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Journal of Functional Analysis 263 (2012) 1129–1146

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

The global existence and convergence of the Calabi flow on $\mathbb{C}^n/\mathbb{Z}^n + i\mathbb{Z}^n$ [☆]

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Received 1 February 2012; accepted 22 May 2012

Available online 29 May 2012

Communicated by F.-H. Lin

Abstract

In this note, we study the long time existence of the Calabi flow on $X = \mathbb{C}^n/\mathbb{Z}^n + i\mathbb{Z}^n$. Assuming the uniform bound of the total energy, we establish the non-collapsing property of the Calabi flow by using Donaldson's estimates and Streets' regularity theorem. Next we show that the curvature is uniformly bounded along the Calabi flow on X when the dimension is 2, partially confirming Chen's conjecture. Moreover, we show that the Calabi flow exponentially converges to the flat Kähler metric for arbitrary dimension if the curvature is uniformly bounded, partially confirming Donaldson's conjecture.

Keywords: Calabi flow; Global existence and convergence

1. Introduction

The Calabi flow was invented by Calabi [6] to search for the canonical metrics in a given Kähler class. Let φ be a Kähler potential and S be the scalar curvature of φ, its equation is

$$\frac{\partial \varphi}{\partial t} = S - \underline{S}, \tag{1}$$

☆ The research of the second named author is financially supported by CIRGET (Centre interuniversitaire de recherches en géométrie et topologie), Montreal, Canada and FMJH (Fondation mathématique Jacques Hadamard), Paris, France.

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0022-1236/$ – see front matter
http://dx.doi.org/10.1016/j.jfa.2012.05.017

where $\underline{S}$ is the average of the scalar curvature. Since it is a 4th order parabolic equation, its long time existence and convergence are hard to study. The Riemann surface case is settled down by Chrusciél [15] and reproved by Chen [9]. The study of the Calabi flow on the ruled manifolds is elaborated by Guan [24] and Székelyhidi [33]. Later, Chen and He prove the short time existence of the Calabi flow in [10]. In the same paper, they prove that the obstruction of the long time existence of the Calabi flow is the Ricci curvature. Furthermore, they establish the stability property of the Calabi flow near a cscK metric. The stability property is generalized by Zheng and the second named author in [26] for the case of extremal metrics. Tosatti and Weinkove [34] also prove the stability the Calabi flow when the first Chern class $c_1 = 0$ or $c_1 < 0$. The stability problem is further studied in Chen and Sun's work [13], they prove that constant scalar curvature Kähler metric "adjacent" to a fixed Kähler class is unique up to isomorphism.

The long time existence problem of the Calabi flow largely remains open. Assuming the long time existence, Donaldson describes the limiting behavior of the Calabi flow in [17]. Székelyhidi [32] shows that if the Calabi flow exists for all time in toric varieties, then the infimum of the Calabi energy is equal to the supremum of the normalized Futaki invariant over all destabilizing test-configurations, partially confirming Donaldson's conjecture in [19].

The global convergence problem of the Calabi flow also largely remains open. An application of the global convergence of the Calabi flow is to solve a conjecture proposed by Apostolov, Calderbank, Gauduchon and Tønnesen-Friedman [2]: A projective bundle $(M, J) = P(E)$ over a compact curve of genus $\geqslant 2$ admits an extremal Kähler metric in some Kähler class if and only if E decomposes as a direct sum of stable sub-bundles.

One of the methods in studying the long time existence problem is the blow-up analysis. It is firstly adopted in Chen and He's work [12]. They establish the following **weak regularity theorem**: Suppose the L^∞ norm of Riemann curvature tensor of the Calabi flow is bounded by 1 in the time interval $[-1, 0]$, then

$$\int_X \left|\nabla^k Rm(0,x)\right|^2 d\omega < C\left(n, k, \int_X \left|Rm(-1,x)\right|^2 d\omega\right). \tag{2}$$

Remark 1. Streets also obtains a similar result in [31].

When the Calabi energy is small in certain toric Fano surfaces, Chen and He are able to obtain a uniform Sobolev constant along the Calabi flow [11]. Hence they derive the **regularity theorem** of the Calabi flow: Suppose the L^∞ norm of Riemann curvature tensor of the Calabi flow is bounded by 1 in the time interval $[-1, 0]$, then

$$\max_{x\in X}\left|\nabla^k Rm(0,x)\right| < C(n,k). \tag{3}$$

Remark 2. The regularity theorem is called Shi's estimate in the Ricci flow [29].

After obtaining the uniform bound of Sobolev constant and the regularity theorem, Chen and He rule out the singularities along the Calabi flow and show that the Calabi flow converges to an extremal metric in the Cheeger–Gromov sense. This result gives us a better understanding of Chen's conjecture (see e.g. [11]):

Conjecture. *The Calabi flow exists for all time.*

Motivated by Donaldson's work in [16,18,20,21], the second named author studies the classification of singularities of the Calabi flow on toric varieties by assuming the total energy bound, the regularity theorem and the non-collapsing property [25]. Later, Streets proves the regularity theorem for the Calabi flow [30]. Then the remaining obstacle for the long time existence of the Calabi flow on a toric variety is the non-collapsing property of the Calabi flow.

The Calabi flow on toric varieties is a parabolic version of the linearized Monge–Ampère equation. The linearized Monge–Ampère equation is studied in the work of Caffarelli and Gutiérrez [5]. Their work has been used in Trudinger–Wang's solution of the Bernstein problem [35] and the affine Plateau problem [36]. In Donaldson's work on the existence of cscK metrics on toric surfaces, he also uses Caffarelli and Gutiérrez's work to obtain the interior regularity of his continuous method [18] and the M-condition near the boundary in order to resolve the non-collapsing issue [21]. Caffarelli and Gutiérrez's work also finds applications in Chen, Li and Sheng's work on the existence of extremal metrics on toric surfaces [14]. Székelyhidi and the first named author apply the ideas of Trudinger and Wang and Donaldson to solve the Abreu's equation on Abelian varieties [22].

The difficulty of the long time existence of the Calabi flow is to show the non-collapsing property, i.e., the injectivity radius has a uniform lower bound in the blow-up analysis. More details can be found in [4,8] and [28] where authors explain how to use the non-collapsing property of Ricci flow to classify the singularities of the Ricci flow in a 3-manifold.

1.1. Main results

For simplicity, we only consider the long time existence and the global convergence of the Calabi flow on $X = \mathbb{C}^n/\Lambda$, where $\Lambda = \mathbb{Z}^n + i\mathbb{Z}^n$. There is a natural T^n action on X via the translation in the Lagrangian subspace $i\mathbb{R}^n \subset \mathbb{C}^n$. Let ω_0 be a flat metric. We consider the space of T^n-invariant Kähler metric (Section 2):

$$\mathcal{H}_{T^n} = \left\{\phi \in C_{T^n}(X)\colon \omega_\phi = \omega_0 + \partial\bar{\partial}\phi > 0\right\}.$$

Then we can prove the following non-collapsing theorem along the Calabi flow in the space $\mathcal{H}_{T^n}$.

Theorem 1.1. *Let $\omega_{\phi_{(-1)}} \in \lambda\mathcal{H}_{T^n}$ be an initial metric, where $\lambda > 1$ is an arbitrary rescaling factor. Suppose that*:

- *The Calabi flow exists for $t \in [-1, 0]$ in $\lambda\mathcal{H}_{T^n}$ and the L^∞ norm of Riemann curvature tensor of the Calabi flow is uniformly bounded by 1 on $X \times [-1, 0]$.*
- *The total energy is bounded at the end point $t = 0$, i.e.,*

$$\int_X \left|Rm(0,x)\right|^n \omega_{\phi_0}^n < C,$$

  where C is a positive constant.
- *There is a constant M such that the Legendre transform of the Kähler potential of ω_{ϕ_0} satisfies the M-condition.*
- *$|Rm(0,x)| = 1$ for some $x \in X$.*

Then the injectivity radius of x at time $t=0$ is bounded from below by C_1 depending only on n, M, and C.

Next we obtain the following long time existence result of the Calabi flow which partially confirms Chen's conjecture.

Theorem 1.2. *In dimension 2, given any initial data in $\mathcal{H}_{T^n}$, the Calabi flow exists for all time in $\mathcal{H}_{T^n}$ and the curvature is uniformly bounded along the flow.*

For the global convergence, Donaldson has the conjecture that: If the Calabi flow exists for all time and there exists a cscK metric in the Kähler class, then the Calabi flow converges to a cscK metric [17]. Berman [3] proves that the Calabi flow converges to a Kähler Einstein metric in the weak topology of currents if the Calabi flow exists for all time. Our following result confirms Donaldson's conjecture in X for arbitrary dimension.

Theorem 1.3. *If the Calabi flow exists for all time in $\mathcal{H}_{T^n}$ and the curvature is uniformly bounded, then it converges to ω_0 which is a flat metric.*

2. Abelian varieties

Let $X=\mathbb{C}^n/\Lambda$ where $\Lambda=\mathbb{Z}^n+i\mathbb{Z}^n$. We write each point as $z=\xi+i\eta$, where ξ and $\eta\in\mathbb{R}^n$ and can be viewed as the periodic coordinates of X. Let

$$\omega_0=\frac{\sqrt{-1}}{2}\sum_{\alpha=1}^{n}dz_\alpha\wedge d\bar{z}_\alpha=\sum_{\alpha=1}^{n}d\xi_\alpha\wedge d\eta_\alpha$$

be the standard flat metric with associated local Kähler potential $\frac{1}{2}|z|^2$. The group T^n acts on X via translation in η variable in the Lagrangian subspace $i\mathbb{R}^n\subset\mathbb{C}^n$, thus we can consider the space of torus invariant Kähler metrics in the fixed class $[\omega_0]$:

$$\mathcal{H}_{T^n}=\left\{\phi\in C^\infty_{T^n}(M):\ \omega_\phi=\omega_0+\frac{\sqrt{-1}}{2}\partial\bar{\partial}\phi>0\right\}.$$

Functions invariant under the translation of T^n are independent of η, so they are smooth functions on $X/T^n\cong T^n$, i.e., $\phi(\xi)$ is a periodic and smooth function on $\mathbb{R}^n$. Without loss of generality, we can assume that the fundamental domain for the periodicity of ϕ is $[-\frac{1}{2},\frac{1}{2}]^n$. We can write the local Kähler potential in $\mathcal{H}_{T^n}$ in complex coordinates as

$$v(\xi)=\frac{1}{2}|\xi|^2+\phi(\xi)$$

and the scalar curvature is

$$S=-\sum_{i,j}v^{i\bar{j}}\log\bigl[\det(v_{a\bar{b}})\bigr]_{i\bar{j}}$$

where v is a convex function on $\mathbb{R}^n$ since it is a local Kähler potential. Then we can take the Legendre transform of v, with dual coordinate $x = \nabla v(\xi)$. In fact, x induces a Lie group moment map: $X \to T^n$. The transformed function $u(x)$ is defined by

$$u(x) + v(\xi) = x \cdot \xi.$$

The image of the Lie group moment is isomorphic to $X/T^n \cong T^n$. We denote $P = [-\frac{1}{2}, \frac{1}{2}]^n$ as the fundamental domain of T^n. Let $\underline{u} = \frac{1}{2}|x|^2$. One can check that $u - \underline{u}$ is a periodic function in $\mathbb{R}^n$ with fundamental domain P.

A calculation in [1] gives

$$S = -\sum_{i,j} \frac{\partial^2 u^{ij}(x)}{\partial x_i \partial x_j}$$

which is called Abreu's equation, i.e., the expression of scalar curvature under symplectic coordinates. Notice that in our case, the average of S is 0, thus we can rewrite the Calabi flow in terms of Abreu's equation as

$$\frac{\partial u}{\partial t} = \sum_{i,j} \frac{\partial^2 u^{ij}(x)}{\partial x_i \partial x_j} \tag{4}$$

where we use the fact that $\frac{\partial v(t,\xi)}{\partial t} = -\frac{\partial u(t,x)}{\partial t}$ [23]. In fact, by the proof of the short time existence of the Calabi flow in [10], if the initial metric is in $\mathcal{H}_{T^n}$, then the Calabi flow will stay in $\mathcal{H}_{T^n}$ for a short time.

3. Calabi flow and M-condition

First, we want to introduce Donaldson's M-condition which is crucial in controlling the injectivity radius.

For any line segment $\overline{p_0 p_3} \subset P$, let $p_1, p_2 \in P$ be two points in P such that the lengths of $\overline{p_0 p_1}$, $\overline{p_1 p_2}$, $\overline{p_2 p_3}$ are the same. Let ν be the unit vector parallel to the vector $p_3 - p_0$. We say that u satisfies the M-condition on $\overline{p_0 p_3}$ if

$$\left|\nabla_\nu u(p_1) - \nabla_\nu u(p_2)\right| < M.$$

Definition 3.1. If for any line segment $l \subset P$, u satisfies the M-condition on l, then we say that u satisfies the M-condition on P.

The goal of this section is to prove the following proposition:

Proposition 3.2. *The M-condition is preserved under the Calabi flow.*

To achieve this goal, we prove C^0 and C^1 bounds of the solution of the Calabi flow. Thus, there is a uniform constant M such that the M-condition holds for all time.

In Calabi and Chen's work [7], they show that the Calabi flow decreases the distance. We will reproduce this result in our settings.

Proposition 3.3. *The L^2 norm of $u(t,x)-\underline{u}$ is decreasing along the Calabi flow.*

Proof. Let ν be the outward normal vector along the boundary of P and ds be the boundary measure. We have

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_P \big(u(t,x)-\underline{u}\big)^2\,dx\\
&=2\int_P \big(\underline{S}-S(t)\big)\big(u(t,x)-\underline{u}\big)\,dx\\
&=2\int_P \big(u^{ij}_{\ ij}(t,x)-\underline{u}^{ij}_{\ ij}\big)\big(u(t,x)-\underline{u}\big)\,dx\\
&=2\int_{\partial P} \big(u^{ij}_{\ i}(t,x)-\underline{u}^{ij}_{\ i}\big)\nu_j\big(u(t,x)-\underline{u}\big)\,ds-2\int_P \big(u^{ij}_{\ i}(t,x)-\underline{u}^{ij}_{\ i}\big)\big(u_j(t,x)-\underline{u}_j\big)\,dx\\
&=-2\int_P \big(u^{ij}_{\ i}(t,x)-\underline{u}^{ij}_{\ i}\big)\big(u_j(t,x)-\underline{u}_j\big)\,dx\\
&=-2\int_{\partial P} \big(u^{ij}(t,x)-\underline{u}^{ij}\big)\nu_i\big(u_j(t,x)-\underline{u}_j\big)\,ds+2\int_P \big(u^{ij}(t,x)-\underline{u}^{ij}\big)\big(u_{ij}(t,x)-\underline{u}_{ij}\big)\,dx\\
&=2\int_P \big(u^{ij}(t,x)-\underline{u}^{ij}\big)\big(u_{ij}(t,x)-\underline{u}_{ij}\big)\,dx\\
&\leqslant 0.
\end{aligned}$$

The boundary integrals vanish due to the fact that $u-\underline{u}$ is a periodic function on $\mathbb{R}^n$ with fundamental domain P. For the last step, notice that

$$\sum_{ij}\big(u^{ij}(t,x)-\underline{u}^{ij}\big)\big(u_{ij}(t,x)-\underline{u}_{ij}\big)=Trace\big(\big(u^{ij}(t,x)-I\big)\big(u_{ij}(t,x)-I\big)\big).$$

Thus at each point we can choose an orthonormal basis such that $(u_{ij}(t,x))=diag(\lambda_1,\dots,\lambda_n)>0$ since u is convex, then

$$\sum_{ij}\big(u^{ij}(t,x)-\underline{u}^{ij}\big)\big(u_{ij}(t,x)-\underline{u}_{ij}\big)=\sum_i(\lambda_i-1)\big(\lambda_i^{-1}-1\big)\leqslant 0. \qquad \square$$

As an immediate corollary, we have

Corollary 3.4. *The L^2 norm of $u(t,x)$ depends only on the initial metric and is bounded independent of t.*

Proof.

$$\int_P u^2(t,x)\,dx \leqslant 2\int_P \underline{u}^2 + \big(u(t,x)-\underline{u}\big)^2\,dx \leqslant 2\int_P \underline{u}^2 + \big(u(0)-\underline{u}\big)^2\,dx \leqslant C_0. \qquad \square$$

Then we have the following observation:

Proposition 3.5. *The L^∞ norm of $u(t,x)$ in P depends only on the initial metric and is bounded independent of t.*

Proof. By the periodicity of $u(t,x) - \underline{u}(x)$, we can do the estimates in larger domains than P: we control the upper bound of $u(t,x)$ in $[-2,2]^n$ and the lower bound of $u(t,x)$ in $[-1,1]^n$. Thus we are able to control the gradient of $u(t,x)$ in P.

Notice that the L^2 norm of $u(t,x)$ in domain $[-3,3]^n$ is bounded C_1 which depends only on the initial metric and is independent of t. Let us temporarily suppress the t variable, we will write $u(t,x)$ as $u(x)$ in the proof. First, we prove that $u(x)$ has an upper bound. Suppose not, then the maximum of $u(x)$ in $[-2,2]^n$ reaches at one of its vertices. Without loss of generality, we can assume that the maximum of $u(x)$ is reached at $(2,\dots,2)$. We consider the supporting plane of $u(x)$, i.e. $l(x)$, at $(2,\dots,2)$. Then it is easy to see that there is an area larger than 1 such that the value of $l(x)$ in this area is greater or equal to its value at $(2,\dots,2)$. Since the value of $u(x)$ is always greater than $l(x)$, we conclude that the L^2 norm of $u(x)$ is greater than C_1. A contradiction.

Next we show that $u(x)$ is bounded from below in $[-1,1]^n$. Let x be the point in $[-1,1]^n$ reaching the minimum of $u(x)$ in $[-1,1]^n$. Notice that $u(x)$ is bounded from above along the boundary of $[-2,2]^n$. If $u(x)$ is very negative, then there is an area larger than 2^n such that the value of $u(x)$ in this area is less than $u(x)/3$. It also contradicts to the fact that the L^2 norm of $u(x)$ is bounded by C_1. $\square$

Since u is convex, we have

Corollary 3.6. *The derivative of $u(t,x)$ with respect to x in P is bounded by C_2 which depends only on the initial metric and is independent of t.*

Now we give a proof of Proposition 3.2.

Proof of Proposition 3.2. It is easy to see that if the derivative of u is bounded, then the M-condition holds automatically by its definition. $\square$

4. Non-collapsing

In this section, we apply Donaldson's estimates and the regularity theorem to obtain the lower bound of the injectivity radius in Theorem 1.1. Since the estimates are done in $t=0$, we will write $u(x)$ instead of $u(0,x)$ for convenience. Also we write P instead of λP in this section.

Notice that although the evolution equation of the Riemann curvature tensor of the Calabi flow is expressed in terms of holomorphic coordinates, Streets' estimates which are done in terms of real coordinates can still go through:

(1) The integral estimates in [31] can go through because

$$\frac{1}{2}\triangle_d = \triangle_\partial = \triangle_{\bar\partial}$$

in Kähler manifolds.

(2) To do the analysis in [30], we not only need to lift up the metric g to the tangent bundle at x, i.e. $T_x X$, but also we need to lift up the holomorphic structure J and the Kähler form ω. This can also be done.

Donaldson proves the following lemma in dimension 2. With some modifications, the second named author generalizes Donaldson's result to any dimension.

Lemma 4.1. *If the square of the Riemannian curvature norm*

$$|Rm|^2(x) = \sum_{i,j,k,l} u^{ij}_{\ kl} u^{kl}_{\ ij}(x) \leqslant 1$$

pointwisely and there is a constant M such that u satisfies the M-condition, then for any point $x \in P$, we have

$$\big(u_{ij}(x)\big) < C I_n,$$

where C is a constant depending only on M.

Proof. See Lemma 4 in [20] and Lemma 4.4 in [25]. □

Remark 3. Once we have the upper bound of $(D^2 u)$, we can give a proof of the regularity theorem from the weak regularity theorem, as shown in Appendix A.

The following lemma which obtains the lower bound of (u_{ij}) at one point is established by Donaldson in dimension 2. The second named author generalizes it to higher dimensions.

Lemma 4.2. *Suppose in P, the L^∞ norm of Riemannian curvature*

$$|Rm|_{L^\infty} \leqslant 1$$

and

$$\int_P \big|Rm(x)\big|^n \, dx < C_1.$$

If there is a point $x \in P$ such that $|Rm|(x) = 1$. Then the regularity theorem tells us that

$$\big(u_{ij}(x)\big) > C_2 I_n,$$

where C_2 depends only on M, C_1 and n.

Proof. See Proposition 11 in [20] and Section 5 in [25]. □

By applying Donaldson's estimates, we can control the lower bound of $(u_{ij}(y))$ for any other point y.

Lemma 4.3. *Suppose in the polytope P, $|Rm|_{L^\infty} \leqslant 1$. For any point $y \in P$, let the Riemannian distance between x and y be d. We have*

$$\big(u_{ij}(y)\big) \geqslant \frac{1}{e^{2d}}\big(u_{ij}(x)\big).$$

Proof. We apply Lemma 7 in [20]. Since we do not have boundaries in our case, we can let the boundary distance α go to ∞. Hence we obtain the result. □

Notice that if (u_{ij}) is bounded from above, then the geodesic distance is bounded above by the Euclidean distance multiplying a constant. Thus we obtain the lower bound of $(u_{ij})(y)$ depending on the Euclidean distance between y and x.

Once we obtain the upper bound and lower bound of (u_{ij}) of points around x. Applying Donaldson's arguments in Lemma 11 of [20], we see that the injectivity radius at x is bounded from below. For reader's convenience, we repeat his arguments here.

Proof of Theorem 1.1. Since we already control the upper bound and lower bound of (D^2u), we only need to control the injectivity radius of $P \times \mathbb{R}^n$. Moreover, we only need to consider cut points because the curvature is bounded. Applying Lemma 8 of [20] (Lemma 4.9 of [25]), we conclude that the Euclidean metric can be comparable with the Riemannian metric. Then Lemma 10 of [20] shows that the injectivity radius at x has a lower bound. □

5. Singularity analysis

Since we have controlled the M-condition along the Calabi flow, thus we are ready to prove Theorem 1.2 by the blow-up analysis. First, we study the formation of singularities of the Calabi flow under the assumption that the total energy, i.e.,

$$\int_P \big|Rm(t,x)\big|^n\,dx$$

is uniformly bounded along the flow.

By Chen and He's result [10], we know that if the Calabi flow cannot extend over time T, then there is a sequence of times $t_i \to T$ and a sequence of points $p_i \in P$ such that $|Ric|(t_i, p_i) \to \infty$. Since we are dealing with the global convergence, we also need to rule out the case that there is a sequence of points (t_i, p_i) where t_i may approach to ∞ such that $|Rm(t_i, p_i)| \to \infty$. We will prove this by the contradiction arguments.

Suppose not, then without loss of generality, we can assume that

$$\big|Rm(t_i, p_i)\big| = \max_{t \leqslant t_i,\ p \in P}\big|Rm(t, p)\big|.$$

Denote $\lambda_i = |Rm(t_i, p_i)|$, rescaling the original Calabi flow $u(t)$ by λ_i, i.e.,

$$P_i = \lambda_i P,$$
$$u_i(t,x) = \lambda_i u\left(\frac{t-T_i}{\lambda_i^2}, \frac{x-p_i}{\lambda_i}\right).$$

Then we get a sequence of the Calabi flows $u_i(t)$ such that $|Rm_i(0,0)| = 1$ and

$$\max_{t\leqslant 0,\ p\in P_i} |Rm|(t,p) = 1.$$

We try to show that a subsequence of the Calabi flow converges to a limiting Calabi flow and the limiting Calabi flow cannot exist. Then we can conclude that the Riemannian curvature is uniformly bounded along the Calabi flow.

We apply the regularity theorem now. Notice that the first Euclidean derivative of u is unchanged by rescaling, thus the M-condition is preserved under the rescaling. Also the total energy is preserved under the rescaling because the contribution from the $|Rm|^n$ cancels with the contribution from the volume. Thus we obtain the non-collapsing property of $(P_i, u_i(0))$. By [18], Abreu's equation can be rewritten as

$$U^{ij}\left(\frac{1}{\det(u_{kl})}\right)_{ij} = -S,$$

where S is the scalar curvature. Notice that (U^{ij}) is an elliptic operator. In order to apply Shauder's estimates to control the derivatives of u, we need to control the derivatives of S in the Euclidean sense. Using the regularity theorem, we can show that the scalar curvature S has a uniform C^k bound in the Euclidean sense.

Proposition 5.1. *Let f be any smooth function on X and be invariant under the torus action. Then*

$$\left|\partial^k f\right|_g^2 = \sum_{i_1, j_1, \ldots, i_k, j_k} u^{i_1 j_1} \cdots u^{i_k j_k} f_{i_1\cdots i_k} f_{j_1\cdots j_k}.$$

Proof. By definition,

$$\left|\partial^k f\right|_g^2 = g^{i_1\bar{j}_1} \cdots g^{i_k\bar{j}_k} f_{,\, i_1\cdots i_k} f_{,\, \bar{j}_1\cdots\bar{j}_k}.$$

A direct calculation shows

$$\begin{aligned}
&g^{i_1\bar{j}_1} \cdots g^{i_k\bar{j}_k} f_{,\, i_1\cdots i_k} f_{,\, \bar{j}_1\cdots\bar{j}_k} \\
&= g^{i_k\bar{j}_k} \frac{\partial}{\partial z_{i_k}} \left(g^{i_{k-1}\bar{j}_{k-1}} \cdots g^{i_1\bar{j}_1} f_{,\, i_1\cdots i_{k-1}}\right) f_{,\, \bar{j}_1\cdots\bar{j}_k} \\
&= g^{i_k\bar{j}_k} \frac{\partial}{\partial z_{i_k}} \left(g^{i_{k-1}\bar{j}_{k-1}} \frac{\partial}{\partial z_{i_{k-1}}} \left(\cdots g^{i_2\bar{j}_2} \frac{\partial}{\partial z_{i_2}} \left(g^{i_1\bar{j}_1} f_{i_1}\right) \cdots\right)\right) f_{,\, \bar{j}_1\cdots\bar{j}_k} \\
&= \frac{\partial^k f}{\partial x_{j_k} \cdots \partial x_{j_1}} f_{,\, \bar{j}_1\cdots\bar{j}_k}
\end{aligned}$$

$$
\begin{aligned}
&= \delta^{i_1}_{j_1} \cdots \delta^{i_k}_{j_k} \frac{\partial^k f}{\partial x_{i_k} \cdots \partial x_{i_1}} f_{,\ \bar{j}_1 \cdots \bar{j}_k} \\
&= u^{i_1 \alpha_1} \cdots u^{i_k \alpha_k} \frac{\partial^k f}{\partial x_{i_k} \cdots \partial x_{i_1}} g^{\alpha_1 \bar{j}_1} \cdots g^{\alpha_k \bar{j}_k} f_{,\ \bar{j}_1 \cdots \bar{j}_k} \\
&= u^{i_1 j_1} \cdots u^{i_k j_k} f_{i_1 \cdots i_k} f_{j_1 \cdots j_k}. \qquad \square
\end{aligned}
$$

Corollary 5.2. *If $(D^2 u)$ is bounded from above, then $D^k S$ is bounded.*

Proof. We choose an orthonormal basis such that $(D^2 u) = diag(\lambda_1, \dots, \lambda_n)$. Since $(D^2 u)$ is bounded from above, we have $\lambda_i \leqslant C$ for all i. Thus

$$
\begin{aligned}
u^{i_1 j_1} \cdots u^{i_k j_k} S_{i_1 \cdots i_k} S_{j_1 \cdots j_k} &= \sum_{i_1, \dots, i_k} \frac{S^2_{i_1 \cdots i_k}}{\lambda_{i_1} \cdots \lambda_{i_k}} \\
&\geqslant \frac{1}{C^k} \sum_{i_1, \dots, i_k} S^2_{i_1 \cdots i_k}.
\end{aligned}
$$

Since $|\nabla^k S|$ is bounded by the regularity theorem, we conclude that $|D^k S|$ is bounded. $\square$

We normalize $u_i(0, \cdot)$ at the origin by subtracting an affine function such that

$$
u_i(0, 0) = 0, \qquad D_x u_i(0, 0) = 0.
$$

Since for any t, $(D^2 u_i(t, \cdot))$ is bounded from above, we obtain the C^k bound of S in the Euclidean sense. Thus there is a constant $\delta_0 > 0$ such that

$$
C_1 < \left(D^2 u_i(t, 0)\right) < C_2
$$

for all $-\delta_0 \leqslant t \leqslant 0$. Hence for any p and $t \in [-\delta_0, 0]$, $(D^2 u_i(t, p))$ is bounded from below where the lower bound depends on the Euclidean distance from p to the origin. So we obtain the C^k bound of $u_i(t, p)$ in the Euclidean sense for $t \in [-\delta_0, 0]$.

Next we calculate the derivatives of $u_i(t, p)$ in time and the mixed derivatives in time and space. Notice that the first derivative of $u_i(t, \cdot)$ in t is just the scalar curvature $S_i(t, \cdot)$. We use u instead of $u_i(t, \cdot)$ in the following calculations for convenience. The second derivative of u in t is

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= -\frac{\partial S}{\partial t} \\
&= \sum_{ij} \left(\frac{\partial u^{ij}}{\partial t} \right)_{ij} \\
&= \sum_{ij} \left(u^{ik} S_{kl} u^{jl} \right)_{ij} \\
&= -\left(u^{i\alpha} u_{\alpha\beta i} u^{\beta k} S_{kl} u^{jl} \right)_j + \left(u^{ik} S_{kli} u^{jl} \right)_j - \left(u^{ik} S_{kl} u^{j\alpha} u_{\alpha\beta i} u^{\beta l} \right)_j
\end{aligned}
$$

$$
\begin{aligned}
&= u^{i\gamma}u_{\gamma\delta j}u^{\delta\alpha}u_{\alpha\beta i}u^{\beta k}S_{kl}u^{jl} - u^{i\alpha}u_{\alpha\beta ij}u^{\beta k}S_{kl}u^{jl} \\
&\quad + u^{i\alpha}u_{\alpha\beta i}u^{\beta\gamma}u_{\gamma\delta j}u^{\delta k}S_{kl}u^{jl} - u^{i\alpha}u_{\alpha\beta i}u^{\beta k}S_{klj}u^{jl} \\
&\quad + u^{i\alpha}u_{\alpha\beta i}u^{\beta k}S_{kl}u^{j\gamma}u_{\gamma\delta j}u^{\delta l} - u^{i\alpha}u_{\alpha\beta j}u^{\beta k}S_{kli}u^{jl} \\
&\quad + u^{ik}S_{klij}u^{jl} - u^{ik}S_{kli}u^{j\alpha}u_{\alpha\beta j}u^{\beta l} + u^{i\gamma}u_{\gamma\delta j}u^{\delta k}S_{kl}u^{j\alpha}u_{\alpha\beta i}u^{\beta l} \\
&\quad - u^{ik}S_{klj}u^{j\alpha}u_{\alpha\beta i}u^{\beta l} + u^{ik}S_{kl}u^{j\gamma}u_{\gamma\delta j}u^{\delta\alpha}u_{\alpha\beta i}u^{\beta l} \\
&\quad - u^{ik}S_{kl}u^{j\alpha}u_{\alpha\beta ij}u^{\beta l} + u^{ik}S_{kl}u^{j\alpha}u_{\alpha\beta i}u^{\beta\gamma}u_{\gamma\delta j}u^{\delta l}. \qquad (*)
\end{aligned}
$$

Lemma 5.3. *If we change the coordinate system by an orthonormal transformation, the value of $(*)$ remains unchanged at the origin.*

Proof. Let $O = (a_{ij})$ be an orthonormal matrix and $v(x) = u(x\ O)$, $A(x) = S(x\ O)$. Following the calculations in Claim 4.1 of [25], we have

$$
\frac{\partial v}{\partial x_i}(0) = \sum_{\alpha} a_{i\alpha}u_{\alpha}(0), \qquad \frac{\partial^2 v}{\partial x_i \partial x_j}(0) = \sum_{\alpha,\beta} a_{i\alpha}a_{j\beta}u_{\alpha\beta}(0),
$$
$$
\frac{\partial^3 v}{\partial x_i \partial x_j \partial x_k}(0) = \sum_{\alpha,\beta,\gamma} a_{i\alpha}a_{j\beta}a_{k\gamma}u_{\alpha\beta\gamma}(0),
$$
$$
\frac{\partial^4 v}{\partial x_i \partial x_j \partial x_k \partial x_l}(0) = \sum_{\alpha,\beta,\gamma,\delta} a_{i\alpha}a_{j\beta}a_{k\gamma}a_{l\delta}u_{\alpha\beta\gamma\delta}(0),
$$

and

$$
\frac{\partial A}{\partial x_i}(0) = \sum_{\alpha} a_{i\alpha}S_{\alpha}(0), \qquad \frac{\partial^2 A}{\partial x_i \partial x_j}(0) = \sum_{\alpha,\beta} a_{i\alpha}a_{j\beta}S_{\alpha\beta}(0),
$$
$$
\frac{\partial^3 A}{\partial x_i \partial x_j \partial x_k}(0) = \sum_{\alpha,\beta,\gamma} a_{i\alpha}a_{j\beta}a_{k\gamma}S_{\alpha\beta\gamma}(0),
$$
$$
\frac{\partial^4 A}{\partial x_i \partial x_j \partial x_k \partial x_l}(0) = \sum_{\alpha,\beta,\gamma,\delta} a_{i\alpha}a_{j\beta}a_{k\gamma}a_{l\delta}S_{\alpha\beta\gamma\delta}(0).
$$

Also we have

$$
v^{ij}(0) = \sum_{\alpha,\beta} a_{i\alpha}a_{j\beta}u^{\alpha\beta}(0).
$$

By routine calculations, we can check that, for example,

$$
u^{i\gamma}u_{\gamma\delta j}u^{\delta\alpha}u_{\alpha\beta i}u^{\beta k}S_{kl}u^{jl} = v^{i\gamma}v_{\gamma\delta j}v^{\delta\alpha}v_{\alpha\beta i}v^{\beta k}A_{kl}v^{jl}.
$$

Thus we obtain the conclusion. $\square$

To show that $(*)$ is bounded, we can assume that we are in the origin. By an orthonormal transformation of the coordinate system, we can assume that $(u_{ij}) = diag(\lambda_1, \dots, \lambda_n)$ is a diagonal matrix. Then

$$u^{i\gamma} u_{\gamma\delta j} u^{\delta\alpha} u_{\alpha\beta i} u^{\beta k} S_{kl} u^{jl} = \sum_{i,j,k,\delta} \frac{1}{\lambda_i \lambda_j \lambda_k \lambda_\delta} u_{i\delta j} u_{\delta ki} S_{kj}.$$

The right hand side is obviously bounded. Thus we conclude that $\frac{\partial^2 u}{\partial t^2}(t,p)$, $t \in [-\delta_0, 0]$ is bounded by a constant depending only on the Euclidean distance between p and the origin. Similar arguments show that

Proposition 5.4.

$$\frac{\partial^k u}{\partial t^k}(t,p), \qquad \frac{\partial^{k+l} u}{\partial t^k \partial x^l}(t,p)$$

is bounded for all $t \in [-\delta_0, 0]$, $k \geqslant 1$, $l \geqslant 0$. *The bounds depend on* k *or* k, l *and the Euclidean distance between* p *and the origin.*

So there is a subsequence of $u_i(t)$ converging to a limiting Calabi flow $u_\infty(t)$, $-\delta_0 \leqslant t \leqslant 0$. Moreover, $u_\infty(0)$ is a smooth convex function $\bar{u}$ in $\mathbb{R}^n$ with the following property:

(1) The L^∞ norm of Riemannian curvature is bounded by 1, i.e.,

$$|Rm|_{L^\infty} = \max_{p \in \mathbb{R}^n} \sum_{i,j} \bar{u}^{ij}_{\ \ kl}(p) \bar{u}^{kl}_{\ \ ij}(p) \leqslant 1.$$

(2) The Euclidean derivative of $\bar{u}$ is bounded by M which is the same constant in the M-condition, i.e.,

$$|D\bar{u}| < M.$$

We want to rule out the above singularity by the following nonexistence lemma:

Lemma 5.5. *If in addition* $\bar{S} = 0$, *then such* $\bar{u}$ *cannot exist.*

Proof. For dimensional 2, see Theorem 2 in [20]. For higher dimension, see Proposition 5.2 in [25]. □

Remark 4. In dimensional 2, Jia and Li prove a more general result in [27]: the solution to the equation

$$\sum_{ij} u^{ij}_{\ \ ij} = 0$$

in $\mathbb{R}^2$ must be a quadratic function.

5.1. The case of dimension 2

In dimension 2, we will prove that the scalar curvature of the limiting Calabi flow $u_\infty(t)$ is 0. For each $u_i(t)$, we temporarily suppress the index i. Let

$$E^{ij} = \frac{\partial u^{ij}}{\partial t}$$

and notice that the derivative of the Calabi energy

$$Ca_i(t) = \int_{P_i} S(t,x)^2\,dx$$

with respect to t is

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{P_i} S(t,x)^2\,dx &= -2\int_{P_i} (S-\underline{S})E^{ij}_{\ ij}\,dx \\
&= -2\int_{\partial P_i} (S-\underline{S})E^{ij}_{\ i}\nu_j\,ds + 2\int_{P_i} S_j E^{ij}_{\ i}\,dx \\
&= 2\int_{P_i} S_j E^{ij}_{\ i}\,dx \\
&= 2\int_{\partial P_i} S_j E^{ij}\nu_i\,ds - 2\int_{P_i} S_{ij}E^{ij}\,dx \\
&= -2\int_{P_i} S_{ij}E^{ij}\,d\mu \\
&= -2\int_{P_i} S_{ij}u^{ia}S_{ab}u^{bj}\,dx \\
&\leqslant 0.
\end{aligned}$$

Hence

$$Ca_i(-\delta_0) - Ca_i(0) = 2\int_{\delta_0}^{0}\int_{P_i} S_{ij}u^{ia}S_{ab}u^{bj}\,dx\,dt.$$

For the limiting Calabi flow, we have

$$0 = \lim_{i\to\infty} Ca_i(-\delta_0) - Ca_i(0) \geqslant 2\int_{-\delta_0}^{0}\int_{\mathbb{R}^n} S_{ij}u^{ia}S_{ab}u^{bj}\,dx\,dt.$$

That is to say, $S(t)$ must be an affine function on $\mathbb{R}^n$. Since

$$Ca_\infty(0) = \lim_{i\to\infty} Ca_i(0) < C,$$

$S(t)$ must be 0.

Proof of Theorem 1.2. In [6], Calabi shows that the Calabi flow decreases the Calabi energy. Moreover, he shows that in dimension 2, the total energy is equivalent to the Calabi energy. Combining the above results, we obtain that the curvature is uniformly bounded along the Calabi flow. □

6. Exponential convergence

Suppose that the Calabi flow exists for all time and the curvature is uniformly bounded. The remaining question is whether the Calabi flow converges to the flat Kähler metric. A well-known fact is that the Calabi flow decreases the Mabuchi energy. In our case, the Mabuchi energy can be explicitly written as

$$Ma(u) = -\int_P \log\det(u_{ij})\,dx.$$

Taking the derivative with respect to the time variable t, we obtain

$$\frac{\partial}{\partial t} Ma\big(u(t,x)\big) = \int_P u^{ij}(t,x) S_{ij}(t,x)\,dx = -\int_P S^2(t,x)\,dx \leqslant 0.$$

Since the curvature is bounded uniformly and the M-condition is preserved along the Calabi flow, we know that $(u_{ij}(t,x))$ is bounded from above. The fact that the Mabuchi energy is decreasing along the Calabi flow shows that for any t, there is at least one point $p \in P$ such that

$$\det\big(D^2 u(t,x)(p)\big) > C$$

for some constant C. Hence

$$\big(D^2 u(t,x)(p)\big) > C I_n.$$

Thus $(D^2 u(t,x))$ is bounded from below point-wisely. Notice that $u(t,x)$ has a priori C^0 and C^1 bound. Hence if we take a sequence of time $t_i \to \infty$, applying Corollary 5.2 and Proposition 5.4, we can show that there is a subsequence of t_i such that the Calabi flow

$$u_i(t,x) = u(t - t_i, x)$$

in the interval $[-1,0]$ converges to a limiting flow $u_\infty(t,x)$, $t \in [-1,0]$. Following the arguments as in the previous section, we conclude that $A_\infty(t,x)$ must be an affine function with respect to x. Hence $u_\infty(t,x)$ must be the flat Kähler metric.

To show the exponential convergence, by the stability result of [26], we only need to show that the corresponding Kähler potential $\phi(t_i, \xi)$ satisfies the following conditions:

- $\omega(t_i) \geqslant C_1 \omega_\infty$.
- $|\phi(t_i, \xi)|_{C^{2,\alpha}(\omega_\infty)} < C_2$.
- $\lim_{i\to\infty} dist(\phi(t_i), \phi_\infty) = 0$.

It is easy to see that the first and the last conditions are satisfied. Since ω_∞ is a flat metric, to control the $C^{2,\alpha}$ norm of $\phi(t_i, \xi)$, by compact embedding $C^3 \hookrightarrow C^{2,\alpha}$, we only need to control the third derivative of $\phi(t_i, \xi)$,

$$\frac{\partial^3 \phi(t_i, \xi)}{\partial \xi_j \partial \xi_k \partial \xi_l} < C.$$

We can obtain this inequality by the following formula:

$$\begin{aligned}
\frac{\partial^3 \phi(t_i, \xi)}{\partial \xi_j \partial \xi_k \partial \xi_l} &= \frac{\partial u^{jk}(t_i, x)}{\partial \xi_l} \\
&= \frac{\partial x_\alpha}{\partial \xi_l} \frac{\partial u^{jk}(t_i, x)}{\partial x_\alpha} \\
&= u^{\alpha l}(t_i, x) \frac{\partial u^{jk}(t_i, x)}{\partial x_\alpha} \\
&= -u^{j\alpha}(t_i, x) u^{k\beta}(t_i, x) u^{l\gamma}(t_i, x) u(t_i, x)_{\alpha\beta\gamma}.
\end{aligned}$$

Appendix A

In this appendix, we want to show that the M-condition with the weak regularity theorem can give us the strong regularity theorem.

Theorem A.1. *Suppose in P, the L^∞ norm of Riemann curvature tensor is bounded by* 1 *and the symplectic potential u satisfies the M-condition. If*

$$\int_P \left|\nabla^k Rm\right|^2(x)\, dx < C(k),$$

for all k, then

$$\left|\nabla^k Rm\right|(x) < C(k, M),$$

for all k and $x \in P$.

Proof. Let $F^k(x) = |\nabla^k Rm|(x)$. Since the curvature is bounded and u satisfies the M-condition, we conclude that

$$\left(u_{ij}(x)\right) < C$$

for all $x \in P$. It is easy to see that for any $x \in P$,

$$\left|\nabla F^k(x)\right| \leqslant \left|F^{k+1}(x)\right|.$$

Since

$$\left|\nabla F^k(x)\right|^2 = 2\sum_{i,j} u^{ij} F_i^k F_j^k \geqslant C\left|\nabla_E F^k\right|_E^2,$$

where $|\nabla_E F^k|_E$ is the Euclidean norm of the Euclidean derivative. Thus we have

$$\int_P \left|\nabla_E F^k\right|_E^2 dx < C(k).$$

The Sobolev embedding theorem tells us that

$$\int_P \left(F^k(x)\right)^q dx < C(k),$$

where $q = \frac{2n}{n-2}$ if $n > 2$ or $q = 4$ if $n = 2$. It is easy to see that after finite steps, we could reach the conclusion. □

Acknowledgments

The authors would like to express their gratitude to the anonymous referee for his numerous suggestions. The second named author is grateful for the consistent support of Professor Xiuxiong Chen, Pengfei Guan, Vestislav Apostolov and Paul Gauduchon. He also benefited from the discussion with Professor Shing-Tung Yau during his visit at Harvard University. He wants to thank Joel Fine, Si Li and Jeffrey Streets for useful conversations. Part of this work was done while the second named author was visiting the Northwestern University, he would like to thank Professor Steve Zelditch for his warm hospitality.

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