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SEMICOMPACTNESS AND DIMENSION OF INCREMENTS

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We construct two examples the first of which is a Lindelöf, separable and strongly zero-dimensional space the increment of which in any compactification is collectionwise normal, countably paracompact and infinite dimensional. The second example is a Lindelöf, separable, non-semicompact space that has a compactification with discrete increment.

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compactification	covering dimension
increment	Čech-complete
semicompact	Lindelöf
collectionwise normal spaces	countably paracompact

1. Introduction

To put the examples described in the abstract in perspective, we recall some definitions and results.

A space is called semicompact (rimcompact or peripherally compact) if it has a base consisting of open sets with compact boundary.

Theorem 1. *A space X satisfying the bicompact axiom of countability has a compactification Y with* $\dim(Y-X)\leq 0$ *iff X is semicompact* [7, 8, 12, 14].

A space X satisfies the bicompact axiom of countability if every compact subset of X is contained in another compact subset of X possessing a countable base of neighbourhoods. This is in fact a necessary and sufficient condition for the increment $Y-X$ to be Lindelöf for every compactification Y of X [14].

Theorem 2. *Every semicompact space X has a compactification Y with* $\operatorname{ind}(Y-X)\leq 0$ [7, 9].

Example 1. There is a non-semicompact space X with $\operatorname{ind}(\beta X-Y)=0$ (but $\dim(\beta X-X)\neq 0$) [13].

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Example 2. There is a non-semicompact space X having a compactification Y with $\dim(Y-X)=0$ [9].

Example 3. There is a semicompact space X such that for every compactification Y of X, $Y-X$ is normal and $\dim(Y-X)=\infty$ [2].

Icd X (resp. cmp X) is defined inductively as follows. Icd $X=-1$ (resp. cmp $X=-1$) iff X is Čech-complete (resp. X is compact). For $n=0,1,2,\ldots$, Icd $X\leqslant n$ (resp. cmp $X\leqslant n$) if whenever F is a closed (resp. singleton) set and G an open set of X with $F\subset G$, then there is an open set U of X with $F\subset U\subset G$ and Icd $B(U)\leqslant n-1$ (resp. cmp $B(U)\leqslant n-1$), where $B(U)$ denotes the boundary of U. Of course, Icd $X=n$ (resp. cmp $X=n$) if Icd $X\leqslant n$ and Icd $X\nleqslant n-1$ (resp. cmp $X\leqslant n$ and cmp $X\nleqslant n-1$) and Icd $X=\infty$ (resp. cmp $X=\infty$) if Icd $X\nleqslant n$ (resp. cmp $X\nleqslant n$) for each positive integer n.

De Groot's conjecture.[1] *If X is separable metrisable,* cmp $X\leqslant n$ *iff X has a compactification Y with* $\dim(Y-X)\leqslant n$ [8].

Note that if this conjecture is true, it generalises Theorem 1 for X metric separable as cmp $X\leqslant 0$ iff X is semicompact.

Theorem 3. *For a metrisable space X,* Icd $X\leqslant n$ *iff X has a completion Y with the* $\dim(Y-X)\leqslant n$ [1].

Example 4. There is a Lindelöf, separable space X of weight ω_1, such that every two disjoint closed sets of X can be separated by a locally compact closed subset of X and if Y is a Čech-complete extension of X, then $\dim(Y-X)=\operatorname{ind}(Y-X)=\infty$ [10].

The space X of Example 4 satisfies Icd $X=0$ and so shows that Theorem 3 is not valid outside metrisable spaces. As cmp $X=1$, it also shows that de Groot's conjecture is certainly false outside metrisable spaces. Our first example demonstrates these two points more dramatically as we show that the constructed space X satisfies $\dim(Y-X)=\infty$ for every Čech-complete extension Y of X in addition to the properties announced in the abstract.

Eric Van Douwen is reported in [10] to be of the opinion that Example 1 can be modified to produce a space Y with cmp $Y=\infty$ and $\dim(\beta Y-Y)=0$. The space X of our second example is Lindelöf, separable and has a compactification with discrete increment. Moreover, $\dim(\beta X-X)=0$ while cmp $X=\infty$. X can be used to construct a space Z with cmp $Z=\infty$ and $\beta Z-Z$ discrete, but this last property disqualifies Z from being Lindelöf.

[1] The referee has kindly informed the author that this conjecture has now been answered in the negative by R. Pol in his paper "On de Groot's conjecture cmp = def", Bull. Acad. Pol. Sci., 30 (1982) 461–464.

Our method of construction, which bears no relation to those employed in the examples listed above, is that of producing new topologies from standard ones by assigning limit points to certain sequences so as to induce the required results [cf, 4, 11, 3].

In this paper, all spaces are at least Tychonoff, $|X|$ denotes the cardinality of X, βX the Stone-Čech compactification of X, c the cardinality of the continuum, $\omega(c)$ the first ordinal of cardinality c and ω_1 the first uncountable ordinal. N denotes the set of positive integers and I the unit interval with their usual topologies. The product of countably infinite copies of I is denoted by I^N, its usual topology by τ and its usual metric by d.

For standard results in General Topology or Dimension Theory we refer to [5, 6].

2. The first example

Let $\{S_\alpha = (S_{\alpha 1}, S_{\alpha 2}, \ldots): \alpha < \omega(c)\}$ be an enumeration of the collection of all sequences of countable subsets of I^N with $|\bigcap_{i=1}^{\infty} \bar{S}_{\alpha i}| = c$ such that for each $\alpha < \omega(c)$, there is a limit ordinal β and a non-limit ordinal γ with $\alpha < \beta < \gamma < \omega(c)$ and $S_\alpha = S_\beta = S_\gamma$. Let P be a dense countable subset of I^N and $\lhd$ a well-ordering on I^N of the same type as $\omega(c)$ and such that $x < y$ if $x \in P$ and $y \notin P$. For each $\alpha < \omega(c)$, we choose a point x_α in $\bigcap_{i=1}^{\infty} \bar{S}_{\alpha i}$ and a sequence $\{x_{\alpha n}\}$ converging to x_α so that

(1) Each $\{x_{\alpha n}\}$ contains infinitely many points from P and from each $S_{\alpha i}$, $i \in N$.

(2) $x \lhd x_\alpha$ if $x = x_{\alpha n}$ or $x \in P$ or $x = x_\beta$ and $\beta < \alpha$.

Let $A = \{x_\alpha : \alpha$ a limit ordinal in $\omega(c)\} \cup P$ and $B = \{x_\alpha : \alpha$ a non-limit ordinal in $\omega(c)\}$. We note that A, B are Bernstein sets of I^N i.e. if F is an uncountable closed subset of I^N then $|A \cap F| = |B \cap F| = c$. For then $|F| = c$ and if S is a countable dense subset of F then $(S, S, \ldots) = S_\alpha$ and hence $x_\alpha \in F$ for continuum many α's that are limit ordinals and continuum many α's that are non-limit ordinals.

We now define a new topology σ on $A \cup B$, finer than τ, by using transfinite induction with respect to $\lhd$ to define 'basic' neighbourhoods according to the following specifications.

(3) If $x \in P$, then $\{x\}$ is a basic neighbourhood of x.

(4) For $\alpha < \omega(c)$, a basic neighbourhood of x_α consists of x_α together with basic neighbourhoods of $x_{\alpha n}$ of d-diameter $\leq 1/n$ for all but finitely many n's.

Transfinite induction readily shows that all basic neighbourhoods are countable, τ-closed and compact. Hence σ is zero-dimensional Tynchonoff, locally compact and locally countable. Furthermore, $\{x_{\alpha n}\}$ converges to x_α with respect to σ and hence, in view of (1), P is dense in σ so that σ is separable.

We implicitly assume henceforth that subsets of $A \cup B$ carry the subspace topology with respect to σ.

Let $Z = \{\infty\} \cup A \cup B$ be the one-point compactification of $A \cup B$. Then $X = Z - B$ is our first example. Clearly X is separable and Z is a compactification of X. In the sequel, Y denotes an arbitrary compactification of X, $g: \beta X \to Z$ and $f: \beta X \to Y$ the canonical extensions of the identity map $X \to X$.

Claim 1. *X is Lindelöf with* $\dim X = 0$.

Proof. The fact that each point of $A \cup B$ has a countable neighbourhood implies that a compact subset of $A \cup B$ is countable. Hence X is Lindelöf. For if G is an open neighbourhood of ∞, then $X - G$ is countable. Furthermore, $X - G$ is contained in an open countable set U of $A \cup B$ and since X is normal on account of the fact that it is Lindelöf, there is an open set V of X with $X - G \subset V \subset \bar{V} \subset U$. Now as U is countable, $\dim U \leq 0$, and hence there is a clopen set H of U with $X - G \subset H \subset V$. It is readily seen that H is clopen in X and hence $X - H$ is a clopen neighbourhood of ∞ inside G. Since the basic neighbourhoods of each point of $A \cup B$ are clopen, we deduce that $\operatorname{ind} X \leq 0$ and since X is Lindelöf and non-empty, $\dim X = 0$. □

Claim 2. *If $B_1, B_2, \ldots$ are closed subsets of $Y - X$ with $\bigcap_{i=1}^{\infty} B_i = \emptyset$, then $\bigcap_{i=1}^{\infty} \overline{gf^{-1}(B_i)}^{\tau}$ is countable.*

Proof. Suppose this is false and let S_i be a countable τ-dense subset of $gf^{-1}(B_i)$. Then $|\bigcap_{i=1}^{\infty} \bar{S}_i^{\tau}| = c$ and $(S_1, S_2, \ldots) = S_\alpha$ for uncountably many limit ordinals $\alpha < \omega(c)$. For each such α, in view of (1) and (4), x_α belongs to $\bigcap_{i=1}^{\infty} \overline{gf^{-1}(B_i)} \cap A$ which is therefore uncountable. Let S be a countable τ-dense subset of $\bigcap_{i=1}^{\infty} \overline{gf^{-1}(B_i)} \cap A$. Then for some non-limit ordinal $\beta < \omega(c)$, $(S, S, \ldots) = S_\beta$. If y is a limit point of $\{x_{\alpha n}\} \cap S$ in $g^{-1}(x_\beta)$, it is readily seen that y is in $\bigcap_{i=1}^{\infty} f^{-1}(B_i)$, which implies $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$. □

Claim 3. *$Y - X$ is countably paracompact and collectionwise normal.*

Proof. As $f: \beta X - X \to Y - X$ is perfect and countable paracompactness and collectionwise normality are invariant under perfect mappings, it suffices to prove that $\beta X - X$ has these properties. Let E, F be disjoint closed sets of $\beta X - X$. Then by Claim 2, $\overline{g(E)}^{\tau} \cap \overline{g(F)}^{\tau}$ is countable and hence, as in the proof of Claim 1, it is contained in a countable clopen set H of $A \cup B$. Since metrisable and Lindelöf spaces are normal, there exist disjoint τ-open sets U_1, U_2 of $A \cup B - H$ and disjoint open sets V_1, V_2 of the σ-compact and hence Lindelöf space $g^{-1}(H)$ such that $\overline{g(E)}^{\tau} - H \subset U_1$, $\overline{g(F)}^{\tau} - H \subset U_2$, $E \cap g^{-1}(H) \subset V_1$ and $F \cap g^{-1}(H) \subset V_2$. Then $g^{-1}(U_1) \cup V_1$ and $g^{-1}(U_2) \cup V_2$ are disjoint open sets of βX containing E and F, respectively. Hence $\beta X - X$ is normal.

Let $B_1, B_2, \ldots$ be a decreasing sequence of closed sets of $\beta X - X$ with $\bigcap_{i=1}^{\infty} B_i = \emptyset$. Then $g(B_1), g(B_2), \ldots$ is a decreasing sequence of closed sets of $Z - X = B$. Furthermore, for each $x \in B$, $g^{-1}(x)$ is compact and hence $g^{-1}(x) \cap B_i = \emptyset$ for some

$i \in N$ since $\bigcap_{i=1}^{\infty} B_i = \emptyset$. Hence $\bigcap_{i=1}^{\infty} gB_i = \emptyset$. By Claim 2, we may let $\bigcap_{i=1}^{\infty} \overline{g(B_i)}^{\tau} = \{x_1, x_2, \ldots\}$. Let

$$G_i = \{x \in B\colon d(x, g(B_i)) < 1/i\} - (\{x_1, x_2, \ldots, x_i\} - g(B_i)), \quad i \in N.$$

Clearly, $\bigcap_{i=1}^{\infty} G_i = \emptyset$ so that $g^{-1}(G_i)$ is an open set of $\beta X - X$ containing B_i with $\bigcap_{i=1}^{\infty} g^{-1}(G_i) = \emptyset$. Since $\beta X - X$ is normal, this proves that it is also countably paracompact [5, Corollary 5.2.2].

Collectionwise normality of the normal space $\beta X - X$ will follow if we prove that a discrete subset D of it is countable. Now since $g\colon \beta X - X \to Z - X$ is perfect, then $g(D)$ is discrete in $Z - X = B$. But if $g(D)$ is uncountable and S is a τ-dense countable dense subset of it, then $(S, S, \ldots) = S_\alpha$ for some non-limit ordinal $\alpha < \omega(c)$, which implies that x_α is an accumulation point of $g(D)$ in B. We conclude that $g(D)$ is countable and since for each x in B, $g^{-1}(x)$ is compact and so $g^{-1}(x) \cap D$ is finite, then B is countable and $\beta X - X$ is collectionwise normal. □

Claim 4. *If E, F are disjoint closed sets of $A \cup B$, there is a countable subset H of B such that*

$$fg^{-1}(E - H) \cap fg^{-1}(F - H) = \emptyset.$$

Proof. Otherwise $E^* = \{x \in E\colon fg^{-1}(x) \cap fg^{-1}(F) \neq \emptyset\}$ is uncountable and if S is a τ-dense countable subset of it, then $(S, S, \ldots) = S_\alpha$ for some limit ordinal $\alpha < \omega(c)$. Then any neighbourhood of the point x_α of X in Y contains $fg^{-1}(V)$ for some open neighbourhood V of x_α in Z. Since V contains points of the subset S of E^*, then $fg^{-1}(V)$ contains points of $fg^{-1}(E) \cap fg^{-1}(F)$. This implies that the point x_α of X belongs to the closed set $fg^{-1}(E) \cap fg^{-1}(F)$ of $Y - \{\infty\}$, which is impossible since $E \cap F = \emptyset$ and f, g extend the identity $X \to X$. □

Remark 1. We digress to recall some facts about I^N. Let A_i, B_i, $i \in N$, be the ith pair of opposite faces of I^N and let E_i, F_i be disjoint closed subsets of I^N whose interiors contain A_i, B_i, respectively. It is well known that if for each i in N, Z_i is a partition in I^N between A_i and B_i, then $\bigcap_{i=1}^{\infty} Z_i \neq \emptyset$ and hence $|\bigcap_{i=2}^{\infty} Z_i| = c$. The same conclusion remains valid if Z_i is a partition in some Berstein subset J of I^N between $E_i \cap J$ and $F_i \cap J$. To see this, let L_i be a partition in I^N between A_i, B_i with $J \cap L_i \subset Z_i$ [6, Lemma 1.2.9]. Then $\bigcap_{i=2}^{\infty} L_i$ and hence $J \cap \bigcap_{i=2}^{\infty} L_i$ and $\bigcap_{i=2}^{\infty} Z_i$ have cardinality c.

Claim 5. *If W is the increment of X in a G_δ-set $W \cup X$ of Y, then* $\dim W = \infty$.

Proof. Let H be a countable subset of the locally countable space $A \cup B$ containing the σ-compact subset $gf^{-1}(Y - X \cup W)$ of B. In view of Claim 4, we can also suppose that for each i in N, $fg^{-1}(E_i - H) \cap fg^{-1}(F_i - H) = \emptyset$. Evidently we can further suppose that H is open in Z. Now $fg^{-1}(B - H)$ is a closed subset not only of W but also of the normal $Y-X$. It follows that every bounded real valued continuous function on $fg^{-1}(B-H)$ extends to W and so βW contains $\beta fg^{-1}(B-H)$. Hence $\dim fg^{-1}(B-H) \leq \dim W$, for $\dim X = \dim \beta X$ for all Tychonoff spaces X; in fact, if X is non-normal, $\dim X = \dim \beta X$ by definition. Thus, let us suppose that $\dim W < \infty$. Then there is a partition Z_i in $fg^{-1}(B-H)$ between $fg^{-1}(E_i-H)$ and $fg^{-1}(F_i-H)$, $i=2,3,\ldots$, such that $\bigcap_{i=2}^{\infty} Z_i = \emptyset$. Write $Z_i = X_i \cap Y_i$, where X_i, Y_i are closed sets of $fg^{-1}(B-H)$ disjoint from $fg^{-1}(E_i-H)$, $fg^{-1}(F_i-H)$, respectively, with $X_i \cup Y_i = fg^{-1}(B-H)$. Now, in view of Claim 2, there is a countable subset G of $A \cup B$ containing H such that, if $J = B-G$ and $i=2,3,\ldots$, then $\overline{gf^{-1}(X_i)^{\tau}} \cap \overline{gf^{-1}(Y_i)^{\tau}} \cap J$ is a partition in J between $E_i \cap J$ and $F_i \cap J$ with $\bigcap_{i=2}^{\infty} \overline{gf^{-1}(X_i)^{\tau}} \cap \overline{gf^{-1}(Y_i)^{\tau}} \cap J = \emptyset$. This contradicts Remark 1 since J is a Bernstein subset of I^N as a cocountable subset of a Bernstein subset of I^N. We conclude that $\dim W = \infty$.

Remark 2. The W of Claim 5 is not always normal. This is a consequence of the fact that there exists a non-normal space containing a copy of N as a dense subset and hence there exists some compactification of N which is not hereditarily normal so that some open subset of this compactification is not normal. To see this, consider a non-standard topology on the set of real numbers R such that each point of Q, the set rationals, is isolated and every neighbourhood of a point x in $R-Q$ contains all but finitely many members of some fixed sequence of rationals converging to x with respect to the standard topology on R. With this new topology, R is not normal because, since it is separable, the set of all continuous, bounded real valued functions on R has cardinality c while the corresponding set for its discrete closed subspace $R-Q$ has cardinlity 2^c.

3. The second example

Partition I^N into three disjoint Berstein set A, B, C, and let $\{S_\alpha = (S_{\alpha 1}, S_{\alpha 2}) : \alpha < \omega(c)\}$ be an enumeration of the collection of all pairs of sequences of A with $|\bar{S}_{\alpha 1} \cap \bar{S}_{\alpha 2}| = c$ such that for each $\alpha < \omega(c)$, there is a limit ordinal β and a non-limit ordinal γ with $\alpha < \beta < \gamma < \omega(c)$ and $S_\alpha = S_\beta = S_\gamma$. Let P be a countable dense subset of A and $\lhd$ a well-ordering on $A \cup B$ of the same type as $\omega(c)$ and such tht $x \lhd y$ if $x \in P$ and $y \notin P$. For each $\alpha < \omega(c)$, we choose a point x_α in $\bar{S}_{\alpha 1} \cap \bar{S}_{\alpha 2}$ and a sequence $\{x_{\alpha n}\}$ converging to x_α so that

(1) $\{x_{\alpha n}\}$ consists of infinitely many points from P and from each $S_{\alpha i}$, $i=1,2$.
(2) $x_\alpha \in A$ if α is a limit ordinal.
(3) $x_\alpha \in B$ if α is not a limit ordinal.
(4) $x \lhd x_\alpha$ for $x = x_{\alpha n}$ or $x \in P$ or $x = x_\beta$ with $\beta < \alpha$.

Finally, if x is a point of

$$A \cup B - \{x_\alpha : \alpha < \omega(c)\} \cup P$$

we fix a sequence $\{x_n\}$ in P converging to x.

Now we use transfinite induction with respect $\lhd$ to define 'basic' neighbourhoods of some topology σ on I^N as follows.

(5) If $x \in P$, $\{x\}$ is a basic neighbourhood of x.

(6) A basic neighbourhood of a point x of $A \cup B$ consists of x together with basic neighbourhoods of x_n of d-diameter $\leq 1/n$ for all but finitely many n's.

(7) Basic neighbourhoods of a point of C are all its τ-neighbourhoods.

It is readily seen that σ is a finer topology than τ and is locally compact and locally countable at each point of $A \cup B$. Regularity is easily verified using additionally (7). The fact that C is a Berstein set of I^N ensures that every σ-neighbourhood of C is co-countable in I^N and hence that σ is Lindelöf and therefore Tychonoff on any subset of I^N containing C.

In the sequel, Z denotes I^N with topology σ and Y the Stone-Čech compactification of Z.

Our second example is $X = Y - B$. Evidently, for each x in $A \cup B$, $\{x_n\}$ converges to x and hence, in view of the fact that $\{x_n\}$ contains infinitely many points of P, P is dense in Z, Y and X, which are therefore separable.

Claim 1. *Y is a compactification of X with $Y - X$ discrete.*

Proof. It suffices to observe that for each $\alpha < \omega(c)$, $\{x_{\alpha n}\}$ consists of points of A and transfinite induction establishes that x_α has a basic neighbourhood consisting entirely of points of A apart from x_α itself. □

Claim 2. *X is Lindelöf.*

Proof. Since each point of B has a compact neighbourhood in Z, then $\overline{Y - Z}$ contains no points of B. Hence $X = \overline{Y - Z} \cup (Z - B)$ and is therefore Lindelöf as the union of two Lindelöf spaces. □

Claim 3. cmp $X = \infty$.

Proof. First observe that ind $C = \dim C = \infty$ as can be seen from Remark 1 of Section 2. Hence there is a point x of C and a neighbourhood G of x in X such that for all partitions E of X between x and $X - G$, $\operatorname{ind}(E \cap C) = \infty$.

Now suppose that cmp $X = n < \infty$ and let E_1 be a closed subset of X separating between x and $X - G$ with cmp $E_1 \leq n - 1$. Then there are disjoint open sets U_1, V_1 of X such that $x \in U_1$, $X - G \subset V_1$ and $E_1 = X - U_1 \cup V_1$. We may of course assume that $E_1 = \bar{U}_1 \cap \bar{V}_1$. Now if $S_1 = U_1 \cap P$, $T_1 = V_1 \cap P$, then $\bar{S}_1 \cap \bar{T}_1 = E_1$ and $\bar{S}_1^\tau \cap \bar{T}_1^\tau \cap C = E_1 \cap C$.

Fix a countable τ-dense subset P_1 of $E_1 \cap A$ and let W, H be open neighbourhoods in X of a point of $E_1 \cap C$ with $\bar{H} \subset W$ and $H \cap E_1 \cap C$ uncountable. Then for some limit ordinal $\alpha < \omega(c)$, $(H \cap S_1, H \cap T_1) = S_\alpha$ so that x_α is a point of $\bar{H} \cap E_1 \cap A$ and hence $W \cap P_1 \neq \emptyset$. Noting also that $E_1 \cap C$ contains only countably many points with countable neighbourhoods, because $\operatorname{ind} E_1 \cap C = \infty$, we see that $F_1 = \bar{P}_1$ is a closed subset of E_1 with $\operatorname{ind} F_1 \cap C = \infty$ and $\operatorname{cmp} F_1 \leq n-1$.

The above process can be repeated to obtain for each $i = 2, 3, \ldots, n+1$, closed sets E_i, F_i of X with $F_i \subset E_i \subset F_{i-1}$, $\operatorname{ind} F_i \cap C = \infty$, $\operatorname{cmp} E_i \leq n-i$, a countable subset P_i of A such that $F_i = \bar{P}_i$ and subsets S_i, T_i of P_{i-1} with $E_i = \bar{S}_i \cap \bar{T}_i$. Now $E_{n+1} = \bar{S}_{n+1} \cap \bar{T}_{n+1}$ is a compact subset of X with $\operatorname{ind}(E_{n+1} \cap C) = \infty$. Hence $E_{n+1} \cap C = \bar{S}^{\tau}_{n+1} \cap \bar{T}^{\tau}_{n+1} \cap C$ is uncountable and for some non-limit ordinal $\alpha < \omega(c)$, $S_\alpha = (S_{n+1}, T_{n+1})$ so that the point x_α of B is a limit point of E_{n+1} contradicting the fact that E_{n+1} is a compact subset of X. We must conclude that $\operatorname{cmp} X = \infty$. □

Claim 4. $\dim(\beta X - X) = 0$.

Proof. Each point x of $B = Y - X$ has a countable closed neighbourhood H in Y. If $g: \beta X \to Y$ is the canonical extension of the identity $X \to X$, then $g^{-1}(x)$ is a closed subset of $\beta(H - \{x\})$ and hence $\dim g^{-1}(x) = 0$. The result then follows from the fact that $g^{-1}(x)$ is clopen in $\beta X - X$. □

Remark 1. Consider $W = [0, \omega_1] \times Y - \{\omega_1\} \times B$. If f: $W \to I$ is continuous, it follows from the fact that the countable set P is dense in Y that there is some $\alpha < \omega_1$ such that f is constant on $[\alpha, \omega_1] \times \{p\}$ for each p in P. Hence f extends to $[0, \omega_1] \times Y$, $\beta W = [0, \omega_1] \times Y$, $\beta W - W = \{\omega_1\} \times B$ is discrete while W is not semicompact, infact, $\operatorname{cmp} W = \operatorname{cmp}(\{\omega_1\} \times X) = \operatorname{cmp} X = \infty$.

We note, however, that if a space S is Lindelöf and non-compact, then $\beta S - S$ is not discrete. For in such a case, S is not sequentially compact and contains an infinite sequence with no accumulation point in S. Hence, since S is normal, $\beta S - S$ contains a copy of $\beta N - N$, which is not discrete.

The same argument shows that W is not normal because $\{\omega_1\} \times X$ contains a discrete infinite sequence converging to a single point of $\{\omega_1\} \times B$.

Remark 2. The following question presents itself. If S is a countable space, is it true that $\dim \beta S - S \leq 0$? If this fails, does S have a compactification T with $\dim T - S \leq 0$? Note that Theorem 1 does not apply as S need not satisfy the bicompact axiom of countability. For example, let $S = \{\infty\} \cup \bigcup_{n=1}^{\infty} S_n$ where each S_n is a copy of N and a neighbourhood of ∞ is cofinite in each S_n.

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