Some Properties of the Second Conjugate of a Tauberian Operator

Beatriz Hernando*

Departamento de Matemáticas Fundamentales, Facultad de Ciencias, Universidad Nacional de Educación a Distancia, 28040 Madrid, Spain

Submitted by Joel H. Shapiro

Received February 17, 1998

A bounded linear operator \( T: X \to Y \) (Banach spaces) is defined to be Tauberian provided whenever \( \{x_n\} \subset X \) is bounded and \( \{T(x_n)\} \subset Y \) is weakly convergent, then \( \{x_n\} \) has a weakly convergent subsequence. Hence, they appear as opposite to weakly compact operators. In 1991 a Tauberian operator \( T \) between separable Banach spaces was found such that its second conjugate \( T^{**} \) is not Tauberian. Though \( T^{**} \) might not be Tauberian, in this paper we prove that it satisfies the following property when \( X \) is separable: whenever \( \{x_n^{**}\} \subset X^{**} \) is bounded and \( \{T^{**}(x_n^{**})\} \subset Y^{**} \) is weakly convergent, then \( \{x_n^{**}\} \) has a \( w^* \)-convergent subsequence. Other properties of \( T^{**} \) are proved and the nonseparable case is also studied.

INTRODUCTION

Tauberian operators appeared in response to a problem in summability (see [7]). They were first defined as those operators \( T: X \to Y \) that satisfy \( T^{**}(x^{**}) \in Y \) implies \( x^{**} \in X \). Since then, they have been applied in many different situations, for instance, to study the isomorphic properties of Banach spaces [2, 8], the equivalence between the Radon–Nikodym property and the Krein–Milman property [11], and the factorization of operators [9]; see also [1, 3–6]. One of the most significant characterizations of Tauberian operators was given by Neidinger and Rosenthal in [9]. They proved that a nonzero operator is Tauberian if and only if it maps weakly closed bounded (convex) sets to closed sets. In this paper we prove that although the second conjugate \( T^{**} \) of a Tauberian operator \( T \):

---

* Partially supported by DGICYT Grant PB94-0243. E-mail: bhernan@mat.uned.es.

0022-247X/98 $25.00
Copyright © 1998 by Academic Press
All rights of reproduction in any form reserved.
$X \to Y$ might not be Tauberian, it maps $w^*$-sequentially closed bounded (convex) sets to closed sets, when $X$ is separable.

We shall say an operator $T: X \to Y$ preserves a certain family of sets $P$ provided, for all bounded sets $A \subseteq X$, $A \in P$ if and only if $T(A) \in P$. From the beginning, Tauberian and semi-Fredholm operators have been compared because both can be characterized as preserving certain families of sets: Tauberian operators preserve weakly relatively compact sets and semi-Fredholm operators preserve relatively compact sets [3–9]. The class of operators that preserve conditionally weakly compact sets (a set $A$ is conditionally weakly compact if each sequence in $A$ has a weakly Cauchy subsequence) is studied in [4, 5] in the context of semigroups of generalized Fredholm operators and also in [2, 3] where they are called semi-Tauberian and are applied to study isomorphic properties of $L_1(\mu, X)$. The semi-Fredholm property is stronger than the Tauberian, and the Tauberian property is stronger than the semi-Tauberian.

The main result of this paper proves some characterizations of semi-Tauberian operators acting on a separable Banach space. These characterizations give rise to new properties of the second conjugate of a Tauberian operator. Kalton and Wilansky [7] proved that the second conjugate $T^{**}$ of a Tauberian operator $T$ is Tauberian, when $T$ has closed range, and they asked when this property is also true. In [1] a Tauberian operator $T: X \to X$ is constructed such that $X$ is separable and $T^{**}$ is not even semi-Tauberian.

We now give the main result which will be proved in Section 1 and will be extended to the nonseparable case in Section 2.

**Main Result.** Let $X$ be a separable Banach space and $T: X \to Y$ a nonzero operator. The following are equivalent:

(i) Every bounded sequence $\{x_n^{**}\}$ in $X^{**}$ with $(T^{**}(x_n^{**}))$ $w^*$-Cauchy in $Y^{**}$ has a $w^*$-Cauchy subsequence.

(ii) $T^{**}(A)$ is closed for every $w^*$-sequentially closed bounded (convex) set $A$ in $X^{**}$.

(iii) $T(A)$ is closed for every closed bounded (convex) set $A$ contained in a subspace $Z$ of $X$ which has Schur property.

(iv) Every bounded sequence $\{x_n\}$ in $X$ with $(T(x_n))$ weakly Cauchy in $Y$ has a weakly Cauchy subsequence; i.e., $T$ is semi-Tauberian.

Any Tauberian operator satisfies (iv) (see [5]) and therefore all the other properties. A truly interesting semi-Tauberian operator must have a domain $X$ such that $X$ has a copy of $l_1$ but it is not hereditarily $l_1$, because when $X$ is hereditarily $l_1$ every semi-Tauberian operator from $X$ is semi-Fredholm [3, Remark 1.4(a)]. On the other hand, for every subspace
1. PROOF OF THE MAIN RESULT

Recall from [10] that, given a topological space \( K \), a nonempty subset \( L \) of \( K \), and a real-valued function \( f \) defined on \( K \), it is said that \( f \) satisfies the discontinuity criterion on \( L \) provided there are two real numbers \( r, \delta \) with \( \delta > 0 \) so that, for every nonempty relatively open subset \( U \) of \( L \), there are \( y \) and \( z \) in \( U \) such that \( f(y) > r + \delta \) and \( f(z) < r \). In [10], Rosenthal proved the following result for separable Banach spaces: every bounded sequence \( x; \) \( X** \) without weak* Cauchy subsequences has a subsequence point-wise convergent on some subset \( L \) of \( B_X \). Our main result follows from Rosenthal's result and Lemma 1.1.

**Lemma 1.1.** Let \( T: X \to Y \) be an operator and \( f: (B_X^*, \sigma(X^*, X)) \to \mathbb{R} \) a function that satisfies the discontinuity criterion on some subset \( L \) of \( B_X^* \). If there exists a bounded net \( (x_n^**) \subset X** \) point-wise convergent to \( f \) on \( L \) such that \( (T^**(x_n^**)) \) is weak*-convergent to 0, then there exists an \( l_1 \)-sequence \( (x_n) \subset X \) such that \( \|T(x_n)\| \to 0 \).

**Proof.** To prove that \( (x_n) \) is an \( l_1 \)-sequence, we shall use the criterion about Boolean independent sequences of sets \( ((A_n, B_n)) \) developed by Rosenthal (see [3]).

Since \( f \) satisfies the discontinuity criterion on \( L \), there exist \( x_1^* \) and \( x_2^* \) in \( L \) with \( f(x_1^*) > r + \delta \) and \( f(x_2^*) < r \). Since the net \( (x_n^**) \) converges to \( f \) on \( \{x_1^*, x_2^*\} \), some \( i_0 \) exists such that, for all \( i \geq i_0 \), \( x_i^** \) belongs to the weak* closure of the following set \( C_1 \) (we shall write \( x_i^** \in w^*-C_1 \)):

\[
C_1 = B_X \cap \{x** \in X**: x**(x_1^*) > r + \delta \} \cap \{x** \in X**: x**(x_2^*) < r \}
\]

(we have assumed that \( (x_n^**) \subset B_X^** = w^* - B_X \)). Moreover, the hypothesis that \( (T^**(x_i^**)) \) is weak*-convergent to 0 yields that \( 0 \in T^{**}(w^* - C_1) = w^* - T(C_1) \), and then \( 0 \) belongs to the weak closure of \( T(C_1) \) or, equivalently, to the norm closure of \( T(C_1) \). Thus, some \( x_1 \in C_1 \) exists such that \( \|x_1\| \leq 1 \) and \( \|T(x_1)\| \leq 1 \).
Denote by $A_t$ the set \( \{ x^* \in B_X^* : x^*(x_1) > r + \delta \} \) and by $B_1$ the set \( \{ x^* \in B_X^* : x^*(x_1) < r \} \). These are weak* open sets, $x_1^* \in A_1 \cap L$ and $x_2^* \in B_1 \cap L$, and $f$ satisfies the discontinuity criterion on $L$; consequently, some points $x_{11}^*, x_{12}^* \in A_1 \cap L$ and $x_{21}^*, x_{22}^* \in L \cap B_1$ exist with $f(x_{1k}^*) > r + \delta$ and $f(x_{2k}^*) < r$ for $k = 1, 2$. Repeating the preceding argumentation with the set

\[
C_2 = B_X \cap \{ x^{**} \in X^{**} : x^{**}(x_{1k}^*) > r + \delta, k = 1, 2 \}
\]

\[
\cap \{ x^{**} \in X^{**} : x^{**}(x_{2k}^*) < r, k = 1, 2 \},
\]

it is possible to find some $x_2 \in C_2$ such that $\|x_2\| \leq 1$, $\|T(x_2)\| \leq 1/2$, $x_{11}^* \in A_1 \cap A_2 \cap L$, $x_{12}^* \in A_1 \cap B_2 \cap L$, $x_{21}^* \in B_1 \cap A_2 \cap L$, and $x_{22}^* \in B_1 \cap B_2 \cap L$, where $A_2$ and $B_2$ are the weak* open sets: $A_2 = \{ x^* \in B_X^* : x^*(x_2) > r + \delta \}$ and $B_2 = \{ x^* \in B_X^* : x^*(x_2) < r \}$.

Proceeding inductively in the same form one arrives at the desired sequence.

**Proof of the Main Result.** (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (iv) Suppose $T$ is not semi-Tauberian. Then there is an $l_1$-sequence $\{x_n\} \subset X$ such that $\{T(x_n)\}$ is weakly Cauchy. Since $\{T(x_2 - x_{2n+1})\}$ is weakly null, 0 belongs to the weak closure of the convex hull of $\{T(x_2 - x_{2n+1}) : n \geq p\}$ for all $p$. Hence, a convex block sequence $\{y_n\}$ of $\{x_2 - x_{2n+1}\}$ can be chosen such that $\|T(y_n)\| \to 0$ and, obviously, $\{y_n\}$ is also an $l_1$-sequence. Taking a suitable subsequence, we can assume that one of the following assertions holds: $T(y_n) \to x$ for all $n$, or else, $T(y_n) = 0$ for all $n$. Let $M$ be a positive constant such that $\|\sum a_n y_n\| \geq 2M \|a_n\|_{l_1}$ for all $(a_n) \in l_1$. Since $T \neq 0$, there exist $x \in X$ and $y^* \in Y^*$ with $\|x\| = M$ and $y^*(T(x)) > 0$. Then, for every null sequence $(r_n)$ in $(0, 1]$, the sequence $\{z_n = y_n + r_n x\}$ is also an $l_1$-sequence and $\|T(z_n)\| \to 0$. Furthermore, we can take in both cases a suitable sequence $(r_n)$ such that $y^*(T(z_n)) > 0$ for all $n$. On the other hand, as $(z_n)$ is equivalent to the canonical basis of $l_1$, the $w^*$-sequential closure of the convex hull of $(z_n)$ is the set $A = \{ \sum t_n z_n : t_n \geq 0 $ for all $n $ and $\sum t_n = 1 \}$. Thus, $T^{**}(A) = T(A)$ is not closed, since $y^*(\sum t_n z_n) = \sum t_n y^*(T(z_n)) \neq 0$.

(iv) $\Rightarrow$ (i) Suppose (i) does not hold. Then there is a bounded sequence $\{x_n^{**}\} \subset X^{**}$ without $w^*$-Cauchy subsequences such that $\{T^{**}(x_n^{**})\}$ is $w^*$-Cauchy. Let $y^{**}$ be the $w^*$-limit of $\{T^{**}(x_n^{**})\}$. Since $\{x_n^{**}\}$ is bounded, $y^{**} \in T^{**}(w^* B_X^*)$ and so we may assume that $y^{**} = 0$. Hence, by [10, Theorem 2] and Lemma 1.1, there is an $l_1$-sequence $\{x_n\} \subset X$ with $\|T(x_n)\| \to 0$.

Another property of $T^{**}$ was proved in [8]. If $T : X \to Y$ is Tauberian and $X$ is separable, then $T^{**}(B_t(Y))^{-1} = B_t(X)$, $B_t(X)$ being the family
of the Baire-1 elements of $X$, i.e., the $w^*$-limits of sequences of elements of $X$. In fact, this property of $T^{**}$ characterizes semi-Tauberian operators; see [3].

2. NONSEPARABLE CASE

Since the Rosenthal result from [10] is not applicable to the nonseparable case, the properties described in the main result might not be equivalent in this case. We shall consider some weaker properties. Let us start by observing that property (i) seems to be stronger than being semi-Tauberian. Consider the operator $T: c_0(\mathcal{H}) \to l_2$ given by $T(x) = \sum(1/2^n)x(n)e_n$, where $\{e_n\}$ is the canonical basis of $l_2$ and $x(n)$ is the value of the function $x \in c_0(\mathcal{H})$ in $n$. $T$ is semi-Tauberian but $T^{**}: l_2(\mathcal{H}) \to l_2$ does not satisfy property (i) because there exist bounded sequences in $l_2(\mathcal{H})$ without $w^*$-Cauchy subsequences. Since semi-Tauberian operators from nonseparable spaces do not satisfy property (i), they need not satisfy property (ii); i.e., $T^{**}(A)$ need not be closed for every $w^*$-sequentially closed bounded subset $A$ of $X^{**}$. However, $T^{**}(A)$ is closed when $A$ is the $w^*$-sequential closure of a bounded subset of $X$. To prove this, we shall need some notation: given $A$ a subset of $X^{**}$, $K(A)$ shall denote the set of elements of $X^{**}$ which are $w^*$-limits of sequences in $A$. Let $K_0(A) = A$ and inductively let $K_\alpha(A) = K(U_{< \alpha}K_\beta(A))$ for $0 < \alpha \leq \omega$, where $\omega$ is the first uncountable ordinal. $K_\omega(A)$ is the $w^*$-sequential closure of $A$.

**Proposition 2.1.** Let $T: X \to Y$ be a nonzero operator. The following are equivalent:

(i) $T^{**}(K_\omega(A))$ is closed for every bounded (convex) set $A \subset X$.

(ii) $T(A)$ is closed for every closed bounded (convex) set $A$ contained in a subspace $Z$ of $X$ which has Schur property.

(iii) Every bounded sequence $\{x_n\}$ in $X$ with $\{T(x_n)\}$ weakly Cauchy in $Y$ has a weakly Cauchy subsequence; i.e., $T$ is semi-Tauberian.

**Proof.** (i) $\Rightarrow$ (ii) is trivial and (ii) $\Rightarrow$ (iii) was already proved in the main result.

(iii) $\Rightarrow$ (i) observe that for every subset $A \subset X$ and for every ordinal number $0 \leq \alpha \leq \omega$ the following are satisfied:

$$K_\alpha(A) = \bigcup \{K_\alpha(A \cap Z) : Z \text{ is a closed separable subspace of } X\}$$
and
\[ K_w(T(A)) = \bigcup \{ K_w(T(A \cap Z)) : Z \text{ is a closed separable subspace of } X \}, \]
because the closed linear span of a countable union of separable subspaces is also separable. Thus if \( T \) is semi-Tauberian for every bounded subset \( A \) of \( X \),
\[
T^{**}(K_w(A)) = T^{**}\left( \bigcup \{ K_w(A \cap Z) : Z \text{ is a closed separable subspace of } X \} \right)
= \bigcup \{ T^{**}(K_w(A \cap Z)) : Z \text{ is a closed separable subspace of } X \},
\]
and, by the main result, for every closed separable subspace \( Z \) of \( X \),
\[
T^{**}(K_w(A \cap Z)) = K_w(T(A \cap Z)). \text{ Therefore, } T^{**}(K_w(A)) = \bigcup \{ K_w(T(A \cap Z)) : Z \text{ is a closed separable subspace of } X \} = K_w(T(A)). \]

REFERENCES