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Calogero–Moser Space and Kostka Polynomials

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We consider the canonical map from the Calogero–Moser space to symmetric powers of the affine line, sending conjugacy classes of pairs of $n \times n$-matrices to their eigenvalues. We show that the character of a natural $\mathbb{C}^*$-action on the scheme-theoretic zero fiber of this map is given by Kostka polynomials. A similar result is proved for a cyclic version of the Calogero–Moser space. © 2002 Elsevier Science (USA)

1. INTRODUCTION

1.1. The aim of this paper is to prove a refined version of Conjecture 17.14 of [EG]. To explain our result, recall the so-called *Calogero–Moser space* $\mathcal{C}_n$, a $2n$-dimensional complex algebraic manifold introduced by Kazhdan–Kostant–Sternberg [KKS], and studied further by Wilson [W]. It is defined as $\mathcal{C}_n := \mathcal{CM}_n /\!\!/ \mathbf{PGL}_n$, the quotient by the natural (free) conjugation-action of the group $\mathbf{PGL}_n$ on the set

$$\mathcal{CM}_n := \{(X, Y) \in \mathsf{Mat}_n \times \mathsf{Mat}_n \mid [X, Y] + \mathsf{Id} = \text{rank } 1 \text{ matrix}\}. \quad (1.1)$$

Let $\mathbb{A}^{(n)}$ denote the space of unordered n-tuples of complex numbers. The assignment $(X, Y) \mapsto (Spec(X), Spec(Y))$, sending a pair of $(n \times n)$-matrices to the corresponding pair of n-tuples of their eigenvalues gives a map $p : \mathcal{C}_n \to \mathbb{A}^{(n)} \times \mathbb{A}^{(n)}$. The zero fiber $p^{-1}(0, 0)$ of this map is formed by the

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conjugacy classes of *nilpotent* pairs $(X, Y) \in \mathcal{CM}_n$. This fiber is known to be a finite set labelled naturally by partitions of n. Given such a partition λ, let $p^{-1}(0,0)_\lambda$ be the corresponding point in the zero fiber.

Conjecture 17.14 of [EG] states that, for any partition λ, the corresponding point in the (scheme-theoretic) zero fiber of p comes with *multiplicity* equal to $(\mathit{dim}\, V_\lambda)^2$, where V_λ is an irreducible representation of the symmetric group $\mathfrak{S}_n$ attached to the partition λ in the standard way, see [M].

1.2. In the present paper, we propose and prove the following q-analogue of the above conjecture. Observe that the complex torus $\mathbb{C}^*$ acts naturally on $\mathcal{CM}_n$ by $z : (X, Y) \mapsto (z^{-1} \cdot X, z \cdot Y)$, $\forall z \in \mathbb{C}^*$. This $\mathbb{C}^*$-action descends to the Calogero–Moser space $\mathcal{C}_n$ and preserves the zero fiber $p^{-1}(0,0)$. Now given λ, a partition of n, let $p^{-1}(0,0)_\lambda$ be the corresponding irreducible component of the zero fiber viewed as a *nonreduced* scheme (set theoretically concentrated at one point). The $\mathbb{C}^*$-action keeps these points (set theoretically) fixed hence, for each λ, induces a $\mathbb{C}^*$-action on the coordinate ring of the scheme $p^{-1}(0,0)_\lambda$, a finite-dimensional vector space. The character of this finite-dimensional $\mathbb{C}^*$-module may be viewed as a Laurent polynomial $ch_\lambda \in \mathbb{Z}[q, q^{-1}]$. Now, recall that for each partition λ one defines the *Kostka polynomial* $\mathsf{K}_\lambda(q) \in \mathbb{Z}[q]$ which is a certain q-analogue[2] of $\mathit{dim}\, V_\lambda$, the dimension of the corresponding irreducible $\mathfrak{S}_n$-representation, see e.g. [M, III.6].

Our result reads

THEOREM. *For any partition λ (of n), we have*: $ch_\lambda = \mathsf{K}_\lambda(q) \cdot \mathsf{K}_\lambda(q^{-1})$.

1.3. This result has a natural generalization to other finite complex reflection groups W in a vector space $\mathfrak{h}$. In more details, in [EG] the authors associate to a pair $(\mathfrak{h}, W)$ a *Calogero–Moser space* $\mathcal{C}_W$ together with a finite map $p : \mathcal{C}_W \to \mathfrak{h}/W \times \mathfrak{h}/W$. In the special case $\mathfrak{h} = \mathbb{C}^n$ and $W = \mathfrak{S}_n$, the space $\mathcal{C}_W$ reduces to the variety $\mathcal{C}_n$, and the map $p : \mathcal{C}_W \to \mathfrak{h}/W \times \mathfrak{h}/W$ reduces to the map $p : \mathcal{C}_n \to \mathbb{A}^{(n)} \times \mathbb{A}^{(n)}$ considered above.

More generally, in this paper we will consider the case where $\mathfrak{h} = \mathbb{C}^n$ and $W = \Gamma \sim \mathfrak{S}_n$ is a wreath product of $\mathfrak{S}_n$ and $\Gamma = \mathbb{Z}/N\mathbb{Z}$, a cyclic group of some fixed order N (thus, $W = \mathfrak{S}_n \ltimes (\mathbb{Z}/N\mathbb{Z})^n$), acting naturally in $\mathfrak{h}$. It has been proved in [EG] that the corresponding Calogero–Moser space $\mathcal{C}_{\Gamma,n} := \mathcal{C}_W$ is a smooth affine algebraic variety isomorphic to a certain Nakajima's Quiver variety for a cyclic quiver.

[2] In the main body of the paper, we use a minor modification $K_\lambda(q)$ of the standard $\mathsf{K}_\lambda(q)$ possessing a property $K_\lambda(q) \cdot K_\lambda(q^{-1}) = \mathsf{K}_\lambda(q) \cdot \mathsf{K}_\lambda(q^{-1})$.

Conjecture 17.14 of [EG] states that the reduced fiber of p over $(0,0) \in \mathfrak{h}/W \times \mathfrak{h}/W$ can be identified with the set $\mathsf{Irrep}(W)$ of isomorphism classes of irreducible representations of W, and the multiplicity of the point in this fiber corresponding to $\rho \in \mathsf{Irrep}(W)$ equals $(dim\,\rho)^2$.

It is well known that the irreducible $\Gamma \sim \mathfrak{S}_n$-modules are naturally parametrized by the set $\mathfrak{P}_\Gamma(n)$ of $\Gamma^\vee$-partitions of n, see e.g. [M, Part I, Appendix B]. Here $\Gamma^\vee$ is the set of irreducible characters of Γ, and a $\Gamma^\vee$-partition Λ is a collection $(\lambda_\chi,\ \chi \in \Gamma^\vee)$ of ordinary partitions such that $\sum_\chi |\lambda_\chi| = n$. It is known that the points of reduced fiber of $\mathscr{C}_{\Gamma,n}$ over $(0,0)$ are also naturally numbered by $\mathfrak{P}_\Gamma(n)$ (in case of trivial Γ it was proved in [W], and in the general case in [K2]). By abuse of notation we will denote the point in the fiber corresponding to $\Lambda \in \mathfrak{P}_\Gamma(n)$ by Λ as well.

1.4. The cyclic Calogero–Moser space $\mathscr{C}_{\Gamma,n}$ has a natural $\mathbb{C}^*$-action, such that its fixed point set $\mathscr{C}_{\Gamma,n}^{\mathbb{C}^*}$ coincides with the reduced zero fiber. We consider the character of induced $\mathbb{C}^*$-action in the Artin coordinate ring $\mathscr{O}_\Lambda$ of the component $p^{-1}(0,0)_\Lambda$ of the fiber concentrated at the point $\Lambda \in \mathscr{C}_{\Gamma,n}^{\mathbb{C}^*}$.

1.5. For an arbitrary cyclic group $\Gamma = \mathbb{Z}/N\mathbb{Z}$ and $\Lambda \in \mathfrak{P}_\Gamma(n)$, we introduce a polynomial $\mathsf{K}_\Lambda(q) \in \mathbb{Z}[q]$ which is a q-analogue of $dim\, V_\Lambda$, the dimension of the corresponding irreducible $\Gamma \sim \mathfrak{S}_n$-module, see 5.4, and prove the following.

THEOREM. *The character of* $\mathbb{C}^*$*-module* $\mathscr{O}_\Lambda$ *equals* $\mathsf{K}_\Lambda(q) \cdot \mathsf{K}_\Lambda(q^{-1})$.

1.6. Our proof is a straightforward application of the remarkable work [W]. Wilson has studied the reduced fibers of the second projection $p_2 : \mathscr{C}_n \to \mathbb{A}^{(n)}$ and identified them as certain products of Schubert cells in Grassmannians. His results reduce our problem to some classical computations in Grassmannians.

One ingredient in the proof of Theorem 1.2 is a relative *Drinfeld compactification* $\bar{\mathscr{C}}_n$ of the Calogero–Moser space $\mathscr{C}_n$ (such that the projection p_2 extends to the proper projection $p_2 : \bar{\mathscr{C}}_n \to \mathbb{A}^{(n)}$, see 2.5) and its cyclic version, see 5.3. Though it enters our proof only at some technical point, we believe that $\bar{\mathscr{C}}_n$ is a very interesting object in itself.

The space $\bar{\mathscr{C}}_n$ was, in fact, implicitly introduced in [W] where Wilson studied the embedding of $\mathscr{C}_n$ into the *adelic Grassmannian* $\mathsf{Gr}_{\mathrm{ad}}$ (the cyclic version of this embedding is studied in [BGK]). Wilson constructed a set-theoretic partition $\mathsf{Gr}_{\mathrm{ad}} = \bigsqcup_{k\in\mathbb{N}} \mathscr{C}_n$. However, it turns out that the union $\bigsqcup_{0\leqslant k\leqslant n} \mathscr{C}_k$ *cannot* be equipped with the structure of an algebraic variety. The algebraic variety $\bar{\mathscr{C}}_n$ has, on the other hand, a natural partition $\bar{\mathscr{C}}_n = \bigsqcup_{0\leqslant k\leqslant n} \mathscr{C}_k \times \mathbb{A}^{(n-k)}$, see 2.6, into smooth locally closed strata (similar in spirit to the stratification used in [K1]) and may be viewed as an algebraic "resolution" of $\bigsqcup_{0\leqslant k\leqslant n}\mathscr{C}_k$, a nonalgebraic substack of $\mathsf{Gr}_{\mathrm{ad}}$. The name "*Drinfeld's compactification*" is suggested by a close analogy with Drinfeld's quasimap spaces, cf. [K1].

In 2.7 we propose an alternative conjectural definition of $\bar{\mathscr{C}}_n$ as a step towards its generalization for other Nakajima quiver varieties.

2. WILSON'S EMBEDDING INTO A RELATIVE GRASSMANNIAN

2.1. *The Calogero–Moser space.* Fix a positive integer n and consider the space $\mathscr{CM}_n$ defined in (1.1). Then $\mathscr{CM}_n$ is smooth, and the action of $\mathbf{PGL}_n$ by the simultaneous conjugation is free (see [W]). The quotient space $\mathscr{C}_n := \mathscr{CM}_n/\mathbf{PGL}_n$ is a $2n$-dimensional smooth affine algebraic variety, the *Calogero–Moser space*. For $n=0$ we define $\mathscr{C}_0$ to be a point.

Recall that $\mathbb{A}^{(n)} := \mathbb{A}^n/\mathfrak{S}_n$. The assignment $Y \mapsto \mathit{Spec}(Y)$, sending a matrix $Y \in \mathsf{Mat}_n$ to the n-tuple of its eigenvalues viewed as a finite subscheme of $\mathbb{A}^1$ given by zeros of the characteristic polynomial of Y, yields an isomorphism of algebraic varieties: $\mathsf{Mat}_n /\!\!/ \mathbf{PGL}_n \xrightarrow{\sim} \mathbb{A}^{(n)}$ (where $\mathsf{Mat}_n /\!\!/ \mathbf{PGL}_n$ denotes the categorical quotient). The second projection $\mathscr{CM}_n \to \mathsf{Mat}_n, (X,Y) \mapsto Y$, descends to the projection $\pi_n : \mathscr{C}_n \to \mathbb{A}^{(n)}$. Wilson has determined all the reduced fibers of π_n. Namely, he constructed an embedding of any fiber into a certain product of (finite dimensional) Grassmann varieties, and identified the image with a union of products of certain Schubert cells. Let us formulate his results more precisely. Till the end of this section *fiber* means *reduced fiber*, and we write $\pi^{-1}(-)$ instead of $\pi^{-1}(-)_{\mathsf{reduced}}$.

2.2. Theorem (Wilson [W, 7.1]). *Suppose a divisor $D = D_1 + D_2 \in \mathbb{A}^{(n)}$ is a sum of divisors $D_1 \in \mathbb{A}^{(m)}$, $D_2 \in \mathbb{A}^{(k)}$ with disjoint supports. Then there is a canonical isomorphism $\pi_n^{-1}(D) \simeq \pi_m^{-1}(D_1) \times \pi_k^{-1}(D_2)$.*

We will refer to this result as the *factorization property* of the projection π_n (or rather of the collection of maps π_n over $n \in \mathbb{N}$).

2.3. In view of the above theorem, in order to describe an arbitrary fiber of π_n, it suffices to describe the fiber over the principal diagonal, $\pi_n^{-1}(ny)$, $y \in \mathbb{A}^1$. To this end, consider the polynomial algebra $\mathbb{C}[z]$ and, for any $y \in \mathbb{C}$ write $\mathfrak{m}_y = (z-y)\cdot\mathbb{C}[z]$ for the corresponding maximal ideal. Let $\mathsf{Gr}(n,y) \simeq \mathsf{Gr}_n(\mathbb{C}^{2n})$ be the Grassmannian of n-dimensional subspaces in the $2n$-dimensional vector space $\mathbb{C}[z]/\mathfrak{m}_y^{2n}$. The vector space $\mathbb{C}[z]/\mathfrak{m}_y^{2n}$ comes equipped with a distinguished complete flag

$$0 \subset \mathfrak{m}_y^{2n-1}/\mathfrak{m}_y^{2n} \subset \mathfrak{m}_y^{2n-2}/\mathfrak{m}_y^{2n} \subset \cdots \subset \mathfrak{m}_y/\mathfrak{m}_y^{2n} \subset \mathbb{C}[z]/\mathfrak{m}_y^{2n}$$

(quotients of ideals). This flag defines the *Schubert stratification* of $\mathsf{Gr}(n, y)$. Let $\mathsf{Sch}_n(y) \subset \mathsf{Gr}(n, y)$ denote the locally closed subvariety formed by all the Schubert cells (the strata) of dimension n.

THEOREM (Wilson [W, 6.4]). *There is a canonical isomorphism* $\pi_n^{-1}(ny) \simeq \mathsf{Sch}_n(y)$.

2.4. Wilson also describes the way the above fibers glue together. In order to formulate his result, we recall that $\mathbb{A}^{(n)}$ may be viewed as the space of all codimension n ideals $I \subset \mathbb{C}[z]$, and introduce the following definition.

DEFINITION. The *relative Grassmannian* $\mathscr{G}_n$ is the space of pairs (I, W) where $I \subset \mathbb{C}[z]$ is a codimension n ideal, and $W \subset \mathbb{C}[z]/I^2$ is an n-dimensional linear subspace.

Clearly, $\mathscr{G}_n$ is a quasiprojective variety equipped with a projection $\pi_n : \mathscr{G}_n \twoheadrightarrow \mathbb{A}^{(n)}, (I, W) \mapsto I$. For any $I \in \mathbb{A}^{(n)}$ we have: $\pi_n^{-1}(I) \simeq \mathsf{Gr}_n(\mathbb{C}^{2n})$.

Wilson considers an open subset $\mathscr{C}_n^{\mathsf{reg}} \subset \mathscr{C}_n$ formed by the (conjugacy classes of) pairs (X, Y) such that Y is diagonalizable and has pairwise distinct eigenvalues. Each element in $\mathscr{C}_n^{\mathsf{reg}}$ has a unique representative of the form $Y = diag(y_1, \dots, y_n)$, $X = \|x_{ij}\|$, with $x_{ij} = (y_i - y_j)^{-1}$, for $i \neq j$, and $x_{ii} = \alpha_i$. To the n-tuple $(y_1, \dots, y_n)$ we associate the n-tuple of lines $W_i = \langle 1 - \alpha_i(z - y_i)\rangle \subset \mathbb{C}[z]/\mathfrak{m}_{y_i}^2$, $i = 1, \dots, n$, in the corresponding 2-planes. Wilson defines an embedding $\beta : \mathscr{C}_n^{\mathsf{reg}} \to \mathscr{G}_n$ by the formula $\beta : (X, Y) \mapsto (I, W)$ where $I = (z - y_1) \cdot \dots \cdot (z - y_n)$, and W is set to be a direct sum of the lines W_i, that is,

$$I = \mathfrak{m}_{y_1} \cdot \dots \cdot \mathfrak{m}_{y_n}, \qquad W = \oplus_i W_i \subset \mathbb{C}[z]/I^2 \equiv \mathbb{C}[z]/\mathfrak{m}_{y_1}^2 \oplus \cdots \oplus \mathbb{C}[z]/\mathfrak{m}_{y_n}^2.$$

THEOREM (Wilson [W, 5.1]). (i) *The map β extends to an embedding $\beta : \mathscr{C}_n \hookrightarrow \mathscr{G}_n$ commuting with the projections π_n.*

(ii) *Given $D = \sum_{k=1}^{l} n_k y_k \in \mathbb{A}^{(n)}$ and $C \in \pi_n^{-1}(D) \subset \mathscr{C}_n$ write $C = (W_1, \dots, W_l)$, $W_k \in \mathsf{Sch}_{n_k}(y_k)$. Then, under the natural identification $\mathbb{C}[z]/\prod_{k=1}^{l} \mathfrak{m}_{y_k}^{2n_k} \equiv \oplus_{k=1}^{l} \mathbb{C}[z]/\mathfrak{m}_{y_k}^{2n_k}$, we have*

$$\beta : C \mapsto \left(\prod_{k=1}^{l} \mathfrak{m}_{y_k}^{n_k}, \bigoplus_{k=1}^{l} W_k \right).$$

2.5. *Drinfeld relative compactification.* We define $\bar{\mathscr{C}}_n \subset \mathscr{G}_n$ as the closure of $\beta(\mathscr{C}_n)$ or, equivalently, of $\beta(\mathscr{C}_n^{\mathsf{reg}})$. Specifically, consider the open stratum of the diagonal stratification $\mathring{\mathbb{A}}^{(n)} \subset \mathbb{A}^{(n)}$ formed by all the n-tuples of pairwise distinct points. Consider the locally closed subvariety $\bar{\mathscr{C}}_n^{\mathsf{reg}} \subset \pi_n^{-1}(\mathring{\mathbb{A}}^{(n)}) \subset \mathscr{G}_n$ formed by all pairs

$$\left\{ \begin{array}{l} (I, W) \mid I = \mathfrak{m}_{y_1} \cdot \ldots \cdot \mathfrak{m}_{y_n}, \quad W \subset \mathbb{C}[z]/I^2 \equiv \\ \qquad \mathbb{C}[z]/\mathfrak{m}_{y_1}^2 \oplus \cdots \oplus \mathbb{C}[z]/\mathfrak{m}_{y_n}^2), \text{ such that } W \cap (\mathbb{C}[z]/\mathfrak{m}_{y_i}^2) \neq 0, \forall i. \end{array} \right\}$$

Thus, W is a direct sum of lines $W_i \subset \mathbb{C}[z]/\mathfrak{m}_{y_i}^2$.

DEFINITION. The *Drinfeld compactification* $\bar{\mathscr{C}}_n \subset \mathscr{G}_n$ is defined as the closure of $\bar{\mathscr{C}}_n^{\mathsf{reg}}$ in $\mathscr{G}_n$. The restriction of $\pi_n : \mathscr{G}_n \to \mathbb{A}^{(n)}$ to $\bar{\mathscr{C}}_n$ is also denoted by π_n.

Clearly, $\pi_n : \bar{\mathscr{C}}_n \to \mathbb{A}^{(n)}$ is a projective morphism.

2.6. *Twist by a divisor.* The rest of this section will not be used elsewhere in the paper but it helps to understand better the structure of $\bar{\mathscr{C}}_n$.

For $0 \leqslant k \leqslant n$ we will define a map $\mathsf{twist}_k^n : \bar{\mathscr{C}}_k \times \mathbb{A}^{(n-k)} \to \bar{\mathscr{C}}_n$ (twist by a divisor). To this end, given an ideal $I \subset \mathbb{C}[z]$ of codimension $n-k$, and $(J, W) \in \bar{\mathscr{C}}_k$, take the preimage of W under the natural projection $\mathbb{C}[z]/IJ^2 \to \mathbb{C}[z]/J^2$, and let $W' \subset I/I^2J^2 \subset \mathbb{C}[z]/I^2J^2$ correspond to this preimage under the natural identification $I/I^2J^2 \simeq \mathbb{C}[z]/IJ^2$. We set $\mathsf{twist}_k((J, W), I) := (IJ, W')$.

From now on we will identify $\mathscr{C}_n$ with its image $\beta(\mathscr{C}_n) \subset \bar{\mathscr{C}}_n \subset \mathscr{G}_n$. Given $y \in \mathbb{A}$, write $\overline{\mathsf{Sch}}_m(y) \subset \mathsf{Gr}(m, \mathbb{C}[z]/\mathfrak{m}_y^{2m})$ for the union of Schubert cells of dimension $\leqslant m$. Wilson's Theorem 2.4 yields

THEOREM. (i) *Let* $D = \sum_{k=1}^{l} n_k y_k \in \mathbb{A}^{(n)}$. *Then* $\pi_n^{-1}(D) \subset \bar{\mathscr{C}}_n$ *equals* $\prod_{k=1}^{l} \overline{\mathsf{Sch}}_{n_k}(y_k)$. *Specifically, under the natural identification* $\mathbb{C}[z]/\prod_{k=1}^{l} \mathfrak{m}_{y_k}^{2n_k} \equiv \oplus_{k=1}^{l} \mathbb{C}[z]/\mathfrak{m}_{y_k}^{2n_k}$, *a point* $(W_1, \ldots, W_l)$, $W_k \in \overline{\mathsf{Sch}}_{n_k}(y_k)$, *corresponds to* $\oplus_{k=1}^{l} W_k$.

(ii) $\bar{\mathscr{C}}_n \backslash \mathscr{C}_n = \mathsf{twist}_{n-1}^n(\bar{\mathscr{C}}_{n-1} \times \mathbb{A}^1)$, *where the RHS is a closed subvariety.*

(iii) $\bar{\mathscr{C}}_n$ *is a disjoint union of the locally closed subvarieties*:

$$\bar{\mathscr{C}}_n = \bigsqcup_{k=1}^{n} \mathsf{twist}_k^n(\mathscr{C}_k \times \mathbb{A}^{(n-k)}).$$

Part (i) implies, in particular, that the map $\pi_n : \bar{\mathscr{C}}_n \to \mathbb{A}^{(n)}$ enjoys the factorization property.

2.7 *Remark.* One would like to find a construction of $\bar{\mathscr{C}}_n$ in the ordinary Calogero–Moser setup of 2.1, avoiding the use of adelic Grassmannian. Here is a conjectural definition. Recall that $\mathscr{C}\mathscr{M}_n \subset \mathsf{Mat}_n \times \mathsf{Mat}_n$ is a smooth closed subvariety. Now Mat_n can be viewed as an open subset of $\mathsf{Gr}(n, 2n)$ via identifying a matrix X with the graph $W_X \subset \mathbb{C}^n \oplus \mathbb{C}^n$ of the corresponding linear map $\mathbb{C}^n \to \mathbb{C}^n$. Let $\mathscr{CM}'_n$ be the closure of $\mathscr{CM}_n$ in $\mathsf{Gr}(n, 2n) \times \mathsf{Mat}_n$. The group $\mathbf{PGL}_n$ acts on $\mathscr{CM}'_n$ naturally: $g(W, Y) = (gW, gYg^{-1})$. Let $\bar{\mathscr{C}}'_n$ be the GIT quotient of $\mathscr{CM}'_n$ with respect to $\mathbf{PGL}_n$.

Question. Is there an isomorphism $\bar{\mathscr{C}}'_n \simeq \bar{\mathscr{C}}_n$ extending the identity isomorphism on the common open subset $\bar{\mathscr{C}}'_n \supset \mathscr{C}_n \subset \bar{\mathscr{C}}_n$?

3. $\mathbb{C}^*$-ACTION ON SCHUBERT CELLS

3.1. For $y = 0$, we write $\mathsf{Gr}(n)$ instead of $\mathsf{Gr}(n, y)$ for the Grassmannian of n-dimensional subspaces of $\mathbb{C}[z]/(z^{2n})$. We have the standard complete flag in $\mathbb{C}[z]/(z^{2n})$ (see 2.3):

$$0 \subset \mathfrak{m}_0^{2n-1}/\mathfrak{m}_0^{2n} \subset \mathfrak{m}_0^{2n-2}/\mathfrak{m}_0^{2n} \subset \cdots \subset \mathfrak{m}_0/\mathfrak{m}_0^{2n} \subset \mathbb{C}[z]/\mathfrak{m}_0^{2n}.$$

Recall that $\mathsf{Sch}_n \subset \mathsf{Gr}(n)$ is a disjoint union of the n-dimensional cells, which are known to be exactly the cells Sch_λ numbered by the set $\mathfrak{P}(n)$ of partitions of n. In more detail, given a partition $\lambda = (l_1, \ldots, l_n)$, $0 \leqslant l_1 \leqslant \cdots \leqslant l_n$, $l_1 + \ldots + l_n = n$, we have

$$\begin{aligned}\mathsf{Sch}_\lambda = \{ W \in \mathsf{Gr}(n) \mid &\, dim\, (W \cap (\mathfrak{m}_0^{2n-l_i-i}/\mathfrak{m}_0^{2n})) = i, \ \forall i = 1, \ldots, n; \\ &\text{and } dim\, (W \cap (\mathfrak{m}_0^{j}/\mathfrak{m}_0^{2n})) = i, \ \forall j \text{ such that} \\ &2n - l_i - i > j > 2n - l_{i+1} - i - 1\}.\end{aligned}$$

The multiplicative group $\mathbb{C}^*$ acts on $\mathbb{C}[z]$ by $(c, z^i) \mapsto c^{-i} z^i$. This action induces a natural action on $\mathsf{Sch}_\lambda \subset \mathsf{Gr}(n)$ contracting this Schubert cell to the unique fixed point $W_\lambda := \langle z^{2n-l_1-1}, z^{2n-l_2-2}, \ldots, z^{2n-l_n-n} \rangle$ (we think of the point $W_\lambda \in \mathsf{Gr}(n)$ as a vector subspace spanned by the base vectors $z^{2n-l_1-1}, \ldots, z^{2n-l_n-n}$). The tangent space $T_{W_\lambda}\mathsf{Sch}_\lambda$ at the point W_λ is naturally isomorphic to the direct sum of the following vector spaces of linear maps:

$$\begin{aligned}&Hom(\mathbb{C}z^{2n-l_1-1}, \langle z^{2n-1}, \ldots, z^{2n-l_1} \rangle) \oplus \\ &Hom(\mathbb{C}z^{2n-l_2-2}, \langle z^{2n-1}, \ldots, \widehat{z^{2n-l_1-1}}, \ldots, z^{2n-l_2-1} \rangle) \oplus \cdots \oplus \\ &Hom(\mathbb{C}z^{2n-l_n-n}, \langle z^{2n-1}, \ldots, \widehat{z^{2n-l_1-1}}, \ldots, \widehat{z^{2n-l_i-i}}, \ldots, z^{2n-l_n-n+1} \rangle),\end{aligned}$$

where ^ means omission of an element. From this we read off easily the character of $\mathbb{C}^*$ on $T_{W_\lambda}\mathsf{Sch}_\lambda$. Specifically, write $h_\lambda(u)$ for the hook length of a box u in the Young diagram attached naturally to a partition λ. Below, we use the notation q^i for the character $\mathbb{C}^* \to \mathbb{C}^*, c \mapsto c^i$, and write $ch\, V$ for the character of a finite-dimensional $\mathbb{C}^*$-module V.

3.2. Lemma. *We have*: $ch(T_{W_\lambda}\mathsf{Sch}_\lambda) = \sum_{u\in\lambda} q^{-h_\lambda(u)}$.

4. NILPOTENT EXTENSIONS OF SCHUBERT CELLS

4.1. Recall the map $\pi_n : \mathscr{C}_n \to \mathbb{A}^{(n)},\ (X, Y) \mapsto Spec(Y)$. Denote this map by p_2, and similarly, consider the other projection $p_1 : \mathscr{C}_n \to \mathbb{A}^{(n)},\ (X, Y) \mapsto Spec(X)$. Note that there is an involution ω on $\mathscr{C}_n$ such that $\omega : (X, Y) \mapsto (Y^t, X^t)$, and we have: $p_1 = p_2 \circ \omega$. Let $p = (p_1, p_2)$ stand for the simultaneous projection $(p_1, p_2) : \mathscr{C}_n \to \mathbb{A}^{(n)} \times \mathbb{A}^{(n)}$. To distinguish between the two copies of $\mathbb{A}^{(n)}$ we will use the notation $p : \mathscr{C}_n \to \mathbb{A}_1^{(n)} \times \mathbb{A}_2^{(n)}$. According to [EG], the map p is a finite morphism.

The scheme theoretic fiber $p_2^{-1}(0)$ is a disjoint union of schemes $p_2^{-1}(0)_\lambda$ such that the underlying reduced scheme is Sch_λ, to be denoted Sch_λ^2 from now on. Similarly, the scheme theoretic fiber $p_1^{-1}(0)$ is a disjoint union of schemes $p_1^{-1}(0)_\lambda$ such that the underlying reduced scheme is denoted by Sch_λ^1.

Our goal is to compute the scheme theoretic fiber $p^{-1}(0, 0)$. It is well known that the corresponding reduced scheme is a disjoint union of points: the $\mathbb{C}^*$-fixed points of Sch_n^1 (or equivalently, Sch_n^2). Abusing the language we will denote the $\mathbb{C}^*$-fixed point of Sch_λ^2 by λ; thus $\mathsf{Sch}_\lambda^1 \cap \mathsf{Sch}_\lambda^2 = \lambda$. We will denote the (scheme-theoretic) connected component of $p^{-1}(0, 0)$ concentrated at λ by $p^{-1}(0, 0)_\lambda$.

Note that $p^{-1}(0, 0)_\lambda$ is the fiber over $0 \in \mathbb{A}_1^{(n)}$ with respect to the projection $p_1 :\ p_2^{-1}(0)_\lambda \to \mathbb{A}_1^{(n)}$. Our first step will be to compute the fiber over $0 \in \mathbb{A}_1^{(n)}$ with respect to the projection $p_1 :\ \mathsf{Sch}_\lambda^2 \to \mathbb{A}_1^{(n)}$.

4.2. Recall that the Kostka polynomial associated to a Young diagram λ is a polynomial in the variable 'q' given by the formula: $q^{m(\lambda)}(1 - q) \ldots (1 - q^n) \prod_{u\in\lambda}(1 - q^{h_\lambda(u)})^{-1}$, where $m(\lambda)$ is a certain positive integer, see [M, p. 243, Example 2]. This is a q-analogue of the dimension $dim\, V_\lambda$ of the irreducible representation V_λ of the symmetric group $\mathfrak{S}_n$. We will consider a version of Kostka polynomial with the lowest term equal to 1, that is, we put

$$K_\lambda(q) := (1 - q) \ldots (1 - q^n) \prod_{u\in\lambda} (1 - q^{h_\lambda(u)})^{-1}.$$

PROPOSITION. *For scheme-theoretic intersections, we have*

$$ch\,\mathcal{O}(\mathsf{Sch}^2_\lambda \cap p_1^{-1}(0)_\lambda) = K_\lambda(q) \quad \textit{and} \quad ch\,\mathcal{O}(\mathsf{Sch}^1_\lambda \cap p_2^{-1}(0)_\lambda) = K_\lambda(q^{-1}).$$

Proof. The two formulas are analogous, so we only prove the first one. We compute the geometric fiber of the sheaf $(p_1)_*\mathcal{O}_{(\mathsf{Sch}^2_\lambda)}$ at the point $0 \in \mathbb{A}_1^{(n)}$. This is a locally free coherent sheaf, that is a (trivial) vector bundle, so to compute the character of its geometric fiber at 0 it suffices to know the character $ch\,\mathcal{O}(\mathsf{Sch}^2_\lambda)$ of its space of global sections, and the character of $\mathcal{O}(\mathbb{A}_2^{(n)})$. Now we pass to the formal completions at 0 and λ. Thus, we are reduced to finding the characters of tangent spaces $T_0\mathbb{A}_1^{(n)}$ and $T_\lambda\mathsf{Sch}^2_\lambda$. The former character equals $1 + q^{-1} + \cdots + q^{-n}$, while the latter character was computed in Lemma 3.2. We conclude that $ch\,\hat{\mathcal{O}}_{\mathbb{A}_1^{(n)},0} = (1-q)^{-1}\ldots(1-q^n)^{-1}$, and $ch\,\hat{\mathcal{O}}_{\mathsf{Sch}^2_\lambda,\lambda} = \prod_{u\in\lambda}(1-q^{h_\lambda(u)})^{-1}$. Thus, we get

$$ch\,\mathcal{O}(\mathsf{Sch}^2_\lambda \cap p_1^{-1}(0)_\lambda) = ch\,\hat{\mathcal{O}}_{\mathsf{Sch}^2_\lambda,\lambda}/ch\,\hat{\mathcal{O}}_{\mathbb{A}_1^{(n)},0} = K_\lambda(q) \quad ∎$$

4.3. We are going to compute $ch\,\mathcal{O}(p^{-1}(0,0)_\lambda)$ along similar lines. To this end, it suffices to compute the character of the completion $ch\,\hat{\mathcal{O}}_{p_2^{-1}(0)_\lambda,\lambda}$. We will prove that $ch\,\hat{\mathcal{O}}_{p_2^{-1}(0)_\lambda,\lambda} = K_\lambda(q^{-1})\prod_{u\in\lambda}(1-q^{h_\lambda(u)})^{-1}$. Hence, arguing exactly as in the proof of Proposition 4.2 we will be able to conclude that $ch\,\mathcal{O}(p^{-1}(0,0)_\lambda) = K_\lambda(q)K_\lambda(q^{-1})$, as required in Theorem 1.2. Thus, to prove the theorem it suffices to prove the following.

PROPOSITION. $ch\,\hat{\mathcal{O}}_{p_2^{-1}(0)_\lambda,\lambda} = K_\lambda(q^{-1})\prod_{u\in\lambda}(1-q^{h_\lambda(u)})^{-1}$.

4.4. We start the proof of the proposition with the following.

LEMMA. *The smooth varieties* Sch^1_λ *and* Sch^2_λ *are transversal at* λ.

Proof. The varieties Sch^1_λ and Sch^2_λ are smooth of complementary dimensions. Moreover, the character of $T_\lambda\mathsf{Sch}^2_\lambda$ is a polynomial in q^{-1} without a constant term, while the character of $T_\lambda\mathsf{Sch}^1_\lambda$ is a polynomial in q without a constant term. Hence, these two tangent spaces must have zero intersection, and we are done. ∎

Thus, the formal completion of $\mathscr{C}_n$ at λ is isomorphic to a product of formal completions of Sch^1_λ and Sch^2_λ at λ. We will denote by pr_1 and pr_2 the projections to the corresponding factor. The fiber over λ of the restriction of pr_2 to the formal completion of $p_2^{-1}(0)_\lambda$ equals: $pr_2^{-1}(\lambda) = \mathsf{Sch}^1_\lambda \cap p_2^{-1}(0)_\lambda$. We already know formulas for $ch\,\mathcal{O}(\mathsf{Sch}^1_\lambda \cap p_2^{-1}(0)_\lambda)$ and $ch\,\hat{\mathcal{O}}_{\mathsf{Sch}^2_\lambda,\lambda}$, so to complete the proof it suffices to show that $pr_{2*}\hat{\mathcal{O}}_{p_2^{-1}(0)_\lambda,\lambda}$ is a (trivial) vector bundle on the completion of Sch^2_λ at λ. To this end, it suffices to show that the dimension of the generic fiber of $(pr_2)_*\hat{\mathcal{O}}_{p_2^{-1}(0)_\lambda,\lambda}$ equals $dim\,\mathcal{O}(\mathsf{Sch}^1_\lambda \cap p_2^{-1}(0)_\lambda) = d_\lambda\ (= dim\ V_\lambda)$. But the dimension of the generic fiber equals m_λ, the multiplicity of the scheme $p_2^{-1}(0)$ at the generic point of its reduced subscheme Sch^2_λ.

To compute this multiplicity m_λ we may as well work in the Drinfeld compactification $\bar{\mathscr{C}}_n$ embedded into the relative Grassmannian $\mathscr{G}_n$ over $\mathbb{A}_2^{(n)}$. For a general point $\underline{y}$, the fiber $p_2^{-1}(\underline{y}) \subset \mathsf{Gr}(n, \underline{y})$ is reduced at the generic point, so m_λ is the coefficient of the cycle class $[p_2^{-1}(\underline{y})]$ with respect to the Schubert basis $\{[\overline{\mathsf{Sch}}_\lambda],\ \lambda \in \mathfrak{P}(n)\}$ of the degree $2n$ homology group of $\mathsf{Gr}_n(\mathbb{C}^{2n})$.

Now recall that a general n-tuple $\underline{y} = (y_1, \ldots, y_n) \in \mathbb{A}_2^{(n)}$ of pairwise distinct points gives rise to a direct sum decomposition $\mathbb{C}[z]/\mathfrak{m}_{y_1}^2 \cdot \ldots \cdot \mathfrak{m}_{y_n}^2 = \oplus_i \mathbb{C}[z]/\mathfrak{m}_{y_i}^2$, and $p_2^{-1}(\underline{y}) \subset \mathsf{Gr}(n, \underline{y})$ is the product of corresponding projective lines: $p_2^{-1}(\underline{y}) = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \subset \mathsf{Gr}_n(\mathbb{C}^{2n})$. It is the classical result of Schubert calculus that for the corresponding homology classes one has an expansion: $[\mathbb{P}^1 \times \cdots \times \mathbb{P}^1] = \sum_\lambda m_\lambda \cdot [\overline{\mathsf{Sch}}_\lambda]$; moreover, the coefficients m_λ can be read off from the formula: $\mathbf{p}_1^n = \sum_\lambda m_\lambda \cdot s_\lambda$, an expansion of the nth power of the first symmetric function $\mathbf{p}_1$ with respect to the basis of Schur functions s_λ. The coefficients in the latter expansion are well-known to be equal to $d_\lambda = K_\lambda(1)$, see e.g. [M, p. 114].

This completes the proof of Proposition 4.3. and the proof of Theorem 1.2.

5. CYCLIC CALOGERO–MOSER SPACE

5.1. Consider the action of $\Gamma = \mathbb{Z}/N\mathbb{Z} \subset \mathbb{C}^*$ on the Calogero–Moser space $\mathscr{C}_{nN}$. The fixed-point subvariety $\mathscr{C}_{nN}^\Gamma$ consists of various connected components. There is a single component characterized by the property that the representation of Γ in the fiber of the tautological bundle at any point in this component is a multiple of the regular representation, see [K2]. We will call this connected component $\mathscr{C}_{\Gamma,n}$. According to [M], $\mathscr{C}_{\Gamma,n}$ is a special case of Nakajima's Quiver variety (corresponding to N-cyclic quiver with n-dimensional spaces at all "finite" vertices, one-dimensional space at an "extended" vertex, and a nonzero value of the diagonal moment map).

We have the natural projection $p = (p_1, p_2) :\ \mathscr{C}_{\Gamma,n} \to (\mathbb{A}_1^{(nN)} \times \mathbb{A}_2^{(nN)})^\Gamma$. Note that

$$(\mathbb{A}_1^{(nN)} \times \mathbb{A}_2^{(nN)})^\Gamma = (\mathbb{A}_1^{(nN)})^\Gamma \times (\mathbb{A}_2^{(nN)})^\Gamma \qquad \text{and} \qquad (\mathbb{A}_\imath^{(nN)})^\Gamma = (\mathbb{A}_\imath^1/\Gamma)^{(n)}$$

to be denoted $\mathbb{A}_{\Gamma,\imath}^{(nN)} := (\mathbb{A}_\imath^{(nN)})^\Gamma, \imath = 1, 2$, and view p as a projection $p = (p_1, p_2) :\ \mathscr{C}_{\Gamma,n} \to \mathbb{A}_{\Gamma,1}^{(n)} \times \mathbb{A}_{\Gamma,2}^{(n)}$. The natural $\mathbb{C}^*$-action on $\mathscr{C}_{nN}$ when restricted to $\mathscr{C}_{\Gamma,n}$ factors through $\mathbb{C}^* \overset{c \mapsto c^N}{\to} \mathbb{C}^*$, and we will consider the resulting $\mathbb{C}^*$-action on $\mathscr{C}_{\Gamma,n}$ (which is generically free).

5.2. Wilson's embedding $\beta : \mathscr{C}_{nN} \hookrightarrow \mathscr{G}_{nN}$ is Γ-equivariant, and its image lands into a connected component $\mathscr{G}_{\Gamma,n} \subset \mathscr{G}^{\Gamma}_{nN}$ characterized by the property that the representation of Γ in the fiber of the tautological bundle at any point of this component is a multiple of the regular representation (to see the inclusion: $\mathscr{C}_{\Gamma,n} \subset \mathscr{G}_{\Gamma,n}$ it suffices to check it at any $\mathbb{C}^*$-fixed point, e.g. $\lambda = (nN)$). We will denote by $\beta : \mathscr{C}_{\Gamma,n} \hookrightarrow \mathscr{G}_{\Gamma,n}$ this Γ-version of Wilson's embedding, and we will use it to describe the reduced fibers of p_2.

First of all, the action of Γ on $\mathbb{C}[z]/(z^{2nN})$ yields a weight space decomposition: $\mathbb{C}[z]/(z^{2nN}) = \oplus_{\chi \in \Gamma^\vee} (\mathbb{C}[z]/(z^{2nN}))_\chi$, according to the characters of Γ. Each weight space is $2n$-dimensional. Note that we can canonically identify $\Gamma^\vee$ with $\mathbb{Z}/N\mathbb{Z}$, and then $(\mathbb{C}[z]/(z^{2nN}))_\chi$ is spanned by $\{z^k,\ k \equiv \chi \,(\mathrm{mod}\, N)\}$. The fiber of $\mathscr{G}_{\Gamma,n}$ over $nN \cdot 0 \in (\mathbb{A}_2^{(nN)})^\Gamma$ is given by

$$\left\{ W \subset \mathbb{C}[z]/(z^{2nN}) \ \middle|\ W = \bigoplus_{\chi \in \Gamma^\vee} W_\chi,\ \ W_\chi \subset (\mathbb{C}[z]/(z^{2nN}))_\chi,\ \mathit{dim}\, W_\chi = n \right\}$$

(hence, each vector space W has dimension nN). Thus, this fiber equals $\prod_{\chi \in \Gamma^\vee} \mathsf{Gr}(n, (\mathbb{C}[z]/(z^{2nN}))_\chi)$. Each space $(\mathbb{C}[z]/(z^{2nN}))_\chi$ has a distinguished complete flag (given by the intersections with powers of the maximal ideal). Thus, each variety $\mathsf{Gr}(n, (\mathbb{C}[z]/(z^{2nN}))_\chi)$ has a natural stratification into Schubert cells numbered by partitions.

Set $\mathfrak{P}_\Gamma(n) := \{\lambda_\chi \mid \chi \in \Gamma^\vee, \sum_\chi |\lambda_\chi| = n\}$, and given $\Lambda \in \mathfrak{P}_\Gamma(n)$ put $\mathsf{Sch}^2_\Lambda := \prod_\chi \mathsf{Sch}_{\lambda_\chi} \subset \prod_{\chi \in \Gamma^\vee} \mathsf{Gr}(n, (\mathbb{C}[z]/(z^{2nN}))_\chi)$. Now Wilson's Theorem 2.3 together with [K2, Corollary 4.18, Theorem 5.3] yield the following.

Proposition. *The reduced fiber of $\mathscr{C}_{\Gamma,n}$ over $0 \in \mathbb{A}^{(n)}_{\Gamma,2}$ is canonically isomorphic to $\coprod_{\Lambda \in \mathfrak{P}_\Gamma(n)} \mathsf{Sch}^2_\Lambda$.*

Corollary. (i) *Each component Sch^2_Λ contains a unique $\mathbb{C}^*$-fixed point $\Lambda \in \mathscr{C}_{\Gamma,n}$.*

(ii) *The reduced fiber of $p : \mathscr{C}_{\Gamma,n} \to \mathbb{A}^{(n)}_{\Gamma,1} \times \mathbb{A}^{(n)}_{\Gamma,2}$ over $(0,0)$ coincides with the set $\mathscr{C}^{\mathbb{C}^*}_{\Gamma,n} = \mathfrak{P}_\Gamma(n)$.*

We will denote by $p^{-1}(0,0)_\Lambda$ the connected component of the scheme theoretic fiber concentrated at the point Λ, and we will write $p_1^{-1}(0)_\Lambda$, resp. $p_2^{-1}(0)_\Lambda$, for the connected component of the scheme theoretic fiber concentrated at Sch^1_Λ, resp. Sch^2_Λ.

5.3. We define the Drinfeld compactification $\bar{\mathscr{C}}_{\Gamma,n} \supset \mathscr{C}_{\Gamma,n}$ as the closure of $\mathscr{C}_{\Gamma,n}$ inside $\mathscr{G}_{\Gamma,n}$.

We will need a description of a general fiber of $p_2: \bar{\mathscr{C}}_{\Gamma,n} \to \mathbb{A}^{(n)}_{\Gamma,2}$. Choose ζ, a primitive Nth root of unity. Then a general point $\underline{y} \in \mathbb{A}^{(n)}_{\Gamma,2}$ can be represented by a collection

$$\underline{y} = (y_1, \zeta y_1, \dots, \zeta^{N-1} y_1, y_2, \dots, \zeta^{N-1} y_2, \dots, y_n, \dots, \zeta^{N-1} y_n)$$

of nN pairwise distinct points of $\mathbb{A}^1_2$. The $2nN$-dimensional vector space $V = \mathbb{C}[z]/\mathfrak{m}^2_{y_1} \dots \mathfrak{m}^2_{\zeta^{N-1} y_n}$ is acted upon by Γ, and has a weight space decomposition $V = \oplus_{\chi \in \Gamma^\vee} V_\chi$ according to the characters of Γ. Thus, this decomposition has N direct summands, each of dimension $2n$. We also have a direct sum decomposition $V = U_1 \oplus \dots \oplus U_n$ where $U_i = \mathbb{C}[z]/\mathfrak{m}^2_{y_i} \dots \mathfrak{m}^2_{\zeta^{N-1} y_i}$. Note that for any $i = 1, \dots, n$, and $\chi \in \Gamma^\vee$, the intersection $U_i \cap V_\chi$ is two dimensional. We will denote this intersection by $V_{i,\chi}$.

The fiber of the projection $p_2 : \mathscr{G}_{\Gamma,n} \to \mathbb{A}^{(n)}_{\Gamma,2}$ over $\underline{y} \in \mathbb{A}^{(n)}_{\Gamma,2}$ equals $\prod_{\chi \in \Gamma^\vee} \mathsf{Gr}(n, V_\chi)$. The fiber over $\underline{y}$ of the restriction of this projection to $\bar{\mathscr{C}}_n \subset \mathscr{G}_{\Gamma,n}$ is isomorphic to $\prod_{1 \leqslant i \leqslant n} \mathbb{P}^1$, where the space $\prod_{1 \leqslant i \leqslant n} \mathbb{P}^1$ is embedded into $\prod_{\chi \in \Gamma^\vee} \mathsf{Gr}(n, V_\chi)$ as follows. We have a direct sum decomposition $U_i = \oplus_{k \in \mathbb{Z}/N\mathbb{Z}} \mathbb{C}[z]/\mathfrak{m}^2_{\zeta^k y_i}$, and the action of Γ on U_i permutes the summands. Hence $\mathbb{C}[z]/\mathfrak{m}^2_{y_i}$ projects isomorphically onto any $V_{i,\chi}$. Given a line $\ell_i \in \mathbb{P}^1(\mathbb{C}[z]/\mathfrak{m}^2_{y_i})$ we denote by $\ell_{i,\chi} \subset V_{i,\chi}$ its image under the bijective projection above. Finally, for a collection $\{\ell_i\} \in \prod_{1 \leqslant i \leqslant n} \mathbb{P}^1(\mathbb{C}[z]/\mathfrak{m}^2_{y_i})$ the corresponding point of $\prod_{\chi \in \Gamma^\vee} \mathsf{Gr}(n, V_\chi)$ is the collection of subspaces $\{\oplus_i \ell_{i,\chi} \subset V_\chi\}_{\chi \in \Gamma^\vee}$.

5.4. Our aim is to compute the character of $\mathbb{C}^*$-action on the Artin ring $\mathcal{O}(p^{-1}(0,0)_\Lambda)$, that is, to prove Theorem 1.5. The proof is entirely similar to that of 1.2. Let us spell out the intermediate steps. First, we define:

$$K_\Lambda(q) := (1-q)\dots(1-q^n) \prod_{u \in \lambda_\chi, \chi \in \Gamma^\vee} (1 - q^{h_\lambda(u)})^{-1}.$$

Analogous to Proposition 4.2, we obtain

PROPOSITION. *We have*

$$ch\, \mathcal{O}(\mathsf{Sch}^2_\Lambda \cap p_1^{-1}(0)_\Lambda) = K_\Lambda(q) \quad \textit{and} \quad ch\, \mathcal{O}(\mathsf{Sch}^1_\Lambda \cap p_2^{-1}(0)_\Lambda) = K_\Lambda(q^{-1}).$$

Further, an analogue of Proposition 4.3 reads

5.5 Proposition. $ch\,\hat{\mathcal{O}}_{p_2^{-1}(0)_\Lambda,\Lambda} = K_\Lambda(q^{-1}) \cdot \prod_{u\in\lambda_\chi,\chi\in\Gamma^\vee}\ (1-q^{h_\lambda(u)})^{-1}$.

To prove this last Proposition we argue as in 4.4. It suffices to show that the generic multiplicity m_Λ of $p_2^{-1}(0)_\Lambda$ equals $d_\Lambda := K_\Lambda(1)$. To this end, we turn to the cyclic version of Drinfeld compactification $\mathscr{C}_{\Gamma,n}$, see 5.3. A general fiber $p_2^{-1}(\underline{y})$ being reduced at the generic point, m_Λ are the coefficients of the fundamental class $[p_2^{-1}(\underline{y})]$ with respect to the Schubert basis $\{[\overline{\mathsf{Sch}}_\Lambda],\ \Lambda\in\mathfrak{P}_\Gamma(n)\}$ of the degree $2n$ homology group of $\prod_{\chi\in\Gamma^\vee}\mathsf{Gr}_n(\mathbb{C}^{2n})$. Our description of the general fiber $p_2^{-1}(\underline{y})$ in 5.3 boils down to the following.

Take the diagonal embedding $\mathbb{P}^1 = \Delta_{\mathbb{P}^1} \hookrightarrow \prod_{\chi\in\Gamma^\vee}\mathbb{P}^1_\chi$. For each $\chi\in\Gamma^\vee$ we have an embedding $(\mathbb{P}^1_\chi)^n \hookrightarrow \mathsf{Gr}_n$ as in 4.4. Now form the composition

$$(\mathbb{P}^1)^n = (\Delta_{\mathbb{P}^1})^n \hookrightarrow \prod_{\chi\in\Gamma^\vee}(\mathbb{P}^1_\chi)^n \hookrightarrow \prod_{\chi\in\Gamma^\vee}\mathsf{Gr}(n, V_\chi).$$

The homology class of $[\Delta_{\mathbb{P}^1}]$ in the second homology group of $\prod_{\chi\in\Gamma^\vee}\mathbb{P}^1_\chi$ equals $\sum_\chi\ [\mathbb{P}^1_\chi]$, the sum of degree 2 generators of the homology groups of the factors. As in 4.4, we conclude that $[(\Delta_{\mathbb{P}^1})^n] = \sum_\Lambda m_\Lambda\cdot[\overline{\mathsf{Sch}}_\Lambda]$, where the coefficients m_Λ are equal to the coefficients in the expansion of $(\sum_\chi \mathbf{p}_{1,\chi})^n$ with respect to the basis of Schur functions S_Λ. (Here $\mathbf{p}_{1,\chi}$ is the first power sum symmetric function in the variables $x_{i,\chi}$, $1\leqslant i<\infty$, and $S_\Lambda := \prod_\chi s_{\lambda_\chi}(x_{i,\chi})$, see [M, part I, Appendix B].) The latter coefficients are in turn equal to: $n!/\prod_{u\in\lambda_\chi,\chi\in\Gamma^\vee}h_\lambda(u) = d_\Lambda = K_\Lambda(1)$, see [M, (9.6) on p. 178].

This completes the proof of Proposition 5.5, hence the proof of Theorem 1.5.

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