $(\nu', J') = \overline{\phi}(y \otimes b_3 \otimes b')$ has no singular strings in the first rigged partition of length smaller than $s - 1$. Denote the lengths of the strings selected by $\bar{\delta}$ associated with the letter y by $\ell_y^{(k)}$ and $\bar{\ell}_y^{(k)}$. Then in particular $\ell_y^{(1)} \geqslant s - 1$. "Unsplitting" yields on the path side $b_2 \otimes b'$ and on the rigged configuration side (ν', J') with a change in the vacancy numbers by $-\delta_{a,1}\chi$ $(i < s - 1)$. Since $x \leqslant y$ it follows that $\ell_x^{(k)} > \ell_y^{(k)}$ and $\bar{\ell}_x^{(k)} > \bar{\ell}_y^{(k)}$, where $\ell_x^{(k)}$ and $\bar{\ell}_x^{(k)}$ are the lengths of the strings selected by $\bar{\delta}$ associated with x. This shows in particular that $\ell_x^{(1)} \geqslant s$, and from the change in vacancy numbers from $\overline{\phi}(b_2 \otimes b')$ to $\overline{\phi}(x \otimes b_2 \otimes b')$ it follows that there are no singular strings in the first rigged partition of $\overline{\phi}(x \otimes b_2 \otimes b')$ of length smaller than s.

Conversely, assume that $(\nu, J) \in \mathrm{RC}(\mathrm{ls}(L), \lambda)$ is in the image if $\overline{j}$. We need to show that then $b = \overline{\phi}^{-1}(\nu, J)$ has the property that $x \leqslant y$ in the above notation. Call the strings selected by $\bar{\delta}$ in (ν, J) $\ell_x^{(k)}$ and $\bar{\ell}_x^{(k)}$. By assumption $(\nu, J)^{(1)}$ has no singular string of length smaller than s. Hence $\ell_x^{(1)} \geqslant s$. By the definition of $\overline{j}$, we have that the first rigged partition of $(\nu', J') = \overline{j} \circ \bar{\delta}(\nu, J)$ has no singular strings of length smaller than $s - 1$. Hence $s - 1 \leqslant \ell_y^{(1)} < \ell_x^{(1)}$, where $\ell_y^{(k)}$ and $\bar{\ell}_y^{(k)}$ are the lengths of the strings selected by $\bar{\delta}$ on (ν', J'). The algorithm of $\bar{\delta}$ implies that $\ell_y^{(k)} < \ell_x^{(k)}$ and $\bar{\ell}_y^{(k)} < \bar{\ell}_x^{(k)}$, so that $x \leqslant y$ as desired. $\square$

8.2. Coquantum number bijection

Let $\tilde{\phi} = \theta \circ \overline{\phi}$; it can be characterized as follows.

Proposition 8.3. *There exists a unique family of bijections* $\tilde{\phi}: P(B,\lambda) \to \mathrm{RC}(L,\lambda)$ *with the same properties as in Proposition* 8.1 *except that* $\bar{\delta}$, $\bar{j}$ *and* $\bar{i}$ *are replaced by* $\tilde{\delta}$, $\tilde{j}$ *and* $\tilde{i}$ *in* (8.1), (8.2) *and* (8.3), *respectively.*

8.3. Commutations of the basic steps

We record the commutations among the basic steps of the path–RC bijection. Here $r = 1$ for $\mathcal{C}$.

Theorem 8.4.

(1) $[\bar{\delta}, \tilde{\delta}] = 0$.
(2) $[\tilde{j}, \bar{\delta}] = 0$ *and* $[\bar{j}, \tilde{\delta}] = 0$.
(3) $[\bar{j}, \tilde{j}] = 0$.
(4) $[\tilde{i}, \bar{\delta}] = 0$ *and* $[\bar{i}, \tilde{\delta}] = 0$.
(5) $[\tilde{i}, \bar{j}] = 0$ *and* $[\bar{i}, \tilde{j}] = 0$.

The proof of part (1) for type $A_n^{(1)}$ is given in [15, Appendix A]. The proof of part (1) for type $D_n^{(1)}$ is quite technical and follows similar arguments as [15, Appendix A] (see also [19, Appendix C]). Details are available upon request. Parts (2) and (3) follow easily from the definitions. Parts (4) and (5) only apply for $\mathcal{C}^A$ and follow from [15].

For type $D_n^{(1)}$, there is an analogue of Proposition 5.11 for the commutation of $\bar{\delta}$ and $\tilde{\delta}$. Let $(\nu, J) \in \mathrm{RC}(L,\lambda)$, $\tilde{\delta}(\nu, J) \in \mathrm{RC}(\mathrm{rh}(L),\alpha)$, $\bar{\delta}(\nu, J) \in \mathrm{RC}(\mathrm{lh}(L),\beta)$ and $\tilde{\delta}(\bar{\delta}(\nu, J)) = \bar{\delta}(\tilde{\delta}(\nu, J)) \in \mathrm{RC}(\mathrm{lh}(\mathrm{rh}(L)),\gamma)$. Then α is uniquely determined by λ, β, and γ.

Proposition 8.5. *For* λ, α, β, *and* γ *as above the statements of Proposition* 5.11 *hold.*

The proof is an easy consequence of the commutation $[\bar{\delta}, \tilde{\delta}] = 0$ and is available upon request.

8.4. The bijection and the various operations

Theorem 8.6. *Under the family of bijections* $\bar{\phi}$ *the following operations correspond*:

(1) ls *with* $\bar{j}$.
(2) lh *with* $\bar{\delta}$.
(3) rs *with* $\tilde{j}$.
(4) rh *with* $\tilde{\delta}$.
(5) $*$ *with* θ.
(6) R *with the identity.*
(7) lb *with* $\bar{i}$ *and* rb *with* $\tilde{i}$.

Example 8.7. To illustrate point (4) of the above theorem, take

$$b = \boxed{\bar{3}} \otimes \boxed{2|3} \otimes \boxed{1|2} \otimes \boxed{1}$$

of type $D_4^{(1)}$. Then

$$\mathrm{rh}(b) = \begin{array}{|c|}\hline 3\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&2\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1&1\\\hline\end{array}$$

and

$$\bar{\phi}(b) = \begin{array}{|c|c|l}\cline{1-2} & & 0 \\ \cline{1-2} & \multicolumn{2}{l}{0} \\ \cline{1-1} & \multicolumn{2}{l}{0} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|l}\hline & 1 \\ \hline & 0 \\ \hline\end{array} \quad \begin{array}{|c|l}\hline & 0 \\ \hline\end{array} \qquad \begin{array}{|c|l}\hline & 0 \\ \hline\end{array}$$

$$\bar{\phi}\big(\mathrm{rh}(b)\big) = \tilde{\delta}\big(\bar{\phi}(b)\big) = \begin{array}{|c|c|l}\cline{1-2} & & 0 \\ \cline{1-2} & \multicolumn{2}{l}{0} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|l}\hline & 0 \\ \hline\end{array} \quad \emptyset \qquad \emptyset.$$

Proof. Everything is proved for $\mathcal{C}^A$ in [15], including part (7), which only applies in that case. We assume that $B \in \mathcal{C}$ for type $D_n^{(1)}$. Parts (1) and (2) hold by Proposition 8.1. We prove parts (3)–(5) simultaneously by induction. The induction is based first on the quantity $\sum_i s_i$ for the crystal $\bigotimes_i B^{s_i}$, and then by decreasing induction on the number of tensor factors.

Consider part (3). Suppose first that $B = B^s$ for some $s \geqslant 2$. Then $P(B)$ has only one element 1^s. It is easy to show that $\bar{\phi}(1^s)$ is the empty RC and that (3) holds. Suppose next that $B = B^1 \otimes B' \otimes B^s$. Consider the diagram

$$\begin{tikzcd}
P(B) \arrow[rrr, "\mathrm{rs}"] \arrow[dr, "\bar{\phi}"'] \arrow[ddd, "\mathrm{lh}"'] & & & P(\mathrm{rs}(B)) \arrow[dl, "\bar{\phi}"] \arrow[ddd, "\mathrm{lh}"] \\
& \mathrm{RC}(L) \arrow[r, "\tilde{j}"] \arrow[d, "\bar{\delta}"'] & \mathrm{RC}(\mathrm{rs}(L)) \arrow[d, "\bar{\delta}"] & \\
& \mathrm{RC}(\mathrm{lh}(L)) \arrow[r, "\tilde{j}"'] & \mathrm{RC}(\mathrm{rs}(\mathrm{lh}(L))) & \\
P(\mathrm{lh}(B)) \arrow[ur, "\bar{\phi}"] \arrow[rrr, "\mathrm{rs}"'] & & & P(\mathrm{rs}(\mathrm{lh}(B))). \arrow[ul, "\bar{\phi}"']
\end{tikzcd} \tag{8.4}$$

Here L, $\mathrm{rs}(L)$, $\mathrm{lh}(L)$, $\mathrm{rs}(\mathrm{lh}(L))$ are the multiplicity arrays corresponding to B, $\mathrm{rs}(B)$, $\mathrm{lh}(B)$, $\mathrm{rs}(\mathrm{lh}(B))$, respectively. We shall view such a diagram as a cube in which the small square is in the background. The left and right faces commute by Proposition 8.1. The front and back faces commute by Proposition 5.9(2) and Theorem 8.4(2), respectively. The bottom face commutes by induction. It follows that the top face "commutes up to $\bar{\delta}$," that is, $\bar{\delta} \circ \tilde{j} \circ \bar{\phi} = \bar{\delta} \circ \bar{\phi} \circ \mathrm{rs}$. But all maps in the top face preserve the highest weight. By Proposition 6.3 it follows that the top face commutes.

The remaining case is $B = B^{s'} \otimes B' \otimes B^{s}$ for $s, s' \geqslant 2$. Consider the diagram below, where $\mathrm{rs}(\mathrm{ls}(L))$ is obtained from $\mathrm{ls}(L)$ by splitting a $B^{s'}$ into $B^{s'-1}$ and B^{1}.

$$\begin{tikzcd}
P(B) \arrow[rrr, "\mathrm{rs}"] \arrow[dr, "\bar{\phi}"'] \arrow[ddd, "\mathrm{ls}"'] & & & P(\mathrm{rs}(B)) \arrow[dl, "\bar{\phi}"] \arrow[ddd, "\mathrm{ls}"] \\
& \mathrm{RC}(L) \arrow[r, "\tilde{j}"] \arrow[d, "\bar{j}"'] & \mathrm{RC}(\mathrm{rs}(L)) \arrow[d, "\bar{j}"] & \\
& \mathrm{RC}(\mathrm{ls}(L)) \arrow[r, "\tilde{j}"'] & \mathrm{RC}(\mathrm{rs}(\mathrm{ls}(L))) & \\
P(\mathrm{ls}(B)) \arrow[rrr, "\mathrm{rs}"'] \arrow[ur, "\bar{\phi}"] & & & P(\mathrm{rs}(\mathrm{ls}(B))). \arrow[ul, "\bar{\phi}"']
\end{tikzcd} \tag{8.5}$$

The left and right faces commute by Proposition 8.1. The front and back faces commute by Proposition 5.9(4) and Theorem 8.4(3), respectively. The bottom face commutes by induction. Since $\bar{j}$ is injective, it follows that the top face commutes. This finishes the proof of part (3).

We now prove part (4). The proof is trivial for the base case $B = B^{1}$. Suppose next that $B = B^{1} \otimes B' \otimes B^{1}$.

$$\begin{tikzcd}
P(B) \arrow[rrr, "\mathrm{rh}"] \arrow[dr, "\bar{\phi}"'] \arrow[ddd, "\mathrm{lh}"'] & & & P(\mathrm{rh}(B)) \arrow[dl, "\bar{\phi}"] \arrow[ddd, "\mathrm{lh}"] \\
& \mathrm{RC}(L) \arrow[r, "\tilde{\delta}"] \arrow[d, "\bar{\delta}"'] & \mathrm{RC}(\mathrm{lh}(L)) \arrow[d, "\bar{\delta}"] & \\
& \mathrm{RC}(\mathrm{lh}(L)) \arrow[r, "\tilde{\delta}"'] & \mathrm{RC}(\mathrm{rh}(\mathrm{lh}(L))) & \\
P(\mathrm{lh}(B)) \arrow[rrr, "\mathrm{rh}"'] \arrow[ur, "\bar{\phi}"] & & & P(\mathrm{rh}(\mathrm{lh}(B))). \arrow[ul, "\bar{\phi}"']
\end{tikzcd} \tag{8.6}$$

The left and right faces commute by Proposition 8.1. The front and back faces commute by Proposition 5.9(1) and Theorem 8.4(1), respectively. The bottom face commutes by induction. Thus the top face commutes up to $\bar{\delta}$. By Proposition 6.3 it suffices to show that both ways around the top face, result in elements with the same highest weight. This follows from Propositions 5.11 and 8.5.

The remaining case is $B = B^s \otimes B' \otimes B^1$. Consider the diagram

$$
\begin{tikzcd}
P(B) \arrow[rrr, "\mathrm{rh}"] \arrow[dr, "\overline{\phi}"'] \arrow[ddd, "\mathrm{ls}"'] & & & P(\mathrm{rh}(B)) \arrow[dl, "\overline{\phi}"] \arrow[ddd, "\mathrm{ls}"] \\
& \mathrm{RC}(L) \arrow[r, "\tilde{\delta}"] \arrow[d, "\overline{j}"'] & \mathrm{RC}(\mathrm{rh}(L)) \arrow[d, "\overline{j}"] & \\
& \mathrm{RC}(\mathrm{ls}(L)) \arrow[r, "\tilde{\delta}"'] & \mathrm{RC}(\mathrm{rh}(\mathrm{ls}(L))) & \\
P(\mathrm{ls}(B)) \arrow[ur, "\overline{\phi}"] \arrow[rrr, "\mathrm{rh}"'] & & & P(\mathrm{rh}(\mathrm{ls}(B)))\mathrm{.} \arrow[ul, "\overline{\phi}"']
\end{tikzcd}
\tag{8.7}
$$

The left and right faces commute by Proposition 8.1. The front and back faces commute by Proposition 5.9(3) and Theorem 8.4(2), respectively. The bottom face commutes by induction. Since $\overline{j}$ is injective it follows that the top face commutes. This proves part (4).

For part (5) the proof of the base case $B = B^s$ is trivial. Suppose next that $B = B^1 \otimes B' \otimes B^1$. Consider the diagram

$$
\begin{tikzcd}
P(B) \arrow[rrr, "*"] \arrow[dr, "\overline{\phi}"'] \arrow[ddd, "\mathrm{rh}"'] & & & P(B^*) \arrow[dl, "\overline{\phi}"] \arrow[ddd, "\mathrm{lh}"] \\
& \mathrm{RC}(L) \arrow[r, "\theta"] \arrow[d, "\tilde{\delta}"'] & \mathrm{RC}(L) \arrow[d, "\bar{\delta}"] & \\
& \mathrm{RC}(\mathrm{rh}(L)) \arrow[r, "\theta"'] & \mathrm{RC}(\mathrm{rh}(L)) & \\
P(\mathrm{rh}(B)) \arrow[ur, "\overline{\phi}"] \arrow[rrr, "*"'] & & & P(\mathrm{rh}(B)^*)\mathrm{.} \arrow[ul, "\overline{\phi}"']
\end{tikzcd}
\tag{8.8}
$$

The right face commutes by Proposition 8.1. The left commutes by part (4) which was just proved above. The back face commutes by the definition of $\tilde{\delta}$. The commutation of the front face is given by Proposition 5.9(5). The bottom face commutes by induction. It follows that the top face commutes up to $\bar{\delta}$. Again it suffices to show that both ways around the top face produce elements of the same highest weight. But this holds since $\overline{\phi}$, θ, and $*$ preserve the highest weight. Here we are using the fact that for $\lambda \in \overline{P}^+$, $V_\lambda^* \cong V_\lambda$.

Next let $B = B' \otimes B^s$ with $s \geqslant 2$.

$$\begin{tikzcd}
P(B) \arrow[rrr, "*"] \arrow[dr, "\bar{\phi}"'] \arrow[ddd, "\mathrm{rs}"'] & & & P(B^*) \arrow[dl, "\bar{\phi}"] \arrow[ddd, "\mathrm{ls}"] \\
& \mathrm{RC}(L) \arrow[r, "\theta"] \arrow[d, "\tilde{j}"'] & \mathrm{RC}(L) \arrow[d, "\bar{j}"] & \\
& \mathrm{RC}(\mathrm{ls}(L)) \arrow[r, "\theta"'] & \mathrm{RC}(\mathrm{ls}(L)) & \\
P(\mathrm{rs}(B)) \arrow[rrr, "*"'] \arrow[ur, "\bar{\phi}"] & & & P(\mathrm{rs}(B)^*). \arrow[ul, "\bar{\phi}"']
\end{tikzcd} \tag{8.9}$$

The right face commutes by Proposition 8.1. The left face commutes by part (3) which was proved above. The back face commutes by the definition of $\tilde{j}$. The commutation of the front face is given by the definition of ls in (5.3). The bottom face commutes by induction. Since $\bar{j}$ is injective, the top face commutes.

For $B = B^s \otimes B'$ with $s \geqslant 2$ the proof is similar to the previous case.

This concludes the proof of part (5).

For the proof of part (6), let $B = B_k \otimes B_{k-1} \otimes \cdots \otimes B_1$. We may assume that $R = R_j$ is the R-matrix being applied at tensor positions j and $j+1$ (from the right). By induction we may assume that $j = k-1$, that is, R acts at the leftmost two tensor positions. By part (5) and Proposition 2.16 we may assume that $j = 1$. Again by induction we may assume that $k = 2$. Let $B = B^t \otimes B^s$ (of type $D_n^{(1)}$). By Lemma 4.5 B is multiplicity-free as a $U_q(D_n)$-crystal. Since R preserves weights it follows that $R(v_{p,q}^{t,s}) = v_{p,q}^{s,t}$. A direct computation shows that $\bar{\phi}(v_{p,q}^{t,s}) = \bar{\phi}(v_{p,q}^{s,t})$. $\square$

8.5. $X = M$ for types $A_n^{(1)}$ and $D_n^{(1)}$

In this subsection we will show that $X_{B,\lambda} = M_{L,\lambda}$ for $B \in \mathcal{C}^A$ for type $A_n^{(1)}$ and $B \in \mathcal{C}$ for type $D_n^{(1)}$. By Proposition 8.1 there is a bijection between the sets $P(B,\lambda)$ and $\mathrm{RC}(L,\lambda)$. Hence it remains to show that the statistics is preserved.

Theorem 8.8. *Let $B \in \mathcal{C}^A$ be a crystal of type $A_n^{(1)}$ or $B \in \mathcal{C}$ a crystal of type $D_n^{(1)}$ and λ a dominant integral weight. The coquantum number bijection $\tilde{\phi}$ preserves the statistics, that is $D_B(b) = cc(\tilde{\phi}(b))$ for all $b \in P(B,\lambda)$.*

Proof. For type $A_n^{(1)}$ the theorem follows from [15, Theorem 9.1]. Hence assume that $B \in \mathcal{C}$ of type $D_n^{(1)}$. By Theorem 8.6(3) and Eqs. (5.3) and (6.2) the maps rs and $\bar{j}$ correspond under $\tilde{\phi}$. By Theorem 4.3 we have $D(\mathrm{rs}(b)) = D(b)$. Similarly, it follows immediately from the definition of $\bar{j}$ in Proposition 6.4 that $cc(\bar{j}(\nu, J)) = cc(\nu, J)$. The maps R

and the identity also correspond under $\tilde{\phi}$ by Theorem 8.6(6), and neither of them changes the statistics.

There exists a sequence $\mathcal{S}_P$ of maps rs and R which transforms a path $b \in P(B,\lambda)$ into a path of single boxes. By Theorem 8.6 there exists a corresponding sequence $\mathcal{S}_{\mathrm{RC}}$ of maps $\bar{j}$ and the identity. Since neither of these maps changes the statistics it follows that

$$D\big(\mathcal{S}_P(b)\big) = cc\big(\mathcal{S}_{\mathrm{RC}}\big(\tilde{\phi}(b)\big)\big) \quad \text{implies that} \quad D(b) = cc\big(\tilde{\phi}(b)\big).$$

The theorem for the case $B = (B^{1,1})^{\otimes N}$ has already been proven in [18]. □

Corollary 8.9. *For $B \in \mathcal{C}^A$ of type $A_n^{(1)}$ or $B \in \mathcal{C}$ of type $D_n^{(1)}$, L the corresponding multiplicity array and λ a dominant integral, we have*

$$X_{B,\lambda}(q) = M_{L,\lambda}(q).$$

Proof. This follows from Theorem 8.8, (7.2) and (2.5). □

9. Type $A_n^{(1)}$ dual bijection

For this section we assume type $A_n^{(1)}$. We define and study the properties of a dual analogue $\bar{\delta}^\vee$ of the $\bar{\delta}$ map that corresponds to removing a tensor factor $B^{1\vee}$ from the left. This is used to prove a duality symmetry (Theorem 9.4) for the path–RC bijection in type $A_n^{(1)}$. This, in turn, is useful for establishing the virtual bijections in Section 10.

Let $\mathcal{C}^{A\vee} \subset \mathcal{C}^A$ be the category of tensor products of crystals of the form $B^{1,s}$ and $B^{1,s\vee}$.

One goal of this section is to give a simpler way to compute $\bar{\phi}$ for $B \in \mathcal{C}^{A\vee}$. Since $\mathcal{C}^{A\vee} \subset \mathcal{C}^A$, Proposition 8.1 gives the definition of $\bar{\phi}$. By (2.6) $B^{1,s\vee}$ is isomorphic to $B^{n,s}$. The definition of $\bar{\phi}$ involves left-splitting $B^{n,s}$, which produces columns $B^{n,1}$, each of which have to be "box split" into boxes $B^{1,1}$ and removed by lh.

We introduce a dual analogue $\bar{\delta}^\vee$ of $\bar{\delta}$, which removes an entire column $B^{n,1}$ in a single step whose computation is entirely similar to a single $\bar{\delta}$ (rather than n of them).

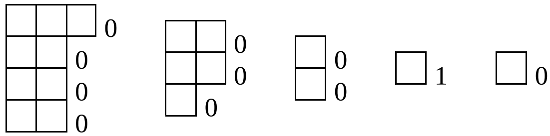
Using $\bar{\delta}^\vee$, we can compute $\bar{\phi}$ for $B \in \mathcal{C}^{A\vee}$ using essentially single row techniques.

9.1. *Dual left hat*

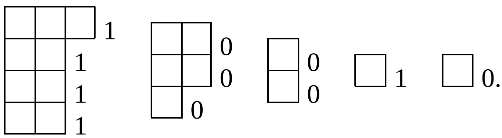
Suppose that $B = B^{1\vee} \otimes B'$. In this particular case we write $\mathrm{lh}^\vee(B) = B'$. By Lemma 5.6 there is a map $\mathrm{lh}^\vee : P(B) \to P(\mathrm{lh}^\vee(B))$ given by removing the left tensor factor. Let $\mathrm{lh}^\vee(L)$ be the multiplicity array of $\mathrm{lh}^\vee(B)$.

The following algorithm is the same as $\bar{\delta}$ except that it starts from large indices instead of small. The map $\bar{\delta}^\vee : \mathrm{RC}(L) \to \mathrm{RC}(\mathrm{lh}^\vee(L))$ is defined as follows. Let $(\nu, J) \in \mathrm{RC}(L)$. Initialize $\ell^{(n+1)} = 0$ and $\ell^{(0)} = \infty$. For i from n down to 1, assuming that $\ell^{(i+1)}$ has already been defined, let $\ell^{(i)}$ be the smallest integer such that $(\nu, J)^{(i)}$ has a singular string of length $\ell^{(i)}$ and $\ell^{(i)} \geqslant \ell^{(i+1)}$. If no such singular string exists, let $\ell^{(j)} = \infty$ for $1 \leqslant j \leqslant i$. Let $\mathrm{rk}^\vee(\nu, J) = (i+1)^\vee \in B^{1\vee}$ where i is the maximum index i such that $\ell^{(i)} = \infty$.

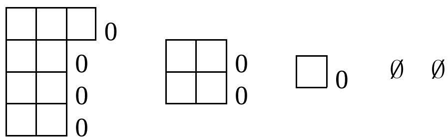
Example 9.1. For $B = B^{1\vee} \otimes (B^1)^{\otimes 2} \otimes (B^2)^{\otimes 3} \otimes (B^3)^{\otimes 2}$ of type $A_5^{(1)}$ and $\lambda = \overline{\Lambda}_1 + \overline{\Lambda}_2 + 2\overline{\Lambda}_3 + \overline{\Lambda}_4 + \overline{\Lambda}_6$ the rigged configuration

$(\nu, J) =$ 

is in $\mathrm{RC}(L, \lambda)$ with L the multiplicity array corresponding to B. The same configuration now written with the vacancy number next to each part is



Then

$\bar{\delta}^{\vee}(\nu, J) =$ 

and $\mathrm{rk}^{\vee}(\nu, J) = 2^{\vee}$.

Given $\mu \in \lambda^-$, there is also an inverse of the dual algorithm $\bar{\delta}^{\vee}$ associated with the weight $(\lambda - \mu)^{\vee}$ similar to the inverse of $\bar{\delta}$ as defined in Section 6.3.

Proposition 9.2. $\bar{\delta}^{\vee} : \mathrm{RC}(L) \to \mathrm{RC}(\mathrm{lh}^{\vee}(L))$ *is a well-defined injective map such that the diagram commutes*:

$$\begin{tikzcd} P(B) \arrow[r, "\bar{\phi}"] \arrow[d, "\mathrm{lh}^{\vee}"'] & \mathrm{RC}(L) \arrow[d, "\bar{\delta}^{\vee}"] \\ P(\mathrm{lh}^{\vee}(B)) \arrow[r, "\bar{\phi}"'] & \mathrm{RC}(\mathrm{lh}^{\vee}(L)). \end{tikzcd} \tag{9.1}$$

Moreover, if $\overline{\phi}(b_1 \otimes b) = (\nu, J)$ *then* $\overline{\phi}(b) = \bar{\delta}^{\vee}(\nu, J)$ *and* $b_1 = \mathrm{rk}^{\vee}(\nu, J)$.

Proof. The map $\mathrm{lh}^{\vee}$ removes $B^{1\vee} \cong B^{n,1}$. This may be achieved by n applications of $\mathrm{lh} \circ \mathrm{lb}$, which splits a box from a column and then removes it. Let Δ be the corresponding n-fold composition of maps $\bar{\delta} \circ \bar{i}$. It must be shown that $\Delta(\nu, J) = \bar{\delta}^{\vee}(\nu, J)$.

Let $a^\vee = b_1$. The letters $1 \leqslant a_1 < a_2 < \cdots < a_n \leqslant n+1$ in b_1 (where b_1 is viewed as an element of $B^{n,1}$) satisfy

$$a_i = \begin{cases} i, & \text{for } 1 \leqslant i < a, \\ i+1, & \text{for } a \leqslant i \leqslant n. \end{cases}$$

It is clear from Proposition 6.5 and the algorithm for $\bar\delta$ of Section 6.3 that the composition $\bar\delta \circ \bar{i}$ corresponding to the letter a_i for $a \leqslant i \leqslant n$ in b_1 removes a box from one string of length $s^{(i)}$ in the ith rigged partition and leaves all other strings unchanged. It also follows from the algorithms and change of vacancy numbers that $s^{(i)} \geqslant s^{(i+1)}$ and that $s^{(i)}$ is the length of the smallest singular string in $(\nu, J)^{(i)}$ with this property. For $1 \leqslant i < a$ the composition $\bar\delta \circ \bar{i}$ leaves the rigged configuration unchanged with $s^{(i)} = \infty$. It follows by induction that $s^{(i)} = \ell^{(i)}$, where $\ell^{(i)}$ as in the definition of $\bar\delta^\vee$, and hence that $\Delta(\nu, J) = \bar\delta^\vee(\nu, J)$. $\square$

9.2. *Dual left split*

We restate left splitting for a special case. Suppose $B = B^{s\vee} \otimes B'$ for $s \geqslant 2$. Define $\mathrm{ls}^\vee : B^{s\vee} \to \mathrm{ls}^\vee(B) := B^{1\vee} \otimes B^{s-1\vee}$ to be the composite map

$$B^{s\vee} \xrightarrow{\sim} B^{n,s} \xrightarrow{\mathrm{ls}} B^{n,1} \otimes B^{n,s-1} \xrightarrow{\sim} B^{1\vee} \otimes B^{s-1\vee}.$$

By Example 2.7 we may write $b \in B^{s\vee}$ as a word of length s in the dual alphabet. Computing $\mathrm{ls}^\vee$ using Example 2.6, it is seen that $\mathrm{ls}(b) = b_2 \otimes b_1$ where b_2 is the leftmost dual letter in b and b_1 is the remaining word of length $s-1$ in the dual alphabet.

Let $\mathrm{ls}^\vee(L)$ be the multiplicity array for $\mathrm{ls}^\vee(B)$. Let us denote by $\bar{j}^\vee$ the map on RCs which corresponds to $\mathrm{ls}^\vee$ under the path–RC bijection $\bar\phi$. It is the map $\bar{j}$ with respect to $B^{n,s}$ and is therefore inclusion (with some changes in vacancy numbers). With these definitions the following diagram commutes by Proposition 8.1 for $A_n^{(1)}$:

$$\begin{tikzcd} P(B,\lambda) \arrow[r, "\bar\phi"] \arrow[d, "\mathrm{ls}^\vee"'] & \mathrm{RC}(L,\lambda) \arrow[d, "\bar{j}^\vee"] \\ P(\mathrm{ls}^\vee(B),\lambda) \arrow[r, "\bar\phi"'] & \mathrm{RC}(\mathrm{ls}^\vee(L),\lambda). \end{tikzcd} \tag{9.2}$$

9.3. *The bijection $\bar\phi$ for $\mathcal{C}^{A\vee}$*

The results of this section to this point may be summarized as follows.

Proposition 9.3. *There is a unique bijection $\bar\phi : P(B) \to \mathrm{RC}(L)$ satisfying the following properties. It sends the empty path to the empty rigged configuration, and if the leftmost tensor factor in B is*:

(1) B^1: (8.1) *holds.*
(2) B^s *for* $s \geqslant 2$: (8.2) *holds.*
(3) $B^{1\vee}$: (9.1) *holds.*
(4) $B^{s\vee}$ *for* $s \geqslant 2$: (9.2) *holds.*

9.4. Duality on paths and the bijection

Let $B \in \mathcal{C}^{A\vee}$. Let L and $L^\vee$ be the multiplicity arrays for B and $B^\vee$, respectively. Explicitly, $L_i^{\vee(a)} = L_i^{(n+1-a)}$ for $1 \leqslant a \leqslant n$. Given a classical highest weight λ, let $\lambda^\vee = -w_0\lambda$ be the highest weight of the contragredient dual module to the A_n-module highest weight λ. There is a bijection $\vee : \mathrm{RC}(L, \lambda) \to \mathrm{RC}(L^\vee, \lambda^\vee)$ given by $(\nu, J) \mapsto (\nu', J')$ where $\nu'^{(a)} = \nu^{(n+1-a)}$ and $J'^{(a,i)}$ is obtained from $J^{(n+1-a,i)}$ by complementation within the $m_i^{(n+1-a)}(\nu) \times p_i^{(n+1-a)}(\nu)$ rectangle.

Theorem 9.4 [16]. *Let $B \in \mathcal{C}^A$, $B^\vee$ its contragredient dual, and L and $L^\vee$ their respective multiplicity arrays. The diagram commutes*:

$$\begin{tikzcd} P(B) \arrow[r, "\overline{\phi}"] \arrow[d, "\vee"'] & \mathrm{RC}(L) \arrow[d, "\vee"] \\ P(B^\vee) \arrow[r, "\overline{\phi}"'] & \mathrm{RC}(L^\vee). \end{tikzcd}$$

10. Virtual bijection

In this section we will prove $X = M$ for the category $\mathcal{C}$ for the nonsimply-laced algebras. For the simply-laced types $A_n^{(1)}$ and $D_n^{(1)}$ this was proved in Corollary 8.9. For the nonsimply-laced affine families it suffices to prove the following theorem.

Theorem 10.1. *For $B \in \mathcal{C}$, let $\Psi : B \to \hat{V}$ be the virtual crystal embedding, L and $\hat{L}$ the multiplicity arrays for B and $\hat{V}$, respectively. Then the simply-laced bijection $\overline{\phi}_{\hat{L}} : P(\hat{V}) \to \mathrm{RC}(\hat{L})$ restricts to a bijection $\overline{\phi}^v : P^v(B) \to \mathrm{RC}^v(L)$.*

As an immediate corollary we obtain:

Corollary 10.2. *For $\lambda \in \overline{P}^+$, $B \in \mathcal{C}$ and L the corresponding multiplicity array we have*

$$X_{B,\lambda}(q) = VX_{B,\lambda}(q) = VM_{L,\lambda}(q) = M_{L,\lambda}(q).$$

Proof. The left and right equalities were proven in Theorem 3.8 and Corollary 7.3, respectively. The middle equality follows from Theorems 8.8 and 10.1. $\square$

The remainder of this section is occupied with the proof of Theorem 10.1.

10.1. *Virtual* lh

Suppose $B = B_X = B_X^1 \otimes B_X' \in \mathcal{C}$ with virtual crystal embeddings $\Psi : B_X \to \hat{V}$ and $\Psi' : B_X' \to \hat{V}'$. By abuse of notation we write $\widehat{\mathrm{lh}}(\hat{V}) = \hat{V}'$. The map $\widehat{\mathrm{lh}} : \hat{V} \to \hat{V}'$ is defined by:

(1) if $Y = A_{2n-1}^{(1)}$ then $\widehat{\mathrm{lh}} : B_Y^{1\vee} \otimes B_Y^1 \otimes \hat{V}' \to \hat{V}'$ is defined by $\widehat{\mathrm{lh}} = \mathrm{lh} \circ \mathrm{lh}^\vee$, which drops the two leftmost factors in $\hat{V}$;
(2) if $Y = D_{n+1}^{(1)}$ and $X = B_n^{(1)}$ then $\widehat{\mathrm{lh}} : B_Y^2 \otimes \hat{V}' \to \hat{V}'$ is defined by $\widehat{\mathrm{lh}} = \mathrm{lh} \circ \mathrm{lh} \circ \mathrm{ls}$. This accomplishes the same thing as deleting the tensor factor B_Y^2;
(3) if $Y = D_{n+1}^{(1)}$ and $X = A_{2n-1}^{(2)}$ then $\widehat{\mathrm{lh}} : B_Y^1 \otimes \hat{V}' \to \hat{V}'$ is defined by $\widehat{\mathrm{lh}} = \mathrm{lh}$.

Note that in each case the total effect of the map $\widehat{\mathrm{lh}} : \hat{V}^1 \otimes \hat{V}' \to \hat{V}'$ is to drop the tensor factor $\hat{V}^1$. Therefore the following diagram commutes trivially:

$$\begin{tikzcd} B_X^1 \otimes B_X' \arrow[r, "\Psi\otimes\Psi"] \arrow[d, "\mathrm{lh}"'] & \hat{V}^1 \otimes \hat{V}' \arrow[d, "\widehat{\mathrm{lh}}"] \\ B_X' \arrow[r, "\Psi"'] & \hat{V}'. \end{tikzcd}$$

10.2. *Virtual* ls

Let $s \geqslant 2$. Recall the virtual rs map $\widehat{\mathrm{rs}} : \hat{V}^s \to \hat{V}^{s-1} \otimes \hat{V}^1$ defined in the proof of Proposition 4.6. Define the virtual ls map $\widehat{\mathrm{ls}} : \hat{V}^s \to \hat{V}^1 \otimes \hat{V}^{s-1}$ by

$$\widehat{\mathrm{ls}} = * \circ \widehat{\mathrm{rs}} \circ *. \tag{10.1}$$

Proposition 10.3. *The map $\widehat{\mathrm{ls}} : \hat{V}^s \to \hat{V}^1 \otimes \hat{V}^{s-1}$ is described explicitly as follows.*

(1) *If $Y = A_{2n-1}^{(1)}$ then $\widehat{\mathrm{ls}} : B_Y^{s\vee} \otimes B_Y^s \to B_Y^{1\vee} \otimes B_Y^1 \otimes B_Y^{s-1\vee} \otimes B_Y^{s-1}$ is the composition*

$$\begin{aligned} B_Y^{s\vee} \otimes B_Y^s &\xrightarrow{\mathrm{ls}_Y^\vee \otimes 1} B_Y^{1\vee} \otimes B_Y^{s-1\vee} \otimes B_Y^s \xrightarrow{R} B_Y^s \otimes B_Y^{1\vee} \otimes B_Y^{s-1\vee} \\ &\xrightarrow{\mathrm{ls}_Y \otimes 1 \otimes 1} B_Y^1 \otimes B_Y^{s-1} \otimes B_Y^{1\vee} \otimes B_Y^{s-1\vee} \xrightarrow{R} B_Y^{1\vee} \otimes B_Y^1 \otimes B_Y^{s-1\vee} \otimes B_Y^{s-1}. \end{aligned}$$

(2) *If $Y = D_{n+1}^{(1)}$ and $X = B_n^{(1)}$ then $\widehat{\mathrm{ls}} : B_Y^{2s} \to B_Y^2 \otimes B_Y^{2s-2}$ is the map that splits off the first two symbols, that is, $uv \mapsto u \otimes v$ where $uv \in B_Y^{2s}$, $u \in B_Y^2$, and $v \in B_Y^{2s-2}$.*
(3) *If $Y = D_{n+1}^{(1)}$ and $X = A_{2n-1}^{(2)}$ then define $\widehat{\mathrm{ls}} = \mathrm{ls}_Y : B_Y^s \to B_Y^1 \otimes B_Y^{s-1}$.*

Proof. It is enough to check these on classical highest weight vectors. This is easy because the various crystals are multiplicity-free as classical crystals. $\square$

Remark 10.4. Let $B = B_X = B^s \otimes B'$ and let $\Psi : B \to \hat{V}$ and $\Psi' : B' \to \hat{V}'$ be the virtual crystal, realizations. By abuse of notation we write $\widehat{\mathrm{ls}}(\hat{V}) = \hat{V}^1 \otimes \hat{V}^{s-1} \otimes \hat{V}'$. We also use the notation $\widehat{\mathrm{ls}}$ for the map $\widehat{\mathrm{ls}} \otimes 1_{\hat{V}'} : \hat{V}^s \otimes \hat{V}' \to \hat{V}^1 \otimes \hat{V}^{s-1} \otimes \hat{V}'$. It also satisfies (10.1).

10.3. Virtual $\bar{\delta}$ and $\bar{j}$

Given the virtual crystal embedding $\Psi : B_X \to \hat{V}$, let L and $\hat{L}$ be the multiplicity arrays for B_X and $\hat{V}$, respectively. The maps $\hat{\delta}$ and $\hat{j}$ are defined to be the maps on rigged configurations which correspond under $\bar{\phi}$ to the maps $\widehat{\mathrm{lh}}$ and $\widehat{\mathrm{ls}}$. More precisely, since $\bar{\phi}$ is a bijection for type Y there are unique maps $\hat{\delta}$ and $\hat{j}$ defined by the commutation of the diagrams

$$\begin{array}{ccc} P(\hat{V}) & \xrightarrow{\bar{\phi}_{\hat{L}}} & \mathrm{RC}(\hat{L}) \\ \Big\downarrow{\scriptstyle \widehat{\mathrm{lh}}} & & \Big\downarrow{\scriptstyle \hat{\delta}} \\ P(\widehat{\mathrm{lh}}(\hat{V})) & \xrightarrow[\bar{\phi}_{\widehat{\mathrm{lh}}(\hat{L})}]{} & \mathrm{RC}(\widehat{\mathrm{lh}}(\hat{L})) \end{array} \quad \text{and} \quad \begin{array}{ccc} P(\hat{V}) & \xrightarrow{\bar{\phi}_{\hat{L}}} & \mathrm{RC}(\hat{L}) \\ \Big\downarrow{\scriptstyle \widehat{\mathrm{ls}}} & & \Big\downarrow{\scriptstyle \hat{j}} \\ P(\widehat{\mathrm{ls}}(\hat{V})) & \xrightarrow[\bar{\phi}_{\widehat{\mathrm{ls}}(\hat{L})}]{} & \mathrm{RC}(\widehat{\mathrm{ls}}(\hat{L})) \end{array} \tag{10.2}$$

where $\widehat{\mathrm{lh}}(\hat{L})$ and $\widehat{\mathrm{ls}}(\hat{L})$ are the multiplicity arrays for $\widehat{\mathrm{lh}}(\hat{V})$ and $\widehat{\mathrm{ls}}(\hat{V})$, respectively.

For $\lambda \in \overline{P}^+(X)$ let $\widehat{\mathrm{rk}} : \mathrm{RC}(\hat{L}, \Psi(\lambda)) \to \hat{V}^1$ be the map which gives the tensor product of the ranks of the sequence of rigged configurations that occur during the computation of $\hat{\delta}$.

Lemma 10.5. *$\hat{\delta}$ maps $\mathrm{RC}^v(L)$ into $\mathrm{RC}^v(\mathrm{lh}(L))$ and $\widehat{\mathrm{rk}}$ maps $\mathrm{RC}^v(L)$ into $\mathrm{Im}(\Psi : B_X^1 \to \hat{V}^1)$.*

Proof. The proof proceeds by cases.

$X = C_n^{(1)}$ and $Y = A_{2n-1}^{(1)}$. According to Definition 7.1 the elements $(\hat{\nu}, \hat{J}) \in \mathrm{RC}^v(L)$ have the following properties:

(1) $\hat{m}_i^{(a)} = \hat{m}_i^{(2n-a)}$ and $\hat{J}^{(a,i)} = \hat{J}^{(2n-a,i)}$;
(2) $\hat{m}_i^{(n)} = 0$ if i is odd;
(3) the parts of $\hat{J}^{(n,i)}$ are even.

From (10.2) and Proposition 9.2 it is clear that $\hat{\delta} = \bar{\delta} \circ \bar{\delta}^\vee$. It must be shown that $\hat{\delta}(\hat{\nu}, \hat{J})$ also possesses the three properties (1)–(3). Let $\ell^{\vee(a)}$ the lengths of the strings selected by $\bar{\delta}^\vee$ and $\ell^{(a)}$ be the lengths of the strings selected by the subsequent application of $\bar{\delta}$. Let $\mathrm{rk}^\vee(\hat{\nu}, \hat{J}) = (2n + 1 - r)^\vee$ for some $1 \leqslant r \leqslant 2n$. If $r \leqslant n$, it is clear from the definitions that $\ell^{(a)} = \ell^{\vee(2n-a)}$ for $1 \leqslant a < r$, so that points (1)–(3) still hold. Here $\widehat{\mathrm{rk}}(\hat{\nu}, \hat{J}) = (2n + 1 - r)^\vee \otimes r = \Psi(r)$. For $r = n + 1$, we must have $\ell^{\vee(n+1)} < \ell^{\vee(n)}$ since otherwise by the symmetry (1) $\ell^{\vee(n-1)} = \ell^{\vee(n)} = \ell^{\vee(n+1)} < \infty$ which contradicts the assumption that

$r = n+1$. However, this implies that $\ell^{(a)} = \ell^{\vee(2n-a)}$ for $1 \leqslant a < n$ and $\ell^{(n)} = \ell^{\vee(n)} - 1$. Since the vacancy numbers are all even (1)–(3) remain valid. One has $\widehat{\mathrm{rk}}(\hat{\nu}, \hat{J}) = n^\vee \otimes (n+1) = \Psi(\bar{n})$. Finally let $r > n+1$ and let $r' \leqslant n$ be minimal such that $\ell^{\vee(2n-r')} = \ell^{\vee(n)}$. By symmetry (1) we have $\ell^{\vee(a)} = \ell^{\vee(n)}$ for all $r' \leqslant a \leqslant 2n-r'$. By the algorithms for $\bar{\delta}^\vee$ and $\bar{\delta}$ and properties (1)–(3) for $(\hat{\nu}, \hat{J})$ it follows that $\ell^{(a)} = \ell^{\vee(2n-a)}$ for $1 \leqslant a < r'$ and $2n - r' < a < r$, and $\ell^{(a)} = \ell^{\vee\,(2n-a)} - 1$ for $r' \leqslant a \leqslant 2n - r'$. Again this implies that properties (1)–(3) hold for $\hat{\delta}(\hat{\nu}, \hat{J})$. Then $\widehat{\mathrm{rk}}(\hat{\nu}, \hat{J}) = (2n+1-r)^\vee \otimes r = \Psi(\overline{2n+1-r})$.

$X = A_{2n}^{(2)}$ and $Y = A_{2n-1}^{(1)}$. The elements in $\mathrm{RC}^v(L)$ are characterized by points (1) and (3). Everything goes through as for the case $X = C_n^{(1)}$ except that, since $\hat{\nu}^{(n)}$ may contain odd parts, it is possible that $\ell^{\vee(n)} = 1$. In this case $\ell^{\vee(a)} = 1$ for all $1 \leqslant a \leqslant 2n-1$ by point (1). Then $\ell^{(a)} = \infty$ for all $1 \leqslant a \leqslant 2n-1$, so that $\hat{\delta}(\hat{\nu}, \hat{J})$ again satisfies (1) and (3). Then $\widehat{\mathrm{rk}}(\hat{\nu}, \hat{J}) = 1^\vee \otimes 1 = \Psi(\emptyset)$.

$X = D_{n+1}^{(2)}$ and $Y = A_{2n-1}^{(1)}$. The elements in $\mathrm{RC}^v(L)$ are characterized by point (1). The proof goes through as before except that $\hat{J}^{(n,i)}$ could have an odd part. This could only change the computation of $\bar{\delta} \circ \bar{\delta}^\vee$ if such an odd part were selected. Recall that $p_i^{(n)}$ is even for all i. Therefore the odd part cannot be selected by $\bar{\delta}^\vee$. It can only be selected by $\bar{\delta}$ if $\mathrm{rk}^\vee(\hat{\nu}, \hat{J}) = (n+1)^\vee$ and the odd part has size $p_i^{(n)} - 1$ for some $i \geqslant \ell^{\vee(n+1)}$. By point (1) and the fact that $(\hat{\nu}, \hat{J})^{(a)}$ is unchanged by $\bar{\delta}^\vee$ for $1 \leqslant a \leqslant n-1$, we have $\ell^{(a)} = \ell^{\vee(2n-a)}$ for $1 \leqslant a \leqslant n-1$ and $\ell^{(n)}$ is the odd (now singular) part. Thus after applying $\bar{\delta} \circ \bar{\delta}^\vee$ point (1) still holds. $\widehat{\mathrm{rk}}(\hat{\nu}, \hat{J}) = (n+1)^\vee \otimes (n+1) = \Psi(0)$. Note that $\ell^{(n+1)} = \infty$ since $\bar{\delta}^\vee$ caused the strings in the $(n+1)$th rigged partition that were longer than $\ell^{\vee(n+1)}$, to become nonsingular.

$X = A_{2n}^{(2)\dagger}$ and $Y = A_{2n-1}^{(1)}$. The elements in $\mathrm{RC}^v(L)$ are characterized by (1) and:

(3′) the parts of $\hat{J}^{(n,i)}$ have the same parity as i.

Let $\mathrm{rk}^\vee(\hat{\nu}, \hat{J}) = (2n+1-r)^\vee$ for some $1 \leqslant r \leqslant 2n$. If $r \leqslant n$, we have as for the case $X = C_n^{(1)}$ that $\ell^{(a)} = \ell^{\vee(2n-a)}$ for $1 \leqslant a < r$, so that (1), and (3′) still hold and $\widehat{\mathrm{rk}}(\hat{\nu}, \hat{J}) = (2n+1-r)^\vee \otimes r = \Psi(r)$.

If $r = n+1$, note that $\ell^{\vee(n)} \in 2\mathbb{Z}$ since all vacancy numbers $p_i^{(n)}$ are even, so that by (3′) only the riggings for i even can possibly be singular. As in case $C_n^{(1)}$ we must have $\ell^{\vee(n+1)} < \ell^{\vee(n)}$. By symmetry (1) we have $\ell^{(a)} = \ell^{\vee(2n-a)}$ for $1 \leqslant a < n$. The application of $\bar{\delta}^\vee$ changes the vacancy numbers in the nth rigged partition corresponding to the strings of length i for $\ell^{\vee(n+1)} \leqslant i < \ell^{\vee(n)}$ by -1, which makes these vacancy numbers odd. In particular, the rigging of the new string of length $\ell^{\vee(n)} - 1$ is odd. In addition, $\ell^{\vee(n+1)} \leqslant \ell^{(n)} < \ell^{\vee(n)}$ and by (3′) $\ell^{(n)}$ must be odd. By the change in vacancy number after the application of $\bar{\delta}$, the new rigging of the string of length $\ell^{(n)} - 1$ must be even. Hence (1) and (3′) hold for $\hat{\delta}(\hat{\nu}, \hat{J})$ and $\widehat{\mathrm{rk}}(\hat{\nu}, \hat{J}) = (n+1)^\vee \otimes (n+1) = \Psi(0)$.

If $r > n+1$, let $r' \leqslant n$ be defined as for the case $C_n^{(1)}$. As before $\ell^{\vee(a)} = \ell^{\vee(n)}$ for $r' \leqslant a \leqslant 2n - r'$. If $r' < n$ everything goes through as in case $C_n^{(1)}$. If $r' = n$ (which means that $\ell^{\vee(n+1)} < \ell^{\vee(n)}$), by the same arguments as for $r = n+1$, we have $\ell^{(a)} = \ell^{\vee(2n-a)}$ for $a \neq n$, $\ell^{\vee(n+1)} \leqslant \ell^{(n)} < \ell^{\vee(n)}$ and (3′) holds for the new riggings. Hence properties (1) and (3′) hold for $\hat{\delta}(\hat{\nu}, \hat{J})$ and $\widehat{\mathrm{rk}}(\hat{\nu}, \hat{J}) = (2n+1-r)^\vee \otimes r = \Psi(\overline{2n+1-r})$.

$X = B_n^{(1)}$ *and* $Y = D_{n+1}^{(1)}$. The elements in $\mathrm{RC}^v(L)$ are characterized by

(1) $m_i^{(n)} = m_i^{(n+1)}$ and $J^{(n,i)} = J^{(n+1,i)}$ for all $i > 0$;
(2) $\nu^{(a)}$ and $J^{(a,i)}$ have only even parts for $1 \leqslant a < n$.

By Section 10.1 and (10.2) we have $\hat{\delta} = \bar{\delta} \circ \bar{\delta} \circ \bar{j}$. Let $\ell^{(a)}$ and $\bar{\ell}^{(a)}$ (respectively $s^{(a)}$ and $\bar{s}^{(a)}$) be the length of the selected strings for the right (respectively left) $\bar{\delta}$. Then it follows from the definition of $\bar{j}$, $\bar{\delta}$ and point (2) that $s^{(a)} = \ell^{(a)} - 1$ for $1 \leqslant a < n$. Furthermore, from point (1) we obtain that $\ell^{(n)} = \ell^{(n+1)} > s^{(n)} = s^{(n+1)}$, and again by point (2) that $\bar{s}^{(a)} = \bar{\ell}^{(a)} - 1$ for $1 \leqslant a < n$. This implies that points (1) and (2) hold for $\hat{\delta}(\hat{\nu}, \hat{J})$. Moreover, let $x = \mathrm{rk}(\hat{\nu}, \hat{J})$ and $y = \mathrm{rk}(\bar{\delta}(\hat{\nu}, \hat{J}))$. Note that $x, y \neq n+1, \overline{n+1}$ because of point (1). Also $x = y$ except possibly $x = n$ and $y = \bar{n}$. Then $\widehat{\mathrm{rk}}(\hat{\nu}, \hat{J}) = xx = \Psi(x)$ if $x = y$ or $\widehat{\mathrm{rk}}(\hat{\nu}, \hat{J}) = n\bar{n} = \Psi(0)$ if $x \neq y$.

$X = A_{2n-1}^{(2)}$ *and* $Y = D_{n+1}^{(1)}$. The elements in $\mathrm{RC}^v(L)$ are characterized by point (1). It is obvious from its definition that $\hat{\delta} = \bar{\delta}$ preserves this property. Let $x = \mathrm{rk}(\hat{\nu}, \hat{J})$. As before $x \neq n+1, \overline{n+1}$ because of point (1). Then $\widehat{\mathrm{rk}}(\hat{\nu}, \hat{J}) = x = \Psi(x)$. □

Thus we may define the virtual rank map $\mathrm{rk}^v : \mathrm{RC}^v(L) \to B_X^1$ by $\mathrm{rk}^v(\hat{\nu}, \hat{J}) = x$ where $\Psi(x) = \widehat{\mathrm{rk}}(\hat{\nu}, \hat{J})$ for all $(\hat{\nu}, \hat{J}) \in \mathrm{RC}^v(L)$. Then we have:

Proposition 10.6. *The map* $(\hat{\delta}, \mathrm{rk}^v) : \mathrm{RC}^v(L, \lambda) \to \bigcup_{\mu \in \lambda^-} \mathrm{RC}^v(\mathrm{lh}(L), \mu) \times B_X^1$ *is injective.*

For the proof of Theorem 10.1 we also need the inverse to Lemma 10.5 which involves the inverse of $\hat{\delta}$. Let $\lambda \in \overline{P}_X^+$, $L = (L_1, L_2, \ldots)$ a multiplicity array and $\mathrm{lh}^{-1}(L) = (L_1 + 1, L_2, L_3, \ldots)$. Denote by $\widetilde{\mathrm{RC}}^v(L, \lambda)$ the subset of $\mathrm{RC}^v(L, \lambda) \times B^1$ given by $((\nu, J), b)$ such that $\lambda + \mathrm{wt}(b) \in \overline{P}^+$ and if $b = 0$ then also $\lambda_n > 0$. Let $\hat{b} = \Psi(b)$. By abuse of notation we define

$$\hat{\delta}^{-1} : \widetilde{\mathrm{RC}}^v(L, \lambda) \to \bigcup_{\beta \in \lambda^+} \mathrm{RC}\big(\widehat{\mathrm{lh}^{-1}(L)}, \Psi(\beta)\big).$$

If $Y = A_{2n-1}^{(1)}$, let $\hat{b} = b_1 \otimes b_2$. Then $\hat{\delta}^{-1}((\nu, J), b) = \bar{\delta}^{\vee^{-1}}(\bar{\delta}^{-1}((\nu, J), b_2), b_1)$, with $\bar{\delta}^{-1}$ as defined in Section 6.3 and $\bar{\delta}^{\vee^{-1}}$ as defined in Section 9.1. If $Y = D_{n+1}^{(1)}$ and $X = B_n^{(1)}$, let $\hat{b} = xy$. Then $\hat{\delta}^{-1}((\nu, J), b) = \bar{\delta}^{-1}(\bar{\delta}^{-1}((\nu, J), y), x)$. Finally for $Y = D_{n+1}^{(1)}$ and $X = A_{2n-1}^{(2)}$, let $\hat{b} = x$. Then $\hat{\delta}^{-1}((\nu, J), b) = \bar{\delta}^{-1}((\nu, J), x)$.

Lemma 10.7. *Given* λ, L, $\mathrm{lh}^{-1}(L)$, b *and* $\hat{b}$ *as above the map* $\hat{\delta}^{-1}$ *maps* $\widetilde{\mathrm{RC}}^{v}(L,\lambda)$ *into* $\bigcup_{\beta\in\lambda^+}\mathrm{RC}^v(\mathrm{lh}^{-1}(L),\beta)$.

Proof. The proof is very similar to the proof of Lemma 10.5. $\square$

Lemma 10.8. $\hat{j}$ *maps* $\mathrm{RC}^v(L)$ *into* $\mathrm{RC}^v(\mathrm{ls}(L))$.

Proof. Let $Y = A_{2n-1}^{(1)}$. By (10.2) and Section 10.2 we have $\hat{j} = \bar{j}\circ\bar{j}^{\vee}$. Both $\bar{j}$ and $\bar{j}^{\vee}$ are inclusions that do not change the rigged configuration (only certain vacancy numbers). Hence if $(\hat{\nu},\hat{J})\in\mathrm{RC}^v(L)$ has the characterization as stated in the previous lemma, then so does $\hat{j}(\hat{\nu},\hat{J})$.

Let $Y = D_{n+1}^{(1)}$. If $X = A_{2n-1}^{(2)}$, we have $\hat{j} = \bar{j}$. For $X = B_n^{(1)}$, let $B = B^s\otimes B'$ for $s\geqslant 2$ and embeddings $\Psi : B\to\hat{V}$ and $\Psi' : B'\to\hat{V}'$ with $\hat{V} = B_Y^{2s}\otimes\hat{V}'$. It can be shown (using $*$ and properties of rs) that if $x, y\in B_Y^1$ and $u\in B_Y^{2s-2}$ are such that $xyu\in B_Y^{2s}$ then for any $b'\in\hat{V}'$ one has $\widehat{\mathrm{ls}}(xyu\otimes b') = xy\otimes u\otimes b'$. One may show that the corresponding operation on RCs is inclusion. This may be seen by observing that $\mathrm{ls}\circ\widehat{\mathrm{ls}} : B_Y^{2s}\otimes\hat{V}'\to B_Y^1\otimes B_Y^1\otimes B_Y^{2s-2}\otimes\hat{V}'$, which sends $xyu\otimes b'$ to $x\otimes y\otimes u\otimes b'$, can also be computed by a composition of ls maps and R-matrices, whose corresponding maps on RCs are inclusions. This proves that $\hat{j}(\hat{\nu},\hat{J})\in\mathrm{RC}^v(\mathrm{ls}(L))$. $\square$

10.4. Proof of Theorem 10.1

It must be shown that the bijection $\bar{\phi}_{\hat{L}} : P(\hat{V})\to\mathrm{RC}(\hat{L})$ maps $P^v(B)$ (1) into and (2) onto $\mathrm{RC}^v(L)$, thereby defining a bijection $\bar{\phi}_L^v : P^v(B)\to\mathrm{RC}^v(L)$ by restriction. Let $B = B^s\otimes B'$ with $\Psi : B\to\hat{V}^s\otimes\hat{V}'$.

The case $s=1$. For (1) consider a typical element of $P^v(B,\lambda)$, given by $\Psi(b)$ with $b\in P(B,\lambda)$. Write $b = x\otimes b'$ with $x\in B_X^1$ and $b'\in P(B',\mu)$. Then $\Psi(b')\in P^v(\mathrm{lh}(B),\mu)$. Let $(\hat{\nu},\hat{J}) = \bar{\phi}_{\hat{L}}(\Psi(b))\in\mathrm{RC}(\hat{L})$. It must be shown that $(\hat{\nu},\hat{J})\in\mathrm{RC}^v(L,\lambda)$. By (10.2) and induction one has $\hat{\delta}(\hat{\nu},\hat{J})\in\mathrm{RC}^v(\mathrm{lh}(L),\mu)$ and $\widehat{\mathrm{rk}}(\hat{\nu},\hat{J}) = \Psi(x)$. By Lemma 10.7 we can conclude that $(\hat{\nu},\hat{J})\in\mathrm{RC}^v(L,\lambda)$.

For (2) let $(\hat{\nu},\hat{J})\in\mathrm{RC}^v(L)$. Let $\hat{b} = \hat{x}\otimes\hat{b}'\in P(\hat{V})$ (with $\hat{x}\in\hat{V}^1$ and $\hat{b}'\in\hat{V}'$) be such that $\bar{\phi}_{\hat{L}}(\hat{b}) = (\hat{\nu},\hat{J})$. It must be shown that $\hat{b}\in P^v(B)$. By (10.2) we have $\bar{\phi}_{\widehat{\mathrm{lh}}(\hat{L})}(\widehat{\mathrm{lh}}(\hat{b})) = \hat{\delta}(\bar{\phi}_{\hat{L}}(\hat{b})) = \hat{\delta}(\hat{\nu},\hat{J})\in\mathrm{RC}^v(\widehat{\mathrm{lh}}(\hat{L}))$. By induction $\hat{b}' = \widehat{\mathrm{lh}}(\hat{b})\in P^v(\mathrm{lh}(B))$; write $\hat{b}' = \Psi(b')$ for some $b'\in B'$. By Lemma 10.5 and (10.2), $\hat{x} = \Psi(x)$ for $x = \mathrm{rk}^v(\hat{\nu},\hat{J})$. Let $b = x\otimes b'\in B$. By definition $\Psi(b) = \Psi(x)\otimes\Psi(b') = \hat{x}\otimes\hat{b}' = \hat{b}$. Therefore $\hat{b}\in P^v(B)$ as desired.

The case $s\geqslant 2$. For (1), a typical element of $P^v(B)$ has the form $\Psi(b)$ for $b\in P(B)$. Let $\bar{\phi}_{\hat{L}}(\Psi(b)) = (\hat{\nu},\hat{J})\in\mathrm{RC}(\hat{L})$. It must be shown that $(\hat{\nu},\hat{J})\in\mathrm{RC}^v(L)$. Note that $\hat{j}(\hat{\nu},\hat{J}) = \hat{j}(\bar{\phi}_{\hat{L}}(\Psi(b))) = \bar{\phi}_{\hat{L}^s}(\widehat{\mathrm{ls}}(\Psi(b)))\in\mathrm{RC}^v(\mathrm{ls}(L))$ by (10.2) and induction. But $\hat{j}(\hat{\nu},\hat{J}) = (\hat{\nu},\hat{J})$ and $(\hat{\nu},\hat{J})\in\mathrm{RC}(\hat{L})$. It follows that $(\hat{\nu},\hat{J})\in\mathrm{RC}^v(L)$.

For (2), let $(\hat{\nu}, \hat{J}) \in \mathrm{RC}^v(L)$. Let $\hat{b} \in P(\hat{V})$ be such that $\overline{\phi}_{\hat{L}}(\hat{b}) = (\hat{\nu}, \hat{J})$. It must be shown that $\hat{b} \in P^v(B)$. By (10.2) and induction, $\hat{j}(\hat{\nu}, \hat{J}) = \hat{j}(\overline{\phi}_{\hat{L}}(\hat{b})) = \overline{\phi}_{\hat{L}^s}(\widehat{\mathrm{ls}}(\hat{b})) \in \mathrm{RC}^v(\mathrm{ls}(L))$. Therefore $\widehat{\mathrm{ls}}(\hat{b}) \in P^v(\mathrm{ls}(B))$. We conclude that $\hat{b} \in P^v(B)$ by (10.1), Propositions 4.6(3) and 3.9.

This concludes the proof of Theorem 10.1.

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