On Finite Groups with a Sylow $p$-Subgroup of Maximal Class

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INTRODUCTION

We prove the following Theorem.

THEOREM A. Let $G$ be a finite group with a Sylow $p$-subgroup $P$ of maximal class, $|P| = p^n$, $n > p + 3$, $p > 3$. For $i > 2$ let $P_i = K_i(P)$ be the members of the lower central series of $P$ and define $P_1$ by $P_1/P_4 = C_{p^4}(P_2/P_4)$. Assume that $G$ and $P$ satisfy

(i) $|O_p(G)| > p$;

(ii) If $P_1$ is abelian then $N_G(P)/P_1$ is cyclic.

Then one of the following occurs:

(a) $G = O_p(G)N_G(P)$;

(b) $G = O_p(G)N_G(P)$;

and

(α) $P_2$ is abelian, $P_2O_p(G) \triangle G$, every chief factor of $G$ which is involved in $P_1$ is an elementary abelian $p$-group of order $p$ or $p^{p-2}$, $P_1$ is of class 2, there exists a normal subgroup $G_0$ in $G$ such that $O_p(G_0) \neq G_0$ and $|G/G_0|/|N_G(P)/P_1|$ or

(β) $P_1$ is abelian, $G$ has a normal subgroup $G_0$ such that $O_p(G_0) \neq G_0$, $|G/G_0|/|N_G(P)/P_1|$ and $n \equiv 2 \mod p - 1$.

or

(γ) $P_1$ is abelian, $n \equiv 0 \mod p - 1$ or $n \equiv 1 \mod p - 1$ and if $|N_G(P)/P_1| = 2$, $p = 2^m + 1$, then $G/O_p(G) \cong PSL(2, 2^m)$.
or

(δ) \( \overline{G} = G/O_p(G)P_1 \) has a subgroup \( \overline{H} \cong \text{PSL}(2, p) \), \( |G:H| \leq 2 \),

or

(ε) \( |N_G(P)/PC_G(P)| = 2 \), \( p = 2^m + 1 \), \( G/O_p(G)P_1 \cong \text{PSL}(2, 2^m) \) and every chief factor of \( G \) which is involved in \( P_1 \) is an elementary abelian \( p \)-group of order \( p \) or \( p^{m-1} \).

The idea of the proof is roughly this: Assume that Theorem A is false and let \( G \) be a minimal counterexample. We show in Section 6 that \( O_p(G) = 1 \) and \( O_p(G) = P_1 \). Consequently, \( G/P_1 \) acts on certain elementary abelian sections of \( P_1 \). Applying a theorem of Feit [3] to this situation, we derive the desired contradiction in Section 9, by the aid of the results on \( p \)-groups of maximal class and their automorphism groups proved in Sections 1–4.

The main difficulty is in showing that \( O_p(G) = P_1 \). This difficulty is resolved in Section 7 by considering linear groups with quadratic action on the underlying vectorspace.

List of Notations

Notations are standard. We list them here for the sake of completeness.

\( \mathbb{Z} \), the integers;

\( \mathbb{Q} \), the rationals;

\( p \), an odd prime;

\( C_p \), the multiplicative subgroup of order \( p \) of \( \mathbb{C} \setminus \{0\} \), \( \mathbb{C} \) the complex numbers;

\( \mathbb{F}_{p^r} \), the field with \( p^r \) elements.

Let \( G \) be a finite group, \( a, b \) elements of \( G \) and \( S, T \) subsets of \( G \). We denote

\( |G| \), the order of \( G \);

\( \langle S, T \rangle \), the subgroup of \( G \) generated by \( S \) and \( T \);

\( S^T = \langle t^{-1}st \mid t \in T, s \in S \rangle \);

\( [a, b] = a^{-1}b^{-1}ab \).

For every integer \( n \) define \([a, nb] \) inductively as follows: \( [a, 0b] = a \) and for \( n > 0 \), \( [a, nb] = [[a, (n-1)b], b] \).

For every integer \( n \) and elements \( x_1, \ldots, x_n \in G \) define \([x_1, \ldots, x_n] = [\ldots[x_1, x_{n-1}], x_n] \).

For subsets \( S \) and \( T \) of \( G \) define \( [S, T] = \langle |s, t| \mid s \in S, t \in T \rangle \).
In particular,

\[ G' = [G, G]; \]
\[ K_i(G), \text{the } i\text{th member of the lower central series of } G; \]
\[ Z_i(G), \text{the } i\text{th member of the upper central series of } G; \]
\[ F(G), \text{the Fitting subgroup of } G; \]
\[ \Phi(G), \text{the Frattini subgroup of } G; \]
\[ \text{Aut}(G), \text{the group of automorphisms of } G; \]
\[ O_p(G), \text{the largest normal } p\text{-subgroup of } G; \]
\[ O_p'(G), \text{the largest normal } p'\text{-subgroup of } G; \]
\[ O_p^0(G), \text{the smallest normal subgroup of } G \text{ with } p\text{-power index.} \]

Let \( A \) be a subgroup of \( G \). Then we denote

\[ N_G(A) = \{ x \in G | A^K = A \}; \]
\[ C_G(A) = \{ x \in G | a^x = a \text{ for every } a \in A \}. \]

Let \( H \) and \( K \) be finite groups. Then

\[ H \leq K, \text{ } H \text{ is a subgroup of } K; \]
\[ H \hookrightarrow K, \text{ } H \text{ can be embedded in } K; \]
\[ H \triangleleft G, \text{ } H \text{ is a normal subgroup of } G; \]
\[ H \triangleright K, \text{ } H \text{ is a characteristic subgroup of } K; \]
\[ H \wr K, \text{ the wreath product of } H \text{ by } K. \]

Let \( P \) be a \( p \)-group. We denote

\[ J(P), \text{ the Thompson subgroup of } P \text{ (the subgroup generated by the} \]
\[ \text{maximal abelian subgroups of maximal order);} \]
\[ \Omega_n(P) = \langle x \in P | x^n = 1 \rangle; \]
\[ \Omega'_n(P) = \langle x^n | x \in P \rangle; \]
\[ \text{cl}(P), \text{the nilpotency class of } P. \]

Let \( X \) be a \( n \times n \) matrix over a field \( F \). Then \( \text{det}(X) \) stands for the determinant of \( X \).

Let \( A \) be an abelian group. Then we denote by \( \text{End}(A) \) the ring of endomorphisms of \( A \).

For a ring \( R \) and an indeterminate \( x \), denote by \( R[x] \) the ring of polynomials in \( x \) over \( R \).

**1. Basic Properties of \( p \)-Groups of Maximal Class**

We recall some results from Blackburn’s paper [1] and some of their immediate consequences, for the sake of completeness.
(1.0) **Definition** [1, p. 53]. A finite $p$-group $P$ of order $p^n$ is a **$p$-group of maximal class** if $\text{cl}(P) = n - 1$.

From now on $P$ will denote a $p$-group of maximal class, $|P| = p^n$, $n \geq 5$.

(1.1) ([1, p. 53]). For $2 \leq i \leq n - 1$, denote by $P_i$ and $Z_i$ the terms of the lower and upper central series of $P$, respectively. Then

(a) $P_i/P_{i+1}$ is cyclic of order $p$;  
(b) $Z_i = P_{n-i}$.

(1.2) Define the subgroup $P_1$ of $P$ by $P_1/P_4 = C_{p/p_4}(P_2/P_4)$. Then

(a) ([1, p. 54]) $P_1$ is a characteristic subgroup of $P$ of index $p$.
(b) ([1, p. 55]) There exist elements $s$ and $s_1$ in $P$ such that $P_1 = P_{i+1} \cdot \langle s_1 \rangle$ and $P = P_1 \cdot \langle s \rangle$.

In particular, $P$ is generated by two elements (e.g., $s$ and $s_1$). Assertions (c) and (d) are immediate consequences of (a) and (b):

(c) We may choose every element $x \in P \setminus P_1$ for $s$ in (b);
(d) $\Phi(P) = P' = P_2$.

(1.3) (a) ([1, p. 58]) Let $s$ and $s_1$ be as in 1.2 and define elements $s_i$ of $P$ by $s_i = [s_1, (i-1)s]$, $2 \leq i \leq n-1$. Then $P_i = P_{i+1} \cdot \langle s_i \rangle$, for $1 \leq i \leq n - 1$.

The following assertion follows from part (a).

(b) Let $s$ and $s_i$, $1 \leq i \leq n - 1$ be as defined in part (a) and in 1.2(b). Then for every $x \in P$ there exist uniquely defined natural numbers $x_0, \ldots, x_{n-1}$, $0 \leq x_i \leq p - 1$, $1 \leq i \leq n - 1$, such that $x = s^{x_0}s_1^{x_1} \cdots s_1^{x_{n-1}}$.

(1.4) **Definition** ([1, p. 57]). We say that $P$ has degree of commutativity $k$, $k$ a non-negative integer, if $[P_i, P_j] \leq P_{i+j+k}$ for every $1 \leq i, j \leq n - 1$. Here $P_\mu = 1$ for $\mu \geq n$.

(1.5)(a) ([1, p. 73]). If $n \geq p + 2$ then $P$ has degree of commutativity $k \geq 1$;
(b) ([1, p. 71]). If $P$ has degree of commutativity $k$, then $[s_1, s_p] \in P_{k+2+p}$.

(1.6)(a) ([1, pp. 65–66]). For every $i$, $1 \leq i \leq n - 3$ and for every $x \in P \setminus P_1$, $P_i \cdot \langle x \rangle$ is a $p$-group of maximal class of order $n + 1 - i$ and degree of commutativity $i - 1$.

(1.6)(b) is an immediate consequence of the discussion in [1, pp. 65–66]:
(b) Let $H$ be a subgroup of $P$ of maximal class. Then there exists an $x \in P \setminus P_1$ such that $H = P_i \cdot \langle x \rangle$ for a suitable $i$, $1 \leq i \leq n - 1$.

(1.7) ([1, pp. 68–69]). Assume that $n \geq p + 2$. Then
(a) $\mathcal{U}_i(P_j) = P_{j+i(p-1)}$ ($P_{\mu} = 1$ for $\mu \geq n$);

(b) $\mathcal{U}_i(P_i) = \Omega_i(P_i) \cap P_i = P_{n-i(p-1)} \cap P_i$.

(c) $P$ has no elementary abelian section of order $\geq p^n$. (This follows easily from the discussion in [1, p. 69].)

(1.8) ([1, p. 53]). Every normal subgroup of $P$ which lies in $P_1$ is characteristic in $P$ and is a $P_i$ for a suitable $i$, $i \geq 1$.

(1.9) Let $P$ be metabelian. Then

(a) ([1, p. 75]) $|P_i'| \leq p^{n-2}$ for $p \geq 5$ and $|P_1'| \leq 9$ for $p = 3$.

(b) If $[s_1, s_2] = s_n^{x_{k+1}} \cdots s_1^{x_0}$, $0 \leq x_i \leq p-1$, $1 \leq x_k$, then $[s_i, s_j] = s_n^{x_{k+1}} \cdots s_1^{x_{j-1}}$.

Statement (b) follows from the fact that $P$ is metabelian and from Witt's identity (see 1.13(c)).

(c) ([1, p. 54]). $P$ has positive degree of commutativity.

(1.10) ([1, p. 64]). For every $x \in P \setminus P_1$, $C_p(x) = \langle x \rangle \cdot P_{n-1}$ and $|C_p(x)| = p^2$.

(1.11) ([1, p. 65]). If $p = 3$ then $P$ is metabelian.

We also mention the following trivial results.

(1.12)(a) $P$ has a unique maximal abelian normal subgroup $A$.

(b) $A$ of part (a) is a subgroup of $P_1$.

(c) For $A$ in part (a) $|P/A| \leq |A|$.

Proof. (b) First we remark that

If $a \in P \setminus P_1$ then $[a, s_1] \in P_2 \setminus P_3$.

For, by 1.9 $a \equiv s^{x_0} \mod P_1$, $1 \leq x_0 \leq p - 1$, hence $[a, s_1] \equiv [s^{x_0}, s_1] \equiv s_2^{x_0} \mod P_3$ and $[a, s_1] \in P_2 \setminus P_3$.

Let $A$ be a maximal abelian normal subgroup of $P$ and assume that $A \leq P_1$. Then there exists an $x \in (P \setminus P_1) \cap A$. By 1.10 $C_p(x) = \langle x \rangle \cdot P_{n-1}$.

But obviously $A \subseteq C_p(x)$. Hence, $A \subseteq C_p(x) = \langle x \rangle \cdot P_{n-1}$; i.e., $A \subseteq \langle x \rangle \cdot P_{n-1}$. Since $n \geq 4$, $[P_{n-2}, P_{n-2}] \leq P_{2(n-2)} = 1$; i.e., $P_{n-2}$ is abelian.

Hence $A \neq P_{n-1}$. On the other hand, $\langle x \rangle \cdot P_{n-1}$ is not normal in $P$, for if $\langle x \rangle \cdot P_{n-1} \triangle P$, then, in particular, $\langle x \rangle \cdot P_{n-1} \leq P_{2(n-2)} = 1$; i.e., $P_{n-2}$ is abelian. Hence $A \neq P_{n-1}$. On the other hand, $\langle x \rangle \cdot P_{n-1}$ is not normal in $P$, for if $\langle x \rangle \cdot P_{n-1} \triangle P$, then, in particular, $\langle x \rangle \cdot P_{n-1} \leq P_{2(n-2)} = 1$; i.e., $P_{n-2}$ is abelian. Hence $A \neq P_{n-1}$. On the other hand, $\langle x \rangle \cdot P_{n-1}$ is not normal in $P$, for if $\langle x \rangle \cdot P_{n-1} \triangle P$, then, in particular, $\langle x \rangle \cdot P_{n-1} \leq P_{2(n-2)} = 1$; i.e., $P_{n-2}$ is abelian. Hence $A \neq P_{n-1}$. On the other hand, $\langle x \rangle \cdot P_{n-1}$ is not normal in $P$, for if $\langle x \rangle \cdot P_{n-1} \triangle P$, then, in particular, $\langle x \rangle \cdot P_{n-1} \leq P_{2(n-2)} = 1$; i.e., $P_{n-2}$ is abelian. Hence $A \neq P_{n-1}$. On the other hand, $\langle x \rangle \cdot P_{n-1}$ is not normal in $P$, for if $\langle x \rangle \cdot P_{n-1} \triangle P$, then, in particular, $\langle x \rangle \cdot P_{n-1} \leq P_{2(n-2)} = 1$; i.e., $P_{n-2}$ is abelian. Hence $A \neq P_{n-1}$. On the other hand, $\langle x \rangle \cdot P_{n-1}$ is not normal in $P$, for if $\langle x \rangle \cdot P_{n-1} \triangle P$, then, in particular, $\langle x \rangle \cdot P_{n-1} \leq P_{2(n-2)} = 1$; i.e., $P_{n-2}$ is abelian. Hence $A \neq P_{n-1}$. On the other hand, $\langle x \rangle \cdot P_{n-1}$ is not normal in $P$, for if $\langle x \rangle \cdot P_{n-1} \triangle P$, then, in particular, $\langle x \rangle \cdot P_{n-1} \leq P_{2(n-2)} = 1$; i.e., $P_{n-2}$ is abelian. Hence $A \neq P_{n-1}$. On the other hand, $\langle x \rangle \cdot P_{n-1}$ is not normal in $P$, for if $\langle x \rangle \cdot P_{n-1} \triangle P$, then, in particular, $\langle x \rangle \cdot P_{n-1} \leq P_{2(n-2)} = 1$; i.e., $P_{n-2}$ is abelian. Hence $A \neq P_{n-1}$.

(a) Let $A$ be a maximal abelian normal subgroup of $P$. Then by part (b) $A \leq P_1$. Therefore by 1.8, $A = P_i$ for some $i$, $i \geq 1$. If $B$ is another
maximal abelian normal subgroup of $P$, then by the same argument $B = P_j$
for some $j, j \geq 1$.
Since $P_i \leq P_j$ for $i, j \geq 1$, this yields that $A = B$.

(c) If $|A| = p^a$ then by [6, 7.3(c)] $P_{2a} = 1$. Since $P$ has nilpotency
class $n - 1$ this means $2a \geq n$; i.e., $n - a \leq a$. Consequently, $|P/A| = p^{n-a} \leq p^a = |A|$; i.e., $|P/A| \leq |A|$, as required.

(1.13) Remarks. (a) Let $P$ be a $p$-group of maximal class of order $p^n$, $n \geq 5$. By 1.12(a) $P$ has a unique maximal abelian normal subgroup $A$ which
lies in $P_i$ and by 1.8, $A$ is a $P_i$ for some $i, i \geq 1$. We shall denote this $i$ by $w$. So $P_w$ is the unique maximal abelian normal subgroup of $P$.

(b) From now on we shall use the notations and definitions of this
section without further explanation, if not stated differently.

(c) We shall use tacitly the following identities of commutators: Let $a$
and $b$ elements of the group $G$. Then

\begin{align*}
\alpha & \quad [ab, c] = [a, c][b, c]; \\
\beta & \quad [a, bc] = [a, c][a, b]^c; \\
\gamma & \quad [a, b^{-1}, c] [b, c^{-1}, a]^c = 1 \quad \text{(Witt's identity)}; \\
\delta & \quad [a, b^{-1}] = [a, b]^{-b^{-1}}.
\end{align*}

2. Basic Structure of $\text{Aut}(P)$

(2.1) Theorem. Let $P$ be a $p$-group of maximal class of order $p^n$, $n \geq 4$
$p \geq 3$. Denote $A = \text{Aut}(P)$ and let $B$ be a Sylow $p$-subgroup of $A$. Then

\begin{itemize}
  \item[(a)] $|A| \leq p^{2(n-2)+1} \cdot (p-1)^2$
  \item[(b)] $B \triangle A$ and $A$ is a splitting extension of $B$ by a $p'$-Hall subgroup $Q$,
               where $Q$ is isomorphic to a subgroup of $\mathbb{Z}_p \times \mathbb{Z}_p$.
  \item[(c)] $A' \leq B$;
  \item[(d)] $A$ is solvable;
  \item[(e)] $F(A) = B$;
  \item[(f)] $n - 2 \leq \text{cl}(B) \leq n - 1$.
\end{itemize}

Proof: (a) Let $\Psi: \text{Aut}(P) \to \text{Aut}(P/\Phi(P))$ be the natural homomorphism
and denote its kernel by $N$. Since $P$ is 2-generated, by 1.2(h), we may regard
$\text{Aut}(P/\Phi(P))$ as a subgroup of $GL(2, p)$ acting on $P/\Phi(P)$ and may represent
it by regular $2 \times 2$-matrices over $\mathbb{Z}_p$. We claim that

\[ \psi(A) \text{ may be represented by a subgroup of the group } \]
\[ \mathcal{G} = \{(a^c; b^d) \mid a, c, d \in \mathbb{Z}_p, a \cdot d \neq 0\}. \quad (*) \]
For, by 1.2(d) \( P_1 \geq \Phi(P) \) and by 1.2(a) \( P_1 \mathrm{ch} P \). Therefore \( P_1/\Phi(P) \mathrm{ch} P/\Phi(P) \); i.e., \( P_1/\Phi(P) \) is an invariant subspace of \( P/\Phi(P) \) under \( \text{Aut}(P/\Phi(P)) \). Choosing \( \{ s \cdot \Phi(P), s_1 \cdot \Phi(P) \} \) as defined in 1.2(b) for a basis of \( P/\Phi(P) \), we get (*).

Now \( |G| = (p-1)^2p \). Hence, \( |\psi(A)| = (p-1)^2p \) by (*). But \( |A| = p^{2(n-2)^2} |\psi(A)| \) by [6, Th. 3.19, p. 275]. Therefore \( |A| = p^{2(n-2)+1} (p-1)^2 \), as required.

(b) \( N \) is a \( p \)-subgroup of \( A \). Therefore \( N \triangleleft B \) and \( BN/N = B/N \) is a Sylow \( p \)-subgroup of \( A/N \). Since \( A/N \cong \psi(A) \), it follows from (*) that the Sylow \( p \)-subgroup of \( A/N \) is normal in \( A/N \), consequently \( B \triangleleft A \). Therefore \( B \) has a complement \( Q \) in \( A \) (see [6, Th. 18.1, p. 126]) and since \( N \triangleleft B \), \( Q \cong QB/B = A/B = \psi(A)/\psi(B) \). By (*) this yields that \( Q \) is isomorphic to a subgroup of \( \mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1} \) (the diagonal matrices of \( G \)).

(c) It follows from part (b) of the theorem that \( A/B \) is abelian. Therefore \( A' \leq B \).

(d) \( A \) is solvable since \( A/B \) is abelian by part (b) of the theorem and \( B \) is nilpotent.

(e) We recall from [6, Th. 6, p. 268] that

\[
\text{if } 1 = N_1 < N_2 < \cdots < N_k = A \text{ is chief series of } A \text{ then } F(A) = \bigcap_{i=2}^{k} C_A(N_i/N_{i-1}).
\]

Let \( \nu: P \to \text{Aut}(P) \) be the natural homomorphism from \( P \) onto the inner automorphism groups of \( P \) and for every subgroup \( X \) of \( P \) denote \( \nu(X) = \bar{X} \). Then \( A \) has a chief series \( 1_A = \bar{P}_{n-1} < \bar{P}_{n-2} < \cdots < \bar{P}_1 < \bar{P} < \cdots < A \). Therefore, by (**), \( F(A) = \bigcap_{i=0}^{n-1} C_A(\bar{P}_i/\bar{P}_{i+1}) \). (Here \( \bar{P}_0 = \bar{P} \).) But \( C_A(\bar{P}_i/\bar{P}_{i+1}) = C_A(\bar{P}_i/\bar{P}_{i+1}) \) for \( 0 \leq i \leq n-2 \). Hence

If \( x \in F(A) \) then \( x \) stabilizes a chief series of \( P \) of length \( n \). (***) Consequently \( x \) is a \( p \)-element of \( A \). Since \( B \triangleleft A \) by part (b) of the theorem this implies that \( x \in B \). Therefore \( F(A) \leq B \). But as \( B \triangleleft A \), \( B \leq F(A) \) by definition. This proves part (e).

(f) By (***) \( B \) stabilizes a chief series of \( P \) of length \( n \). Hence, by [6, Th. 2.5, p. 264] \( B \) has class \( n-1 \) at most. As \( \bar{P} \) has class \( n-2 \), \( n-2 \leq \text{cl}(B) \leq n-1 \).

3. \( p \)-Groups of Maximal Class Possessing a Non-Trivial Automorphism of Order Prime to \( p \)

In [8, Th. 4, p. 6] Miech showed that in general a \( p \)-group of maximal class \( P \) may have "nearly every possible structure" (for a precise statement
see [8, Th. 4]). However, it turns out that the existence of a non-trivial $p'$-automorphism may impose certain limitations on the structure of $P$ (see Lemma 3.2). Knowing this fact, we locate $J(P)$—the Thompson subgroup of $P$—in a special situation (see Lemma 3.4) which occurs in the course of the proof of Theorem A.

(3.1) **Lemma.** Let $P$ be a $p$-group of maximal class of order $p^n$, $n \geq 5$, $\sigma \neq 1$ an automorphism of $P$ of order prime to $p$. Then

(a) There exist natural numbers $a$ and $b$, $1 \leq a, b \leq p - 1$ and an element $s \in P \setminus P_1$ such that $s^a \equiv s \mod P_2$ and for every $i$, $1 \leq i \leq n - 1$, $s_i \equiv s_i^{a_i^{i-1}b} \mod P_{i+1}$ ($s_i$ are defined in 1.3).

(b) If $P'_1 \neq 1$ then $a$ and $b$ of part (a) satisfy $b \equiv a^t \mod p$, for a certain natural number $t$, $1 \leq t \leq n - 1$.

(c) If $P'_1 \neq 1$ then the $p'$-part of $\text{Aut}(P)$ (see Theorem 2.1(e)) is cyclic of order dividing $p - 1$.

**Proof.** (a) By 1.2(a) $P_1 \leq P$. Hence, as $P_2 \leq P \sigma$ acts on $P_1/P_2$. Since $P_1/P_2$ is cyclic of order $p$, by 1.1(a), there exists a natural number $b$, $1 \leq b \leq p - 1$, such that $s_i^b \equiv s_i \mod P_2$. Now, by 1.2(d) we may regard $P/P_2$ as a two-dimensional vectorspace over $\mathbb{F}_p$. Then $\sigma$ acts on $P/P_2$ and $P_1/P_2$ is a $\langle \sigma \rangle$-invariant subspace of $P/P_2$. Therefore, by Maschke’s theorem and 1.2(c), there exists an element $s \in P \setminus P_1$, such that $P/P_2 = P_1/P_2 \times \langle s \rangle \cdot P_2/P_2$, where $P_1/P_2$ and $\langle s \rangle \cdot P_2/P_2$ are $\langle \sigma \rangle$-invariant subspaces. This implies that $s^a \equiv s^{a_i^{i-1}b} \mod P_{i+1}$ for some natural number $a$ and $1 \leq a \leq p - 1$ by 1.2(a). This proves part (a) for $i = 1$. For $i \geq 2$ we proceed by induction on $i$: By 1.3(a) $s_i^{a_i} = [s_{i-1}, s_i]^o = [s_{i-1}, s_i^o]$. Therefore, $s_i^{a_i} = [s_{i-1}^{a_i^{i-1}b}, s_i^o] \equiv s_i^{a_i^{i-1}b} \mod P_{i+1}$, by the induction hypothesis.

(b) Assume that $P'_1 \neq 1$. Then there exist natural numbers $i$ and $j$, $i < j$, such that $[s_i, s_j] \neq 1$. Hence $[s_i, s_j] \equiv s_i^{a_i} \mod P_{r+1}$ for suitable $r$, $1 \leq r \leq p - 1$, $4 \leq r \leq n - 1$ and by part (a) of the lemma

$[s_i, s_j]^{a_i} \equiv [s_i^{a_i^{i-1}b}, s_j^{a_i^{i-1}b}] \equiv [s_i, s_j]^{a_i^{i-1}b} \equiv s_i^{a_i^{i-1}b} \mod P_{r+1}$.

On the other hand, $[s_i, s_j]^{a_i} \equiv [s_i^{a_i^{i-1}b}]^o \equiv s_i^{a_i^{i-1}b} \mod P_{r+1}$, by part (a) of the lemma. Therefore $b \equiv a_i^{r-1} \mod p$, as required.

(c) Follows from part (b) of the lemma and Theorem 2.1(b).

**Remark.** From now on, for a $p$-group of maximal class $P$ with $P'_1 \neq 1$ and a $\sigma$ as above, we fix the element $s$ to be an element in $P \setminus P_1$ which satisfies $s^o = s \mod P_2$ for a suitable $a \in \mathbb{Z}$, $1 \leq a \leq p - 1$.

(3.2) **Lemma.** Let $P$ be a $p$-group of maximal class of order $p^n$. Assume that $P_1$ is non-abelian and for $1 \leq i$, let $[s_i, s_{i+1}] \equiv s_i^{a_i} \mod P_{r+1}$. 


\[ i \leq r_i \leq n - 1, 1 \leq a_i \leq p - 1. \] If \( P \) has a non-trivial automorphism \( \sigma \) of order prime to \( p \) then

(a) If \( j > i \) then \( r_j \geq r_i + 2 \), provided \( r_i \leq n - 2 \). If \( r_i \geq n - 1 \) then \( r_j = n \).

(b) \( P \) has degree of commutativity \( r_1 = 3 \).

(c) If \( r_2 > r_1 + 2 \) then \( [s_1, s_1] \equiv s_{r_1-2}^{a_1} \mod P_{r_1+1} \), for \( t \geq 2 \).

(d) If \( r_2 > r_1 + 2 \) then \( |P'| \leq p^{a_2-2} \).

Proof: (a) If \( P \) is metabelian then (a) is trivial. Let \( P_w \) be the first abelian \( P_t \) and for every \( t, 0 \leq t \leq w - 1 \), denote \( H_{w-t} = \langle P_{w-t-1}, s \rangle \). Then by 1.6(a) \( H_{w-t} \) is a \( p \)-group of maximal class and \( H_w \) is metabelian. Hence (a) holds for \( t = 0 \) i.e. for the group \( H_w \). We prove by induction on \( t \) that (a) holds for \( H_{w-t} \). For \( t = 0 \) we have just proved. Assume that (a) holds for \( H_{w-t} \) and prove for \( H_{w-t-1} \). Denote \( H = H_{w-t-1} \), for \( i, 1 \leq i \leq n + t + 1 - w \), denote \( H_{t} = P_{w-t-1} + i \) and for \( q, 1 \leq q \leq n - t + 1 - w \), \( h_q = s_{w-t+q-1} \). Then \( \langle H_2, s \rangle = H_{w-t} \) and \( H = \langle H_{w-t}, h_1 \rangle = \langle \langle H_2, s \rangle, h_1 \rangle \).

Therefore, we may assume without loss of generality that (a) holds for \( \langle P_2, s \rangle \) and prove \( r_i \geq r_i + 2 \), if \( r_i \leq n - 2 \) and \( r_i = n \) otherwise. If \( s_{r_i} = 1 \) then by the induction hypothesis \( P_2 \) is abelian and our assertion is trivial. Hence we may assume that \( s_{r_i} \neq 1 \). Let us compute \( [s_3, s_1] \).

\[
[s_3, s_1] = s_3^{-1} [s_2, s_{s_{s_2}^{-1}}] [s_{s_2}, s_{s_2}^{-1}] \\
= s_3^{-1} [s_3 [s_3, s_{s_2}^{-1}]] [s_{s_2}, s_{s_2}^{-1}] \\
= s_3^{-1} [s_3 [s_3, s_{s_1}]] [s_{s_1}, s_{s_1}^{-1}] [s_{s_1}, s_{s_1}] \\
= [s_{s_1}, s_{s_1}^{-1}] [s_{s_1}, s_{s_1}] [s_{s_1}, s_{s_1}].
\]

If \( [s_2, s_1, s] = 1 \) then of course all of \( [s_3, [s_2, s_1]], [s_{s_2}, s_{s_2}^{-1}], [s_{s_1}, s_{s_1}] \), \( [s_{s_2}, s_{s_1}], [s_{s_1}, s_{s_2}] \) are 1, hence \( [s_3, s_1] \equiv [s_{s_2}, s_{s_2}^{-1}] \neq 1 \). But then, by Theorem 3.1(a), \( a^2b = a^2bab \); i.e., \( a = 1 \), contradiction \( (\sigma \neq 1) \). Therefore \( [s_2, s_1, s] \neq 1 \) and \( r_1 \leq n - 2 \). Now, \( [s_3, s_1] \equiv [s_3, s_2^{-1}] [s_2, s_1, s] \mod P_{r_1+2} \), by (*)). If \( r_1 = r_2 \) then \( [s_3, s_1] \equiv [s_3, s_2^{-1}] \mod P_{r_1+1} \) and again \( a = 1 \), a contradiction. If \( r_1 > r_2 \) then \( [s_2, s_1] \equiv [s_2, s_2^{-1}] \mod P_{r_1+1} \) and again \( a = 1 \). Hence \( r_1 < r_2 \). But \( r_2 - r_1 = 2 \mod |\sigma| \), for \( [s_1, s_2] \equiv s_{r_1}^{a_1} \mod P_{r_1+1} \) implies \( b = a^{r_1-1}b \) and \( [s_2, s_1] \equiv s_{r_2}^{a_2} \mod P_{r_2+1} \) implies \( aba^2b = a^{r_2-1}b \). Hence, \( a^{r_2-r_1-2} = 1 \). This implies \( r_2 - r_1 \geq 2 \).

(b) We prove, by induction on \( j \) and \( t \), that \( [s_j, s_{j+t}] \in P_{r_j+t-1} \). The assertion is obviously true for \( j \geq w - 1 \) and \( t \) arbitrary, since \( H_w = \langle P_{w-1}, s \rangle \) is metabelian. Hence, as we have seen above, we may
assume the assertion for \( j \geq 2 \) and every \( t \) and prove for \( j = 1 \) and every \( t \).

We compute \([s_i, s_{i+1}]\) in terms of \([s_j, s_{j+1}]\), \( j \geq 1, t \geq 1 \). Let \( x = [s_i, s_1] \).

Then we may assume that \([s_i, s_1] = s_a^{r_i-2+i} \mod P_r_i+1-i, 0 \leq a \leq p-1 \), by the induction hypothesis. Therefore \([s_{i+1}, s_1] = s_i^{-1}s_i^{r_i+1} = s_i^{-1}[s_i, s_1^{-1}] \equiv s_i^{-1}[s_i s_{r_i-2+i}, s_{r_2}^{-1}] \mod P_r_i+1\). Hence

\[
[s_{i+1}, s_1] \equiv s_i^{-1}[s_i s_{r_i-2+i}, s_{r_2}^{-1}] \mod P_r_i+1 
\] (1)

Now,

\[
[s_i s_{r_i-2+i}, s_{r_2}^{-1}] = [s_i s_{r_i-2+i}, s][s_i s_{r_i-2+i}, s^{-1}]^a \\
= s_i^{-1}[s_i, s_{r_i-2+i}][s_i s_{r_i-2+i}, s][s_i, s_{r_i-2+i}]^a \\
\cdot [s_i s_{r_i-2+i}, s_{r_2}^{-1}]^a \\
[s_i s_{r_i-2+i}, s] [s_i, s_{r_2}^{-1}] \mod P_r_i+1 
\] (2)

Combining (1) and (2) we get

\[
[s_{i+1}, s_1] \equiv [s_i s_{r_i-2+i}, s][s_i, s_{r_2}^{-1}] \mod P_r_i+1. 
\] (3)

But by our hypothesis \([s_i, s_{r_2}^{-1}] \in P_r_i+1-i = P_{r_i+1-1} \), since \( r_1 \leq r_2 - 2 \), by part (a) and \([s_i, s_{r_i-2+i}] \in P_{r_i+1-i} \) too. This proves our assertion. Now let \( k_i = r_i - 2i - 1, 1 \leq i \). Then \( k_i - k_{i-1} = r_i - r_{i-1} + 2 \geq 0 \), or \( r_i \geq n - 1 \) by part (a). We may assume \( r_i \leq n - 2 \). Then \( k_i \geq k_{i-1} \) and

\[
k_i \geq k_1 \quad \text{for} \ i \geq 1
\] (4)

Since \([s_i, s_{i+1}] \in P_r_i+1 \) by the above assumption, it follows from part (a) that \([P_i, P_{i+1}] \leq P_{r_i+1-i} \). But \( P_{r_i+1-i} = P_{2i+l+k_i} \). Hence by (4) \([P_i, P_{i+1}] \leq P_{2i+l+k_i} \) and \( P \) has degree of commutativity \( k_1 = r_1 - 3 \).

(c) We prove part (c) by induction on \( i \). It follows from (3) that if \([s_i, s_1] = s_{r_i-2+i} \mod P_{r_i-1-i} \) and \([s_i, s_1] = [s_{r_i-2+i}, s][s_i, s_{r_i-2+i}] \mod P_{r_i+1-i} \). By what we have proved above, it follows that \([s_i, s_{r_i-2+i}] \in P_{r_i+1-i} \). Hence by assumption \( r_1 \leq r_2 - 3 \). But then \([s_i, s_1] = [s_i, s_1] \mod P_{r_i+1-i} \), as required.

(d) By part (a) of the lemma it suffices to show that \([s_i, s_2] \in P_{n-p+2} \); i.e., \( r_1 \geq n - p + 2 \). Assume by way of contradiction that \( r_1 < n - p + 2 \). Then \( r_1 + p - 2 < n \) and by part (c) of the Lemma \([s_i, s_2] = s_{r_i+p-2} \mod P_{r_i+p-2} \) and \([s_i, s_2] \in P_{r_i+p-2} \). But by 1.5(b) \([s_i, s_2] \in P_{r_i+p-1} \), contradicting \( 1 \leq r_1 < p - 1 \). Therefore \( r_1 \geq n - p + 2 \) and consequently \(|P_i| = p^{n-2} \), as required.

(3.3) Lemma. Let \( P \) be a \( p \)-group of maximal class with \( P_1 \) of class 2 and \(|P_1| = p^d \), \( d \leq p - 1 \). Let \( P_w \) be the unique maximal abelian \( P_i \) (see
1.12(a)). Assume that \( P \) has a nontrivial automorphism of order prime to \( p \) and let \( x \) be an element in \( P_1 \setminus P_2 \). Then

(a) For every \( t, \ 1 \leq t \leq d \), \( [s_1, s_{t+1}] \equiv e^{l(t)} \mod P_{n-d+t-1}, \) where

\[
l(t) \equiv \sum_{j=0}^{w-2} (-1)^j \binom{t-1-j}{j} h_j \mod p, \text{ for certain } h_j \in \mathbb{Z};
\]

(b) Let \( 1 \leq \xi_1 < \xi_2 < \cdots < \xi_c \leq p \) be all the solutions of \( l(t) = 0 \mod p \) in the interval \( 1 \leq t \leq p \), \( t \in \mathbb{Z} \). Then \( c \leq w - 2 \) and to every \( z \in C_{P_2}(x) \setminus P_{d+2} \) there exists a \( \xi_i \) such that \( z \in P_{\xi_i+1} \setminus P_{\xi_i+2} \).

(c) \( |C_{P_2}(x)/P_{d+2}| \leq p^{w-2} \).

Proof. (a) By direct calculation, or by Miech [8, Lemma 1.21], we find

\[
[s_1, s_{t+1}] = \prod_{j=0}^{w-2} \prod_{\omega=0}^{j} [s_{1+j}, s_{2+j}, (t-(1+j+\omega))s] c(t, j, \omega),
\]

\[
c(t, j, \omega) = (-1)^j \binom{t-1-\omega}{j} \binom{\omega}{j} . \quad (**)
\]

By Lemma 3.2(b) \( P \) has degree of commutativity \( k = n - d - 3 \). In particular, \( [s_{1+j}, s_{2+j}, (t-(1+j+\omega))s] \in P_{\beta(t, j, \omega)} \), where \( \beta(t, j, \omega) = n - d - t - 1 + j - \omega \). For fixed \( t \), \( \beta(t, j, \omega) \) takes its least value \( l \) for \( j = \omega = 0 \) and \( l = n - d + t - 1 \). So by (**) if \( n - d + t - 1 \leq n - 1 \); i.e., if \( t \leq d \) then \( s_{1}^{-1}, s_{t+1} \equiv e^{l(t)} \mod P_{n-d+t} \), where

\[
l(t) \equiv \sum_{j=0}^{w-2} (-1)^j \binom{t-1-j}{j} h_j \mod (p)
\]

(b) Let \( x = s_0^\alpha u, \ 1 \leq \alpha \leq p - 1, \ u \in P_2 \) and \( y_i = s_i^\epsilon u_i, \ 1 \leq \epsilon \leq p - 1, \ 2 \leq i < d \), \( u_i \in P_{i+1} \), \( y_i \in C_{P_1}(x) \). Then \( x, y_i = 1 \) and by Lemma 3.2(a) this implies \( [s_1, s_i] \in P_{n-d-1+i} \). Therefore \( 1 - l \) is a solution of \( \overline{l}(t) = 0 \mod p \), by part (a) of the lemma, where \( \overline{l}(t) \) denotes the residue class of \( l(t) \mod p \), for every \( f(t) \in \mathbb{Z}[t] \). But \( \overline{l}(t) \) has degree \( w - 2 \) at most, so \( \overline{l}(t) \) has at most \( w - 2 \) solutions in \( F_p \). This means that there are at most \( w - 2 \) natural numbers \( 1 \leq \xi_1 < \xi_2 < \cdots < \xi_c \leq p \) such that \( l(\xi_i) \equiv 0 \mod p \). Hence \( c \leq w - 2 \). Since \( d + 2 \leq p + 1 \) and \( [y_i, x] = 1 \) only if \( i - 1 \) is a solution of \( l(t) \equiv 0 \mod p \); i.e., \( i - 1 \) is one of the \( \xi_i \)'s, we get the desired result.

(c) Let \( c \) and \( \xi_i, \ 1 \leq i \leq c \) be as in part (b) of the lemma. It suffices to prove that for every \( 1 \leq i \leq c \) \( |C_{P_2}(x) \cap P_{\xi_i+1}/C_{P_2}(x) \cap P_{\xi_i}| \leq p \) and \( |C_{P_2}(x) \cap P_{\xi_i+1}/P_{d+2}| \leq p \). For then

\[
|C_{P_2}(x)/P_{d+2}| \leq \prod_{i=1}^{c} |C_{P_2}(x) \cap P_{\xi_i+1}/C_{P_2}(x) \cap P_{\xi_i}|.
\]
\[ |C_{p_i}(x) \cap P_{d+1}|/|P_{d+2}| \leq p^w, \] by part (b) of the lemma. Let \( y, y' \in C_{p_i}(x) \) such that \( y = s_{d+1}^\alpha u, y' = s_{d+1}^\beta v, u, v \in P_{d+2}, 1 \leq \alpha, \beta \leq p-1 \). Then obviously \( z = y^\alpha y'^{-\alpha} \in C_{p_i}(x) \cap P_{d+2} \). But then by part (b) of the lemma \( z \in P_{d+1} \cap C_{p_i}(x) \), hence \( |C_{p_i}(x) \cap P_{d+1}/C_{p_i}(x) \cap P_{d+1}| = p \). The same argument shows that \( |C_{p_i}(x) \cap P_{d+1}/|P_{d+2}| = p \), as required.

(3.4) Lemma. Let \( P \) be a \( p \)-group of maximal class of order \( p^n \). Let \( N = \mathcal{O}(P^*_1) \cdot K_3(P^*_1) \) and for every subgroup \( X \leq P \) denote \( \bar{X} = XN/N \). Assume that

(i) \( P \) has a nontrivial automorphism \( \sigma \) of order prime to \( p \), and

(ii) \( P \) is nonabelian.

If \( |\overline{P_1}| = p^d \) and \( \overline{P_w} \) is the first abelian \( \overline{P_1} \), then \( p \leq |\overline{P_1}/J(\overline{P_1})| \leq p^{|d+2|/2} \) for \( d \) even and \( w = (d+3)/2 \) or \( p \leq |\overline{P_1}/J(\overline{P_1})| \leq p^{|d+3|/2}, \) for \( d \) odd.

Proof. Without loss of generality, we may assume that \( N = 1 \) and prove the lemma for \( P_1 \) instead of \( \overline{P_1} \). Then \( P_1 \) is of class 2 and \( |P_1'| = p^d \leq p^{p-1} \) by 1.9(a). Hence \( P_1' = P_{n-d} \) and by Lemma 3.2(b), \( P \) has degree of commutativity \( k = n - d - 3 \). Now, \( P_1 \) is abelian only if \( [s_i, s_{i+1}] = 1 \). Since \( k = n - d - 3 \) this happens only if \( 2i + 1 + n - d - 3 \geq n \); i.e., \( i \leq (d + 1)/2 \).

\[ w \leq (d + 3)/2 \tag{5} \]

It follows from Lemma 3.3(c) that \( |C_{p_i}(x)| \leq |P_{d+2}| \cdot p^w \). Consequently, \( |C_{p}(x)| \leq p^{(n-d-2)+(w-1)} = p^{n-d+w-3} \). If \( n - (d + 3)/2 \leq n - d + w - 3 \) then \( w \geq (d+3)/2 \). For \( d \) even this contradicts (5) while for \( d \) odd \( w = (d+3)/2 \), by (5). Hence for \( d \) even \( n - d + w - 3 < n - (d + 3)/2 \) and \( |C_{p}(x)| \leq p^{n-d+w-3} < p^{n-(d+3)/2} \leq |P_w| \); i.e.,

\[ |C_{p}(x)| < |P_w| \text{ for } d \text{ even and } w = (d+3)/2 \]

or \[ |C_{p}(x)| < |P_w| \text{ for } d \text{ odd.} \tag{6} \]

It follows from (6) that for \( d \) even, and if \( w \neq (d+3)/2 \) then for \( d \) odd too, \( P_1 \) has no abelian subgroup \( M \) of maximal order such that \( x \in P_1 \setminus P_2 \). But then certainly \( J(P_1) \leq P_2 \). This and (6) proves Lemma 3.4.

4. The Transfer Theorem

In the sequel we shall need Wielandt's transfer theorem [9], but in a slightly different formulation:
(4.1) THEOREM. Let $G$ be a finite group, $P$ a Sylow $p$-group of $G$ and $P_1$ a weakly closed subgroup of $G$ in $P$. Let $N = N_G(P_1)$ and assume that $P$ has two subgroups $P_0$ and $P_2$ s.t. the following conditions are satisfied:

(i) $P_0$ is strongly closed in $P$;

(ii) The strong closure of $P_2$ in $P$ lies in $O^p(N)$;

(iii) $P_1$ and $P_2$ are normal subgroups of $P_0$.

(iv) If $x \in P_0$, $y \in P_2$ then $\langle x, y \rangle \cdot P_2/P_0$ has no section isomorphic to $C_p \wr C_p$. Then $N/O^p(N) \cong G/O^p(G)$.

This formulation differs from the original one only in (iv), where instead of the requirement that "$\langle x, y \rangle \cdot P_2/P_0$ has no section isomorphic to $C_p \wr C_p$," appears the requirement "$\langle x, y \rangle \cdot P_2/P_0$ has no exceptional section." It is not difficult to prove that a finite $p$-group is exceptional-free iff it is $C_p \wr C_p$-free. But we shall prove Theorem 4.1 in a different way. We recall the definition of an exceptional $p$-group. Let $C^*$ be the multiplicative group of the complex numbers and let $C_p$ be the subgroup of $C^*_p$ of $p$th roots of unity.

Let $S_p$ be the symmetric group on $p$ elements represented by $p \times p$ permutation matrices. A finite transitive $p$-subgroup of $C^*_p \wr S_p$ is an exceptional $p$-group (Ausnahmegruppe) if it has a diagonal element $x \in C_p \wr C_p$ s.t. $\det(x) \neq 1$. Now, by Tate's theorem [6, p. 431] it will suffice to prove Satz 1 of [9] for the case $N/O^p(N)$ is elementary abelian. In the proof of Satz 1 of [9] the coefficient group for the monomial representation of $G$ is chosen from $C_{pn}$, where $C_{pn}$ is a cyclic direct component of $N/O^p(N)$. But as $N/O^p(N)$ is elementary, $n = 1$ and we have to look for exceptional groups only in sections isomorphic to $C_p \wr C_p$.

Since $C_p \wr C_p$ is minimal irregular, the only irregular section of $C_p \wr C_p$ is $C_p \wr C_p$ itself. Hence if $P$ is $C_p \wr C_p$-free, every section of $P$ which is isomorphic to a section of $C_p \wr C_p$ is regular, and, hence, non-exceptional, as required.

The following corollaries are immediate.

(4.2) COROLLARY. Let $G$ be a finite group, $P$ a Sylow $p$-subgroup of $G$. If $P$ has no sections isomorphic to $C_p \wr C_p$ then $N_G(P)/O^p(N_G(P)) \cong G/O^p(G)$.

(4.3) COROLLARY. Let $G$ be a finite group, $P$ a Sylow $p$-subgroup of $G$. If no section $H$ of $P$ which is generated by two elements has elementary abelian section of order $p^n$ then $N_G(P)/O^p(N_G(P)) \cong G/O^p(G)$.

(4.4) COROLLARY. Let $G$ be a finite group, $P$ a Sylow $p$-subgroup of $G$ of maximal class, $p > 2$. If $P \neq C_p \wr C_p$ then $N_G(P)/O^p(N_G(P)) \cong G/O^p(G)$. In particular, if $N_G(P) = PC_G(P)$ then $G$ has a normal $p$-complement.
Proof. It follows from the structure of $p$-groups of maximal class [1, p. 69] that if $P \neq C_p \wr C_p$, then $P$ has no elementary abelian section of order $\geq p^2$. Hence the result follows from Corollary 4.3.

(4.5) Corollary. Let $G$ be a finite group of odd order. Assume that $p$ is the smallest prime in $\pi(G)$ and $G$ has a Sylow $p$-subgroup $P$ of maximal class $P \neq C_p \wr C_p$. Then $G$ has a normal $p$-complement.

We shall refer to Theorem 4.1 and its corollaries as the “Transfer Theorem.”

5. The Structure of a Minimal Counterexample

(5.1) Lemma. Let $G$ be a finite group with a Sylow $p$-subgroup $P$ of maximal class of order $p^n$, $n \geq 5$. Assume that $G$ is a minimal counterexample to Theorem A. Then

(a) $O_p(G) = 1$;
(b) If $|P : O_p(G)| = p$ then $O_p(G) = P_1$.

Proof. (a) Assume $O_p(G) \neq 1$. Then $G/O_p(G)$ satisfies (a) or (b) of Theorem A. Since $O_p(G/O_p(G)) = 1$ $G$ satisfies (a) or (b) of Theorem A. But this contradicts our assumption on $G$.

(b) If $|P : O_p(G)| = p$ then either $O_p(G) = P_1$ or $O_p(G)$ is of maximal class by 1.2(a) and 1.6(b).

Assume $O_p(G) \neq P_1$. Then $O_p(G) = S = \langle x, P_1 \rangle$, $x \in P \setminus P_1$. Since $|P| \geq p^5$, $S$ is a $p$-group of maximal class with $S_1 = P_2$ characteristic and

$$N_G(S)/SC_G(S) \subseteq \left\{ \begin{pmatrix} a & 0 \\ c & b \end{pmatrix} \mid a, b, c \in \mathbb{Z}_p \right\}.$$

But $A = \{(a \ b) \mid a, b, c \in \mathbb{Z}_p \}$ has $A_0 = \{(c \ 0) \mid c \in \mathbb{Z}_p \}$ as the commutator subgroup. Hence $A_0 \triangle A$ and in our case this means $P \triangle G$, contradiction.

(5.2) Lemma. Let $G$ be a finite group with Sylow $p$-subgroup $P$ of maximal class of order $p^n$, $n \geq 5$. Let $K_1$ and $K_2$ be two normal $p$-subgroups of $G$ and let $N$ be a normal subgroup of $G$. Denote $H = N \cdot P$ and assume that the following conditions are satisfied:
(i) $K_2 < K_1 \leq N, K_1 \leq P_1$;
(ii) $|K_1 : K_2| \geq p^2$;
(iii) $[N, K_2] \leq K_1$.

Then $H = O_p(H)P$.

Proof. Since $K_2 \leq P_1$, by assumption (i), $K_2 = P_i$ for a certain $i$, $i \geq 1$, by 1.8 and $s_i, s_{i+1} \notin K_1$ by assumption (ii). Now, assumption (iii) implies that $x^h \equiv x^g \mod K_1$ for every $x \in K_2$, $h \in H$ and a suitable $y \in P$. In particular, for $s_i, s_{i+1}$ and every $h \in N_H(P)$ we have $s_i^h \equiv s_i \mod P_{i+1}$ and $s_{i+1}^h \equiv s_{i+1} \mod P_{i+2}$. Hence, by Lemma 3.1(a), using its notations, $a = b = 1$. This implies that $P$ has no non-trivial $p'$-automorphisms induced by $H$; i.e., $N_H(P) = P \cdot C_H(P)$. But then $H$ has a normal $p$-complement $O_p(H)$, by the Transfer Theorem (see Sect. 4). Consequently $H = O_p(H) \cdot P$, as required.

(5.3) Lemma. Let $G, P, K_1, K_2$ be as in the previous lemma and for a subgroup $H$ of $G$ denote $\bar{H} = HKJK_1$. Assume that

(i) $O_p(G) \leq P_1$, and
(ii) $O_p(G) = 1$.

Then

(a) $C_G(\bar{K}_1) = C_P(\bar{K}_2)$;
(b) If $A^* / K_1$ is the largest abelian normal $p$-subgroup of $P/K_1$ then $\bar{A}^* \leq O_p(G)$;

(c) Let $K_1 = \Phi(O_p(G))$ and $A^*$ as in part (b). If $G$ is a minimal counterexample to Theorem A, then $O_p(G) = A^*$, $|P/O_p(G)| \leq p^{p-1}$ and $|A^*| \geq p^2$.

(d) If $P \triangle G$ then $N_G(P) = PT$, where $T$ is a cyclic group of order dividing $p - 1$.

Proof. (a) Define a subgroup $N$ in $G$ by $N/K_1 = C_G(\bar{K}_2)$. Then $N \triangle G$ and all the assumptions of Lemma 5.2 are satisfied by $K_1, K_2$ and $N$. Hence, denoting $H = N\bar{P}$, Lemma 5.2 implies $H = O_p(H) \cdot P$. Since $O_p(G) = 1$, by assumption (ii) of the lemma, $H = P$; i.e., $N \leq P$. Consequently $C_G(\bar{K}_2) = C_P(\bar{K}_2)$.

(b) If $A^* \not\leq O_p(G)$ then $O_p(G) \leq A^*$, by 1.8 and assumption (i) of the Lemma. Hence, $C_P(O_p(\bar{G})) = A^* \geq A^*$ and, by part (a), $C_P(O_p(\bar{G})) = C_P(O_p(\bar{G})) \triangle \bar{G}$. Consequently, $C_P(O_p(\bar{G})) \leq O_p(\bar{G}) \leq A^*$, contradicting $C_P(O_p(\bar{G})) \geq A^*$. 


By definition, \( O_p(G) \) is elementary abelian of order \( p^r \) and by 1.7(a) \( r \leq p - 1 \). If \( |O_p(G)| = p \) the \( O_p(G) \) is cyclic and \( O_p(G) = Z(P) \), hence \( |O_p(G)| = p \). But this contradicts condition (i) of Theorem A. Hence \( 2 \leq r \leq p - 1 \) and by part (b) \( O_p(G) \supseteq A^* \). Since \( O_p(G) \) is abelian, \( O_p(G) = A^* \), hence \( O_p(G) = A^* \). Finally, \( |P/A^*| \leq |A^*| \) by 1.12(c). As \( |A^*| = |O_p(G)| \) and \( p^r \leq |O_p(G)| \leq p^{p - 1} \), we get \( |P/O_p(G)| = |P/O_p(G)| \leq |A^*| = |O_p(G)| \leq p^{p - 1} \); i.e., \( |P/O_p(G)| \leq p^{p - 1} \).

\[(d) \hspace{1cm} \text{Set } K_1 = 1 \text{ and } K_{i - 1} - P_1 \text{ in part (a). Then } C_{G}(P_i) = C_{C_p}(P_i) \text{ and } PC_{G}(P) \leq PC_{C_p}(P_i) = P. \] But \( N_{G}(P)/PC_{G}(P) \supseteq Aut(P) \). Thus \( N_{G}(P) \) has the desired form by Lemma 2.1(c).

\[(5.4) \hspace{1cm} \text{Remark. Lemma 5.1(b) yields that } O_p(G) \leq P_1 \text{ and by Lemma 5.3(c) } P_{p - 1} \subseteq O_p(G). \text{ So } O_p(G) \text{ is a } P_i \text{ for a natural number } i, 1 \leq i \leq p - 1. \] The next step in the proof of Theorem A is the exact location of \( O_p(G) \): In Section 8 we shall see that \( O_p(G) = P_1 \). To achieve this we shall consider two situations:

(i) \( P_{i - 1} \) is of class 3 at least \( (i \geq 2) \);

(ii) \( P_{i - 1} \) is of class 2 \( (i \geq 2) \).

We shall deal with the necessary preliminaries to case (i) in Section 6 and with case (ii) in Section 7.

6. \( p \)-Groups of Maximal Class with \( P_{p - 1} \) of Class 3 at Least

In order to prove the main result of this section (Theorem 6.2) we have to discuss metabelian \( p \)-groups of maximal class with \( P_1 \) of class 3.

\[(6.0) \hspace{1cm} \text{Let } P \text{ be a metabelian } p \text{-group of maximal class of order } p^n. \text{ Assume that } P_1 \text{ is of class 3 at least and let}
\]

\[ [s_1, s_2] = \prod_{t=0}^{k-1} s_{n-k+t}^x, \quad 0 \leq x_{k-t} \leq p - 1, x_k \neq 0, \text{ as in 1.9(b)} \] (7)

Then by 1.9(b)

\[ [s_1, s_2] = \prod_{t=0}^{k-j+1} s_{n-k+j+t}^x. \] (8)

Hence \( [P_1, P_1] = P_{2(n-k-1)} \) and we may assume without loss of generality that

\[ K_3(P_1) = [P_1, P_1] = P_{2(n-k-1)} = P_{n-1}; \]

i.e.,

\[ n - k = k + 1. \] (9)
(6.1) LEMMA. Let $P$ be a metabelian $p$-group of maximal class of order $p^n$, $n \geq 4$. Assume that $P$ has an automorphism $\sigma$ which satisfies:

1. $s^\sigma = ss_1^{-1}$;
2. $s_i^\sigma = s_1 u$, $u \in P_2$ (and $s_i^\sigma = [s_i^{s_{i-1}}, s^\sigma]$ for $i \geq 2$).

Let $v_i = [v, (i-1)s]$ for $i \geq 1$, and every $v \in P_2$. Then

1. $s_i^\sigma = s_i u_i [s_{i-1}, s_1^{-1}]^{-1} a_i$, where $a_i \in P_{n-k+i}$, $i \geq 2$;
2. $[s_1, v] = \prod_{k=1}^{k-1} v^{s_{k-1}^{-1} - s_{k-1}^{-1}}$, where $s_{k-1}$ are as defined in formula (8) in 6.0 and $v \in P_2$;
3. $[s_i^\sigma, s_2^\sigma] = [s_1, s_2] [s_1, u_2] \mod P_{n-i}$;
4. $n - k \equiv 2 \mod p$.

Proof: (a) Induction on $i$.

\[ s_2^\sigma = [s_1, s]^\sigma = [s_1 u, ss_1^{-1}] = [s_1, ss_1^{-1}]^u [u, ss_1^{-1}] = [s_1, s u_s][u, s][u, s_1] = (s_2[s_2, s_1^{-1}]^u [u, s_1^{-1}] u_2 [u_2, s_1^{-1}]. \]

Since $u \in P_2$ and $P'_2 = 1$,

\[ s_i^\sigma = s_i u_i [s_{i-1}, s_1^{-1}] [s_{i-1}, s_i^{-1}, u_i [u, s_{i-1}^{-1}] u_2 [u_2, s_1^{-1}]. \]

Let $a_2 = [s_2, s_1^{-1}] [u_1, s_1^{-1}] [u_2, s_1^{-1}]$. Then $a_2 \in P_{n-k}$ since $u_1, u_2 \in P_2$. Hence $s_i^\sigma = s_i u_i a_2$, $a_2 \in P_{n-k}$. This shows that our assertion is true for $i = 2$.

\[ s_3^\sigma = s_2 u_i a_2, s_1^{-1} = s_2 u_i a_2, s_1^{-1} [s_2 u_i a_2, s_1^{-1}]^{s_1^{-1}} \]

\[ = s_2 u_3 [s_2, s_1^{-1}] [u_2, s_1^{-1}] [a_2, s_1^{-1}] [s_2, s_1^{-1}] [u_2, s_1^{-1}] [a_2, s_1^{-1}] [s_1, a_1^{-1}] [a_1, s_1^{-1}] \]

where $a'_1 = [a_2, s] \in P_{n-k+1}$. We show that the expression in the round brackets belongs to $P_{n-k+1}$. Since $a'_1 \in P_{n-k}$ and $a'_1 \in P_{n-k+1}$, obviously $[a_2, s_1^{-1}], [s_1, s_1^{-1}], [u_2, s_1^{-1}], [u_2, s_1^{-1}]$ and of course $[a'_1, s_1^{-1}]$ all belong to $P_{n-k+1}$. Therefore (a) is valid for $i = 3$.

Assume $i \geq 3$ and $s_i = s_i u_i [s_{i-1}, s_1^{-1}]^{i-2} a_i$. Then

\[ s_i^{s_{i+1}} = [s_i, u_i] [s_{i+1}, s_1^{-1}]^{i-2} a_i, ss_1^{-1}] \]

\[ = [s_i, s_1^{-1}] [u_i, s_1^{-1}] [s_{i-1}, s_1^{-1}] [s_1, a_1^{-1}] [u_2, s_1^{-1}] [s_1, a_1^{-1}] [a_1, s_1^{-1}] \]
\[ \begin{align*}
&= s_{i+1} u_{i+1} [s_{i+1}, s_1^{-1}] [u_{i+1}, s_1^{-1}] [([s_{i-1}, s_1^{-1}] [a_i, s_1^{-1}])^{-2}, s] \\
&\quad \cdot [a_i, s])^{s_1^{-1}} [([s_{i-1}, s_1^{-1}] [u_i, s_1^{-1}] [s_{i-1}, s_1^{-1}, s_1^{-1}] [a_i, s_1^{-1}])^{-2}, s] \\
&= s_{i+1} u_{i+1} [s_1, s_1^{-1}]^{-1} [([s_{i+1}, s_1^{-1}] [u_{i+1}, s_1^{-1}] [([s_{i-1}, s_1^{-1}, s_1^{-1}] [a_i, s_1^{-1}])^{-2}, s])^{-2}, s] \\
&\quad \cdot [s_1, s_1, s_1]^{-2} [a_i, s_1^{-1}] [([s_{i-1}, s_1^{-1}, s_1^{-1}] [a_i, s_1^{-1}])^{-2}, s],
\end{align*}\]

since \([([s_{i-1}, s_1^{-1}]^{-2}, s]) = [s_1, s_1^{-1}]^{-2}\).

It is easy to check that the expression in the round brackets belongs to \(P_{n-k-2+l+1}\).

(b) Since \(P_2\) is a normal subgroup of \(P\), every element of \(P\) induces an automorphism on \(P_2\) by conjugation and it is an elementary well known fact that the map \(\nu: P \to \text{Aut}(P_2)\) defined by \(\nu: x \to \nu(x)\), \(\nu(ax) = ax\), for every \(a \in P_2\), \(x \in P\), is a homomorphism and its kernel contains \(P_2\). Let \(R\) be the subring of \(\text{End}(P_2)\) generated by \(\nu(s)\) and \(\nu(s_1)\). Then, since \([s, s_1] \in P_2 \subseteq \text{Ker} \nu\), \(R\) is a commutative ring (with unity \(\nu(1)\)) in which every \(r \in R\) has a representation by

\[
r = \sum_{ij} \lambda_{ij} \nu(s)^j \nu(s_1)^i, \quad \lambda_{ij} \in \mathbb{Z}, \quad 0 \leq i, j \leq p - 1, \quad (*)
\]

and \(P_2\), written multiplicatively, becomes a cyclic right \(R\)-module generated by \(s_1\) over \(R\) (see 1.3) with action \(ar = \prod_{k=k-i}^{l-1} \nu(x)^{s_1^{-1}}\) for \(a \in P_2\) and \(r \in R\) as in (*) Now, for every \(u \in P\), \(\nu(\mu) - \nu(1)\) is the commutation endomorphism of \(P_2\) by \(u\). Therefore, we may rewrite formulation (7) in 6.0 as \(s_2(\nu(s_1) - \nu(1)) = s_2(-\sum x_{k-2}^i \nu(s) - \nu(1))^{n-k-2+i}\) and since \(P_2\) is generated by \(s_2\) as an \(R\)-module, \(\nu(s_1) - \nu(1) = -\sum_{t=0}^{k-1} x_{k-t}^i (\nu(s) - \nu(1))^{n-k-2+i}\). In particular, for every \(v \in P_2\), \([v, s_1] = \nu(v(s_1) - \nu(1)) = \nu(-\sum x_{k-t}^i (\nu(s) - \nu(1))^{n-k-2+i}) = \prod_t \nu(x_{k-t}^i)^{s_1^{-1}}\), as required.

(c) \(s_2^o = s_1 u_1\), by definition and \(s_2^o = s_2 u_2 a_2, a_2 \in P_{n-k}\), by part (a) of the lemma. Hence \([s_1^o, s_2^o] = [s_1 u_1, s_2 u_2 a_2] = [s_1, s_2][s_1, u_2][s_2, a_2]\), since \(u_1, s_2 u_2 a_2 \in P_2\) and \([P_2, P_2] = 1\). As \(a_2 \in P_{n-k}\), \([s_1, a_2] \in P_{n-k-2+n-k} = P_{2(n-k-1)}\), and the result follows from formula (9) in 6.0.

(d) We show that \([s_1, s_2]^o = [s_1^o, s_2^o]\) implies \(n - k - 2 \equiv 0 \text{ mod } p\). By formula (7) in 6.0 \([s_1, s_2]^o = \prod_{t=0}^{k-1} (s_{n-k-t}^{x_{n-k-t}})^{a_{n-k-t}}\). Therefore, by part (a) of the Lemma,

\[
[s_1, s_2]^o = \prod_{t=0}^{k-1} s_{n-k-t}^{x_{n-k-t}} \prod_{t=0}^{k-1} u_{n-k-t}^{x_{n-k-t}} \prod_{t=0}^{k-1} a_{n-k-t}^{x_{n-k-t}}
\]

\[
\quad \cdot \prod_{t=0}^{k-1} [s_{n-k-t-1}, s_1^{-1}]^{(n-k-2+t)x_{n-k-t}},
\]

and \(\text{Ker} \nu\), \(R\) is a commutative ring (with unity \(\nu(1)\)) in which every \(r \in R\) has a representation by

\[
r = \sum_{ij} \lambda_{ij} \nu(s)^j \nu(s_1)^i, \quad \lambda_{ij} \in \mathbb{Z}, \quad 0 \leq i, j \leq p - 1, \quad (*)
\]

and \(P_2\), written multiplicatively, becomes a cyclic right \(R\)-module generated by \(s_1\) over \(R\) (see 1.3) with action \(ar = \prod_{k=k-i}^{l-1} \nu(x)^{s_1^{-1}}\) for \(a \in P_2\) and \(r \in R\) as in (*) Now, for every \(u \in P\), \(\nu(\mu) - \nu(1)\) is the commutation endomorphism of \(P_2\) by \(u\). Therefore, we may rewrite formulation (7) in 6.0 as \(s_2(\nu(s_1) - \nu(1)) = s_2(-\sum x_{k-2}^i \nu(s) - \nu(1))^{n-k-2+i}\) and since \(P_2\) is generated by \(s_2\) as an \(R\)-module, \(\nu(s_1) - \nu(1) = -\sum_{t=0}^{k-1} x_{k-t}^i (\nu(s) - \nu(1))^{n-k-2+i}\). In particular, for every \(v \in P_2\), \([v, s_1] = \nu(v(s_1) - \nu(1)) = \nu(-\sum x_{k-t}^i (\nu(s) - \nu(1))^{n-k-2+i}) = \prod_t \nu(x_{k-t}^i)^{s_1^{-1}}\), as required.

(c) \(s_2^o = s_1 u_1\), by definition and \(s_2^o = s_2 u_2 a_2, a_2 \in P_{n-k}\), by part (a) of the lemma. Hence \([s_1^o, s_2^o] = [s_1 u_1, s_2 u_2 a_2] = [s_1, s_2][s_1, u_2][s_2, a_2]\), since \(u_1, s_2 u_2 a_2 \in P_2\) and \([P_2, P_2] = 1\). As \(a_2 \in P_{n-k}\), \([s_1, a_2] \in P_{n-k-2+n-k} = P_{2(n-k-1)}\), and the result follows from formula (9) in 6.0.

(d) We show that \([s_1, s_2]^o = [s_1^o, s_2^o]\) implies \(n - k - 2 \equiv 0 \text{ mod } p\). By formula (7) in 6.0 \([s_1, s_2]^o = \prod_{t=0}^{k-1} (s_{n-k-t}^{x_{n-k-t}})^{a_{n-k-t}}\). Therefore, by part (a) of the Lemma,

\[
[s_1, s_2]^o = \prod_{t=0}^{k-1} s_{n-k-t}^{x_{n-k-t}} \prod_{t=0}^{k-1} u_{n-k-t}^{x_{n-k-t}} \prod_{t=0}^{k-1} a_{n-k-t}^{x_{n-k-t}}
\]

\[
\quad \cdot \prod_{t=0}^{k-1} [s_{n-k-t-1}, s_1^{-1}]^{(n-k-2+t)x_{n-k-t}},
\]
finite groups with a Sylow $p$-subgroup

$a_{n-k+1} \in P_{2(n-k-1)+1}$. Since $P_{2(n-k-1)+1} = P_{n-1}$ by formula (9) in 6.0, $a_{n-k} \in P_{n-1}$, $a_{n-k+t} = 1$ for $t \geq 1$ and

$[s_1, s_2] = [s_1, s_2] \prod_{t=0}^{k-1} u_{n-k+t}^{s_k} [s_{n-k-1}, s_1^{-1}]^{(n-k-2)x_0} \text{mod} P_{n-1}$. (10)

Now,

$[s_{n-k-1}, s_1^{-1}] = [s_{n-k-1}, s_1]^{-s_1^{-1}} = [s_1, s_{n-k-1}]^{s_1^{-1}} = [s_1, s_{n-k-1}, s_1^{-1}] = 1$ and $[s_{n-k-1}, s_1^{-1}] = s_{n-k}^{x_k} \text{mod} P_{n-1}$. Substituting this to (10) we get, noting that $u_0 \equiv 1$ by part (b) of the lemma,

$[s_1, s_2] = [s_1, s_2] [s_1, u_2] s_{n-k}^{x_k(n-k-2)} \text{mod} P_{n-1}$. (11)

But by part (c) of the lemma $[s_0, s_0^2] \equiv [s_1, s_2] [s_1, u_2] \text{mod} P_{n-1}$. Since $\sigma$ is an automorphism of $P$, we may compare (11) with part (c) of the lemma and this yields $x_k^2(n-k-2) \equiv 0 \text{mod} p$; i.e., $n-k-2 \equiv 0 \text{mod} p$, as required.

(6.2) Theorem. Let $P$ be a $p$-group of maximal class of order $p^n$, $n \geq 5$, and let $P_w$ be the maximal abelian normal subgroup of $P$, $w \geq 3$ (see 1.12). If $P_{w-1}$ is of class 3 at least and $[P_w, P_{w-1}] = P_r$ then $w \equiv r \text{mod} p$.

Proof. To prove Theorem 6.2 denote $H = \langle P_w, s \rangle$. Then $H$ is a $p$-group of maximal class of order $p^{n-w+3}$, by 1.6(a), with $h_i = s_{w-2+i}$ and $H_i = P_{w-2+i}$. It is easy to check that $s_{w-2}$ induces an automorphism $\sigma$ on $H$ which satisfies conditions (a) and (b) of Lemma 6.1 and since $H_2 = P_w$ is abelian all the conditions of Lemma 6.1 (for $H$ in place of $P$) are satisfied. If $[h_1, h_2] = h_i^c \text{mod} H_{c+1}$, $1 \leq x_c \leq p-1$, then $c \equiv 2 \text{mod} p$, by Lemma 6.1(d). As $r = w + c - 2$, $c = r + 2 - w$ and $c \equiv 2 \text{mod} p$ implies $r \equiv w \text{mod} p$, as required.

(6.3) Corollary. Let $P$ be a $p$-group of maximal class of order $p^n$, $n \geq 5$ and let $P_w$ be the maximal abelian normal subgroup of $P$. Let $[P_w, P_{w-1}] = P_r$. If $P_{w-1}$ is of class 3 at least and $w \equiv r \text{mod} p$ then $w \leq 2$. For, if $w \geq 3$ then $w \equiv r \text{mod} p$, by Theorem 6.2.

7. Linear Groups with Quadratic Action

(7.0) We introduce here some notations and recall results from Glauberman's paper [5].
(a) Let $V$ be a vector space over a field $F$, $W$ a subspace, $S$ a subset or an element of $V$ and let $G$ be a group of operators on $V$. Denote the action of $g \in G$ on $w \in W$ by $w^g$. We shall use the following notations:

\[
[S, G] = \{ w^g - w \mid w \in S, g \in G \}, \quad [w, g] = w^g - w;
\]

$C_w(G) = \{ w \in W \mid w^g = w, \text{for every } g \in G \}$;

$C_G(W) = \{ g \in G \mid w^g = w, \text{for every } w \in W \}$.

$[W, G]$ and $C_w(G)$ are subspaces of $V$ and $C_G(W)$ is a subgroup of $G$. These objects arise naturally in the semidirect product $V \cdot G$.

In (b), (c) and (d) we recall some notions from Glauberman's paper [5].

(b) ([5, p. 253]) Let $Qd(p)$ be the semidirect product of a two-dimensional vector space $V$ over $F_p$ by the special linear group $SL(V)$ on $V$. Let $F(p)$ be the normalizer of some Sylow $p$-subgroup of $Qd(p)$, thus $|Qd(p)| = p^3(p^3 - 1)$ and $|F(p)| = p^3(p - 1)$.

(c) ([5, p. 255]) Suppose $G$ is an operator group on an abelian group $V$. We say that an element $x$ of $G$ acts quadratically on $V$ if $[V, x] \neq 1$ and $[V, x, x] = 1$.

(d) ([5, p. 155]) Let $G$ be a group. Suppose $G \cong SL(2, p^r)$ and $G$ acts faithfully as an operator group on an elementary abelian group $V$. Let $\tilde{G}$ be the group of automorphisms of $V$ determined by $G$. We say that $V$ is a canonical module for $G$ if there exists a field $F$ of endomorphisms of $V$ such that $V$ is a two-dimensional vector space over $F$ and $\tilde{G} = SL(V, F)$. We say that $V$ is a semi-canonical module for $F$ if $V$ is a non-zero direct sum of a finite number of canonical modules for $G$. Finally, we recall some results from Glauberman's paper [5], for the convenience of the reader.

(e) ([5, Lemma 2.5(b)]) Let $V$ be a finite-dimensional vector space over $F_p$, $p > 2$. Assume that $G$ is an irreducible group of linear transformations of $V$ that is generated by two $p$-elements, $x$ and $y$, that act quadratically on $V$. Then either $G \cong SL(2, p^m)$ for some $m$ and $V$ is a canonical module for $G$ or $|V| = 81$, $|Z(G)| = 2$ and $G/Z(G) \cong A_5$.

(f) ([5, Theorem 3.1]) Consider the following situation.

**Hypothese I'.** Suppose $p$ is a prime, $V$ a finite abelian $p$-group and $G$ is a subgroup of Aut($V$). Assume:

(I'.1) $M$ is a maximal subgroup of $G$;

(I'.2) $S$ is a Sylow $p$-subgroup of $M$;

(I'.3) $A$ is a non-identity abelian subgroup of $S$;
(I'.4) \( A \notin O_p(G) \);
(I'.5) \([V, A, A] = 1\);
(I'.6) whenever \( g \in G \setminus M \), \( \langle A, g \rangle = G \).

Assume Hypothesis I'. Then

(a) \( S \) is a Sylow \( p \)-subgroup of \( G \);
(b) the intersection \( \bigcap_{x \in C} M^x \) is a normal subgroup of \( G \) and has \( O_p(G) \) as a Sylow \( p \)-subgroup; and
(c) there exists \( k \in G \) such that \( A^k \notin M \).

Choose a fixed \( k \in G \) that satisfies (c). Let \( B = A^k \). Let \( \overline{V} = V/C_r(G) \) and \( X = [V, G] \). Then

(d) \( G = \langle A, B \rangle \);
(e) \( X = [\overline{V}, A] \times [\overline{V}, B] \) and \( C_G(X) \leq O_p(G) \);
(f) \( C_x(A) = [V, A] \), \( C_x(B) = [\overline{V}, B] \) and \( C_x(G) = 1 \).

(Since parts (g)-(i) of Theorem 3.1 of [5] is not relevant to our investigation we do not quote them here.)

(g) ([5, Theorem 3.2]) Assume Hypothesis I'. Suppose \( p \) is odd, \( |A| = p \) and \( C_p(G) = 1 \). Then \( G \cong SL(2, p) \) and \( V \) is a semi-canonical module for \( G \).

(h) Finally, we mention the following trivial fact: \( F(p) \) (see 7.0(b)) has for every primitive root \( a \in \mathbb{F}_p \), a normal series

\[ 1 \triangle H_0 \triangle H_1 \triangle H_2^a = F(p) \]

with the following properties:

(a) \( H_0 \) is elementary abelian of order \( p^2 \), \( H_0 = \langle x, y \rangle \)
(b) \( H_1 = H_0 \langle u \rangle \), \( |u| = p \), \( x^u = x y^t \), \( y^u = y \).
(c) \( H_2^a = H_1 \langle t \rangle \), \( |t| = p - 1 \), \( u^t = u^{a^2} \), \( x^t = x^{a^{-1}} \), \( y^t = y^a \).

Proof. Take

\[
\begin{aligned}
x &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
y &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \\
u &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
t &= \begin{bmatrix} a & 0 & 0 \\ 0 & a^{-1} & 1 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
\]

(7.1) THEOREM. Let \( G \) be a finite group of linear transformations acting on a vector space \( V \) of dimension \( d \leq p - 1 \) over \( \mathbb{F}_p \). Let \( \rho(G) \) be the set of all the subgroups of \( G \) and let us define \( \hat{\rho}: G \times G \to \rho(G) \) by \( \hat{\rho}(x, g) = \langle x, x^g \rangle \), \( x, g \in G \). Let \( \Lambda = \{ \hat{\rho}(x, g) | \hat{\rho}(x, g) \text{ is not a } p\text{-group} \} \) and assume that the following condition is satisfied:
(i) There exist elements \(x, g' \in G\) s.t. \(x\) acts quadratically on \(V\) and \(L(x, g') \in \mathcal{A}\).

Then

(a) \(L(x, g')\) contains a minimal element \(L\) of \(\mathcal{A}\) s.t. \(L = L(x, 1)\) for some \(l \in L(x, g')\) and there exists a Sylow \(p\)-subgroup \(B\) of \(L\) which contains \(x\) s.t. \(N_L(C_V(P_0)) \geq N_L(B)\), where \(P_0 = \langle x \rangle\) and \(N_L(B) = RT, T\) cyclic of order \(p - 1\).

(b) Let \(L\) and \(P_0\) be as in part (a). If \(N_L(C_V(P_0)) = N_L(P_0)\) then \(L \cong SL(2, p)\).

We prove Theorem 7.1 through Lemmas 7.2–7.6.

(7.2) Lemma. Assume the notations of Theorem 7.1 and suppose that \(G, B, x, g'\) satisfy condition (i) of Theorem 7.1. Let \(\mathcal{A}_0\) be the set of the minimal elements of \(\mathcal{A}\). Then

(a) Every \(L \in \mathcal{A}_0\) has a normal subgroup \(N\) s.t. \(L/N \cong SL(2, p)\) and there are subgroups \(W_1 < W_2\) of \(V\) such that \(W_2/W_1\) is a canonical module for \(L/N\) (see 7.0(d)).

(b) Let \(L\) and \(N\) be as in (a). Then \(N = O_p(L)\) and \(L = \langle x, x' \rangle, l \in L \setminus N\).

Note. Part (b) agrees with Corollary 4.2 of [5]. But in our case we can prove it more simply.

Proof. (a) By assumption \(\mathcal{A}_0 \neq \emptyset\). Let \(L \in \mathcal{A}_0\). First we prove that \(V\) has at least one \(L\)-composition factor \(W\) s.t. \([W, L] \neq 0\). Assume \([W, L] = 0\) to every \(L\)-composition factor \(W\) of \(V\) and let \(Q \neq 1\) be a Sylow \(q\)-subgroup of \(L\), \(q \neq p, q\) a prime. Since \(L\) stabilizes a normal series of \(V\), \(Q\) stabilizes it too. But this forces \([V, Q] = 0\); i.e., \(Q = 1\), as \(G\) is a group of linear transformations. Therefore \([W, L] \neq 0\) for at least one \(L\)-composition factor \(W\) of \(V\).

Now let \(W\) be an \(L\)-composition factor of \(V\) s.t. \([W, L] \neq 0\). Then \(x\) or \(x^q\) does not act trivially on \(W\). Assume \(x\) acts nontrivially and \(x^q\) acts trivially on \(W\). Then \(x\) and \(x^q\) cannot be conjugate in \(L\). By the minimality of \(L\) in \(\mathcal{A}\), to every \(l \in L\), \(L(l)\) is either a \(p\)-group or is the whole of \(L\). We claim that there is an \(l \in L\) s.t. \(L(l) = L\) and hence \(L\) is generated by two elements of order \(p\), \(x\) and \(x^l\) which act quadratically on \(W\). Indeed, if \(L(l) \neq L\) to every \(l \in L\), then \(L(l)\) is a \(p\)-group to every \(l \in L\) and \(\langle x \rangle^x\) is contained in a normal \(p\)-subgroup \(H\) of \(L\) which obviously doesn't contain \(x\). Hence \(L = \langle x, x^q \rangle = \langle H, x^q \rangle = H \triangle L\). But \(|H| = p^a\) and \(|\langle x \rangle| = p\). This contradicts \(L \in \mathcal{A}\) and proves our claim. Let \(N\) be the kernel of the representation of \(L\) on \(W\). Then by [5, Lemma 2.5(b)] (See 7.0(e)) \(L/N \cong SL(2, p')\) for some \(r\) and \(W\) is a canonical module for \(L/N\). We prove \(r = 1\). \(L/N\) has an element
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\( hN \) of order \( p + 1 \). Let \( L_0 = \langle P_0, P_0^h \rangle \), \( P_0 = \langle x \rangle \). Then \( L_0/N/N \cong SL(2, P) \) and \( L_0 \) is not a \( p \) group. Therefore \( L_0 = L \) by the minimality of \( L \) in \( A \) and \( L/N \cong SL(2, p) \), as required.

(b) Let \( 0 = V_1 < V_2 < \cdots < V_t = V \) be an \( L \)-composition series of \( V \) and let \( \psi_1, \ldots, \psi_t \) be the representations of \( L \) on \( W_i = V_i/V_{i-1} \), \( 2 \leq i \leq t \), with kernels \( N_i \), \( 2 \leq i \leq t \), respectively. If \( Q \) is a sylow \( q \)-subgroup of \( N_0 = \bigcap_{i=2}^{t} N_i \), \( q \neq p \), then \( Q \) centralizes a normal series of \( V \) and \( [V, Q] = 0 \); i.e., \( Q = 1 \). Therefore \( N_t < O_p(L) \leq N_t \), \( 2 \leq i \leq t \). If \( N_t = O_p(L) = N_0 \) we are done. Hence we may assume that \( N_1 \neq N_2 \). By part (a) \( L/N_1 \cong L/N_2 \cong SL(2, p) \). Let \( N = N_1 \cap N_2 \) and to every subgroup \( H \leq L \), denote \( H = HN/N \. \) \( \tilde{N}_1 \equiv \tilde{N}_1 \tilde{N}_2/\tilde{N}_2 \hookrightarrow SL(2, p) \), \( \tilde{N}_2 \equiv \tilde{N}_2 \cdot \tilde{N}_1/\tilde{N}_1 \hookrightarrow SL(2, p) \). Hence we have two possibilities:

\( \tilde{N}_1 \equiv SL(2, p) \times SL(2, p) \), \( \tilde{N}_2 \equiv \tilde{N}_2 \equiv SL(2, p) \).

(\( \beta \)) \( |\tilde{N}_1| = |\tilde{N}_2| = 2 \).

Assume (\( \alpha \)) and let \( \tilde{P}_0 \) be a sylow \( p \)-subgroup of \( \tilde{L} \) that contains \( \tilde{x} \) and let \( \tilde{P}_1 \) and \( \tilde{P}_2 \) be sylow \( p \)-subgroups of \( \tilde{N}_1 \) and \( \tilde{N}_2 \) respectively s.t. \( \tilde{P}_0 = \tilde{P}_1 \times \tilde{P}_2 \). Then there exist elements \( \tilde{x}_i \in \tilde{P}_i \), \( i = 1, 2 \), s.t. \( \tilde{x} = \tilde{x}_1 \cdot \tilde{x}_2 \) and we may assume \( \tilde{x} \neq 1 \). Let \( \tilde{h} \in \tilde{N}_1 \) be an element of order \( p + 1 \). Then \( \tilde{N}_1 = \langle \tilde{x}_1, \tilde{h} \rangle \) and \( \langle \tilde{x}, \tilde{h} \rangle \) \( \cong \tilde{N}_1 \times \langle \tilde{x}_2 \rangle \) \( \cong \tilde{L} \), contradicting \( \langle \tilde{x}, \tilde{x}^h \rangle = L \). Hence we may assume (\( \beta \)). In particular \( \tilde{N}_1 \times \tilde{N}_2 \leq Z(\tilde{L}) \). Let \( \tilde{Q} \) be a sylow 2-subgroup of \( \tilde{L} \). Then \( \tilde{Q}/\tilde{N}_1 = \tilde{Q}_0 \) be a quaternion 2 group. Assume \( \tilde{L}' = L \). Since \( \tilde{Q} \cap \tilde{L}' \cap Z(\tilde{L}) \leq \tilde{Q}' \), \( \tilde{N}_1 \times \tilde{N}_2 \leq \tilde{Q}' \) and \( \tilde{Q}/\tilde{Q}' \cong \tilde{Q}_0/\tilde{Q}_0 \equiv \mathbb{Z}_2 \times \mathbb{Z}_2 \) and \( \tilde{Q} \) is of maximal class, contradicting \( \tilde{N}_1 \times \tilde{N}_2 \leq Z(\tilde{Q}) \). Hence \( \tilde{L}' \neq L \) and all the sylow \( p \)-subgroups of \( \tilde{L} \) are contained in \( \tilde{L}' \) contradicting \( L = \langle \tilde{x}, \tilde{x}^\tilde{h} \rangle \). Therefore (\( \beta \)) is impossible too, and \( \tilde{N}_1 = \tilde{N}_2 \), i.e., \( N_1 = N_2 \). Finally, \( L = \langle x, x^l \rangle \) \( l \in L \) by part (a) and \( l \notin N \). For if \( l \in N \) then \( x^l \equiv x \mod N \) and \( L = \langle x, x^l \rangle = N\langle x \rangle \) is a \( p \)-group, violating \( L \notin A_0 \).

(7.3) Lemma. Let notations be as in Lemma 7.2, let \( L \in A_0 \) and assume (i) of Theorem 7.1. Let \( P_0 = \langle x \rangle \) and define subspaces \( V_i \) and \( X_i \) of \( V \) as follows:

\[ V_0 = 0 \quad \text{and for } i \geq 1 \quad V_i/V_{i-1} = C_{V/V_{i-1}}(L) \]  

(**) \[ X_0 = [V, L] \quad \text{and for } i \geq 1 \quad X_i = V_i + X_0. \]

Then

(a) There exists a natural number \( i_0 \), s.t. \( V_{i_0+l} = V_{i_0} \) for every natural number \( j \) and \( V/V_{i_0} \) is a semi-canonical module for \( L/N \).

(b) \( X_i/V_i = [V, P_0] + V_i/V_i \oplus [V, P_0^8] + V_i/V_i \);

(c) \( [V, P_0] + V_i/V_i = C_{X_i/V_i}(P_0) \) and \( [V, P_0^8] + V_i/V_i = C_{X_i/V_i}(P_0^8) \).
Proof. (a) Assume that the series of the distinct $V_i$'s terminates at $V$. Then $L$ stabilizes a normal series ($L$-invariant series) of $V$ and is a $p$-group, contradicting $L \in A_0$. Hence there is a $V_{i_0} \neq V$ s.t. $C_{V/V_{i_0}}(L) = 0$. Then with $G = L/C_L(V/V_{i_0})$ it follows from [5, Theorem 3.2] see 7.0(g)) that $C_L(V/V_{i_0}) = N$ and $V/V_{i_0}$ is a semi-canonical module for $L/N$.

Parts (b) and (c) follow from [5, Theorem 3.1(c) and 3.1(f)] (see 7.0(f)) if we set $P_0$ for $A, N_0(B)$ for $M, B$ a Sylow $p$-subgroup of $L$, $X_{i+1}/V_{i+1}$ for $X$ and $V/V_{i+1}$ for $V$.

(7.4) Lemma. Let notations be as in Lemma 7.2 and Theorem 7.1 and assume (i) of Theorem 7.1. Let $W_1 < W_2 < W_3$ be a series of $L$-submodules of $V$ s.t. $[W_1, L] \leq W_{i-1}$, $i = 2, 3$ and let $K = C_L(W_3/W_1)$. Then

(a) $K \triangle L$, $KN = L$ and $|L : K| \leq p$

(b) $x \equiv x^p$ mod $K$ and $[w, x] \equiv [w, x^p]$ mod $W_1$, for every $w \in W_3$.

Proof. (a) Since $W_3$ and $W_1$ are $L$-invariant, $K \triangle L$. Hence $KN/N \triangle L/N$. By Lemma 7.2(a) $L/N \cong SL(2, p)$. Therefore either $|KN/N| = 2$ or $KN = L$. But $L/K$ is an abelian $p$-group by [6, Theorem 2.9, p. 264] since it stabilizes the $L$-invariant series $W_3 > W_2 > W_1$. Hence $KN = L$. It remains to show that $|L : K| = p$. Let $N_0 = K \cap N$ and for a subgroup $A$ (or element $y$) of $L$ let $\bar{A} = AN_0/N_0$ ($\bar{y} = \bar{y}N_0$). Then $\bar{L} = \bar{K} \times \bar{N}$. We may assume that $K \neq L$. Then there exist elements $\bar{k} \in \bar{K}$ and $\bar{n} \in \bar{N}$ such that $\bar{x} = \bar{k} \cdot \bar{n}$. We claim that $\bar{N}$ is an elementary abelian $p$-group; $\bar{N} = N/N_0$ stabilizes an $\bar{N}$-invariant series of length 2 at most (namely, $W_3 > W_2 > W_1$), hence $N/N_0$ is abelian. On the other hand, $N/N_0$ is a $p$-group which is isomorphic to a subgroup of $\text{Aut}(W_3/W_1)$. Since dim $V \leq p - 1$, by assumption, obviously dim($W_3/W_1 \leq p - 1$ and exp($N/N_0 \leq p$, by [6, Th. 16.5(a), p. 384]). Hence $\bar{N}$ is elementary abelian and $\bar{N} \leq Z(\bar{L})$. This implies that as $x = k \cdot n$ then for every $\bar{\ell} \in L$, $x^\ell = \bar{k}^\ell \cdot \bar{n}^\ell \in \bar{K} \cdot \langle \bar{n} \rangle$. But then $\langle \bar{x} \rangle \leq \bar{K} \langle \bar{n} \rangle$. Therefore, since $L = \langle x, x' \rangle$ for some $l \in L$ by 7.2(a), $\bar{L} = \bar{K} \langle \bar{n} \rangle$. As $\bar{N}$ is elementary abelian, $|\bar{n}| \leq p$ and $|L : K| = |\bar{L} : \bar{K}| \leq p$.

(b) Follows from part (a) since $|L/K| \leq p$ clearly implies (b).

(7.5) Lemma. Let notations be as in Lemma 7.3 and assume (i) of Theorem 7.1. Then

(a) For every $i \geq 1$ $C_{V_i}(P_0) = C_{V_i}(P_0^g) = V_i$;

(b) For every $i \geq 1$ $C_{X_i}(P_0) = [V, P_0^g] + V_1$ and $C_{X_i}(P_0^g) = [V, P_0^g] + V_1$;

(c) $C_{V}(P_0) = [V, P_0] + V_1$ and $C_{V}(P_0^g) = [V, P_0^g] + V_1$.

Proof. (a) By induction on $i$. For $i = 1$ (a) holds by definition (*) in 7.3. Let $i \geq 2$ and let $C_{V_i/V_{i-2}} = C_{V_i/V_{i-2}}(P_0)$ and $D_{i/V_{i-2}} = C_{V_i/V_{i-2}}(P_0^g)$. By
Lemma 7.4(b) \([v, x] \equiv [v, x']\) mod \(V_{i-2}\) for every \(v \in V_i\). Hence \(C_i = D_i = C_i \cap D_i\). Since \((P_0, P_0') = L\), \(C_i \cap D_i = C_{i/V_{i-2}}(L)\), hence \(C_i \cap D_i \leq V_{i-1}\) and \(C_i = D_i \leq V_{i-1}\). Hence \(C_{i/V_{i-1}}(P_0) \leq V_{i-1}\) and \(C_{i/V_{i-1}}(P_0') \leq V_{i-1}\), as \(C_{i/V_{i-1}}(P_0) \leq C_i\) and \(C_{i/V_{i-1}}(P_0') \leq D_i\). Consequently \(C_{i/V_{i-1}}(P_0) = C_{i/V_{i-1}}(P_0')\), \(C_{i/V_{i-1}}(P_0) = C_{i/V_{i-1}}(P_0)\) and (a) follows by the induction hypothesis.

(b) Let \(v \in X_i\). Then \(v = u + w,\ u \in V_i,\ w \in [V, L]\). If \(v \in C_{i/V_{i-1}}(P_0)\) then \(v^x = u^x + w^x = (u + a) + (w + \beta) = v\), where \(a = [u, x]\), \(\beta = [w, x]\) and \(a \in V_{i-1},\ \beta \in [V, P_0]\). Hence \(a + \beta = 0\) and \(\beta \in C_{i/V_{i-1}}(P_0)\) since \([\beta, P_0] \leq [V, P_0, P_0] = 0\), as \(x\) acts quadratically on \(V\). Therefore \(\beta \in V_{i-1}\) by part (a), \(w + V_{i-1} / V_i \in C_{X_i/V_i} (P_0)\) and \(w = [v_0, x] + v_1\), where \(v_0 \in V_i, v_1 \in V_i\). Hence \(\beta = a = 0\), i.e. \(u \in C_{i/V_{i-1}}(P_0),\ w \in [V, P_0] + V_{i-1}\). Since \(C_{i/V_{i-1}}(P_0) = V_{i-1}\) by part (a), \(v = u + w \in [V, P_0] + V_{i-1}\). But obviously \([V, P_0] + V_{i-1} \leq C_{X_i/V_i} (P_0)\) and \(C_{X_i/V_i} (P_0) = [V, P_0] + V_{i-1}\). By the same argument \(C_{X_i/V_i} (P_0) = [V, P_0'] + V_{i-1}\), as required.

(c) By Lemma 7.3(a) \(V/V_{i_0}\) is a semi-canonical module for \(L/N\). Hence \(V/V_{i_0} / L/N = V/V_{i_0}\) and \([V, L] + V_{i_0} = V\). By definition (***) in 7.3 \([V, L] + V_{i_0} = X_{i_0}\). Hence \(V = X_{i_0}\) and \(C_{i/V_{i_0}} (P_0) = X_{i_0} = [V, P_0'] + V_{i-1}\), by part (b).

(7.6) Lemma. Let notations and assumptions be as in Lemma 7.5, and Theorem 7.1. Then

(a) \(N \leq N_L (C_{i/V_{i_0}} (P_0))\);

(b) \(N_L (C_{i/V_{i_0}} (P_0))\) has a cyclic subgroup \(T\) of order \(p - 1\).

Proof: For every subgroup \(A\) of \(L\) and every element \(u\) of \(L\) let \(T = AN/N\) and \(\bar{u} = uN\). Let \(\langle \bar{t} \rangle\) be a complement of \(\bar{P}_0\) in \(N_L (\bar{P}_0),\ \|t\| = p - 1\). (Remember that \(L \cong SL(2, p)\)) If \(W\) is a nontrivial \(L\)-module then \(C_w (\bar{P}_0)\) is \(\bar{t}\)-invariant. Now, \(C_{X_i/V_1} (L) = N\) and \(X_1/V_1\) is a semi-canonical module for \(L\) by [5, Theorem 3.2] (see 7.0(g)) as \(C_{X_1/V_1} (L) = V_{i_0} / V_1\) by [5, Theorem 3.1(f)] (see 7.0(f)). Hence \(C_{X_1/V_1} (\bar{P}_0)\) is \(\bar{t}\)-invariant and obviously \(N\)-invariant. By [5, Theorem 3.1(f)] \(C_{X_1/V_1} (\bar{P}_0) = [V, P_0] + V_{i_1} / V_1\). Hence \(C_{X_1/V_1} (\bar{P}_0) = C_{X_1/V_1} (\bar{P}_0) = [V, P_0] + V_{i_1} / V_1\). Consequently \([V, P_0] + V_{i_1} / V_1\) is \(\bar{t}\)-invariant and obviously \(N\)-invariant. But \([V, P_0] + V_{i_1} = C_{i/V_{i_1}} (P_0)\) by Lemma 7.5(c). Hence \(N \leq N_L (C_{i/V_{i_1}} (P_0))\) and \(t \in N_L (C_{i/V_{i_1}} (P_0))\). Since \(\|t\| = p - 1\) \((\bar{E} \cong SL(2, p))\) this proves Lemma 7.6.

(7.7) We turn to the proof of Theorem 7.1.

(a) Let \(B\) be a Sylow \(p\) subgroup of \(L\). Then (a) follows from Lemma 7.6.

(b) Let \(L\) be a minimal element of \(A\) which lies in \(L(x, g')\). Then the assumption of Theorem 7.1(b) implies that \(N_L (C_{i/V_{i_0}} (P_0)) = N_L (P_0)\). Hence by
Lemma 7.6 \( t \in N_L(P_0) \) and \( N \leq N_L(P_0) \). Since \( P_0 \) is of order \( p \) and \( N \) is a \( p \)-group, by Lemma 7.2(b), this implies that \( N \leq C_L(P_0) \). By the same argument \( N \leq C_L(P_0^p) \). But since \( L = \langle P_0, P_0^p \rangle \) this means that \( N \leq Z(L) \). In particular, \( N \) is abelian and since \( |P_0| = p \) by [6, Th. 17.4, p. 121] \( L = N \times L_0 \), where \( L_0 \cong SL(2, p) \). Let \( T \) be the subgroup mentioned in Lemma 7.6(b) and let \( T = \langle t \rangle \). Then \( x^t = x^a \), \( 2 \leq a \leq p - 1 \) and \( |x, t| = x^{a-1} \neq 1 \). Since \( L_0 \triangleleft L \) and \( t \in L_0 \), for some \( n \in N \), \( [x, t] \in L_0 \). Thus \( 1 \neq [x, t] \in L_0 \cap P_0 \) and as \( P_0 \) is of order \( p \), \( P_0 \leq L_0 \). Hence by Lemma 7.2(a) \( L = \langle P_0, P_0^n \rangle \leq L_0 \) and \( L = L_0 \) as required.

8. STRUCTURE OF A MINIMAL COUNTEREXAMPLE (CONTINUED)

The main result of this section is that if \( G \) is a minimal counterexample to Theorem A then \( O_p(G) = P_1 \).

(8.1) LEMMA. Let \( G \) be a finite group with a Sylow \( p \)-subgroup \( P \) of maximal class. Suppose that the following hold

(i) \( G \) is a minimal counterexample to Theorem A;
(ii) \( O_p(G) = P_i \), \( i \geq 2 \);
(iii) \( P_{i-1}/\Phi(P_i) \) is of class 2.

Then

(a) \( 2 \leq i \leq p - 2 \);
(b) \( N_G(P_{i-1}) = N_G(P) \) or \( P_1 \triangle N_G(P_{i-1}) \) and \( N_G(P_{i-1})/P_1 \cong PSL(2, p) \) or \( N_G(P_{i-1})/P_1 \cong PSL(2, p)^+ \);
(c) \( P_j = C_{P_j}(P_{i-1}) \). Then \( N_G(P_j) = N_G(P) \) or \( N_G(P_j) = N_G(P_1) \) and \( N_G(P_j)/P_1 \cong PSL(2, p)^+ \);
(d) \( SL(2, p) \xrightarrow{\subset} G/O_p(G) \xrightarrow{\subset} GL(3, p) \).

Proof. (a) Since \( P_{i-1}/\Phi(P_i) \) is not abelian by supposition (iii), \( [P_{i-1}, P_i] \leq \Phi(P_i) \). As \( \Phi(P_i) \supseteq P_{i+p-1} \), this means that if \( P \) has degree of commutativity \( k \) then \( i - 1 + i + k < i + p - 1 \); i.e., \( i < p - k \). Since \( n \geq p + 3 \) by assumption (i), \( k \geq 1 \) by 1.5(a). Hence \( i < p - 1 \). This proves (a).

(b) Since \( O_p(G) = P_i \), obviously \( P_{i-1} \triangle G \), hence if (b) is false then \( N_G(P_i) \) has structure as described in Theorem A(b)(a) or (b) or (γ) or (ε). If \( P_i \) is abelian then \( P_i = O_p(G) \) by Lemma 5.3(c). Hence (b) and (γ) cannot occur. In case (ε) \( i \) must be 2 for by (a) \( 2 \leq i \leq p - 2 \) and by Theorem A(b)(ε) \( i \) can be 1 or 2 or \( p - 1 \). So \( i = 2 \) in both cases (a) and (ε). But then \( \Phi(P_1) = \Phi(P_2) = P_{p+1} \), hence \( C_{P_1/P_{p+1}}(P_1/P_2) \) must be \( P_{p}P_{p+1} \), contradicting
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$|P_1, P_{p-1}| \leq P_{p+1}$ as $P$ has degree of commutativity $\geq 1$ (i.e., $P_{p-1}/P_{p+1} \leq C_{P,p+1}(P_1/P_2)$). Thus (b) holds.

(c) If (c) is false then, as in part (b), only Theorem A(b)(a) and Theorem A(b)(e) may hold for $N_G(P_i)$. In the first case $i$ must be 2, since $P_2$ is abelian. But this leads to a contradiction as in part (b). If Theorem A(b)(e) holds for $N_G(P_i)$ then since $P_i \triangle G$, $i$ must be 2 again, by part (a), and this leads to a contradiction as in part (b). Hence it remains only to show that $N_G(P_i)/P_i \leq \text{PSL}(2, p)$. Set $V = P_i/P_i \Phi(P_i)$, $P_0 = P_{i-1}/P_i$ in Theorem 7.1 and let $L, N$ and $T$ as in Theorem 7.1 and Lemma 7.2. Then by Lemma 7.6(b) $p - 1 | N_T(P_i)$ hence obviously $p - 1 | N_G(P_i)$. This proves (c).

(d) By supposition (i) and Lemma 5.1(a) $N_G(P) = P_{T_1}$, where $T_1$ is a cyclic subgroup of $N_G(P)$ of order dividing $p - 1$ and if $N_G(P_i)/P_i \cong \text{PSL}(2, p)^+$ then $|T_1| = p - 1$. Let $T_1 = \langle t_1 \rangle$ and assume the notations of part (c). Assume that $V$ has a chain of $t_1$-invariant subspaces $0 = V_1 < V_2 < \cdots < V_d = V$, s.t. dim$(V_i/V_{i-1}) = 1$, for $2 \leq i \leq d$ and if $V_i = V_{i-1} + \langle v_i \rangle$ then $v_i = \lambda_i v_i \mod V_{i-1}$, $\lambda_i \in \mathbb{F}_p$. Since $|t_1| = p - 1$, $t_1$ is diagonalizable on $V$ and $V$ has an ordered basis $u_2, \ldots, u_d$ such that $u_i = \lambda_i u_i$, $2 \leq i \leq d$. Therefore, if $|t_1| = p - 1$ it follows from Lemma 3.1(a) and the fact that dim $V \leq p - 1$, that if $E$ is the set of the eigenvalues of $t_i$ on $V$ then $E \subseteq \{b, ab, \ldots, a^{p-2}b\}$ and every eigenvalue appears with multiplicity 1. Thus

(*) dim $V^\lambda(t_i) \leq 1$ for every $\lambda \in \mathbb{F}_p$, where $V^\lambda(t_i) = \{v \in V | v^t = \lambda v\}$ and $|t_i| = p - 1$.

If $T_2$ is another complement of $P$ in $N_G(P)$ and $T_2 = \langle t_2 \rangle$ then $t_2 = t_1^a u$, where $a$ is a natural number, $1 \leq a \leq p - 1$, $(a, p - 1) = 1$ and $u \in P$. Hence $s_i^a \equiv s_i^a u \equiv s_i^a \mod P_{i+1}$ and since $t_1^a$ is a generator of $T_1$, dim $V^\lambda(t_2) \leq 1$, by (*). Hence

(**) For every element $t_2$ of order $p - 1$ in $N_G(P)$ dim $V^\lambda(t_2) \leq 1$. Assume now that $N_G(P_1) = N_G(P_2)$ and $N_G(P)/P_1 \cong \text{PSL}(2, p)^+$. Then, since the normalizers of the $p + 1$ Sylow $p$-subgroups of $N_G(P_i)$ are conjugate and every cyclic subgroup of order $p - 1$ is contained in such a Sylow normalizer, (***) implies that dim $V^\lambda(t) \leq 1$ for every $\lambda \in \mathbb{F}_p$, where $T = \langle t \rangle$ is a cyclic subgroup of order $p - 1$ of $N_G(P)$ s.t. $TP_i/P_i \leq L$ (see Theorem 7.1(a)). But then in particular dim $C_V(T) \leq 1$ and obviously dim $C_V(L) \leq 1$ and $C_V(C_V(L)/C_V(L)) = C_V(L)/C_V(L)$. Moreover as $V/C_V(L)$ is a semicanonical module for $L/N$ by Lemma 7.3(a), this implies that $V/C_V(L)$ is actually a canonical module for $L/M$: For if dim $V/C_V(L) > 2$ then dim $V \geq 3$ and among the eigenvalues $a^{i-1}b, a^ib, a^{i+2}b$ of $t$ on $V$ at least two have order dividing $p - 1$ properly. However, if dim $V/C_V(L) > 2$ this violates 7.0(b). Hence dim $V \leq 3$ and $G/P_i \leq GL(3, p)$ and $N$ is abelian. Therefore, by [6, Th. 17. 4, p. 121] $L = NL_0, L_0 \cong SL(2, p)$. This proves part (d) of the Lemma.
(8.2) **Proposition.** Let $G$ be a finite group with a Sylow $p$-subgroup $P$ of maximal class.

Assume that

(i) $G$ is a minimal counterexample to Theorem A.

(ii) $O_p(G) \neq P_1$.

Then

(a) $P_1$ is of class 3 at least.

(b) $O_p(G) = P_2$.

**Proof:** Assume $\text{cl}(P_1) < 2$. Then by Lemma 8.1(d) $SL(2, p) \subset G/O_p(G) \subset GL(3, p)$. As $O_p(G/O_p(G)) = 1$, $G/O_p(G)$ has no normal subgroup of order prime to $p$ and by assumptions (i) and (ii) $G/O_p(G)$ has no normal subgroup of index prime to $p$. Checking the list of subgroups of $PSL(3, p)$ in [2] we find that no such $G/O_p(G)$ exists. Consequently $\text{cl}(P_1) \geq 3$ and by Corollary 6.3, Assumption (ii) and 1.8 $P_1 = P_2$. This proves parts (a) and (b) of Lemma 8.2.

(8.3) **Theorem.** Let $G$ be a finite group with a Sylow $p$-subgroup $P$ of maximal class of order $p^n$, $n \geq p + 3$. If $G$ is a minimal counterexample to Theorem A then $O_p(G) = P_1$.

**Proof.** By induction on $n$.

$n = p + 3$: If $O_p(G) \neq P_1$ then $O_p(G) = P_2$ by Proposition 8.2(b). Therefore $C_G(\mathcal{U}(P_2)) \triangle G$. By 1.7(a) $\mathcal{U}(P_2) = P_{p+1}$ and since $n = p + 3$, $\mathcal{U}(P_2) = P_{n-2}$. Hence

$$C_G(P_{n-2}) \triangle G. \quad (12)$$

Since $n > p + 2$, $P$ has positive degree of commutativity, by 1.5(a). Hence $|P_{n-2}, P_1| = 1$; i.e., $P_1 \leq C_P(P_{n-2})$. But then 1.10 implies

$$C_P(P_{n-2}) = P_1. \quad (13)$$

By Lemmas 5.1(a) and 5.3(a) $C_P(P_{n-2}) = C_G(P_{n-2})$. Thus (12) and (13) imply $P_1 = C_P(P_{n-2}) = C_G(P_{n-2}) \triangle G$; i.e., $P_1 \triangle G$.

$n > p + 3$: If $O_p(G) \neq P_1$ then we may assume that $n \leq 2p$. For assume $n \geq 2p + 1$. Then either $n = 2p + 1$ or $n \geq 2p + 2$. If $n = 2p + 1$ then by 1.7(a) $\Omega_2(P) = P_{2p} = P_{n-1}$ and since $O_p(G) = P_2$ by Proposition 8.2(b), $P_{n-1}$ is a normal subgroup of $G$ of order $p$. As $|P/P_{n-1}| = p^{n-1} = p^{2p} \geq P^{p+3}$, $P_1 \triangle G$ by the induction hypothesis. Hence $n \geq 2p + 2$ and consequently $n - p + 1 \geq p + 3$. This implies that $|P/P_{n-p+1}| \geq p^{p+3}$. But by 1.7(b) $\Omega(P_2) = P_{n-p+1}$, hence $P_{n-p+1}$ is a nontrivial normal subgroup of $G$. 


Therefore by the induction hypothesis for $G/Q(P_2)$, $P_1/Q(P_2) \triangle G/Q(P_2)$ and $P_1 \triangle G$. Thus we may assume, if $O_p(G) \neq P$, that

$$p + 4 \leq n \leq 2p. \quad (14)$$

We recall from Lemma 3.2(a) that if $[s_1, s_2] = s_1^a \mod P_{r_1+1}$, $1 \leq a \leq p - 1$, and $[s_2, s_1] = s_2^b \mod P_{r_1+1}$, $1 \leq b \leq p - 1$, then $r_2 - r_1 \geq 2$ or $r_1 \geq n - 1$. Assume that $O_p(G) \neq P$. Then $O_p(G) = P_2$ and $\text{cl}(P_2/\Phi(P_2)) \geq 3$ by Proposition 8.2(a) and 8.3(b). In particular,

$$P_1/P_2$$

is of class 3 at least. (15)

This implies that $r_2 - r_1 > 2$. For if $r_2 - r_1 = 2$ then $|P_1/P_2| = p^2$ and $|P_1/P_2| = p^2$. On the other hand, since $P_2/P_1$ is abelian, $P/P_1$ has positive degree of commutativity by 1.9(c) and $|Z(P_1/P_2)| \geq p^2$. But then by 1.8 $(P_1/P_2)' = P_1/P_2 \leq Z(P_1/P_2)$ and $P_1/P_2$ is of class 2, contradicting (15). Thus $r_2 - r_1 > 2$ and by Lemma 3.2(d) $|P_1| \leq p^p - 2$. Hence, as $n \leq 2p$ by (14), $P_1 \leq P_{n+p} + 2$ and $Z(P_1) \geq P_{p+1}$. Therefore $P_{p+1} \leq P_{p-2} \leq Z(P_1)$. Hence, since $n - 2 \geq p - 1$, $C_{p+1}(p_{p+1}) = P_1$ by 1.10. But $P_{p+1} = \text{U}(P_2)$ by 1.7(a) and $C_G(P_{p+1}) = C_P(P_{p+1})$ by Lemmas 5.1 and 5.3(a). Consequently $P_1 = C_p(P_{p+1}) = C_G(P_{p+1}) \triangle G$; i.e., $P_1 \triangle G$, contradicting the assumption $O_p(G) \neq P_1$. Hence $O_p(G) = P_1$ as required.

9. The Proof of Theorem A

(9.0) We recall some results from Feit’s works [3] and [4].

(a) [3, p. 571] We say that a finite group $G$ is of type $L_2(p)$ if every composition factor is either a $p$ group or a $p'$ group or is isomorphic to $PSL(2, p)$.

(b) [3, p. 571] Let $G$ be a finite group with a cyclic Sylow $p$-subgroup $P$ for some prime $p$. Assume that $G$ is not of type $L_2(p)$. Suppose that $G$ has a faithful indecomposable representation $\mathcal{U}$ of degree $d \leq p$ in a field of characteristic $p$. Then $p \neq 2$, $|P| = p$, $\mathcal{U}$ is indecomposable and $C_G(P) = P \times Z(G)$. Furthermore $d \geq \frac{3}{2}(p - 1)$ and $d \geq \frac{7}{10}p - \frac{1}{2}$ in case $p \geq 13$.

(c) [4, p. 395] Suppose that the following conditions are satisfied:

(i) $p$ is a prime and $|P| = p$ where $P$ is a Sylow $p$-subgroup of the group $G$;

(ii) $C = C_G(P)$, $N = N_G(P)$, $Z = Z(G)$ is cyclic;

(iii) $1 < e = |N:C| < p - 1$, $p - 1 = te$, $N = DC$ with $|D| = e$;

(iv) $C = P \times Z$ and $(|Z|, e) = 1$. Thus $N = PD \times Z$.
Assume that \( e = 2 \), \( G \) is simple and \( G \) has a faithful representation of degree \( d < p \) in a field of characteristic \( p \). Then \( p - 1 = 2^a \) with \( a > 1 \) and \( G \cong \text{PSL}(2, p - 1) \).

(9.1) In this and the next subsection we recall some results from the previous sections and some of their immediate consequences to which we shall refer frequently.

(a) Let \( P \) be a \( p \)-subgroup of maximal class of order \( p^n \), \( n \geq p + 2 \) and let \( 1 \leq i \leq n - 1 \). If \( n - i = (r_i - 1)(p - 1) + u_i, \ 1 \leq u_i \leq p - 1 \) then \( \exp(P_i) = r_i \). \((1.7)\)

(b) Let notations be as in (a) and let \( r = r_1 \). Then \( P_1 \) has two series of characteristic subgroups:

\[
\begin{align*}
(\Omega) & : 1 = \Omega_0 < \Omega_1 < \cdots < \Omega_r = P_1, \quad \Omega_i = \Omega_i(P_1), \ 0 \leq i \leq r; \\
(U) & : U_0 > U_1 > \cdots > U_r = 1, \quad U_i = U_i(P_1), \ 0 \leq i \leq r. 
\end{align*}
\]

(c) \( |\Omega_i(P_j)/\Omega_{i-1}(P_j)| \leq p^{n-1} \) and if \( \Omega_i(P_j) \neq P_j \) then \( |\Omega_i(P_j)/\Omega_{i-1}(P_j)| = p^{n-1}. \)

(d) \( |U_i(P_j)/U_{i+1}(P_j)| \leq p^{n-1} \) and if \( U_{i+1}(P_j) \neq 1 \) then \( |U_i(P_j)/U_{i+1}(P_j)| = p^{n-1}. \)

(e) Let \( 1 \neq K \) be a subgroup of \( P_1 \) which is a characteristic subgroup of \( P \). Then \( K = P_k \) for some \( k, 1 \leq k \leq n - 1 \), by 1.8. Therefore, by (16) there exist natural numbers \( i \) and \( j, 0 \leq i, j \leq r - 1 \), such that \( U_{i+1} \leq K \leq U_i \), \( \Omega_j \leq K \leq \Omega_{j+1} \). Let \( c(K) = \{ \exp(U_i/K), \exp(K/U_{i+1}), \exp(\Omega_{j+1}/K), \exp(K/\Omega_j) \} \). Then \( c(K) \subseteq \{ 0, 1, 2, \ldots, n - 1 \} \) by parts (c) and (d). Let \( \kappa \) be a set of subgroups of \( P_1 \) which are characteristic in \( P \). We shall denote \( c(\kappa) = \bigcup_{K \in \kappa} c(K) \setminus \{ 0 \} \). Then by (c) and (d)

\[
c(\kappa) \subseteq \{ 1, 2, \ldots, n - 1 \} \text{ and if } m \in c(\kappa), \\
\text{then } p - 1 - m \in c(\kappa), \ 1 < m < p - 1. 
\]

In (9.8) we shall see that if \( P \) is a Sylow \( p \)-subgroup of a group \( G \) which is a minimal counterexample to Theorem A and \( \kappa \) is the set of all the normal \( p \)-subgroups of \( G \) then \( c(\kappa) \subseteq \{ 1, p - 2, p - 1 \} \).

Remark. From now on \( \Omega_i \) will stand for \( \Omega_i(P_1) \) and \( U_i \) for \( U_i(P_1) \).

(f) Let notations be as in (e) and let \( \kappa \) be the set of all the nontrivial normal \( p \)-subgroups of \( G \). If \( c(\kappa) \subseteq \{ 1, p - 2, p - 1 \} \) then

(i) If \( P_j/P_i \) is a minimal normal \( p \)-subgroup of \( G/P_j, \ i < j \), and \( |P_j/P_i| = p^e \) then \( e = 1 \) or \( e = p - 2 \) or \( e = p - 1 \).

(ii) \( n \equiv 0 \mod p - 1 \) or \( n \equiv 1 \mod p - 1 \) or \( n \equiv 2 \mod p - 1 \).

(g) Let notations be as in (f) and assume that \( c(\kappa) = \{ 1, p - 2 \} \).
Let $P_t = P_{t_1} > P_{t_2} > P_{t_3} > \cdots > P_{t_k} = 1$ be a series of normal $p$-subgroups of $G$ of maximal length (i.e. $P_t/P_{t+1}$ is minimal normal in $G/P_{t+1}$ for $1 \leq t \leq k - 1$). Denote $h_t = |P_t/P_{t+1}|$, $1 \leq t < k - 1$, and let $(h) = \{h_1, h_2, \ldots, h_{k-1}\}$. Then one of the following occurs

(i) $(h) = \{1, p - 2, 1, p - 2, \ldots, 1, p - 2, 1\}$ and $n \equiv 2 \mod p - 1$;
(ii) $(h) = \{1, p - 2, 1, p - 2, \ldots, 1, p - 2\}$ and $n \equiv 1 \mod p - 1$;
(iii) $(h) = \{p - 2, 1, \ldots, p - 2, 1\}$ and $n \equiv 1 \mod p - 1$;
(iv) $(h) = \{p - 2, 1, p - 2, \ldots, p - 2, 1\}$ and $n \equiv 0 \mod p - 1$.

**Remark.** From now on $K$ denotes the set of nontrivial normal $p$-subgroups of $G$.

(9.2) In the following subsections we shall frequently assume

**HYPOTHESIS II.** (i) $G$ is a finite group with a Sylow $p$-subgroup $P$ of maximal class of order $p^n$, $n \geq 7$;
(ii) $O_p(G) = P_1$;
(iii) $O_p(G) = 1$;
(iv) $P \triangleleft G$;
(v) $N_G(P) \neq PC_G(P)$.

(a) Assume Hypothesis II. Then $N_G(P) = PT$, where $T$ is a cyclic group of order $|T| | p - 1$.

(b) Let notations be as in (a) and assume that $N_G(P) = PT$ and $T = \langle t \rangle$. Then there exist natural numbers $a$ and $b$, $1 \leq a, b \leq p - 1$ such that for every $i$, $1 \leq i \leq n - 1$, $s_i \equiv s_i^{i-1} \mod P_{i+1}$ and $s_i \equiv s_i^a \mod P_2$ (Lemma 3.1(a)).

(c) Let notations be as in part (b) and assume that $N_G(P) = P \cdot T$. Then

(i) $|G/O^p(G)| \geq p$ iff $b = 1$
(ii) $|G/O^p(G)| = p$ then $P_2$ is a Sylow $p$-subgroup of $O^p(G)$ for some $x \in P \backslash P_1$.

**Proof.** By the transfer Theorem (see Sect. 4) suffices to show that $|N_G(P) : O^p(N_G(P))| = p$ iff $b = 1$. Let $N = N_G(P)$. Then $N = P \cdot T$, by part (a). It follows from Burnside's Basis Theorem [6, Th. 3.15 p. 273] that $N/O^p(N) \neq 1$ iff $N/O^p(N) \cdot P_2 \neq 1$. Hence we shall show that $N/O^p(N)P_2 \neq 1$ iff $b = 1$. For every element $u \in N$ and subgroup $X$ of $N$ let $\bar{u} = uP_2$ and $\bar{X} = XP_2/P_2$. Assume that $b = 1$. Then $s_i^a \equiv s_i \mod P_2$; i.e., $|\bar{P}_1, \bar{T}| = 1$. Since $|\bar{S}, \bar{P}_1| = \bar{1}$, this yields $\langle s \rangle \bar{T} \vartriangle \bar{N}$. Consequently $O^p(\bar{N}) \subseteq \langle s \rangle \bar{T}$ and $N/O^p(N) \neq 1$. Assume now that $|N/O^p(N)| = p$. Then $\bar{N}/O^p(\bar{N}) \neq 1$. Let $\bar{K}$ be a normal subgroup of $\bar{N}$ of index $p$. Then $\bar{K} = \langle \bar{x} \rangle \cdot \bar{T}$ for some $\bar{x} \in \bar{P}$. By Maschke's theorem there exists an element $y \in P \backslash P_2$ s.t. $\bar{y}$ is $\bar{T}$-invariant and
\( \bar{P} = \langle \bar{x} \rangle \times \langle \bar{y} \rangle \). Therefore \( [\bar{x}, \bar{T}] \leq \langle \bar{x} \rangle \times \langle \bar{y} \rangle \). Since \( \bar{y} \notin \bar{K} \) and \( \bar{y} \) normalizes \( \bar{K} \), \( (\langle \bar{x} \rangle \bar{T})^\bar{y} = \langle \bar{x} \rangle \bar{T} \). But \( (\langle \bar{x} \rangle \bar{T})^\bar{y} = (\langle \bar{x} \rangle \bar{T})^\bar{y} = (\langle \bar{x} \rangle \bar{T}, \bar{y}) \). Hence \( [T, \bar{T}] \in (\langle \bar{x} \rangle \bar{T} \cap \langle \bar{y} \rangle) = 1 \); i.e., \( [T, \bar{T}] = 1 \). Since \( \bar{y} \in P \), \( \bar{y} = \bar{s}^a \bar{s}^\beta \), \( 0 \leq a, \beta \leq p - 1 \) and not both \( a \) and \( \beta \) are zero. Hence \( \bar{y} = \bar{s}^a \bar{s}^\beta = \bar{s}^{a\beta} \); i.e., \( a\beta = a, b\beta = \beta \). If \( a \neq 0 \) then \( a = 1 \) and \( G \) has a normal \( p \)-complement (note that \( |a| = |T| \)) by the Transfer Theorem, contradiction to Hypothesis II therefore \( a = 0, \beta = 0 \). Consequently \( b = 1 \), and \( s, s^i \in O^p(G) \) which proves part (ii) too.

\((9.3)\) Lemma. Assume Hypothesis II and let \( P_i \) and \( P_j \) be normal \( p \)-subgroups of \( G \). \( i < j \). Suppose that \( V_0 = P_i/P_j \) is elementary abelian and regard \( V_0 \) as a vector space of dimension \( d \) over \( F_p \). \((d = j - i)\). Then

(a) \( d \leq p - 1 \).

(b) \( d > 1 \) and \( G/P_1 \) acts on \( V_0 \) then it acts faithfully and indecomposably on \( V_0 \).

(c) \( n \geq p + 2 \) and \( 1 < d \) then there exists a natural number \( t \) such that \( i + t \leq j \) and \( G/P_1 \) acts faithfully and indecomposably on \( V = P_i/P_{i+t} \).

(d) \( P_0/P_i \) is a minimal normal subgroup of \( G/P_j \) and \( j > i + 2 \) then \( G/P_1 \) acts faithfully and indecomposably on \( V_0 = P_0/P_j \).

(e) \( d = 1 \) then either \( G/P_1 \) acts trivially on \( V_0 \), in which case \( a^{i-1}b = 1 \), or \( G \) has a normal subgroup \( G_0 \) such that \( G_0 \supseteq P \) and \( G_0/P_1 \) acts trivially on \( V_0 \) and \( G/G_0 \) is isomorphic to \( T/\langle t' \rangle \) for some \( r \) (here \( \langle t \rangle = T \)). Furthermore, if \( |T| \) is prime then \( G_0 = G \) and \( G/P_1 \) acts trivially on \( V_0 \).

(f) \( P_i \) and \( P_{i+1} \) are normal \( p \)-subgroups of \( G \) then \( P_{i+2} \) cannot be normal in \( G \), provided \( i \leq n - 2 \) and \( n \geq p + 2 \).

Proof. (a) Since \( \Omega(P_i) = P_{i+p-1} \) by 9.1(d). \( P_{i+p-1} \leq \Phi(P_i) \). Consequently

\[ |P_i/P_j| \leq |P_i/\Phi(P_i)| \leq |P_i/P_{i+p-1}| \leq p^{p-1}; \]

i.e., \( |P_i/P_j| \leq p^{p-1} \).

(b) \( N/P_1 \) be the kernel of the action of \( G/P_1 \) on \( V_0 \). Then \( [N, P_1] \leq P_j \) and by Lemma 5.2 \( NP = O_p((NP)P) \). Hence \( N \leq P_j \), by Hypothesis II(iii). Since \( P = N \) implies \( P \triangle G \), contradicting Hypothesis II(iv), we conclude \( N = P_1 \) and \( G/P_1 \) acts faithfully on \( V_0 \). It follows easily from the way \( s \) acts on \( V_0 \) that \( V_0 \) is a cyclic indecomposable \( G/P_1 \)-module.

(c) \( [P_j, P_1] = P_k \) and define \( t \) as follows: \( t = j - i \) if \( k \geq j \) and \( t = k \) otherwise. Then \( t \geq 2 \) by 1.5(a) and \( G/P_1 \) acts on \( V = P_i/P_{i+t} \) faithfully and indecomposably, by part (b).

(d) Since \( [P_j, P_1] \triangle G \), \( P_i^{l+t} = P_j \) by the definition of \( t \) in part (c). Hence (d) follows from part (c).
(e) If \( G/P_i \), acts trivially on \( P_i/P_{i+1} \) then in particular \( s_i \equiv s_i \mod P_{i+1} \), where \( t \) is defined in 9.2(a). Hence \( a^{t-1}b \equiv 1 \), by 9.2(b). If \( G/P_i \) acts nontrivially then there exists a normal subgroup \( G_0 \) of \( G \) such that \( G_0/P_{i+1} = C_{G/P_{i+1}}(P_i/P_{i+1}) \). Since \( P \leq G_0 \), \( G = G_0 N_{G_0}(P) \) by the Frattini argument and \( G/G_0 = G_0 N_{G_0}(P)/N_{G_0}(P) = PT/PT_0 \) by 5.2(a), where \( T_0 \) is a subgroup of \( T \). Hence \( G/G_0 = T/T_0 \) and \( G_0 \triangleright G \setminus T \) for a natural \( r \). Assume that \(|T| \) is prime and \( G \neq G_0 \). Then \( G/G_0 \cong T \) and \( G_0 \cap T = 1 \). Hence \( N_{G_0}(P) = P \) and \( G_0 \) has a normal \( p \)-complement by the Transfer Theorem (see Sect. 4). Hence \( G_0 = P \), by Hypothesis II(iii) and \( G \) violates Hypothesis II(iv).

(f) By part (e) there exist normal subgroups \( G_1 \) and \( G_2 \) in \( G \) such that \( P \leq G_j \), \( j = 1, 2 \), \( G_j/P_i \) acts trivially on \( P_i/P_{i+1} \) and \( G_2/P_i \) acts trivially on \( P_{i+1}/P_{i+2} \). Let \( G_3 = G_1 \cap G_2 \). Then \( G_3 \vartriangleleft G \) and \( P \leq G_3 \). Since \( G_3/P_i \) acts trivially on both \( G_i/G_{i+1} \) and \( G_{i+1}/G_{i+2} \), either \( G_3 \) acts trivially on \( G_i/G_{i+2} \), which contradicts part (b), or \( G_3 = P \), which implies \( P \triangle G \), contradicting Hypothesis II(iv).

(9.4) Proposition. Assume Hypothesis II and suppose that \( P_1 \) is not abelian. Then \( p \geq 5 \).

Proof. Assume \( p = 3 \). Then \( P_2 \) is abelian and \(|P'_1| = p \) or \(|P'_1| = p^2 \) by 1.11 and 1.9(a) and the supposition \( P'_1 \neq 1 \). Since \( P_1 \vartriangle G \) by Hypothesis II(ii), \( \Omega_1 \vartriangle G \) and \( P'_1 \vartriangle G \). Hence in the first case \( P_{n-2} = \Omega_1 \vartriangle G \) and \( P_{n-1} = P'_1 \vartriangle G \), in contrast with Lemma 9.3(f). Therefore \( P'_1 = P_{n-2} \). But then \( J(P_1) = P_2 \) by Lemma 3.4 (or direct calculation). Hence \( P_2 \triangle G \). Since \( \Omega_1 = P_3 \), by 9.1(c), \( P_2 \triangle G \). But then \( P_1 \triangle G \), \( P_2 \triangle G \) and \( P_3 \triangle G \), contradicting Lemma 9.3(f). Consequently \( p > 3 \).

(9.5) Proposition. Assume Hypothesis II and the additional suppositions:

(i) \( K_3(P_1) U_1(P'_1) = 1 \);
(ii) \(|P'_1| = p^{p-1} \);
(iii) \( c(\kappa) \subseteq \{1, p - 2, p - 1\} \).

Then \( J(P_1) = P_2 \).

Proof. By supposition (ii) \( P_1 \) is nonabelian, hence by Proposition 9.4 \( p \geq 5 \). Assume first \( p = 5 \). By Hypothesis II(v) \( P \) has a nontrivial automorphism of order prime to \( p \). Hence we may apply Lemma 3.4 to \( P \). It follows from Lemma 3.4 and suppositions (i) and (ii) that \( p \leq |P_1/J(P_1)| \leq p^3 \). Hence by assumption (iii) either \( J(P_1) = P_2 \) or \( J(P_1) = P_4 \). Assume \( J(P_1) \neq P_2 \). Then \( J(P_1) = P_4 \) and since \( P_1 \triangle G \) by Hypothesis II(ii), \( P_4 \triangle G \). This implies \(|P_1, P_4| \triangle G \). By Lemma 3.2(b) \( P \) has
degree of commutativity \( n - 4 - 3 = n - 7 \) if \( n \geq 8 \) and degree of commutativity 1 if \( n = 7 \), by 1.5(a). Hence \([P_1, P_4] = P_j\), where \( n - 2 \leq j \leq n \), and by supposition (iii) \( j = n \) or \( j = n - 1 \). If \( j = n \) then \( P_4 \leq Z(P_1) \) and \( P_4(s_1) \) is an abelian subgroup of \( P_1 \) of order greater than \( J(P_1) \), a contradiction. Hence \( j = n - 1 \). Now, by supposition (ii) and Lemma 3.2(a) \([s_1, s_2] \equiv s_{n-4}^a \mod P_{n-3}, a \in \mathbb{Z}, 1 \leq a \leq 4\). Hence by 9.2(b) \( bab \equiv a^{-1}b \mod 5 \); i.e., \( b \equiv a^{-2} \mod 5 \) (\( a^4 = b^5 = 1 \)). Similarly \([P_1, P_4] = P_{n-1} \) implies \([s_1, s_2] = s_{n-1}^a \), \( \beta \equiv a_\beta \in \mathbb{Z}, 1 \leq \beta \leq 4 \) and by 9.2(b) \( ba^2b \equiv a^{-2}b \mod 5 \), i.e. \( b \equiv a^{-1} \mod 5 \). But then \( a^{-1} \equiv b \equiv a^{-2} \mod 5 \); i.e., \( a = 1 \). Hence, \( N_G(P) = PC_G(P) \), contradicting Hypothesis II(v). Consequently \( J(P_1) = P_2 \). Assume now that \( p > 5 \). Again, by Lemma 3.4 \( p \leq |P_j/J(P_1)| < p^{(p+1)/2} \). Since \( p > 5 \), \( (p + 1)/2 < p - 2 \). Hence by supposition (iii) \( |P_j/J(P_1)| = p \) and \( J(P_1) = P_2 \), as required.

(9.6) Proposition. Assume Hypothesis II and the additional supposition: \( c(\kappa) \subseteq \{1, p - 2, p - 1\} \). Then \(|P_1| \neq p\).

Proof. Assume \(|P_1| = p\). Then \( P_1 \) is non-abelian, hence by Proposition 9.4, \( p \geq 5 \). Therefore \( 2 < p - 2 \) and by our supposition:

\[
2 \notin c(\kappa).
\]

By Hypothesis II(v) \( P \) has a non-trivial automorphism of order prime to \( p \). Hence by Lemma 3.2(a) \( P_2 = 1 \) and \( Z(P_1) = P_3 \). Since \( P_1 \triangle G \) by Hypothesis II(ii), \( P_3 = Z(P_1) \triangle G \). Therefore \( P_3 \in \kappa \). Thus \(|P_1/P_3| = p^2\) implies \( 2 \in c(K) \), violating (\( \ast \)). Hence \(|P_1| \neq p\).

(9.7) Proposition. Assume Hypothesis II and suppose that \( G \) satisfies the following:

(i) \(|P_1| = p^{n-2}\);

(ii) \( c(K) = \{1, p - 2\}\);

(iii) \( P_2 \triangle G \).

Then \( G \) has a normal subgroup \( G_0 \) such that \( P \leq G_0 \), \( G_0/G_0 \) is cyclic of order dividing \(|N_G(P)/PC_G(P)|, |N_{G_0}(P)/PC_{G_0}(P)| = 2 \) and either \( p = 2^m + 1 \) and \( G_0/P_1 \cong \text{PSL}(2, 2^m) \) or \( G_0 \) has a maximal normal subgroup \( M \) s.t. \(|M|, p) = 1 \) and \( M \neq O^p(G_0) \).

Proof. We recall from 9.1 that because of suppositions (i), (ii) and (iii) only case (iv) of 9.1(g) may occur. Thus

\[
h = (p - 1, p - 2, 1, p - 2) \text{ and } n \equiv 0 \mod p - 1.
\]
Statement 1.8 with supposition (i) yield \( P'_1 = P_{n-p+3} \) and \( [s_1, s_2] \equiv s_{n-p+2}^e \mod P_{n-p+2}, 1 \leq e \leq p - 1 \), by Lemma 3.2(a). Therefore by 9.2(b) \( bab = a^{n-p+1}b \mod p \) and \( b \equiv a^{n-p} \mod p \). Hence by (**) 

\[ b = a^{-1}. \]  

(***)

It follows from (**) that \( P_{p-1} \triangle G \) and \( P_p \triangle G \). Hence by Lemma 9.3(e) there exists a normal subgroup \( G_0 \) in \( G \) s.t. \( P \leq G_0 \), \( G/G_0 \cong T/(t^r) \), \( N_{G_0}(P) = P(t^r) \), for a certain natural number \( r \). Let \( t_0 = t^r \), \( a_0 = a' \) and \( b_0 = b' \). Then it follows from 9.2(b) that \( s_i^{t_0} \equiv s_i^{a^{-1}b_0} \mod P_{i+1}, 1 \leq i \leq n - 1 \) and by Lemma 9.3(e) \( a_0^{-2}b_0 = 1 \). But \( b_0 = a_0^{-1} \) by (***). Hence either \( |a_0| = |b_0| = 2 \) or \( a_0 = b_0 = 1 \). Since \( a_0 = 1 \) would violate Hypothesis II(v) we must have

\[ |a_0| = |b_0| = 2, \quad |T_0| = 2, \quad T_0 = \langle t_0 \rangle. \]  

(*****)

Thus \( G_0/P_1 \) has an irreducible representation of degree \( p - 2 \) over \( \mathbb{Z}_p \), \( (P_1/P_{p-1}) \), \( G_0/P_1 \) has a Sylow \( p \)-subgroup of order \( p \) with a normalizer of order \( 2p \) and centralizer of order \( p \). Hence by a theorem of Feit [3, p. 335] (see 9.0(c)) either \( G_0/P_1 \) is not simple or \( p = 2^{m+1} \) and \( G_0/P_1 \cong PSL(2, 2^m) \). Assume \( G_0/P_1 \) is not simple and for every subgroup \( H \) of \( G_0 \) let \( \bar{H} = HP_1/P_1 \). Let \( \bar{M} \) be a maximal normal subgroup of \( \bar{G}_0 \). If \( p \mid |\bar{M}| \) then by the Frattini argument \( \bar{G}_0 = \bar{M} \cdot N_{\bar{G}_0}(\bar{P}) \). Hence, as \( \bar{M} \neq \bar{G}_0 \), \( N_{\bar{G}_0}(\bar{P}) \leq \bar{M} \). By 9.2(a) \( N_{\bar{G}_0}(\bar{P}) = \bar{P}T_0 \). Therefore, as \( |T_0| = 2 \) by (*****) this implies \( N_{\bar{G}_0}(\bar{P}) = \bar{P} \) and \( \bar{M} \) has a normal \( p \)-complement by the Transfer Theorem (see Sect. 4). Since \( O_m'(G) = 1 \) by Hypothesis II(iii), this implies \( M = \bar{P} \). Consequently \( P \triangle G \), contradicting Hypothesis II(iv). Hence \( (p, |M|) = 1 \) and as \( b_0 \neq 1 \) by (*****), \( M \neq G_0 \) implies \( M \neq O_p(G_0) \), by 9.2(c).

(9.8) Hypothesis III. \( G \) is a finite group with a Sylow \( p \)-subgroup \( P \) of maximal class and order \( p^n, n \geq p + 3 \), which is a minimal counterexample to Theorem A.

**Proposition.** Assume Hypothesis III. Then

(a) Hypothesis II holds;
(b) \( G/P_1 \) is not of type \( L_2(p) \) in the sense of Feit (see 9.0);
(c) \( c(\kappa) \subseteq \{1, p - 2, p - 1\} \).

**Proof.** (a) Parts (ii) and (iii) of Hypothesis II follow from Lemma 5.1(a) and Theorem 8.3 while (iv) and (v) follow from Theorem A(a) and the Transfer Theorem, respectively.

(b) By part (a) \( P_1 \triangle G \). Hence \( G/P_1 \) makes sense. We propose to show that if \( G/P_1 \) would be of type \( L_2(p) \) then \( G \) would have a (normal) subgroup
$H$ of index 2 in $G$ such that $P_1 \leq H$ and $H/P_1 \cong PSL(2, p)$. However this violates Hypothesis II(ii) since in this case $G$ satisfies (5) of Theorem A(b).

Thus assume that $G/P_1$ is of type $L_2(p)$. Then $G/P_1$ has a normal subgroup $H/P_1$, which is a $p'$-group, a $p$-group or is isomorphic to $PSL(2, p)$. Obviously $H/P_1$ is not a $p$-group. For every subgroup $X$ of $G$ let $X = XP_1/P_1$ and assume $H$ is a $p'$-group. $H = H_0P_1$, $H_0 \cap P_1 = 1$ ($H_0$ is a Hall $p'$-subgroup of $H$). We may assume that $H_0 \cap N_G(P) = 1$, otherwise $[H_0 \cap N_G(P), P] \leq H_0 \cap N_G(P) \cap P = 1$ and $H_0 \cap N_G(P) \leq C_G(P) \leq C_G(P_1) \leq P_1$. Hence $N_H(P) = P_1$. Now, $PH$ is a finite group with a Sylow $p'$-subgroup $P$ of maximal class of order $p^{n}+3$ and by the Transfer Theorem $PH$ has a normal $p$-complement $Q$. $(N_H(P) = P_1)$. $P_1 \cap PH \Rightarrow [Q, P_1] \leq Q \cap P_1 = 1$ and $Q \leq C_G(P_1) \leq P_1$; i.e., $Q = 1$ and $H = 1$. Hence $H \cong PSL(2, p)$. By the Frattini argument $G = N_G(P) \cdot H$ and $G/H = |N_G(P)/N_H(P)| \leq 2$. Consequently $G/P_1$ is not of type $L_2(p)$.

(c) It follows from (17) that if $1 < m < p - 2$ than there exists an $m'$ such that $m' \in c(K)$ and $1 < m' < 2/3(p - 1)$. Therefore, by part (a) $G/P_1$ acts faithfully and indecomposably on an elementary abelian section of $P_1$ of dimension $d$, $1 < d < m' < 2/3(p - 1)$, by Lemma 9.3(c). But this violates Feit's Theorem in [3] (see 9.0(b)) by part (b). Hence $c(\kappa) \subseteq \{1, p - 2, p - 1\}$.

(9.9) Proposition. Assume Hypothesis III. Then

(a) $P_1$ is not abelian;
(b) $P_2 \triangle G$;
(c) $P_2$ is abelian;
(d) $P_1$ is of class 2;
(e) $G$ has a normal subgroup $G_0$ such that $|G/G_0| | N_0(P)/PC_0(P) |$ and $|G : O^P(G_0)| = p$;
(f) $c(\kappa) \subseteq \{1, p - 2\}$.

Proof. (a) By Proposition 9.8(c) $c(\kappa) \subseteq \{1, p - 2, p - 1\}$. Hence by $9.1(f)(i) n = 0 \mod p \quad 1$ or $n = 1 \mod p \quad 1$ or $n = 2 \mod p \quad 1$. If $P_1$ is abelian then we must have $n = 2 \mod p - 1$, for in case $n \equiv 0 \mod p - 1$ or $n \equiv 1 \mod p - 1$ conclusion (b) of Theorem A holds for $G$, contradicting Hypothesis III, by Lemma 9.3(b) and Feit [4, p. 395] (see 9.0(c)). Hence $n = 2 \mod p - 1$. So by 9.1(a) $|P_1/\Omega_{r-1}| = p$ and $|\Omega_{r-1}| = p$; i.e., $\Omega_{r-1} = P_2$ and $\mathcal{U}_{r-1} = P_{n-1}$. Consequently $P_2 \triangle G$ and $P_{n-1} \triangle G$. Therefore it follows from Lemma 9.4(e) that there exists a normal subgroup $G_0$ in $G$ such $G_0$ centralizes $P_{n-1}$ and $P_1/P_2$ and $G/G_0 \cong T/\langle t_0 \rangle$. Therefore $P_{n-1} \leq Z(G_0)$. Let $N_{G_0}(P) = P \cdot T_0$, $T_0 \leq T$ and let $s^{0} = s^{a_0} \mod P_2$, $s^{b_0} = s^{b_0} \mod P_2$, $\langle t_0 \rangle = T_0$. (see 9.2(a) and 9.2(b)). Then $b_0 = 1$ by 9.2(b), as $G_0/P_1$ acts trivially on
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\[ P_1/P_2. \] Hence by 9.2(c) \(|G_0/O^p(G_0)| \geq p. \] So conclusion (b) \( \beta \) of Theorem A holds for \( G \), contradicting Hypothesis III. Consequently \( P' \neq 1. \)

(b) Let \( N = K_3(P_1) \cup_1(P_1') \) and for every subgroup \( X \) of \( G \) let \( \bar{X} = XN/N. \) Then by 9.1(d) \(|\bar{P}_1| \leq p^{p-1}\), by part (a) \( P_1 \) is of class 2 and by Proposition 9.8(c) \( c(\kappa) \leq \{1, p - 2, p - 1\}. \) Hence by 9.1(c)(ii) \(|\bar{P}_1| = p \) or \(|\bar{P}_1| = p^{p-2} \) or \(|\bar{P}_1| = p^{p-1} \). By Proposition 9.8 all the assumptions of Propositions 9.5 and 9.6 are satisfied. Therefore \(|\bar{P}_1| \neq p \) by Proposition 9.6 and if \(|\bar{P}_1| = p^{p-1} \) then \( P_1 = \bar{P}_1 \triangle \bar{G} \) by Proposition 9.5. Hence we may assume that \(|\bar{P}_1| = p^{p-2} \) and \( P_2 \triangle \bar{G} \). Then all the assumptions of Proposition 9.7 are satisfied, hence \( G \) has a normal subgroup \( G_0 \) such that \( P \leq G_0, \) \( G/G_0 \) is cyclic of order dividing \(|N_0(G)/PC_0(G)| \) \(|N_0(G)/PC_0(G)| = 2 \) and either \( p = 2^{m+1}, \) and \( G_0/P \leq PSL(2, 2^{m}) \) or \( G_0 \) has maximal normal subgroup \( M \) s.t. \((|M|, p) = 1 \) and \( M \neq O^p(G_0). \) By Hypothesis III only the second possibility may occur. But then \( G_1 = MP \) is a proper subgroup of \( G \) which contains \( P. \) Hence \( G_1 \) satisfies one of the conclusions of Theorem A. Since \( P_1 \) is nonabelian by part (a) and \( P_2 \triangle G \) by assumption, \( G_1 \) satisfies conclusion (a) of Theorem A. Hence \( G_1 = O_p(G_1)N_0(G_1). \) Since \( M \triangle G_0 \triangle G \), we have \( O_p(M) \leq O_p(G) = 1; \) i.e., \( O_p(M) = 1. \) But obviously \( O_p(G_1) \leq M. \) Hence \( O_p(G_1) \leq O_p(M) = 1 \) and \( G_1 = N_0(G_1). \) Since \( N_0(G_1) = P \cdot T_0, \) \( |T_0| = 2, \) \( G_0/P \). \( G_1 = P \cdot T_0 = N_0(G_1). \) However we cannot have \( G_1 = P \cdot T_0 \) for then \( M = P_1 \cdot T_0 \) and as \( M \triangle G_0, \) \((|M|, p) \leq |M \cap G_0| = 1 \) contradicting \( |P_1 \cdot T_0 | = P \) (see 9.2(b)). This means that the second conclusion of Proposition 9.7 does not occur. Consequently \( P_2 \triangle G \), as required.

(c) Let \( N = K_3(P_2) \cup_1(P_2') \) and for every subgroup \( X \) of \( G \) let \( \bar{X} = XN/N. \) (\( \bar{X} \) makes sense by part (b).) Assume that \( P_2 \) is not abelian. Then \( P_2 \) is of class 2 and by Proposition 9.8(c) \( c(\kappa) \leq \{1, p - 2, p - 1\}. \) Hence by 9.1(c)(ii) \(|\bar{P}_1| = p^{p-2} \) or \(|\bar{P}_1| = p^{p-1} \). By Proposition 9.8 all the assumptions of Propositions 9.5 and 9.6 are satisfied. Therefore \(|\bar{P}_1| \neq p \) by Proposition 9.6 and if \(|\bar{P}_1| = p^{p-1} \) then \( P_1 = \bar{P}_1 \triangle \bar{G} \) by Proposition 9.5. Hence we may assume that \(|\bar{P}_1| = p^{p-2} \) and \( P_2 \triangle \bar{G} \). Then all the assumptions of Proposition 9.7 are satisfied, hence \( G \) has a normal subgroup \( G_0 \) such that \( P \leq G_0, \) \( G/G_0 \) is cyclic of order dividing \(|N_0(G)/PC_0(G)| \) \(|N_0(G)/PC_0(G)| = 2 \) and either \( p = 2^{m+1}, \) and \( G_0/P \leq PSL(2, 2^{m}) \) or \( G_0 \) has maximal normal subgroup \( M \) s.t. \((|M|, p) = 1 \) and \( M \neq O^p(G_0). \) By Hypothesis III only the second possibility may occur. But then \( G_1 = MP \) is a proper subgroup of \( G \) which contains \( P. \) Hence \( G_1 \) satisfies one of the conclusions of Theorem A. Since \( P_1 \) is nonabelian by part (a) and \( P_2 \triangle G \) by assumption, \( G_1 \) satisfies conclusion (a) of Theorem A. Hence \( G_1 = O_p(G_1)N_0(G_1). \) Since \( M \triangle G_0 \triangle G \), we have \( O_p(M) \leq O_p(G) = 1; \) i.e., \( O_p(M) = 1. \) But obviously \( O_p(G_1) \leq M. \) Hence \( O_p(G_1) \leq O_p(M) = 1 \) and \( G_1 = N_0(G_1). \) Since \( N_0(G_1) = P \cdot T_0, \) \( |T_0| = 2, \) \( G_0/P \). \( G_1 = P \cdot T_0 = N_0(G_1). \) However we cannot have \( G_1 = P \cdot T_0 \) for then \( M = P_1 \cdot T_0 \) and as \( M \triangle G_0, \) \((|M|, p) \leq |M \cap G_0| = 1 \) contradicting \( |P_1 \cdot T_0 | = P \) (see 9.2(b)). This means that the second conclusion of Proposition 9.7 does not occur. Consequently \( P_2 \triangle G \), as required.
the Transfer Theorem which again would imply $H \triangleleft G$. Thus $G_1$ satisfies Hypothesis II and of course $c(\kappa_0) \subseteq \{1, p - 2, p - 1\}$, where $\kappa_0$ is the set of all the normal $p$-subgroups of $G_1$. Hence, as in part (b), $|H_1'| = p$ or $|\bar{H_1}'| = p^{p-2}$ or $|\bar{H_1}'| = p^{p-1}$. Since either $P_2$ is abelian and $|P_1'| = p$, or $|P_1' : P_1''| \geq p^2$, by Lemma 3.2(a) and since $c(\kappa) \subseteq \{1, p - 2, p - 1\}$ we must have $|P_1' : P_2'| \geq p^i$. Hence by Lemma 3.2(d) $|P_1'| \leq p^{p-2}$. But then $|P_2'| \leq p^{p-4}$ forces $|P_2'| = 1$ or $|P_2'| = p$. However, as $G_0$ satisfies Hypothesis II, $|\bar{H_1}'| \neq p$ by Proposition 9.6. Hence $|\bar{H_1}'| = 1$ and $P_2$ is abelian.

(d) Since $P_2$ is abelian, $|\bar{P_1}'| \leq p^{p-2}$ by 1.9(a). Therefore $P$ has degree of commutativity $k = n - p - 4$ by 1.9(b) and $K_3(P_1) = [P_1', P_1] \leq [P_{n-p+2}P_1'] \leq P_{n-p+2n-p+1} = P_{2(n-p+1)}$. By 9.1(f)(ii) $n \equiv 0 \mod p - 1$ or $n \equiv 1 \mod p - 1$ or $n \equiv 2 \mod p - 1$. Since $n \geq p + 3$ this means that $n \geq 2p - 2$ and $K_3(P_1) \leq P_n = 1$. Thus $P_1$ is of class 2.

(e) Follows from Lemmas 9.3(e) and 9.2(c).

(f) Follows from Proposition 9.8(c).

(9.10) Conclusion. Assume that Theorem A is false. Then there exists a finite group $G$ with a Sylow $p$-subgroup $P$ of maximal class of order $p^n$, $n \geq p + 3$, which is a minimal counterexample for Theorem A; i.e., $G$ satisfies Hypothesis III. By Proposition 9.9 $G$ satisfies conclusion (6) of Theorem A(b). This obviously shows that Theorem A holds.

Remarks. (a) If we require

(i) If $P_2$ is abelian and $|P_2| \leq p^{p-2}$ then $|P_2| \leq p^{2/3(p-1)}$;

(ii) $P \neq C_p \wr C_p$;

then Theorem A remains true for $n < p + 3$ with the additional possibility: $O_{p'}(G)P_2 \triangleleft G$ and $P_1/\Phi(P_2)$ is of class 3 at least.

(b) Let $G$ be a group of linear transformations of a vector space $V$ of dimension $d \leq p - 1$ over $\mathbb{F}_p$ with a Sylow $p$-subgroup $P$ of order $p$. If $G$ acts indecomposably on $V$ then $V : G$ has a Sylow $p$-subgroup $P$ of maximal class with $P_1$ abelian and $N_G(P)/P \cdot C_G(P)$ cyclic of order dividing $p - 1$. Hence the determination of the structure of groups with $P$ of maximal class and $P_1$ abelian is as difficult as the problem of determining the structure of indecomposable linear groups with a Sylow $p$-subgroup of order $p$.

(c) Theorem A is true for other families of Sylow $p$-subgroups $P$ of a finite group $G$ such that $P$ is generated by two elements and has a cyclic lower central series (See [7]). It is interesting that for these families only part (a) of Theorem A holds.
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