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NOTE

On the Size of Set Systems on $[n]$ Not Containing Weak (r, Δ)-Systems

Vojtěch Rödl*

Department of Mathematics and Computer Science, Emory University, Atlanta, Georgia 30322

and

Luboš Thoma†

Let $r \geqslant 3$ be an integer. A weak (r, Δ)-system is a family of r sets such that all pairwise intersections among the members have the same cardinality. We show that for n large enough, there exists a family $\mathscr{F}$ of subsets of $[n]$ such that $\mathscr{F}$ does not contain a weak (r, Δ)-system and $|\mathscr{F}| \geqslant 2^{(1/3) \cdot n^{1/5} \log^{4/5}(r-1)}$. This improves an earlier result of Erdős and Szemerédi (1978, *J. Combin. Theory Ser. A* **24**, 308–313; cf. Erdős, On some of my favorite theorems, *in* "Combinatorics, Paul Erdős Is Eighty," Vol. 2, Bolyai Society Math. Studies, pp. 97–133, János Bolyai Math. Soc., Budapest, 1990). © 1997 Academic Press

1. INTRODUCTION AND RESULTS

In this note we are going to show a lower bound on the maximal size of a set system of subsets of an n-element set without a weak (r, Δ)-system.

Definition 1.1. Let $r \geqslant 3$ be an integer. A set system $\mathscr{H} = \{A_1, ..., A_r\}$ such that the cardinality of the intersection $A_i \cap A_j$ is the same for all $\binom{r}{2}$ pairs i, j, $1 \leqslant i < j \leqslant r$, is called a weak (r, Δ)-system. By a weak Δ-system we understand a weak (r, Δ)-system for some $r \geqslant 3$.

Throughout the paper we use $\log x = \log_2 x$ and $[n] = \{1, ..., n\}$, where n is a positive integer. We do not optimize the constants used in our statements.

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The existence of weak Δ-systems in set systems has been studied since long time ago, cf. [ER 60], [ER 69], [ES 78], [FR 87], [AFK 95].

The best lower bound was proved by P. Erdős and E. Szemerédi in [ES 78]. They showed:

THEOREM 1.2. *There exists a family $\mathscr{F}$ of subsets of a given set S so that $\mathscr{F}$ does not contain a weak Δ-system, where*

$$|S| = n \qquad \text{and} \qquad |\mathscr{F}| \geqslant n^{\log n / 4 \log \log n}.$$

This lower bound is also mentioned in [E 90].

The best known upper bound was proved in the following stronger form by P. Frankl and V. Rödl [FR 87].

THEOREM 1.3. *Given $r \geqslant 3$, there exist constants $v = v(r) > 0$, $\varepsilon = \varepsilon(r) > 0$, so that for every n and l with $|n/4 - l| \leqslant vn$, and for every family $\mathscr{F}$ of subsets of $[n]$ with $|\mathscr{F}| > (2-\varepsilon)^n$, there exist $F_1, ..., F_r \in \mathscr{F}$ with $|F_i \cap F_j| = l$, $1 \leqslant i < j \leqslant r$.*

In Section 2 we are going to prove the following Theorem 1.4.

THEOREM 1.4. *Let $r \geqslant 3$ be an integer. For n large enough, there exists a family $\mathscr{F}$ of subsets of $[n]$, such that $\mathscr{F}$ does not contain a weak (r, Δ)-system and*

$$|\mathscr{F}| \geqslant 2^{(1/3) \cdot n^{1/5} \cdot \log^{4/5}(r-1)}.$$

This bound improves the bound given in [ES 78], as for $r = 3$, Theorem 1.4 implies that there is a set system $\mathscr{F}$ on $[n]$ such that $\mathscr{F}$ does not contain a weak Δ-system and $|\mathscr{F}| \geqslant 2^{(1/3)\, n^{1/5}}$, for n large enough.

Unfortunately, there is still a large gap between the lower and upper bounds.

2. PROOF OF THE LOWER BOUND

In this section, we are going to prove Theorem 1.4. First, we need a definition and some lemmas.

DEFINITION 2.1. *Let $\mathfrak{m}(n, m, l)$ be the maximal cardinality of a family $\mathscr{F}$ of m-element subsets of $[n]$ such that*

$$|F \cap F'| \leqslant l,$$

for any pair $F, F' \in \mathscr{F}$.

The following bound is well known.

LEMMA 2.2. *For any* $n \geqslant m > l \geqslant 0$

$$\mathfrak{m}(n, m, l) \geqslant \binom{n}{m} \Big/ \sum_{i=l+1}^{m} \binom{m}{i}\binom{n-m}{m-i} \tag{1}$$

holds.

Proof. Consider a graph H with the vertex set consisting of all m-element subsets of the set $[n]$ and in which two subsets are adjacent when they intersect in more than l elements. Clearly, the degree of each vertex of H equals

$$\Delta = \sum_{i=l+1}^{m-1} \binom{m}{i}\binom{n-m}{m-i}.$$

Therefore, the graph H can be properly colored by $\Delta + 1$ colors; the largest of the color classes establishes the lemma. ∎

For a special choice of parameters m and l, we obtain the following lemma.

LEMMA 2.3. *Set* $m = \lfloor n^{3/5} \cdot \log^{2/5}(r-1) \rfloor$ *and* $l = \lceil 6n^{1/5} \cdot \log^{4/5}(r-1) \rceil$. *Then for n sufficiently large the following holds*:

$$\mathfrak{m}(n, m, l) \geqslant e^{5n^{1/5} \cdot \log^{4/5}(r-1)}. \tag{2}$$

Proof. We suppose n is large enough and such that the following inequalities hold:

$$n - 2m + l + 1 \geqslant \frac{1}{3} n, \tag{3}$$

$$l! \geqslant 4 \cdot \left(\frac{l}{e}\right)^{l}, \tag{4}$$

$$\frac{n-2m}{e} \geqslant \frac{n}{3}, \tag{5}$$

$$2\left(1 + \frac{m}{n}\right)^{m} \geqslant e^{m^2/n}, \tag{6}$$

and

$$\ln 2 \geqslant \frac{2 \log^{2/5}(r-1)}{n^{2/5}}. \tag{7}$$

We start with the following claim.

CLAIM A. *For $i = l, ..., m-1$, the following holds*:

$$\frac{\binom{m}{i}\binom{n-m}{m-i}}{\binom{m}{i+1}\binom{n-m}{m-i-1}} > 2. \tag{8}$$

Proof of Claim A. Using (3) we bound

$$\frac{\binom{m}{i}\binom{n-m}{m-i}}{\binom{m}{i+1}\binom{n-m}{m-i-1}} = \frac{(i+1)(n-2m+i+1)}{(m-i)^2} > \frac{l \cdot \frac{1}{3} \cdot n}{m^2} > 2. \quad \blacksquare$$

Note that to show the last inequality, we needed $ln > 6m^2$.
Combining Lemma 2.2 and Claim A, we immediately get

$$\mathfrak{m}(n, m, l) \geqslant \frac{\binom{n}{m}}{\binom{m}{l}\binom{n-m}{m-l}}, \tag{9}$$

for our choice of m and l.

In order to show inequality (2) we will further estimate the right hand side of (9) as follows: (In the estimates below, we use inequalities (4)–(7).)

$$\mathfrak{m}(n, m, l) \geqslant \frac{\binom{n}{m}}{\binom{m}{l}\binom{n-m}{m-l}} = \frac{(m-l)!^2}{m!^2} \cdot \frac{n!\, l!(n-2m+l)!}{(n-m)!^2}$$

$$\geqslant \frac{l!}{m^{2l}} \cdot \frac{n(n-1) \cdot \cdots \cdot (n-m+1)}{(n-m)(n-m-1) \cdot \cdots \cdot (n-2m+1)}$$

$$\times (n-2m+l)(n-2m+l-1) \cdot \cdots \cdot (n-2m+1)$$

$$\geqslant 4 \cdot \left(\frac{l}{em^2}\right)^l \cdot (n-2m)^l \cdot \frac{n}{n-m} \cdot \frac{n-1}{n-m-1} \cdot \cdots \cdot \frac{n-(m-1)}{n-m-(m-1)}$$

$$\geqslant 4 \cdot \left(\frac{nl}{3m^2}\right)^l \cdot \left(1+\frac{m}{n-m}\right)^m$$

$$\geqslant 4 \cdot \left(\frac{nl}{3m^2}\right)^l \cdot \left(1+\frac{m}{n}\right)^m$$

$$\geqslant 2 \cdot \left(\frac{nl}{3m^2}\right)^l \cdot e^{m^2/n} = 2e^{(m^2/n)+l\ln(nl/3m^2)}$$

$$\geqslant e^{n^{1/5}\cdot\log^{4/5}(r-1)+6n^{1/5}\cdot\log^{4/5}(r-1)\cdot\ln 2}$$

$$\geqslant e^{5n^{1/5}\cdot\log^{4/5}(r-1)}.$$

This finishes the proof of the lemma. ∎

Proof of Theorem 1.4. Let $\mathscr{G}$ be a system of subsets of $[n]$ establishing

$$\mathfrak{m}(n, \lfloor n^{3/5}\cdot\log^{2/5}(r-1)\rfloor, \lceil 6n^{1/5}\cdot\log^{4/5}(r-1)\rceil) \geqslant e^{5n^{1/5}\cdot\log^{4/5}(r-1)}.$$

Set $k=\lceil n^{1/5}/3\log^{1/5}(r-1)\rceil$.

Consider a rooted $(r-1)$-ary tree T of height k (i.e., with k edges on every branch and with $((r-1)^{k+1}-1)/(r-2)$ vertices). A *branch* in a rooted tree is defined as a path from the root to a leaf.

Let $|T|$ denote the number of vertices of T. Since for n large enough

$$|\mathscr{G}| > e^{5n^{1/5}\cdot\log^{4/5}(r-1)} > (r-1)^{k+1} > |T|, \tag{10}$$

we can associate a set $\lambda(v)\in\mathscr{G}$ to each vertex v of T in such a way that for $v_1\neq v_2$, $\lambda(v_1)\neq\lambda(v_2)$ holds.

Let $\mathfrak{B}$ be the set of all branches of T. Set

$$\mathscr{F} = \left\{\bigcup_{v\in V(B)} \lambda(v) : B\in\mathfrak{B}\right\}. \tag{11}$$

We will prove that $\mathscr{F}$ is a set system establishing the validity of Theorem 1.4. This follows from the following facts.

FACT A. $|\mathscr{F}| \geqslant 2^{(1/3)\cdot n^{1/5}\cdot\log^{4/5}(r-1)}$.

FACT B. *$\mathscr{F}$ does not contain a weak (r, Δ)-system.*

Proof of Fact A. Let B_1, B_2, $B_1 \neq B_2$, be two branches of T. We will show that

$$\bigcup_{v \in V(B_1)} \lambda(v) - \bigcup_{u \in V(B_2)} \lambda(u) \neq \varnothing \tag{12}$$

and, thus, any two sets assigned to different branches of T are distinct.

Since $|\lambda(v)| = m$ for each vertex v of T, and

$$\left| \bigcup_{v \in V(B_1)} \lambda(v) \cap \bigcup_{u \in V(B_2)} \lambda(u) \right| \leqslant (k+1)^2 \cdot l,$$

we infer that for n large enough

$$\begin{aligned} &\left| \bigcup_{v \in V(B_1)} \lambda(v) - \bigcup_{u \in V(B_2)} \lambda(u) \right| \\ &\quad = \left| \bigcup_{v \in V(B_1)} \lambda(v) - \left(\bigcup_{v \in V(B_1)} \lambda(v) \cap \bigcup_{u \in V(B_2)} \lambda(u) \right) \right| \\ &\quad \geqslant m - (k+1)^2 l \\ &\quad \geqslant n^{3/5} \cdot \log^{2/5}(r-1) - 1 \\ &\qquad - \left(\frac{n^{1/5}}{3 \log^{1/5}(r-1)} + 2 \right)^2 \left(6n^{1/5} \log^{4/5}(r-1) + 1 \right) \qquad (13) \\ &\quad > 0. \qquad (14) \end{aligned}$$

Hence, statement (12) holds.

For the size of the family $\mathscr{F}$ we then get

$$|\mathscr{F}| = (r-1)^k \geqslant (r-1)^{n^{1/5}/3 \log^{1/5}(r-1)} = 2^{(1/3)\, n^{1/5} \cdot \log^{4/5}(r-1)}. \quad \blacksquare$$

Proof of Fact B. We are going to show that $\mathscr{F}$ does not contain a weak (r, Δ)-system. First recall that the elements of $\mathscr{F}$ correspond to the branches of the $(r-1)$-ary tree T and consider any r distinct sets $A_1, ..., A_r$ of $\mathscr{F}$. By a simple pigeon-hole type argument this, however, implies that there are three sets F_1, F_2, and F_3 (among $A_1, ..., A_r$) such that if B_1, B_2, B_3 are branches corresponding to F_1, F_2, F_3

$$|V(B_1) \cap V(B_2)| \neq |V(B_1) \cap V(B_3)|$$

holds. Without loss of generality, assume that

$$|V(B_1) \cap V(B_3)| = j > i = |V(B_1) \cap V(B_2)|.$$

Then we can bound

$$|F_1 \cap F_3| \geqslant jm - \binom{j}{2} \cdot l$$

while

$$|F_1 \cap F_2| \leqslant im + [2i(k-i+1) + (k-i+1)^2]\, l = im + [(k+1)^2 - i^2]\, l$$

(since every $F \in \mathscr{F}$ is a union of $k+1$ sets from $\mathscr{G}$). In order to verify that $\mathscr{F}$ does not contain a weak (r, Δ)-system, it is enough to show that

$$|F_1 \cap F_2| \leqslant im + [(k+1)^2 - i^2]\, l < jm - \binom{j}{2} l \leqslant |F_1 \cap F_3|.$$

This will follow from

$$m > (k+1)^2 l \geqslant \frac{(k+1)^2 - i^2 + \binom{j}{2}}{j-i} \cdot l. \tag{15}$$

While the first inequality follows from (13)–(14), we now prove the second. First, we will rewrite it in an equivalent form:

$$(j-i-1)(k+1)^2 \geqslant \binom{j}{2} - i^2. \tag{16}$$

If $j = i+1$, then $\binom{j}{2} - i^2 = (i(i+1)/2) - i^2 = \frac{1}{2} i(1-i) \leqslant 0$ and, hence, (16) holds. If $j > i+1$, then $(j-i-1)(k+1)^2 > k^2 > \binom{j}{2} - i^2$ and the inequality (16) holds as well. ∎

Remark 2.4. Our bound is very likely not the best possible. Possible improvements could be obtained by improving Lemma 2.2 (for the appropriate choice of parameters) or by modifying our construction.

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