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AKCE International Journal of Graphs and Combinatorics 12 (2015) 180–185
AKCE International Journal of Graphs and Combinatorics
www.elsevier.com/locate/akcej

The gracefulness of the join of graphs (II)

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Received 22 December 2014; accepted 3 September 2015
Available online 23 December 2015

Abstract

In this paper, we present some new families of graceful join of graphs and propose a few unsolved problems in this area.

Keywords: Graceful graph; Join of graphs; Eulerian graph; Broken path

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We call G an (n, m)-graph if it is of order n (i.e., $|V(G)| = n$) and size m (i.e., $|E(G)| = m$). Assume that $1 \leq n - 1 \leq m$. A *graceful valuation* of G is an injection $\theta : V(G) \to \{0, 1, \ldots, m\}$ such that the induced mapping π, defined by $\pi(uv) = |\theta(u) - \theta(v)|$ for each edge uv in G, is a bijection between $E(G)$ and $\{1, 2, \ldots, m\}$. We call G a *graceful graph* if it admits a graceful valuation. For a general survey on graceful graphs, see [1].

Let G and H be two given graphs. The *join* of G and H, denoted by $G + H$, is the graph obtained from the disjoint union of G and H by joining each vertex in G to each vertex in H.

In [3], we present a brief survey on some major results about the gracefulness of $G + H$. In this paper, we shall establish some new families of graceful join of graphs and propose a few unsolved problems in this area.

As usual, K_n, O_n, P_n and C_n denote, respectively, the complete graph, empty graph, path and cycle of order n; $K(p, q)$, which can also be written as $O_p + O_q$, denotes the complete bipartite graph with p and q vertices in the respective partite sets, where $1 \leq p \leq q$.

2. Some major existing results

For ease of reference, we first state three earliest basic results on graceful graphs.

Theorem 1 *([2]). The graph K_n is graceful if and only if $2 \leq n \leq 4$.*

Peer review under responsibility of Kalasalingam University.
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http://dx.doi.org/10.1016/j.akcej.2015.11.013
0972-8600/

Fig. 1. Graph of $P(7, 3)$.

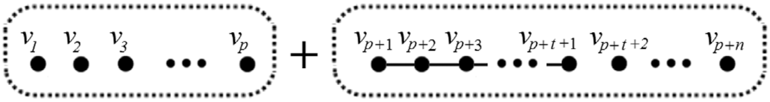

Fig. 2. Graph of $O_p + P(n, t)$.

Theorem 2 ([4]). *The graph $K(p, q)$ is graceful for all $1 \le p \le q$.*

Theorem 3 ([4]). *Let G be an Eulerian graph of size m. If $m \equiv 1$ or 2 (mod 4), then G is not a graceful graph.*

Recall in [3] that for $n \ge t + 2$ and $t \ge 1$, we denote $P(n, t)$ to be the graph of order n consisting of a path of length t and $n - (t + 1)$ isolated vertices, that is, the disjoint union of P_{t+1} and O_{n-t-1}. We call $P(n, t)$ a *broken path*. The graph of $P(7, 3)$ is shown in Fig. 1. Also, for any $r \ge 1$ and any graph H, we denote rH to be the disjoint union of r copies of H.

The gracefulness of the following types of the join of graphs is discussed in [3].

(1) $G + H$, where G be a graceful tree and H is one of the following:
 (i) O_p, (ii) $P(r, 1)$, (iii) $K(1, q)$;
(2) $K_n + O_p$;
(3) $G_f + O_p$, where G_f be the full augmentation of a graceful graph G;
(4) $C_n + O_p$;
(5) $rK_2 + O_p$;
(6) the complete n-partite graphs;
(7) $C_m + P(n, t)$.

3. The join $o_p + P(n, t)$

In this section, we prove the following first result:

Theorem 4. *The join $O_p + P(n, t)$ is graceful for all $p \ge 1$, $n \ge t + 2$ and $t \ge 1$.*

Proof. Let V be the vertex set of the graph $O_p + P(n, t)$ with size $np + t$. Label the vertices in V as shown in Fig. 2.

We define a valuation $f : V \to \{0, 1, \ldots, np + t\}$ as follows:

$$f(v_i) = \begin{cases} in + t & \text{if } 1 \le i \le p \\ \lfloor \frac{t}{2} \rfloor + (-1)^{i+t-p-1} \lfloor \frac{i-p}{2} \rfloor & \text{if } p + 1 \le i \le p + t + 1 \\ i - p - 1 & \text{if } p + t + 2 \le i \le p + n. \end{cases}$$

Clearly f is injective, since $f(v_i) \in \{n+t, 2n+t, 3n+t, \ldots, pn+t\}$ if $1 \le i \le p$, and $f(v_i) \in \{0, 1, 2, \ldots, n-1\}$ if $p + 1 \le i \le p + n$. Also, we observe that

1. $|f(v_a) - f(v_b)| \in \{1, 2, 3, \ldots, t\}$ if $v_a, v_b \in V\big(P(n, t)\big)$ such that $v_a v_b$ is an edge of $P(n, t)$, and
2. $|f(v_a) - f(v_b)| \in \{t + 1, t + 2, t + 3, \ldots, pn + t\}$ if $v_a \in V(O_p)$ and $v_b \in V\big(P(n, t)\big)$.

Since all the edges of $O_p + P(n, t)$ receive distinct labels, f is a graceful valuation. □

As an example, the graceful valuation of $O_4 + P(7, 3)$ shown in the proof for Theorem 4 is given in Fig. 3.

4. The join $o_p + I(n, t)$

For $t \ge 1$ and $n \ge 2t + 1$, we denote $I(n, t)$ to be the disjoint union of tK_2 and O_{n-2t}. The graph $I(10, 4)$ is shown in Fig. 4.

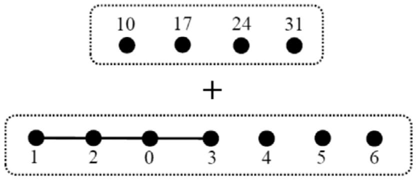

Fig. 3. A graceful valuation of $O_4 + P(7, 3)$.

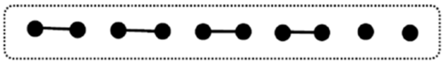

Fig. 4. Graph of $I(10, 4)$.

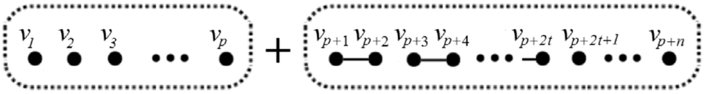

Fig. 5. Graph of $O_p + I(n, t)$.

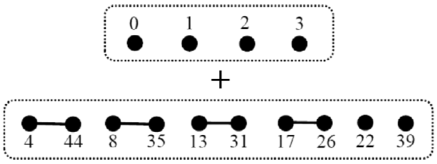

Fig. 6. A graceful valuation of $O_4 + I(10, 4)$.

For graphs of the form $tK_2 + O_p$, where $p \geq 1$, it is shown that not all are graceful (see, for instance, [1] and [3]). The following result says that the situation is different if we add at least one isolated vertex to tK_2.

Theorem 5. *The graph $O_p + I(n, t)$ is graceful for all $p \geq 1$, $n \geq 2t + 1$ and $t \geq 1$.*

Proof. Observe that when $t = 1$, $O_p + I(n, 1) = O_p + P(n, 1)$. Therefore by Theorem 4, result holds. For $t \geq 2$, let V be the vertex set of the graph $O_p + I(n, t)$ with size $np + t$. Label the vertices in V as shown in Fig. 5.

We define a valuation $f : V \to \{0, 1, \ldots, np + t\}$ as follows:

$$f(v_i) = \begin{cases} i - 1 & \text{if } 1 \leq i \leq p + 1 \\ np + t & \text{if } i = p + 2 \\ p + \dfrac{1}{2}\left(1 + (-1)^{i-p-2}\right)(2p + 1)t & \\ \quad +(-1)^{i-p-3}\left(\left\lfloor \frac{i-p-1}{2} \right\rfloor p + \left\lfloor \frac{i-p}{4} \right\rfloor\right) & \text{if } p + 3 \leq i \leq p + 2t + 1 \\ (i - p - 1)p + (t - 1) & \text{if } p + 2t + 2 \leq i \leq p + n. \end{cases}$$

The following observations imply that f is injective.

1. If $1 \leq i \leq p + 1$, then $f(v_i) \in \{0, 1, 2, \ldots, p\}$.
2. If $i = p + 2k + 1$ for some $k \in \{1, 2, 3, \ldots, t\}$, then $f(v_i) \in \left\{2p, 3p + 1, 4p + 1, \ldots, (t + 1)p + \left\lfloor \frac{t}{2} \right\rfloor\right\}$.
3. If $i = p + 2k + 2$ for some $k \in \{1, 2, 3, \ldots, t - 1\}$, then $f(v_i) \in \left\{2tp + t - 1, (2t - 1)p + t - 1, (2t - 2)p + t - 2, \ldots, (t + 2)p + t - \left\lfloor \frac{t}{2} \right\rfloor\right\}$.
4. If $p + 2t + 2 \leq i \leq p + n$, then $f(v_i) \in \{(2t + 1)p + t - 1, (2t + 2)p + t - 1, \ldots, (n - 1)p + t - 1\}$.
5. If $i = p + 2$, then $f(v_i) = np + t$.

Furthermore, it can be shown that all the edges receive distinct labels. It follows that f is a graceful valuation. □

As an illustration, the above graceful valuation of $O_4 + I(10, 4)$ is given in Fig. 6.

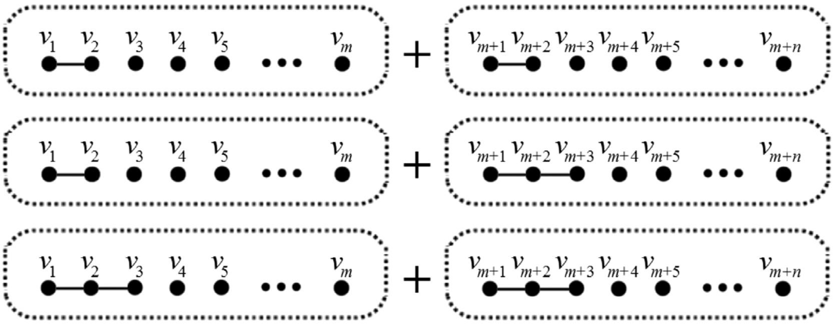

Fig. 7. $P(m, s) + P(n, t)$ when $s = t = 1$ (top), $s = 1$ and $t = 2$ (middle), and $s = t = 2$ (bottom).

5. The join $p(m, s) + p(n, t)$

In this section, we prove the following:

Theorem 6. *If $s, t \in \{1, 2\}$, then $P(m, s) + P(n, t)$ is graceful for all $m \geq s + 2$ and $n \geq t + 2$.*

Proof. Let V be the vertex set of the graph $P(m, s) + P(n, t)$ with size $mn + s + t$. Label the vertices in V as shown in Fig. 7. We define a valuation $f : V \to \{0, 1, \ldots, mn + s + t\}$ as follows:

For $P(m, 1) + P(n, 1)$ with $n \geq m \geq 3$:

i	1	$2 \leq i \leq m$	$m + 1$	$m + 2 \leq i \leq m + n$
$f(v_i)$	0	$(m - i + 1)n + 2$	1	$(i - m + 2) + (m - 1)n$

For $P(m, 1) + P(n, 2)$ with $m \geq 3$ and $n \geq 4$:

i	1	$2 \leq i \leq m$	$m + 1$	$m + 2$	$m + 3$	$m + 4 \leq i \leq m + n$
$f(v_i)$	0	$mn + 5 - i$	$m + 2$	1	$mn - m + 3$	$(i - m - 2)m + 2$

For $P(m, 2) + P(n, 2)$ with $n \geq m \geq 4$:

i	1	2	$3 \leq i \leq m$
$f(v_i)$	n	$mn + 2$	$(m - i + 2)n + 1$

i	$m + 1$	$m + 2$	$m + 3$	$m + 4$	$m + 5 \leq i \leq m + n$
$f(v_i)$	0	$mn + 4$	1	$mn + 3$	$i - m - 3$

It can be shown directly that f is a graceful valuation for each case. □

Other than the graceful valuation, f, that is defined in the proof for Theorem 6, we remark that there is an alternative graceful valuation of $P(m, 2) + P(n, 2)$. We present the alternative graceful valuation $g : V \to \{0, 1, \ldots, mn + 4\}$ in a tabular form as follows:

i	1	2	$3 \leq i \leq m$
$g(v_i)$	2	$(m - 1)n + 4$	$(m - i + 1)n + 3$

i	$m + 1$	$m + 2$	$m + 3$	$m + 4 \leq i \leq m + n$
$g(v_i)$	0	$mn + 4$	1	$(n + 1)m - i + 7$

Fig. 8 shows two graceful valuations of $P(5, 2) + P(6, 2)$.

6. The join $c_5 + P(n, 1)$

It is proved in [3] that the join $C_3 + P(n, t)$ is graceful for all $n \geq t + 2$, where $1 \leq t \leq 3$. In this section, we prove the following:

Theorem 7. *The join $C_5 + P(n, 1)$ is graceful for all $n \geq 3$.*

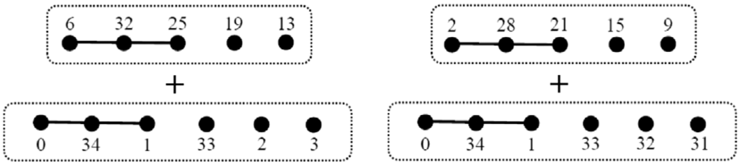

Fig. 8. Two graceful valuations of $P(5, 2) + P(6, 2)$: f as defined in the proof for Theorem 6 (left) and g (right).

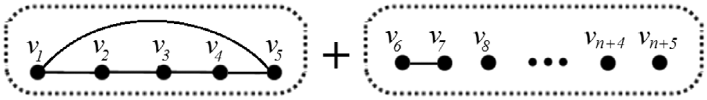

Fig. 9. Graph of $C_5 + P(n, 1)$.

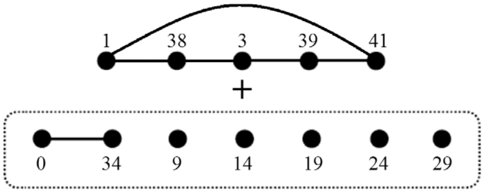

Fig. 10. A graceful valuation of $C_5 + P(7, 1)$.

Proof. Let V be the vertex set of the graph $C_5 + P(n, 1)$ with size $5n + 6$. Label the vertices in V as shown in Fig. 9. We define a valuation $f : V \to \{0, 1, \ldots, 5n + 6\}$ as follows:

i	1 or 3	2	4 or 5	6	7	$8 \le i \le n + 5$
$f(v_i)$	i	$5n + 3$	$5n + 2(i - 2)$	0	$5n - 1$	$5(i - 6) - 1$

It can be shown directly that f is a graceful valuation. □

A graceful valuation for $n = 7$ is shown in Fig. 10.

7. Some unsolved problems

In this final section, we propose some problems for further study.

Problem 1. Is the join $P(m, s) + P(n, t)$ always graceful for all $m \ge s + 2$ and $n \ge t + 2$, where $s \ge 3$ or $t \ge 3$?

Problem 2. As shown in [3], the join $C_3 + P(n, t)$ is always graceful for all $n \ge t + 2$, where $1 \le t \le 3$. How about the case when $t \ge 4$?

Problem 3. As shown in Theorem 7, the join $C_5 + P(n, 1)$ is graceful for all $n \ge 3$. Is the join $C_5 + P(n, t)$ graceful, where $n \ge t + 2$ and $t \ge 2$?

As mentioned in [3], (i) the graph $K_n + O_p$ is graceful for each $n \le 3$ and $p \ge 1$, and (ii) the graphs $K_n + O_1$, $K_n + O_2$ and $K_n + O_3$ are not graceful for all $n \ge 4$. By Theorem 3, it can be checked that for $r \ge 0$, the Eulerian graphs $K_4 + O_{2r+1}$ and $K_6 + O_{2r+1}$ are also not graceful.

Problem 4. Consider the graph $K_n + O_p$.

(i) For $n = 4, 5$ or 6, is the join $K_n + O_4$ graceful?
(ii) Does there exist a positive integer p such that $K_4 + O_p$ is graceful? If the answer is 'yes', find the least value of p.
(iii) Given that $n \ge 5$, can $K_n + O_p$ be graceful if p is sufficiently large?

Problem 5. By Theorem 2, the graph $K(p, q)$ is graceful for all $1 \le p \le q$. Study the gracefulness of $K(p, p) - M$, where $p \ge 3$ and M is a perfect matching in $K(p, p)$.

Let $G = K(p, p) - M$. In Problem 5, we note that if $p = 4r + 3$, where $r \geq 0$, then G is an Eulerian graph of size $(4r + 3)(4r + 2)$, which is not graceful by Theorem 3.

Problem 6. Study the gracefulness of each of the following families of graphs:

(1) $P_m + P_n$, where $m \geq n \geq 2$;
(2) $C_m + P_n$, where $m \geq 3$ and $n \geq 3$;
(3) $C_m + C_n$, where $m \geq n \geq 3$;
(4) $I(m, s) + I(n, t)$, where $m \geq 2s + 1$ and $n \geq 2t + 1$;
(5) $P(m, s) + I(n, t)$, where $m \geq s + 2$ and $n \geq 2t + 1$;
(6) $K(1, p) + I(n, t)$, where $p \geq 3$, $n \geq 2t + 1$ and $t \geq 1$;
(7) $K(1, p) + P(n, t)$, where $p \geq 3$, $n \geq t + 2$ and $t \geq 1$;
(8) More generally, $G + P(n, t)$, where G is a graceful tree and $t \geq 2$.

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