Characterisation of potentially generalised bipartite self-complementary bi-graphic sequences

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Received 2 September 1994; revised 4 April 1996; accepted 11 August 1997

Abstract

In this paper we characterise all graphic bipartitioned sequences for which there is at least one generalised bipartite self-complementary realisation. We also characterise all generalised bipartite self-complementary trees and unicyclic connected graphs. © 1998 Elsevier Science B.V. All rights reserved

Keywords: Trees; Unicyclic graphs; Graphic and bigraphic sequences; Bipartite self-complementary graphs; Potentially bipartite self-complementary bipartitioned sequences

1. Introduction

All graphs considered in this paper are finite undirected graphs without loops and multiple edges. For terminology and notation undefined here the reader is referred to Harary [8].

Let \( G \) be a graph with \( V(G) = \{v_1, v_2, \ldots , v_p\} \) where \( d(v_1) \geq d(v_2) \geq \cdots \geq d(v_p) \). Then the sequence \( (d(v_1), d(v_2), \ldots , d(v_p)) \) is called the degree sequence of \( G \). Conversely, a sequence \( (d_1, d_2, \ldots , d_p) \) with \( d_1 \geq d_2 \geq \cdots \geq d_p \) is called graphic if it is the degree sequence of a graph \( G \). The following result due to Erdős and Gallai [4, 8] on graphic sequences is well known.

Result 1. Let \( \pi = (d_1, d_2, \ldots , d_p) \) be a partition of \( 2q \) into \( p \geq 1 \) parts, \( d_1 \geq d_2 \geq \cdots \geq d_p \). Then \( \pi \) is graphical if and only if for each integer \( r \), \( 1 \leq r \leq p - 1 \),

\[
\sum_{i=1}^{r} d_i \leq r(r-1) + \sum_{i=r+1}^{p} \min\{r, d_i\}.
\]

The complement \( \bar{G} \) of a graph \( G \) is the graph defined on the vertex set \( V(G) \) in which two distinct vertices are adjacent if and only if they are not adjacent in \( G \). A graph \( G \)
with at least two vertices is called self-complementary if \( G \) is isomorphic to \( \bar{G} \). An isomorphism from \( G \) onto \( \bar{G} \) is called a complementing permutation of \( G \) [10].

A graph \( G \) is said to be bipartite if \( V(G) \) can be partitioned into two non-empty sets \( V_1 \) and \( V_2 \) such that every edge in \( G \) joins a vertex in \( V_1 \) to a vertex in \( V_2 \). Such a partition \((V_1, V_2)\) is called a partition of \( G \) and is denoted by \( P = \{V_1, V_2\} \).

A bipartite graph is denoted by \( G(V_1, V_2) \) or \( G(P) \). Given a bipartite graph \( G \), its bipartite complement [5] is defined to be the bipartite graph \( \bar{G}(P) \), where \( V(\bar{G}(P)) = V(G(P)) \) and

\[
E(\bar{G}(P)) = \{uv \mid u \in V_1, v \in V_2 \text{ and } uv \notin E(G)\}.
\]

A bipartite graph \( G(P) \) is said to be bipartite self-complementary (bsc) [5,9] if \( G(P) \) is isomorphic to \( \bar{G}(P) \).

Let \( G(V_1, V_2) \) be a bipartite graph with \( V_1 = \{u_1, \ldots, u_m\} \) and \( V_2 = \{v_1, \ldots, v_n\} \), where \( d(u_1) \geq d(u_2) \geq \cdots \geq d(u_m) \) and \( d(v_1) \geq d(v_2) \geq \cdots \geq d(v_n) \). Let \( d_i = d(u_i) \) and \( e_i = d(v_i) \). Then the bipartitioned sequence \( \pi = (d_1, \ldots, d_m/e_1, \ldots, e_n) \) is called the degree sequence of \( G(P) \) [5].

If \( V_1 = \{u_1, \ldots, u_m\} \) and \( V_2 = \{v_1, \ldots, v_n\} \) then we say that \( S = (u_1, \ldots, u_m/v_1, \ldots, v_n) \) is an ordering of \( G(P) \). The bipartite graph \( G(P) \) with the ordering \( (u_1, \ldots, u_m/v_1, \ldots, v_n) \) is said to be a realisation of the bipartitioned sequence \( \pi = (d_1, \ldots, d_m/e_1, \ldots, e_n) \) if \( d(u_i) = d_i \) and \( d(v_j) = e_j \) for all \( i \) and \( j \). We also say that \( G(P) \) is a realisation of \( \pi \) if \( G(P) \) with some ordering \( S \) is a realisation of \( \pi \). A bipartitioned sequence \( \pi \) is said to be bi-graphic if there is a realisation of \( \pi \). A bipartitioned sequence \( \pi \) is said to be potentially bsc if there exists at least one bsc realisation \( G(P) \) of \( \pi \) [5].

The following result of Gangopadhyay [5] will be used in this paper. We denote by \( C_1 \) the conditions

\[
d_i + d_{m+1-i} = n, \quad 1 \leq i \leq m,
\]

\[
e_j + e_{n+1-j} = m, \quad 1 \leq j \leq n.
\]

**Result 2.** A bi-graphic bipartitioned sequence \( (d_1, \ldots, d_m/e_1, \ldots, e_n) \) is potentially bsc iff it satisfies at least one of the following conditions:

(i) \( C_1 \) holds and exactly one of \( m \) and \( n \) is odd.

(ii) \( C_1 \) holds; both \( m \) and \( n \) are even and either

\[
d_{m/2} = d_{m+1/2} = n/2 \quad \text{or} \quad e_{n/2} = e_{n+1/2} = m/2.
\]

(iii) \( C_1 \) holds; \( m, n \) and \( \sum_{j=1}^{n/2} e_j - \sum_{i=1}^{m/2} d_i - \frac{1}{4}mn \) are all even.

(iv) \( m = n \) is even, \( d_i + e_{m+1-i} = n \) for \( 1 \leq i \leq m \) and \( d_{2i-1} = d_{2i} \) for \( 1 \leq i \leq \frac{1}{2}m \).

Let \( G(V,E) \) be a graph and \( Q = (W_1, \ldots, W_k) \) be a partition of \( V \). The \( k \)-switched graph \( G_k^Q \) (with respect to \( Q \)) is defined by Sampathkumar and Pushpalatha [11] as follows. For all \( W_i \) and \( W_j, \ i \neq j \) in \( Q \) remove the existing edges between \( W_i \) and \( W_j \) and add the missing edges. A generalised \( k \)-self-complementary graph \( G \) has been defined by Sampathkumar and Pushpalatha [1,11] as a graph \( G \) for which there exists a partition \( Q \) of order \( k \) such that \( G_k^Q \simeq G \).
Gangopadhyay and Rao Hebbare [6, 7] have investigated the graphs $G^{Q}_k$ where each set in $Q$ is independent. Sampathkumar and Pushpalatha [11] have considered the graphs $G^{Q}_k$ in a more general setting. In this paper we consider bipartite graphs. A bipartite graph $G$ is said to be generalised bipartite self-complementary (gbsc) if $G^{Q}_2 \cong G$. A bipartitioned sequence $\pi = (d_1, \ldots, d_m/e_1, \ldots, e_n)$ is said to be potentially generalised bipartite self-complementary (pgbsc) if there exists at least one gbsc realisation $G$ of $\pi$.

2. Main results

In this paper we answer the following questions:

(i) When is a bipartitioned sequence $\pi$ bi-graphic, and
(ii) When is a bipartitioned sequence $\pi$ pgbsc?


Theorem 1 (Ryser [2]). Let $\pi = (d_1, \ldots, d_m/e_1, \ldots, e_n)$ be a bipartitioned sequence. Then $\pi$ is bi-graphic iff $\pi$ satisfies the following:\footnote{Theorem 1 is due to Gale-Ryser [2]. This is an alternative proof.}

(i) $n \geq d_1 \geq \cdots \geq d_m, m \geq e_1 \geq \cdots \geq e_n$ and $\sum_{i=1}^m d_i = \sum_{j=1}^n e_j$.

(ii) $\sum_{i=1}^m d_i \leq \sum_{j=1}^n \min(r, e_j), 1 \leq r \leq m$.

Remark. The theorem holds even if condition (ii) is replaced by the equivalent condition.

(ii)' $\sum_{j=1}^k e_j \leq \sum_{i=1}^m \min(k, d_i), 1 \leq k \leq n$.

Proof (Part 1). Let $\pi$ be bi-graphic and let $G(V_1, V_2)$ be a realisation of $\pi$ where $V_1 = \{u_1, \ldots, u_m\}, V_2 = \{v_1, \ldots, v_n\}, d_i = d(u_i), e_j = d(v_j)$ and $n \geq d_1 \geq \cdots \geq d_m, m \geq e_1 \geq \cdots \geq e_n$. Clearly, $\sum_{i=1}^m d_i = \sum_{j=1}^n e_j$. We define a new sequence $\pi' = (f_1, \ldots, f_m, f_{m+1}, \ldots, f_{m+n})$ where

$$f_i = \begin{cases} d_i + m - 1 & \text{if } 1 \leq i \leq m, \\ e_{i-m} & \text{if } m < i \leq m + n. \end{cases}$$

Then $\pi'$ is graphic since it is the degree sequence of the graph obtained from $G(V_1, V_2)$ by joining all the $\binom{n}{2}$ pairs of vertices in $V_1$ by edges. Clearly, $f_1 \geq \cdots \geq f_m$ and $f_{m+1} \geq \cdots \geq f_{m+n}$. Also, $f_m = d_m + m - 1 \geq m \geq e_1 = f_{m+1}$.

By Result 1, for each integer $r$ between 1 and $m + n$, we have

$$E_r = r(r - 1) + \sum_{j=r+1}^{m+n} \min(r, f_j) - \sum_{j=1}^r f_j \geq 0.$$ 

If $r \leq m$,

$$E_r = r(r - 1) + \sum_{j=r+1}^m \min(r, f_j) + \sum_{j=m+1}^{m+n} \min(r, f_j) - \sum_{j=1}^r f_j$$
\[ n = r(r-1) + r(m-r) + \sum_{k=1}^{n} \min(r, e_k) - \sum_{j=1}^{r} d_j - r(m-1) \geq 0 \]

which proves (ii).

Part II: If \( \pi \) satisfies conditions (i) and (ii) \( \pi' \) of part I satisfies the conditions of Result 1. Therefore \( \pi' \) is graphic. Let \( G' \) be a realisation of \( \pi' \) where \( V(G') = \{ u_1, \ldots, u_m, u_{m+1}, \ldots, u_{m+n} \} \) and \( f_i = d(u_i), \ i = 1, \ldots, m+n \). Let \( V_1 = \{ u_1, \ldots, u_m \} \) and \( V_2 = \{ u_{m+1}, \ldots, u_{m+n} \} \). As \( f_i = d_i + m-1, \ i = 1, \ldots, m \) and each vertex in \( V_1 \) can be adjacent to at most \( m-1 \) vertices in \( V_1 \), each vertex \( u_i \) in \( V_1 \) must contribute at least \( d_i \) to the adjacencies in \( V_1 \times V_2 \). However, a vertex \( u_{m+j} \) in \( V_2 \) can contribute no more than \( e_j \) to \( V_1 \times V_2 \). Since \( \sum_{i=1}^{m} d_i = \sum_{j=1}^{n} e_j \), it follows that the induced graph on \( V_2 \) is empty and each vertex \( u_i \) in \( V_1 \) contributes exactly \( d_i \) to the adjacencies in \( V_1 \times V_2 \). Thus each vertex in \( V_1 \) is adjacent to \( m-1 \) vertices in \( V_1 \). Let \( G \) be obtained from \( G' \) by removing these \( \binom{m}{2} \) edges incident at vertices in \( V_1 \). \( G \) is the required realisation of \( \pi \). \( \square \)

We now introduce a symmetric bipartite graph and give a necessary and sufficient condition for a bipartitioned sequence \( \pi \) to be potentially symmetric bi-graphic. We use this in the proofs of the main theorems namely Theorems 3 and 4.

Definition 1. A bipartite graph \( G(V_1, V_2) \) with \( V_1 = \{ u_1, \ldots, u_m \} \), \( V_2 = \{ v_1, \ldots, v_m \} \) is said to be symmetric if \( G = \sigma(G) \) where the permutation \( \sigma \) on \( V_1 \cup V_2 \) is given by

\[ \sigma = (u_1v_1)(u_2v_2)\ldots(u_mv_m). \]

A bipartitioned sequence \( \pi \) is said to be potentially symmetric bi-graphic if it has a realisation which is symmetric.

Theorem 2. A bipartitioned sequence \( \pi = (d_1, \ldots, d_m/e_1, \ldots, e_n) \) is potentially symmetric bi-graphic with a realisation in which a vertex of degree \( d \) is adjacent to its image iff \( \pi \) is bi-graphic, \( m = n \) and \( d_i = e_i, \ 1 \leq i \leq m \).

Proof. Necessity being obvious we prove only sufficiency. So let \( \pi = (d_1, \ldots, d_m/d_1, \ldots, d_m) \) be bi-graphic. To construct a symmetric bipartite graph \( G \) with degree sequence \( \pi \) we proceed by induction on \( m \). It is easy to construct such graphs for \( m = 1 \) or 2. Assuming the construction for \( m-1 \) we start with a sequence \( \pi = (d_1, \ldots, d_m/d_1, \ldots, d_m) \). Consider the sequence

\[ \pi' = (d_2 - 1, d_3 - 1, \ldots, d_{d_1} - 1, d_{d_1+1}, \ldots, d_m/d_2 - 1, \ldots, d_{d_1} - 1, d_{d_1+1}, \ldots, d_m). \]

From Theorem 1 it follows that if \( \pi \) is bi-graphic so is \( \pi' \). Thus, by induction hypothesis we construct a symmetric bipartite graph \( G'(V'_1, V'_2) \) with degree sequence \( \pi' \). \( V'_1 = \{ u_2, \ldots, u_m \} \) and \( V'_2 = \{ v_2, \ldots, v_m \} \). Take new vertices \( u_1 \) and \( v_1 \). Join \( u_1 \) to
The resulting graph $G$ is the required symmetric bipartite graph.

**Definition 2.** Let $G(V_1, V_2)$ be a bipartite graph which is gbsc with partition $Q = (W_1, W_2)$. $G$ is said to be gbsc of type I if at least one of $W_i \cap V_j$, $i, j = 1, 2$ is empty and of type II otherwise.

The following two theorems are the main results of this paper.

**Theorem 3.** A bipartitioned sequence $\pi = (a_1, \ldots, a_m / b_1, \ldots, b_n)$ is pgbsc of type I iff the following conditions hold:

There exist positive integers $m_1, m_2$ such that

(a) $m_1 + m_2 = m$ and one of $m_1$ and $m_2$, say, $m_2$ is $\leq n$.

(b) $m_1$ of $a_i$’s, say $a_1 \geq \cdots \geq a_{m_1}$ (after renaming if necessary) are $\geq m_2$.

(c) $\pi_1 = (a_1 - m_2, \ldots, a_{m_1} - m_2 / b_{m_2 + 1}, \ldots, b_n)$ satisfies one of the conditions (i)–(iii) of Result 2.

(d) $\pi_2 = (a_{m_1 + 1}, \ldots, a_{m_1 + m_2} / b_1 - m_1, \ldots, b_{m_2} - m_1)$ satisfies (i) and (ii) of Theorem 1. Further, $a_{m_1 + i} = b_i - m_1, i = 1, \ldots, m_2$.

**Proof.** Necessity: Let $G(V_1, V_2)$ be a gbsc type I realisation of $\pi$ and $Q = (W_1, W_2)$ be a partition of the vertex set $V$ with respect to which $G^Q_2 \simeq G$. As $G$ is gbsc of type I at least one of $W_i \cap V_j$, $i, j = 1, 2$ is empty. Without loss of generality, let $W_1 \cap V_2 = \emptyset$.

**Case 1:** Exactly one of $W_i \cap V_j$, $i, j = 1, 2$ is empty. So, let

$$V_1 = \{u_1, \ldots, u_{m_1}; x_1, \ldots, x_{m_2}\},$$

$$V_2 = \{y_1, \ldots, y_{m_2}; y_{m_2 + 1}, \ldots, y_n\},$$

$$W_1 = \{u_1, \ldots, u_{m_1}\},$$

and

$$W_2 = \{x_1, \ldots, x_{m_2}; y_1, \ldots, y_{m_2}, \ldots, y_n\}.$$

Observe that in $G^Q_2$ there is a $K_{m_1, m_2}$ joining each $u_i$ to each $x_j$. Thus, there must be a $K_{m_1, m_2}$ in $G$ joining each $u_i$ to $m_2$ of the $y_j$’s say $y_1, \ldots, y_{m_2}$. Thus, $m_2 \leq n$ which proves (a) and (b).

$\pi_2 = (a_{m_1 + 1}, \ldots, a_{m_1 + m_2} / b_1 - m_1, \ldots, b_{m_2} - m_1)$ is the degree sequence of the bipartite graph $H_1$ induced by $\langle x_1, \ldots, x_{m_2}, y_1, \ldots, y_{m_2}\rangle$ (with a reordering of the $y_j$’s if necessary). Thus, it satisfies (i) and (ii) of Theorem 1. In $G^Q_2$ none of the $u_i$’s is adjacent to any of the $y_j$’s, $i = 1, \ldots, m_2$ and each $u_i$ is adjacent to each of $x_j$. If $\sigma$ is the complementing permutation, then it is straightforward that either $\sigma(x_1, \ldots, x_{m_2}) = (y_1, \ldots, y_{m_2})$ or $\pi = ((m/2 + m)^m, (m/2 + m)^m, (m/2 + m)^m)$. In the first case the degree sequence of $y_1, \ldots, y_{m_2}$ in $\sigma(G^Q_2)$ must be the same as the degree sequence of $x_1, \ldots, x_{m_2}$ in $G$. Thus, with a suitable reordering of the $x_j$’s we get

$$a_{m_1 + i} = b_i - m_1, i = 1, \ldots, m_2.$$
Hence (d) is satisfied. If \( m_2 = n \), then \( \pi_1 \) is empty and (c) is proved. If \( m_2 < n \), \( \pi_1 \) is the degree sequence of the bipartite graph \( H_2 \) induced by \( (u_1, \ldots, u_{m_2}, y_{m_2+1}, \ldots, y_n) \). Hence, \( \pi_1 = (a_1 - m_2, \ldots, a_{m_1} - m_2, b_{m_2+1}, \ldots, b_n) \). As \( G \cong G_2^Q \) we get \( H_2 \cong (H_2)_2^Q \) which implies that \( \pi_1 \) satisfies one of the conditions (i)–(iii) of Result 2. Thus (c) is proved. In the second case the conditions (c) and (d) are trivially satisfied.

Case 2: Exactly two of \( W_j \cap V_j, i, j = 1, 2 \) are empty.

Without loss of generality, let \( W_1 = V_1 \) and \( W_2 = V_2 \). Thus we get \( m_1 = m, m_2 = 0 \). Thus (a) and (b) are trivially true. \( \pi_1 \) becomes \( (a_1, \ldots, a_{m_1}/b_1, \ldots, b_n) \). \( G \cong G_2^Q \) implies that \( \pi_1 \) satisfies at least one of (i)–(iii) of Result 2. \( \pi_2 \) is empty. Thus (c) and (d) are satisfied.

**Proof.** Sufficiency: By (a) \( m_2 \leq n \). Firstly, let \( m_2 = n \). Here \( \pi_1 \) is empty and \( \pi_2 = (a_{m_1+1}, \ldots, a_{m_1+m_2}/b_1 - m_1, \ldots, b_{m_2} - m_1) \). By (d) and Theorem 2 we construct \( H(\{x_1, \ldots, x_{m_2}\}, \{y_1, \ldots, y_{m_2}\}) \) such that \( \sigma(H) = \tilde{H} \) where \( \sigma = (x_1 y_1) \ldots (x_{m_2} y_{m_2}) \). Take new vertices \( u_1, \ldots, u_{m_1} \) and join each \( u_i \) to each \( y_j, j = 1, \ldots, m_2 \). The resulting graph \( G \) is gbsc with complementing permutation \( \sigma^* \) given by \( \sigma^*(u_i) = u_i, i = 1, \ldots, m_1 \) and \( \sigma^* \) is \( \sigma \) on \( V(H) \).

Bipartition of \( V(G) \) is

\[
V_1 = \{u_1, \ldots, u_{m_1}, x_1, \ldots, x_{m_2}\}, \quad V_2 = \{y_1, \ldots, y_{m_2}\}.
\]

\[Q = \{W_1, W_2\}, \quad \text{where} \quad W_1 = \{u_1, \ldots, u_{m_1}\}\]

and

\[W_2 = \{x_1, \ldots, x_{m_2}, y_1, \ldots, y_{m_2}\}\].

Next, let \( m_2 < n \).

Here \( \pi_1 \) satisfies one of the conditions (i)–(iii) of Result 2. Therefore \( \pi_1 \) is pbsc. Let \( H_1 \) be a realisation and \( V(H_1) = \{u_1, \ldots, u_{m_1}\} \cup \{y_{m_2+1}, \ldots, y_n\} \). Let \( \sigma_1: V(H_1) \to V(\tilde{H}_1) \) be such that \( \sigma_1(H_1) = \tilde{H}_1 \). Again, for \( \pi_2 \), as in the first part, we construct \( H_2(\{x_1, \ldots, x_{m_2}\}, \{y_1, \ldots, y_{m_2}\}) \) such that \( \sigma_2(H_2) = H_2 \) where \( \sigma_2 = (x_1 y_1) \ldots (x_{m_2} y_{m_2}) \).

Now, join each \( u_i \) to each \( y_j, 1 \leq i \leq m_1, 1 \leq j \leq m_2 \). The resulting graph \( G \) is gbsc of type I with complementing permutation \( \sigma \) on \( V(G) \) such that \( \sigma(G) = G_2^Q \) where \( \sigma/V(H_1) = \sigma_1 \) and \( \sigma/V(H_2) = \sigma_2 \).

Bipartition of \( V(G) \) is

\[
V_1 = \{u_1, \ldots, u_{m_1}, x_1, \ldots, x_{m_2}\}, \quad V_2 = \{y_1, \ldots, y_{m_2}, y_{m_2+1}, \ldots, y_n\}.
\]

\[Q = \{W_1, W_2\}, \quad \text{where} \quad W_1 = \{u_1, \ldots, u_{m_1}\}\]

and

\[W_2 = \{x_1, \ldots, x_{m_2}, y_1, \ldots, y_{m_2}, y_{m_2+1}, \ldots, y_n\}\].

**Theorem 4.** A bipartitioned sequence \( \pi = (a_1, \ldots, a_{m_1}/b_1, \ldots, b_n) \) is pgbsc of type II iff the following conditions hold: There exist integers \( m_1, m_2, n_1, n_2 \) all \( > 0 \), \( m_1 + m_2 = m, n_1 + n_2 = n \) such that

(a) one of \( m_1, m_2 \) is equal to one of \( n_1, n_2 \), say, \( m_2 = n_2 \).
(b) If the $a_i$'s say, $a_1 \geq \ldots \geq a_m$ (after renaming if necessary) are $\geq m_2$ and the remaining $a_{m+1} \geq \ldots \geq a_n$ (after renaming if necessary) are $\geq n_1$.

c) If the $b_i$'s say, $b_1 \geq \ldots \geq b_n$ (after renaming if necessary) are $\geq n_2$ and the remaining $b_{n+1} \geq \ldots \geq b_m$ (after renaming if necessary) are $\geq m_1$.

d) If $\pi_1 = (a_1 - m_2, a_m - m_2/a_1 - m_2, b_{n_1} - m_2)$ and $\pi_2 = (a_{m+1} - n_1, a_{m+1} - n_1, b_{n+1} - m_1)$ satisfy (i) and (ii) of Theorem 1. Further, $a_{m+i} - n_1 = b_{n+i} - m_i$, $1 \leq i \leq m_2$.

**Proof.** Necessity: Let $G(V_1, V_2)$ be a gbse type II realisation of $\pi$ and $Q = (W_1, W_2)$ be a partition of the vertex set $V$ with respect to which $G_2^Q \simeq G$. As $G$ is gbse of type II, $W_i \cap V_j \neq \emptyset$ for $i,j = 1, 2$. Let

\[
V_1 = \{u_1, \ldots, u_{m_1}, x_1, \ldots, x_{m_2}\}, \\
V_2 = \{v_1, \ldots, v_{n_1}, y_1, \ldots, y_{n_2}\}, \\
W_1 = \{u_1, \ldots, u_{m_1}, v_1, \ldots, v_{n_1}\}, \\
W_2 = \{x_1, \ldots, x_{m_2}, y_1, \ldots, y_{n_2}\}.
\]

None of $m_1, n_1, m_2, n_2$ is zero. Existence of $K_{m_1 m_2}$ and $K_{n_1 n_2}$ in $G_2^Q$ implies $m_2 = n_2$. Thus,

\[
V_1 = \{u_1, \ldots, u_{m_1}, x_1, \ldots, x_{m_2}\} \quad \text{and} \quad V_2 = \{v_1, \ldots, v_{n_1}, y_1, \ldots, y_{n_2}\}
\]

Clearly, in $G$ each $u_i$ is adjacent to each $y_j$ and each $v_i$ is adjacent to each $x_j$. Thus, (a)–(c) are true. $\pi_1$ is the degree sequence of the bipartite graph induced by $W_1$ and $\pi_2$ is the degree sequence of the bipartite graph induced by $W_2$. Hence $\pi_1$ and $\pi_2$ satisfy (i) and (ii) of Theorem 1. If $\sigma$ is the isomorphism from $G$ to $G_2^Q$ it is straightforward that either $\sigma(x_1, \ldots, x_{m_2}) = (y_1, \ldots, y_{n_2})$ or $\sigma(u_1, \ldots, u_{m_1}) = (v_1, \ldots, v_{n_1})$ or $\pi = (m_2 m / m_2 m)$ and as in Theorem 3 in each case $a_{m+i} - n_1 = b_{n+i} - m_i$, $1 \leq i \leq m_2$. Thus (d) is satisfied.

**Proof.** Sufficiency: By (d) and Theorem 1 we construct $H_1(u_1, \ldots, u_{m_1}, v_1, \ldots, v_{n_1})$, a bipartite realisation of $\pi_1$. Also, by (d) and Theorem 2 we construct $H_2(x_1, \ldots, x_{m_2}, y_1, \ldots, y_{n_2})$ such that $\sigma_2(H_2) = H_2$ where $\sigma_2 = (x_1 y_1) \ldots (x_{m_2} y_{n_2})$.

Now, join each $u_i$ to each $y_j$ giving a $K_{m_1 m_2}$ and each $v_i$ to each $x_j$ giving a $K_{n_1 n_2}$. The resulting graph is gbse of type II with

\[
V_1 = \{u_1, \ldots, u_{m_1}, x_1, \ldots, x_{m_2}\}, \quad V_2 = \{v_1, \ldots, v_{n_1}, y_1, \ldots, y_{n_2}\},
\]

\[
W_1 = \{u_1, \ldots, u_{m_1}, v_1, \ldots, v_{n_1}\}, \quad W_2 = \{x_1, \ldots, x_{m_2}, y_1, \ldots, y_{n_2}\}.
\]

$\sigma: V(G) \to V(G)$ is given by $\sigma(W_1) = \text{Identity}$, $\sigma(W_2) = \sigma_2 \quad \Box$. 

3. Applications

In the following two theorems we give characterisations of gbzc trees and unicyclic connected bipartite graphs.

The dotted lines in a figure indicate the presence of a number of such edges/paths. Each graph represents actually a family of graphs. For example, the graph of Fig. 2 includes $T_7$, i.e. a $P_5$ with a pendant vertex at the second vertex and a pendant vertex at the third vertex and other such graphs. The vertices in the top row represent $V_1$ and the ones at the bottom represent $V_2$. Vertices enclosed in a rectangle represent either $W_1$ or $W_2$ as indicated while the remaining vertices represent $W_2$ or $W_1$ as the case may be.

**Theorem 5.** A tree $T$ of order $p$ is gbzc iff $T$ is a star, $P_7$ or one of the graphs in Figs. 1–8.

**Proof.** Let $T = T(V_1, V_2)$ be a gbzc tree. We consider three cases.

**Case I:** $n_1 = 0$ and $m_2 \leq n_2$.

Subcase (i): $n_1 = 0$ and $m_2 = n_2$. Since $T$ is a tree if $m_1 \geq 2$ then $m_2 = n_2 = 1$ and hence $T$ must be a star.

If $m_1 = 1$, $\pi_2$ of Theorem 3 becomes $(a_2, \ldots, a_{m_2+1}, b_1 - 1, \ldots, b_{m_2} - 1)$. By (d) of Theorem 3 the realisation of $\pi_2$ must be $m_2$ copies of $K_2$. Hence, $T$ must be as in Fig. 1, where the dotted lines indicate the presence of a number of such paths.

Subcase (ii): $n_1 = 0$ and $m_2 < n_2$. Here $n \geq 2$. As $T$ is acyclic, this forces $m_1$ to be at most 2. If $m_1 = 2$ then $m_2$ must be 1 and $T$ is in Fig. 2. On the other hand, $m_1 = 1, n_1 = 0$ and $m_2 < n_2 = n$ is impossible.

**Case II:** $n_1 = m_2 = 0$. Here $\pi_1$ of Theorem 3 becomes $(a_1, \ldots, a_m, b_1, \ldots, b_n)$ and satisfies one of the conditions (i)–(iv) of Result 2.

Suppose condition (i) holds and say $m$ is odd. Since $T$ is a tree we have

$$m(2 - n) = 2(1 - n).$$

(1)

Clearly, $m = 1$ is impossible. Also, if $m = 2r - 1$ (1) simplifies to $n = 2 + 2/(2r - 3)$ which is not an integer for $r \geq 3$. Thus, $m = 2r - 1$ is not possible if $r \geq 3$.

If $r = 2$ then $m = 3$ and $n = 4$. Thus $\pi_1 = (a_1, a_2, a_3, b_1, b_2, b_3, b_4)$.
Now $C1$ gives $a_1 + a_3 = 4$, $a_2 = 2$, $b_1 + b_4 = b_2 + b_3 = 3$. Thus, $\pi_1$ is either

\[(321/2211)\]  

or

\[(222/2211).\]

In (3) $T$ has exactly two pendant vertices and hence is the path $P_7$. 
In (2) there are only two realisations, namely Figs. 3 and 4, of which Fig. 3 is not gbsc of type I while Fig. 4 is gbsc of type I. However, it will be seen later (Case III) that Fig. 3 is gbsc of type II. In fact, it is a particular case of Fig. 7 with $n_1 = 3$, $m_1 = 2$, $m_2 = n_2 = 1$.

Next, suppose condition (ii) holds. Taking $m = 2r$, $n = 2s$ in (1) we get that $2r - 1 = 2s(r - 1)$ which is impossible. The same argument shows that conditions (iii) and (iv) of Result 2 also do not arise.

Case III: None of $m_1, n_1, m_2, n_2$ is zero. Here $m_2 = n_2$. If $m_2 > 2$, $m_1 = n_1 = 1$ and then $T$ is as in Figs. 5 or 6. If $m_2 = n_2 = 1$, we again make two cases. If $m_1 = n_1$ we interchange $W_1$ and $W_2$ and get Figs. 5 and 6. If $m_1 \neq n_1$ we get Figs. 7 and 8.

Here it should be observed that if we do not make these two cases then Figs. 5 and 6 are particular cases of Figs. 7 and 8, respectively. 

**Theorem 6.** A unicyclic connected bipartite graph $G$ of order $p$ is gbsc iff $G$ is one of the graphs shown in Figs. 9–36.

**Proof.** Let $G(V_1, V_2)$ be a unicyclic connected bipartite graph which is gbsc. As in Theorem 5 here also we make three cases.

Case I: $n_1 = 0$ and $m_2 \leq n_2 (= n)$.

Subcase (i): $n_1 = 0$ and $m_2 = n_2$. If $m_1 = 1$ then $a_1 \geq m_2 = m - 1 = n$. Also, $a_1 \leq n$, i.e. $a_1 = n$. $\pi_2$ of Theorem 3 becomes $(a_2, \ldots, a_{m_2+1}/b_1 - 1, \ldots, b_{m_2} - 1)$. By (d) of
Fig. 13. gbse unicyclic connected graphs.

Fig. 14. gbse unicyclic connected graphs.

Fig. 15. gbse unicyclic connected graphs.

Fig. 16. gbse unicyclic connected graphs.

Fig. 17. gbse unicyclic connected graphs.

Fig. 18. gbse unicyclic connected graphs.

Fig. 19. gbse unicyclic connected graphs.

Fig. 20. gbse unicyclic connected graphs.
Fig. 21. gbic unicyclic connected graphs.

Fig. 22. gbic unicyclic connected graphs.

Fig. 23. gbic unicyclic connected graphs.

Fig. 24. gbic unicyclic connected graphs.

Fig. 25. gbic unicyclic connected graphs.

Fig. 26. gbic unicyclic connected graphs.

Fig. 27. gbic unicyclic connected graphs.

Fig. 28. gbic unicyclic connected graphs.
Theorem 3 the realisations of $\pi$ are those shown in Figs. 9 and 10 and both are gbsc.

If $m_1 = 2$ then $m_2 = n_2 = 2$. Since $G$ is connected it must be the one in Fig. 11 and it is gbsc. $m_1 \geq 3$ is impossible here.

Subcase (ii): $n_1 = 0$ and $m_2 < n_2$. We consider different possible values of $m_1$. At the outset $m_1$ cannot be 1 and it is easily verified that $m_1$ cannot be greater than 4. If $m_1 = 2$ then $m_2 = 2$ and here the only realisation is the graph of Fig. 12 which is again gbsc. It should be pointed out that Fig. 11 is a particular case of Fig. 12. If
Fig. 35. gbse unicyclic connected graphs.

Fig. 36. gbse unicyclic connected graphs.

$m_1 = 3$ then $m_2 = 1$ and $\pi_1$ of Theorem 3 is $(a_1 - 1, a_2 - 1, a_3 - 1/b_2, \ldots, b_n)$. By (c) of Theorem 3 as $m_1$ is odd $\pi_1$ satisfies (i) of Result 2 and, hence $n - 1$ must be even. Also, as $G$ and $G^Q$ are both unicyclic, $n - 1$ cannot be greater than 2. For $n - 1 = 2$ the corresponding graphs are given in Figs. 13 and 14 which are gbse. If $m_1 = 4$ then $m_2 = 1$. It can be seen that here $n_2 = 2$. The corresponding graph which is gbse is given in Fig. 15.

Case II: $n_1 = m_2 = 0$. Here $\pi_1$ of Theorem 3 becomes $(a_1, \ldots, a_m/b_1, \ldots, b_n)$ and satisfies one of the conditions (i)–(iv) of Result 2. If $\pi_1$ satisfies (i) of Result 2 then

$$m(n - 2) = 2n.$$  \hspace{1cm} (4)

$m = 1$ is clearly impossible. If $m = 2r - 1$, (4) simplifies to $n = 2 + 4/(2r - 3)$ which is not an integer for $r \geq 3$. Thus, only the case $m = 3$ remains.

If $m = 3$ then $n = 6$ and

$$\pi_1 = (a_1 a_2 a_3/b_1 b_2 b_3 b_4 b_5 b_6).$$

Now C1 gives $a_1 + a_3 = 6$, $a_2 = 3$, $b_1 + b_6 = b_2 + b_5 = b_3 + b_4 = 3$. Therefore, $\pi_1$ is one of the following

$$(531/222111),$$  \hspace{1cm} (5)

$$(432/222111),$$  \hspace{1cm} (6)

$$(333/222111).$$  \hspace{1cm} (7)

There are three realisations of the sequence in (5) of which only one is gbse and this is shown in Fig. 16. In (6) there are six realisations of which only the graphs shown in Figs. 17 and 18 are gbse. In (7) there are three realisations of which only graphs of Figs. 19 and 20 are gbse. If $\pi_1$ satisfies condition (ii) of Result 2 and if in (4) we put $m = 2r$ and $n = 2s$ the only integral values of $r$ and $s$ are 2 and 2, respectively,
i.e. \( m = 4, n = 4 \). Then \( \pi_1 = (a_1a_2a_3a_4/b_1b_2b_3b_4) \), where

\[
a_1 + a_4 = a_2 + a_3 = b_1 + b_4 = b_2 + b_3 = 4.
\]

The possibilities for \( \pi_1 \) are then

\[(3311/3311), \quad (3311/3221), \quad (3311/2222), \quad (3221/3221), \quad (3221/2222), \quad (2222/2222).\]

There are in all 16 realisations of these sequences of which only the graphs in Figs. 21–28 are gbsc.

**Case III:** None of \( m_1, n_1, m_2, n_2 \) is zero. Here \( m_2 = n_2 \). If \( m_2 = 2 \) and \( m_1 \geq 3 \) or \( n_1 \geq 3 \) then \( G^0_2 \) is not unicyclic. Hence this is impossible.

If \( m_2 = 2 \) then both \( m_1 \) and \( n_1 \) cannot be 2 simultaneously. So let \( m_1 = 2 \) and \( n_1 = 1 \), say. The corresponding graphs are as in Figs. 29 and 30 and both are gbsc. Suppose now \( m_2 \geq 3 \). Then \( G \) being unicyclic \( m_1 \) and \( n_1 \) must be 1. The corresponding graphs are as in Figs. 31–33 and are all gbsc.

Finally, let \( m_2 = n_2 = 1 \). As in the last part of Theorem 5 we make two cases. If \( m_1 = n_1 \) we interchange \( W_1 \) and \( W_2 \) and get Figs. 31–33. If \( m_1 \neq n_1 \) we get Figs. 34–36. Here it should be observed that if we do not make these two cases then Figs. 31–33 are particular cases of Figs. 34–36 respectively.

**Acknowledgements**

The author is grateful to the referee for the helpful comments.

**References**