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Twists of Newforms

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Let $S_k^0(N, \psi)$ denote the subspace generated by newforms in the space of cuspforms of weight k and character ψ on $\Gamma_0(N)$. In this paper we study decompositions of $S_k^0(N, \psi)$ into direct sums of twists (by Dirichlet characters) of other spaces of newforms. Applied to individual newforms, these results immediately yield information on the behavior of newforms under character twists. Most of the results follow from applications of the Eichler Selberg formula for the traces of the Hecke operators. A version of this formula is given in the paper. A sample result is: Let p be a prime and let M be a positive integer prime to p. Let ω be a character mod p^ν with $e = \operatorname{ord}_p f(\omega) > \nu/2$ and let ϕ be a character mod M. Then $S_k^0(p^\nu M, \omega\phi) = \bigoplus_\chi S_k^0(p^e M, \omega\chi^2\phi)^{\bar{\chi}}$ where the sum $\bigoplus_\chi$ is over all primitive characters χ modulo $p^{\nu-e}$ and where $S_k^0(N, \psi)^\chi$ denotes the twist of $S_k^0(N, \psi)$ by χ. © 1990 Academic Press, Inc.

INTRODUCTION

Let $S_k^0(N, \psi)$ denote the space of newforms of weight k and character ψ on $\Gamma_0(N)$. If $F(\tau)$ is a newform in $S_k^0(N, \psi)$ and χ is a Dirichlet character, then it is well known that $F_\chi(\tau)$, the twist of $F(\tau)$ by χ, is a cuspform of weight k and character $\psi\chi^2$ on $\Gamma_0(N')$ for some N'. Furthermore, if $f(\chi)$,

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the conductor of χ, is relatively prime to N, then (see [11, 19]), $F_\chi(\tau)$ is again a newform. If $f(\chi)$ is not relatively prime to N, the situation is more complicated and has been studied by Atkin and Li in [2]. In this paper we study this and related questions. However, our point of view, methods, and results are quite different from those of [2]. Rather than study how a particular newform $F(\tau)$ behaves under character twists, we seek to decompose an entire space $S_k^0(N, \psi)$ of newforms into a direct sum of twists of other spaces of newforms. Of course, any such decomposition immediately yields information on the behavior of newforms under character twists.

Our motivation in studying the decomposition of spaces of newforms was to develop results which would help us in solving the so-called "Basis Problem" (see Eichler [4, p. 77]). Roughly speaking, this problem is to construct an explicit basis for $S_k^0(N, \psi)$ from theta series attached to positive definite rational quaternion algebras. The problem is solved (in so far as is possible) in [9]. A major ingredient in the solution is to reduce the consideration of a space of newforms $S_k^0(N, \psi)$ to spaces $S_k^0(N', \psi')$ with $f(\psi')$ "small." Thus many of the results in this paper express a given space $S_k^0(N, \psi)$ as a direct sum of twists of other spaces $S_k^0(N', \psi')$ with N' and/or $f(\psi')$ smaller than N and/or $f(\psi)$.

1. Preliminaries

In this section we introduce notation and basic results that will be used throughout the paper.

Let N be a positive integer and denote by $\Gamma_0(N)$ the congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z) \mid c \equiv 0 \pmod{N} \right\}.$$

For a positive integer $k \geqslant 2$ and a Dirichlet character ψ on Z modulo N, let $S_k(N, \psi)$ denote the space of all cuspforms of weight k and character ψ on $\Gamma_0(N)$ (see, e.g., [16] or [11]). In particular for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and $F(\tau) \in S_k(N, \psi)$ we have

$$F\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k \, \psi(d) \, F(\tau). \tag{1.1}$$

Note that $S_k(N, \psi) \neq 0$ only if $\psi(-1)(-1)^k = 1$.

Let ψ be a Dirichlet character modulo N (a "character mod N" for short). The natural isomorphism $Z/NZ \cong \bigoplus_{l \mid N} Z/l^{\sigma(l)}Z$ where the sum is over all primes l dividing N and $\sigma(l) = \mathrm{ord}_l(N)$ gives a decomposition $\psi = \prod_{l|N} \psi_l$ where for each prime $l|N$, ψ_l is a character mod $l^{\sigma(l)}$. We denote by $f(\psi)$ the *conductor* of ψ. If ω is a character modulo a power of prime p, we define the *exponential conductor* $e(\omega)$ of ω by $f(\omega) = p^{e(\omega)}$. We say the conductor of the trivial character 1 is 1 so that $e(1) = 0$.

If $F(\tau) \in S_k(N, \psi)$, we let $x = e^{2\pi i\tau}$ and denote the Fourier series expansion of $F(\tau)$ at $i\infty$, i.e., $F(\tau) = \sum_{n=1}^{\infty} a(n)\, e^{2\pi i n\tau}$, by $F(\tau) = \sum_{n=1}^{\infty} a(n)\, x^n$. If (as is usually the case) the indices in a Fourier series expansion run from 1 to ∞, we often drop the limits and write $F(\tau) = \sum a(n)\, x^n$ to mean $F(\tau) = \sum_{n=1}^{\infty} a(n)\, x^n$.

The Hecke operators $T_k(m)$ for $(m, N) = 1$ act on $S_k(N, \psi)$ as follows: If $F(\tau) = \sum a(n)\, x^n \in S_k(N, \psi)$, then $F \mid T_k(m) = \sum b(n)\, x^n$ where

$$b(n) = \sum_{c \mid (n,m)} \psi(c)\, c^{k-1} a(nm/c^2). \tag{1.2}$$

See [5, Nos. 35 and 36] or [16, p. 80].

Let $F(\tau) \in S_k(N, \psi)$ be a common eigenform for all the Hecke operators $T_k(n)$ with $(n, N) = 1$. $F(\tau)$ is called a *newform* of weight k, level N, and character ψ if there are no other linearly independent eigenforms in $S_k(N, \psi)$ having the same eigenvalues as $F(\tau)$ for every $T_k(n)$ with $(n, N) = 1$ (see [1, 12, 11]). In this paper we assume all newforms are "normalized," i.e., their first Fourier coefficient is 1 (see [1, Lemma 19; 11, p. 294]). Denote by $S_k^0(N, \psi)$ the subspace of $S_k(N, \psi)$ generated by newforms.

For a cuspform $F(\tau) = \sum a(n)\, x^n$ and a primitive character χ mod M, the twist of F by χ, denoted by F_χ, is defined by

$$F_\chi(\tau) = \sum_{n=1}^{\infty} a(n)\, \chi(n)\, x^n.$$

PROPOSITION 1.1. *Let $F \in S_k(N, \psi)$ and let χ be a primitive character mod M. Then $F_\chi \in S_k(N', \psi\chi^2)$ where N' is the least common multiple of N, M^2, and $f(\psi) M$.*

Proof. See Proposition 3.64 of [16] or Proposition 3.1 of [2]. ∎

If ψ and χ are as in Proposition 1.1, we denote by $S_k^0(N, \psi)^\chi$ the space $\{F_\chi \mid F \in S_k^0(N, \psi)\}$. Note that by Proposition 1.1 we have

$$S_k^0(N, \psi)^\chi \subseteq S_k(N', \psi\chi^2). \tag{1.3}$$

In regard to a space $S_k(N, \psi)$ or $S_k^0(N, \psi)$ we denote by H the Hecke Algebra generated by all Hecke operators $T_k(n)$, $(n, N) = 1$ acting on

$S_k(N, \psi)$. If there are several spaces under consideration, H will denote the algebra generated by $T_k(n)$ with n prime to the least common multiple of the levels of the various spaces. The $T_k(n)$ with $(n, N)=1$ are a commuting family of "ψ-Hermitian" (see [13, pp. IV–14, IV–24]) operators on $S_k(N, \psi)$ and hence H is a commutative semi-simple algebra. $S_k^0(N, \psi)^\chi$ is an H-submodule of $S_k(N', \psi\chi^2)$ (see, e.g., Proposition 3.2 of [2]) where the notation is as in (1.3). All isomorphisms between spaces of cuspforms in this paper will be H-module isomorphisms. Thus, for example, when we write in Theorem 3.2 below that

$$S_k^0(p^\nu M, \bar{\omega}\phi)^\omega \cong S_k^0(p^\nu M, \omega\phi)$$

we mean that they are isomorphic as H-submodules of $S_k(N', \omega\phi)$ where $N' = p^{2\nu}M$ and where H is generated by all $T_k(n)$ with $(n, pM)=1$.

As H is a semi-simple algebra, to show two H-modules A and B are isomorphic, we need only show that the trace of $T_k(n)$ on A equals the trace of $T_k(n)$ on B for all $T_k(n) \in H$—see, e.g., [18, p. 174]. This is our main tool (and has in fact been one of the main tools in the theory of modular forms since the discovery of the Eichler–Selberg Trace formula for $T_k(n)$ in the early fifties).

An important result from the theory of newforms is the decomposition

$$S_k(N, \psi) \cong \bigoplus_{f(\psi)\mid a \mid N} \delta(N/a)\, S_k^0(a, \psi), \tag{1.4}$$

where $\bigoplus_{f(\psi)\mid a\mid N}$ means the direct sum over all positive integers a with $f(\psi)\mid a$ and $a \mid N$. Here $\delta(s)$ denotes the number of positive integers dividing s and $2A = A \oplus A$, etc. In particular if $N = p^\nu M$ with p a prime, $p \nmid M$ and $\psi = \omega\phi$ with ω a character mod p^ν and ϕ a character mod M, then (1.4) becomes

$$S_k(p^\nu M, \omega\phi) \cong \bigoplus_{i=e(\omega)}^{\nu} (\nu - i + 1) \bigoplus_{f(\phi)\mid a \mid M} \delta(M/a)\, S_k^0(p^i a, \omega\phi). \tag{1.5}$$

The decomposition (1.4) follows from Lemma 15 and Theorem 5 of [1] in the case $\psi = 1$. For the general case see Section 2 of [12] or Section 2 of [11] (and also Theorem B of [10]).

We need the following

LEMMA 1.2. *Let the notation be as in* (1.3). *Then for* $(n, N')=1$ *the trace of* $T_k(n)$ *on* $S_k^0(N, \psi)^\chi$ *considered as a submodule of* $S_k^0(N', \psi\chi^2)$ *is equal to* $\chi(n)$ *times the trace of* $T_k(n)$ *on* $S_k^0(N, \psi)$.

Proof. See Lemma 8.1 of [14]. ∎

We say that cuspforms F and G in $S_k(N, \psi)$ are *equivalent* and we write $F \sim G$ if they are both eigenforms for all $T_k(n)$ with $(n, N) = 1$ with the same eigenvalues. If F and G belong to $S_k(N, \psi)$ with G a newform in $S_k^0(N, \psi)$, then $F \sim G$ implies $F = aG$ for some $a \in \mathbb{C}$ (see Theorem 5 of [1], the introduction to [11], and also [12]). We will often need this result and we usually indicate the places we are using it by saying that our result "follows from the theory of newforms" (see, e.g., the proof of Corollary 3.4 below).

2. The Trace Formula for Hecke Operators

In this section we state the formula for the trace of the Hecke operator $T_k(n)$. Denote by $\mathrm{tr}_{N,\psi} T_k(n)$ the trace of $T_k(n)$ acting on the space $S_k(N, \psi)$.

Lemma 2.1. *Let p be a prime and let ω be a character modulo some power of p. Let $e = e(\omega)$ be the exponential conductor of ω. If σ and τ are non-negative integers with $\sigma + \tau \geqslant e$ and $2\tau \geqslant e$ and u is a unit* mod p, *then*

$$\sum_{z \in Z/p^\sigma Z} \omega(u + zp^\tau) = \begin{cases} p^\sigma \omega(u) & \text{if } e \leqslant \tau \\ 0 & \text{if } e > \tau. \end{cases}$$

Proof. We can assume $e > \tau$ since the result is clear otherwise. Let $G = \{1 + zp^\tau \mid z \in Z/p^\sigma Z\}$. Then G is well defined subgroup of $(Z/p^e Z)^\times$. Since $e > \tau$, there exist α_1 and $\alpha_2 \in (Z/p^e Z)^\times$ with $\alpha_1 \equiv \alpha_2 \pmod{p^\tau}$ and $\omega(\alpha_1) \neq \omega(\alpha_2)$. Thus ω restricted to G is non-trivial and we have $\sum_{z \in Z/p^\sigma Z} \omega(u + zp^\tau) = \omega(u) \sum_{g \in G} \omega(g) = 0$. ∎

Hijikata in [6] computed the traces of the Hecke operators in a quite general setting—see the Theorem on p. 57 of [6]. We copy here (with a few changes) the case of this theorem which we require.

Theorem 2.2 (Hijikata). *Let k be an integer $\geqslant 2$. Let ψ be a character* mod N *and assume $(-1)^k \psi(-1) = 1$. Write $\psi = \prod_{l \mid N} \psi_l$ where for each prime l dividing N, ψ_l is a character* mod l^ν, $\nu = \mathrm{ord}_l(N)$. *Then for $(n, N) = 1$ we have*

$$\mathrm{tr}_{N,\psi} T_k(n) = -\sum_s a(s) \sum_f b(s, f) \prod_{l \mid N} c'_\psi(s, f, l) + \delta(\psi) \deg T_k(n)$$
$$+ \delta(\sqrt{n}) \frac{k-1}{12} N \prod_{l \mid N} (1 + 1/l) - \delta(\sqrt{n}) \frac{\sqrt{n}}{2} \prod_{l \mid N} \mathrm{par}(l),$$

where

$$\delta(\psi)=\begin{cases}1 & \text{if } k=2 \text{ and } \psi \text{ is trivial}\\ 0 & \text{otherwise}\end{cases}$$

$$\delta(\sqrt{n})=\begin{cases}n^{k/2-1}\psi(\sqrt{n}) & \text{if } n \text{ is a perfect square}\\ 0 & \text{otherwise}\end{cases}$$

$$\operatorname{par}(l)=\begin{cases}2l^{\nu-e} & \text{if } \quad e\geqslant\rho+1\\ l^{\rho}+l^{\rho-1} & \text{if } \quad e\leqslant\rho \text{ and } \nu \text{ is even}\\ 2l^{\rho} & \text{if } \quad e\leqslant\rho \text{ and } \nu \text{ is odd.}\end{cases}$$

Here for fixed $l\mid N$, $\nu=\operatorname{ord}_l(N)$, $\rho=[\nu/2]$, and $e=e(\psi_l)$.

The meaning of s, $a(s)$, $b(s,f)$, and $c'_\psi(s,f,l)$ are given as follows.

Let s run over all integers such that s^2-4n is negative or a positive square. Hence by some positive integer t and squarefree negative integer m, s^2-4n has one of the following forms which we classify into the cases (h) or (e) as follows:

$$s^2-4n=\begin{cases}t^2 & & (h)\\ t^2m, & 0>m\equiv 1 \pmod 4 & (e)\\ t^2 4m, & 0>m\equiv 2,3 \pmod 4 & (e).\end{cases}$$

Let $\Phi(X)=\Phi_s(X)=X^2-sX+n$ and let x and y be the roots in $\mathbb{C}$ of $\Phi(X)=0$. Corresponding to the classification of s put

$$a(s)=\begin{cases}(\operatorname{Min}\{|x|,|y|\})^{k-1}|x-y|^{-1}\operatorname{sgn}(x)^k & (h)\\ 1/2(x^{k-1}-y^{k-1})/(x-y) & (e).\end{cases}$$

For each fixed s let f run over all positive divisors of t and let

$$b(s,f)=\begin{cases}1/2\phi((s^2-4n)^{1/2}/f) & (h)\\ h((s^2-4n)/f^2)/\omega((s^2-4n)/f^2) & (e),\end{cases}$$

where ϕ is Euler's function and $h(d)$ (resp. $\omega(d)$) denotes the class number of locally principal ideals (resp. 1/2 the cardinality of the unit group) of the order of $Q(\sqrt{d})$ with discriminant d.

For a pair (s,f) fixed and a prime divisor l of N, let $\nu=\operatorname{ord}_l(N)$, $b=\operatorname{ord}_l(f)$, and put $\tilde{A}=\{x\in Z\mid\Phi(x)\equiv 0 \pmod{l^{\nu+2b}},\ 2x\equiv s \pmod{l^b}\}$ and $\tilde{B}=(x\in\tilde{A}\mid\Phi(x)\equiv 0 \pmod{l^{\nu+2b+1}})\}$. Let $A=A(s,f,l)$ (resp. $B=B(s,f,l)$) be a complete set of representatives of $\tilde{A}$ (resp. $\tilde{B}$) mod $l^{\nu+b}$ and let $B'=B'(s,f,l)=\{s-z\mid z\in B\}$. Then

$$c'_\psi(s,f,l)=\begin{cases}\sum_x\psi_l(x) & \text{if} \quad (s^2-4n)/f^2\not\equiv 0 \pmod l\\ \sum_x\psi_l(x)+\sum_y\psi_l(y) & \text{if} \quad (s^2-4n)/f^2\equiv 0 \pmod l),\end{cases}$$

where x runs over all elements of $A(s, f, l)$ and y runs over all elements of $B'(s, f, l)$.

Proof. The trace formula established in the Theorem on p. 57 of [6] is essentially the formula given in Theorem 2.2. To obtain the exact formula in Theorem 2.2 we first have to translate the formula in [6] to our setting and then we also have to explicitly evaluate the contribution of the parabolic terms which occur when n is perfect square. A translation is necessary for two reasons. First our space $S_k(N, \psi)$ is denoted by $S_0(\Gamma_0(N), k, \bar{\psi})$ in [6] (compare (1.1) above with Section 5.5 of [7] and also (S2) on p. 60 of [6]). Our notation $S_k(N, \psi)$ is consistent with that used by Shimura (see [16, p. 79]) and Atkin and Li [2]. Also it is a natural choice from the point of view of theta series—see [8, 9]. On the other hand, the notation used in [6] is consistent with that used by Eichler (see [3, p. 77]). Secondly the definition of the Hecke operators differs in the present paper from that used in [6]. Our present definition agrees with that used by Shimura, Atkin and Li, and Eichler while the definition used in [6] agrees with that used by Shimizu (see [17]). Specifically, fixing k and N and following [16] let

$$\Delta' = \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2, Z) \mid (a, N) = 1,\ c \equiv 0 \ (\mathrm{mod}\ N),\ \det(\gamma) > 0\right\}$$

and for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta'$ let $a(\gamma) = a$, $d(\gamma) = d$, and

$$f|_{[\gamma]_k}(\tau) = (\det(\gamma))^{k/2} (c\tau + d)^{-k} f(\gamma(\tau)),$$

where $f \in S_k(N, \psi)$ (see [16, p. 28]). Now fix a positive integer n prime to N. Define

$$\Xi(n) = \{\gamma \in \Delta' \mid \det(\gamma) = n\}$$

and let $\Xi(n) = \bigcup_\nu \Gamma_0(N)\,\alpha_\nu$ be some disjoint decomposition of $\Xi(n)$ into *right cosets*. Then for $f \in S_k(N, \psi)$ we have

$$f \mid T_k(n) = n^{k/2-1} \sum_\nu \psi(a(\alpha_\nu))\, f|_{[\alpha_\nu]_k}. \tag{2.1}$$

This follows from (3.5.5) on p. 79 of [16] and the definition of $T'(n)$ on p. 70 of [16]. Note that what we call $T_k(n)$ Shimura calls $T'(n)_{k,\psi}$. From (2.1) one easily obtains (see [16, pp. 79–80]) the action of $T_k(n)$ on Fourier coefficients (see (3.5.12) of [16] and (1.2) above) which we have used in our definition of $T_k(n)$. On the other hand the Hecke operators in [6] are defined using *left cosets*. In [6] the Hecke operators are denoted by $T(n)$ but we will denote them here by $\tilde{T}_k(n)$. They are defined as follows (see (1) on p. 61 of [6]): let $\Xi(n)=\bigcup_\nu \beta_\nu \Gamma_0(N)$ be some disjoint decomposition of $\Xi(n)$ into *left cosets*. Then for $f\in S_k(N,\bar\psi)$ (which is the space $S_0(\Gamma_0(N),k,\psi)$ of [6]) we have

$$\tilde{T}_k(n)f=\sum_\nu \psi(a(\beta_\nu))\,f|_{[\beta_\nu^{-1}]_k}. \tag{2.2}$$

We claim that for $f\in S_k(N,\bar\psi)$

$$n^{k/2-1}\tilde{T}_k(n)f=\psi(n)f|\,T_k(n). \tag{2.3}$$

To show this first note that $\Xi(n)$ is left invariant by the involution $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\mapsto\left(\begin{smallmatrix} d & -b\\ -c & a\end{smallmatrix}\right)$ of $\mathrm{Mat}(2,Z)$. Hence

$$\Xi(n)=\{n\gamma^{-1}\,|\,\gamma\in\Xi(n)\}=n\Xi^{-1}(n)\qquad\text{(say)}.$$

Thus if $\Xi(n)=\bigcup_\nu \Gamma_0(N)\,\alpha_\nu$, then $\Xi^{-1}(n)=\bigcup_\nu \alpha_\nu^{-1}\Gamma_0(N)$ and $\Xi(n)=n\Xi^{-1}(n)=\bigcup_\nu (n\alpha_\nu^{-1})\,\Gamma_0(N)$. Letting $\beta_\nu=n\alpha_\nu^{-1}$ we have

$$\begin{aligned}
\tilde{T}_k(n)f&=\sum_\nu \psi(a(\beta_\nu))\,f|_{[\beta_\nu^{-1}]_k}\\
&=\sum_\nu \psi(d(\alpha_\nu))\,f|_{[\alpha_\nu]_k}\\
&=\sum_\nu \psi(na(\alpha_\nu)^{-1})\,f|_{[\alpha_\nu]_k}\qquad \text{since } a(\alpha_\nu)\,d(\alpha_\nu)\equiv n \pmod N\\
&=\psi(n)\sum_\nu \bar\psi(a(\alpha_\nu))\,f|_{[\alpha_\nu]_k}\\
&=\psi(n)\,n^{1-k/2}f|T_k(n)
\end{aligned}$$

which establishes (2.3). Let $\gamma=\left(\begin{smallmatrix} 0 & -1\\ N & 0\end{smallmatrix}\right)$ and let $f|E=f|_{[\gamma]_k}$. Then it is well known and not difficult to check (see, e.g., Propositions 1.1 and 1.2 of [2]) that $f\mapsto f|E$ is a linear isomorphism of $S_k(N,\psi)$ onto $S_k(N,\bar\psi)$ and that for $f\in S_k(N,\psi)$

$$f\,|T_k(n)|\,E=\psi(n)\,f\,|E|\,T_k(n). \tag{2.4}$$

By (2.4) the trace of $T_k(n)$ on $S_k(N,\psi)$ equals $\psi(n)$ times the trace of $T_k(n)$ on $S_k(N,\psi)$ which by (2.3) equals $n^{k/2-1}$ times the trace of $\tilde{T}_k(n)$ on $S_k(N,\bar\psi)$ which equals $n^{k/2-1}$ times the trace of $T(n)$ on $S_0(\Gamma_0(N),k,\psi)$ in the notation of [6]. Thus after multiplying by $n^{k/2-1}$ the formula in the Theorem on p. 57 of [6] (with $M=N$ and $\mathscr{H}=(Z/NZ)^\times$) gives the trace of $T_k(n)$ on $S_k(N,\psi)$.

All that remains to do is to explicitly evaluate the contribution of the parabolic terms to the trace formula in [6]. This goes as follows. Fix s with

$s^2=4n$. The corresponding contribution to the trace formula (see [6, pp. 57–58]) is

$$-n^{k/2-1}\frac{\sqrt{n}}{4N}(\operatorname{sgn}(s))^k\sum_{f=1}^{N}\prod_{l\mid N}c'_\psi(s,f,l), \tag{2.5}$$

where $c'_\psi(s,f,l)=\sum_x\psi_l(x)+\sum_y\psi_l(y)$ where x (resp. y) runs over elements of $A(s,f,l)$ (resp. $B'(s,f,l)$). Fix $l\mid N$ temporarily and let $v=\operatorname{ord}_l(N)$. If v is odd, say $v=2\rho+1$, then $A(s,f,l)=B'(s,f,l)=\{s/2+zp^{\rho+b+1}\mid z\in Z/p^\rho Z\}$ and if v is even, say $v=2\rho$, then $A(s,f,l)=\{s/2+zp^{\rho+b}\mid z\in Z/p^\rho Z\}$ and $B'(s,f,l)=\{s/2+zp^{\rho+b+l}\mid z\in Z/^{\rho-1}Z\}$. Let $e=e(\psi_l)$. Then by Lemma 2.1 we find that for $v=2\rho+1$

$$c'_\psi(s,f,l)=\begin{cases}2l^\rho\psi_l(s/2) & \text{if } e\leqslant\rho+b+1\\ 0 & \text{if } e\geqslant\rho+b+2\end{cases} \tag{2.6}$$

and for $v=2\rho$

$$c'_\psi(s,f,l)=\begin{cases}(l^\rho+l^{\rho-1})\,\psi_l(s/2) & \text{if } e\leqslant\rho+b\\ l^{\rho-1}\psi_l(s/2) & \text{if } e=\rho+b+1\\ 0 & \text{if } e\geqslant\rho+b+2.\end{cases} \tag{2.7}$$

Thus fixing s and l, $c'_\psi(s,f,l)$ depends only on $b=\operatorname{ord}_l(f)$ so we let $c''(s,b,l)=c'_\psi(s,f,l)$. Further since $e\leqslant v$, $c''(s,b,l)=c''(s,v,l)$ if $b\geqslant v$. Now let $N=l_1^{v_1}\cdots l_r^{v_r}$ with $v_i\geqslant 0$ and l_i distinct primes. For each i, $1\leqslant i\leqslant r$, let $b_i\in Z$ with $0\leqslant b_i\leqslant v_i$. Then the number of f, $1\leqslant f\leqslant N$, with $\operatorname{ord}_{l_i}(f)=b_i$ if $b_i<v_i$ or $\operatorname{ord}_{l_i}(f)\geqslant b_i$ if $b_i=v_i$ for all $i=1,\dots,r$, is $\prod_{i=1}^r(l_i^{v_i-b_i}-l_i^{v_i-b_i-1})$ where we use the convention that $l_i^{-1}=0$. Hence

$$\begin{aligned}&-n^{k/2-1}\frac{\sqrt{n}}{4N}(\operatorname{sgn}(s))^k\sum_{f=1}^{N}\prod_{l\mid N}c'_\psi(s,f,l)\\ &\quad=-n^{k/2-1}\frac{\sqrt{n}}{4}(\operatorname{sgn}(s))^k\\ &\qquad\times\prod_{i=1}^{r}\left(l_i^{-v_i}\sum_{b=0}^{v_i}(l_i^{v_i-b}-l_i^{v_i-b-1})\,c''(s,b,l_i)\right).\end{aligned}$$

If $v=2\rho+1$, from (2.6) we see that

$$l^{-v}\sum_{b=0}^{v}(l^{v-b}-l^{v-b-1})\,c''(s,b,l)=\begin{cases}2l^\rho\psi_l(s/2) & \text{if } e\leqslant\rho\\ 2l^{v-e}\psi_l(s/2) & \text{if } e\geqslant\rho+1\end{cases}$$

while if $\nu = 2\rho$, from (2.7) we see that

$$l^{-\nu} \sum_{b=0}^{\nu} (l^{\nu-b} - l^{\nu-b-1})\, c''(s, \rho, l)$$
$$= \begin{cases} (l^{\rho} + l^{\rho-1})\, \psi_l(s/2) & \text{if } e \leqslant \rho \\ 2l^{\nu-e}\psi_l(s/2) & \text{if } e \geqslant \rho + 1. \end{cases}$$

Noting that we must consider both $s = 2\sqrt{n}$ and $s = -2\sqrt{n}$ and also that $(-1)^k \psi(-1) = 1$, we obtain a total contribution of

$$-n^{k/2-1} \frac{\sqrt{n}}{2} \psi(\sqrt{n}) \prod_{l \mid N} \mathrm{par}(l). \quad \blacksquare$$

We will require a few technical lemmas. Let K be an imaginary quadratic number field. For any order $\mathfrak{o}$ of K, denote by $h(\mathfrak{o})$ the ideal class number of locally principal $\mathfrak{o}$-ideals, i.e., $h(\mathfrak{o}) = [J_K\colon \mathscr{U}(\mathfrak{o})\, K^\times]$ where J_K is the idèle group of K and $\mathscr{U}(\mathfrak{o}) = \{\tilde{a} = (a_l) \in J_K \mid a_l \in U(\mathfrak{o}_l) \text{ for all } l < \infty\}$ where $U(\mathfrak{o}_l)$ denotes the unit group of $\mathfrak{o}_l$. Further let $\omega(\mathfrak{o}) = \frac{1}{2}|U(\mathfrak{o})|$. Then we have

LEMMA 2.3. *Let the notation be as above. Let $\mathfrak{o}$ be an order of K of discriminant Δ and let $\mathfrak{o}'$ be the suborder of $\mathfrak{o}$ of index f. Then*

$$\frac{h(\mathfrak{o}')}{\omega(\mathfrak{o}')} = \frac{h(\mathfrak{o})}{\omega(\mathfrak{o})} f \prod_{l \mid f} \left(l - \left\{\frac{\Delta}{l}\right\} 1/l\right),$$

where

$$\left\{\frac{\Delta}{l}\right\} = \begin{cases} 0 & \text{if } l^2 \mid \Delta \text{ and } l^{-2}\Delta \equiv 0 \text{ or } 1 \pmod 4 \\ (\Delta/l) & \text{the Kronecker symbol, otherwise.} \end{cases}$$

Proof. This is well known and easy. See, e.g., Lemma 4.16 of [14]. $\blacksquare$

LEMMA 2.4. *Let the notation be as in Theorem 2.2. In particular set*

$$s^2 - 4n = \begin{cases} t^2 & & (h) \\ t^2 m, & 0 > m \equiv 1 \pmod 4 & (e) \\ t^2 4m, & 0 > m \equiv 2, 3 \pmod 4 & (e). \end{cases}$$

Let l be a prime dividing N and put $t = l^a t_0$ with $(l, t_0) = 1$. Let $f \mid t$ and put $f = l^b f_0$ with $(l, f_0) = 1$. Then

$$b(s,f)=\begin{cases}(l^{a-b}-l^{a-b-1})\,b(s,l^a f_0) \\ \qquad \text{if } s^2-4n=l^{2a}d^2,\ d \text{ a unit of } Z_l \\ (l^{a-b}+l^{a-b-1})\,b(s,l^a f_0) \\ \qquad \text{if } s^2-4n=l^{2a}d \text{ where } d \text{ is a unit of } Z_l \\ \qquad \text{with } (d/l)=-1 \text{ (i.e., } d\equiv 5 \pmod 8 \text{) if } l=2) \\ l^{a-b}b(s,l^a f_0) \\ \qquad \text{if } l\neq 2 \text{ and } s^2-4n=l^{2a+1}d,\ d \text{ a unit of } Z_l \text{ or} \\ \qquad \text{if } l=2 \text{ and } s^2-4n=l^{2a+2}d,\ d\equiv 2 \text{ or } 3 \pmod 4).\end{cases}$$

Here if $b=a$, we use the convention that $l^{a-b-1}=0$.

Proof. If $s^2-4n=t^2$, then $b(s,f)=\frac{1}{2}\phi(t/f)$ and the result is clear. If s^2-4n is not a perfect square, the result follows directly from Lemma 2.3. ∎

Since we will be employing the trace formula in Theorem 2.2 many times, it will simplify the exposition to explicitly calculate the sets $A(s,f,l)$ and $B'(s,f,l)$.

LEMMA 2.5. *Let $A(s,f,l)$ and $B'(s,f,l)$ be the sets appearing in Theorem 2.2. For fixed N, n, s, and l, $A(s,f,l)$ and $B'(s,f,l)$ depend only on* $\mathrm{ord}_l(f)$ *and we will write $A_b=A(s,f,l)$ and $B'_b=B'(s,f,l)$ where $b=\mathrm{ord}_l(f)$. Let $v=\mathrm{ord}_l(N)$. Then the sets A_b and B'_b are given as follows.*

Case A. $s^2-4n=l^{2a}d^2$, l odd, d a unit of Z_l.

If v is odd, $v=2\rho+1$ we have

if $a-b\leqslant\rho$

$$A_b=\left\{\frac{s\pm l^a d}{2}+zl^{2\rho+2b-a+1}\,\middle|\, z\in Z/l^{a-b}Z\right\}$$

$$B'_b=\left\{\frac{s\pm l^a d}{2}+zl^{2\rho+2b-a+2}\,\middle|\, z\in Z/l^{a-b-1}Z\right\}.$$

Here $B'_b=\phi$ if $b=a$

if $a-b\geqslant\rho+1$

$$A_b=B'_b=\left\{\frac{s}{2}+zl^{\rho+b+1}\,\middle|\, z\in Z/l^{\rho}Z\right\}.$$

If v is even, $v=2\rho$ we have

if $a-b\leqslant\rho-1$

$$A_b=\left\{\frac{s\pm l^a d}{2}+zl^{2\rho+2b-a}\,\middle|\,z\in Z/l^{a-b}Z\right\}$$

$$B'_b=\left\{\frac{s\pm l^a d}{2}+zl^{2\rho+2b-a+1}\,\middle|\,z\in Z/l^{a-b-1}Z\right\}$$

Here $B'_b=\phi$ if $b=a$

if $a-b=\rho$

$$A_b=\left\{\frac{s}{2}+zl^a\,\middle|\,z\in Z/l^\rho Z\right\}$$

$$B'_b=\left\{\frac{s\pm l^a d}{2}+zl^{a+1}\,\middle|\,z\in Z/l^{\rho-1}Z\right\}$$

if $a-b\geqslant\rho+1$

$$A_b=\left\{\frac{s}{2}+zl^{\rho+b}\,\middle|\,z\in Z/l^\rho Z\right\}$$

$$B'_b=\left\{\frac{s}{2}+zl^{\rho+b+1}\,|\,z\in Z/l^{\rho-1}Z\right\}.$$

Case B. $s^2-4n=l^{2a}u$, l odd, u a non-square unit of Z_l.

If v is odd, $v=2\rho+1$ we have

if $a-b\leqslant\rho$ $\quad A_b=B'_b=\phi$

if $a-b\geqslant\rho+1$

$$A_b=B'_b=\left\{\frac{s}{2}+zl^{\rho+b+1}\,\middle|\,z\in Z/l^\rho Z\right\}.$$

If v is even, $v=2\rho$ we have

if $a-b\leqslant\rho-1$ $\quad A_b=B'_b=\phi$

if $a-b=\rho$

$$A_b=\left\{\frac{s}{2}+zl^a\,\middle|\,z\in Z/l^\rho Z\right\}$$

$$B'_b=\phi$$

if $a-b \geqslant \rho+1$

$$A_b = \left\{ \frac{s}{2} + zl^{\rho+b} \,\middle|\, z \in Z/l^{\rho}Z \right\}$$

$$B'_b = \left\{ \frac{s}{2} + zl^{\rho+b+1} \,\middle|\, z \in Z/l^{\rho-1}Z \right\}.$$

Case C. $s^2-4n = l^{2a+1}d$, l odd, d a unit of Z_l.

If v is odd, $v=2p+1$ we have

if $a-b \leqslant \rho-1$ $\quad A_b = B'_b = \phi$

if $a-b=\rho$

$$A_b = \left\{ \frac{s}{2} + zl^{a+1} \,\middle|\, z \in Z/l^{\rho}Z \right\}$$

$$B'_b = \phi$$

if $a-b \geqslant \rho+1$

$$A_b = B'_b = \left\{ \frac{s}{2} + zl^{\rho+b+1} \,\middle|\, z \in Z/l^{\rho}Z \right\}.$$

If v is even, $v=2\rho$ we have

if $a-b \leqslant \rho-1$ $\quad A_b = B'_b = \phi$

if $a-b \geqslant \rho$

$$A_b = \left\{ \frac{s}{2} + zl^{\rho+b} \,\middle|\, z \in Z/l^{\rho}Z \right\}$$

$$B'_b = \left\{ \frac{s}{2} + zl^{\rho+b+1} \,\middle|\, z \in Z/l^{\rho-1}Z \right\}.$$

Case D. $s^2-4n = 2^{2a}d^2$, $l=2$, d a unit of Z_2.

If v is odd, $v=2\rho+1$ we have

if $a-b \leqslant \rho$

A_b and B'_b are as in Case A with $l=2$

if $a-b=\rho+1$

$$A_b = B'_b = \left\{ \frac{s+2^a d}{2} + z2^a \,\middle|\, z \in Z/2^{\rho}Z \right\}$$

if $a-b \geqslant \rho+2$

A_b and B'_b are as in Case A with $l=2$.

If v is even, $v = 2\rho$ we have

if $a - b \leqslant \rho - 1$

A_b and B'_b are as in Case A with $l = 2$

if $a - b = \rho$

$$A_b = \left\{ \frac{s + 2^a d}{2} + z2^a \,\middle|\, z \in Z/2^\rho Z \right\}$$

$$B'_b = \left\{ \frac{s \pm 2^a d}{2} + z2^{a+1} \,\middle|\, z \in Z/2^{\rho-1} Z \right\}$$

if $a - b = \rho + 1$

$$A_b = \left\{ \frac{s}{2} + z2^{a-1} \,\middle|\, z \in Z/2^\rho Z \right\}$$

$$B'_b = \left\{ \frac{s + 2^a d}{2} + z2^a \,\middle|\, z \in Z/2^{\rho-1} Z \right\}$$

if $a - b \geqslant \rho + 2$

A_b and B'_b are as in Case A with $l = 2$.

Case E. $s^2 - 4n = 2^{2a} u$, $l = 2$, $u \in Z_2$, $u \equiv 5 \pmod 8$.

If v is odd, $v = 2\rho + 1$ we have

if $a - b \leqslant \rho$ $\quad A_b = B'_b = \phi$

if $a - b \geqslant \rho + 1$

A_b and B'_b are as in Case D with $d = 1$

If v is even, $v = 2\rho$ we have

if $a - b \leqslant \rho - 1$ $\quad A_b = B'_b = \phi$

if $a - b = \rho$

$$A_b = \left\{ \frac{s + 2^a}{2} + z2^a \,\middle|\, z \in Z/2^\rho Z \right\}$$

$B'_b = \phi$

if $a - b \geqslant \rho + 1$

A_b and B'_b are as in Case *D* with $d = 1$.

Case F. $s^2-4n=2^{2a}4w$, $l=2$, $w\in Z_2$, $w\equiv 3 \pmod 4$).

If v is odd, $v=2\rho+1$ we have

if $a-b\leqslant\rho-1$ $\quad A_b=B'_b=\phi$

if $a-b=\rho$

$$A_b=\left\{\frac{s+2^{a+1}}{2}+z2^{a+1}\,\middle|\,z\in Z/2^\rho Z\right\}$$

$$B'_b=\phi$$

if $a-b\geqslant\rho+1$

$$A_b=B'_b=\left\{\frac{s}{2}+z2^{\rho+b+1}\,\middle|\,z\in Z/2^\rho Z\right\}.$$

If v is even, $v=2\rho$ we have

if $a-b\leqslant\rho-1$ $\quad A_b=B'_b=\phi$

if $a-b=\rho$

$$A_b=\left\{\frac{s}{2}+z2^{a}\,\middle|\,z\in Z/2^\rho Z\right\}$$

$$B'_b=\left\{\frac{s+2^{a+1}}{2}+z2^{a+1}\,\middle|\,z\in Z/2^{\rho-1} Z\right\}$$

if $a-b\geqslant\rho+1$

$$A_b=\left\{\frac{s}{2}+z2^{\rho+b}\,\middle|\,z\in Z/2^\rho Z\right\}$$

$$B'_b=\left\{\frac{s}{2}+z2^{\rho+b+1}\,\middle|\,z\in Z/2^{\rho-1} Z\right\}.$$

Case G. $s^2-4n=2^{2a}4c$, $l=2$, $c\in Z_2$, $c\equiv 2 \pmod 4$).

If v is odd, $v=2\rho+1$ we have

if $a-b\leqslant\rho-1$ $\quad A_b=B'_b=\phi$

if $a-b=\rho$

$$A_b=\left\{\frac{s}{2}+z2^{a+1}\,\middle|\,z\in Z/2^\rho Z\right\}$$

$$B'_b=\phi$$

if $a-b\geqslant\rho+1$

A_b and B'_b are as in Case F.

If v is even, $v=2\rho$ we have

if $a-b\leqslant\rho-1$ $\quad A_b=B'_b=\phi$

if $a-b=\rho$

$$A_b=\left\{\frac{s}{2}+z2^a\,\middle|\, z\in Z/2^\rho Z\right\}$$

$$B'_b=\left\{\frac{s}{2}+z2^{a+1}\,\middle|\, z\in Z/2^{\rho-1}Z\right\}$$

if $a-b\geqslant\rho+1$

A_b and B'_b are as in Case F.

Proof. This follows directly from the definition of A_b and B'_b by easy but tedious calculations. We leave them to the reader. ∎

3. Twisting Newforms

In this section we begin our study of the behavior of newforms under character twists. We are interested in finding isomorphisms between the twists of various spaces of newforms and also decompositions of spaces of newforms as direct sums of twists of other spaces of newforms.

LEMMA 3.1. *Let p be a prime and M a positive integer prime to p. Assume ω is a character* mod p^v *and ϕ is a character* mod M. *Then*

$$\omega(n)\,\mathrm{tr}_{p^vM,\bar\omega\phi}\,T_k(n)=\mathrm{tr}_{p^vM,\omega\phi}\,T_k(n) \tag{3.1}$$

for all n with $(n, pM)=1$.

Proof. We will employ the trace formula (Theorem 2.2) and will use the notation given there. Since there is nothing to prove if ω is trivial, we assume ω is non-trivial. Hence there are no degree terms, $\deg T_k(n)$. Consider the "mass terms," i.e., those with $\delta(\sqrt{n})((k-1)/12)$. These occur only if n is a perfect square. Their contribution to the L.H.S. (left hand side) of (3.1) is

$$\omega(n)\,n^{k/2-1}\bar\omega(\sqrt{n})\,\phi(\sqrt{n})\,\frac{k-1}{12}\,p^v(1+1/p)\,M\prod_{l\mid M}(1+1/l).$$

Their contribution to the R.H.S. (right hand side) of (3.1) is

$$n^{k/2-1}\omega(\sqrt{n})\,\phi(\sqrt{n})\,\frac{k-1}{12}\,p^v(1+1/p)M\prod_{l\mid M}(1+1/l)$$

which equals the contribution to the L.H.S. of (3.1). Next consider the parabolic terms, i.e., those with $-\delta(\sqrt{n})(\sqrt{n}/2)$. They also occur only if n is a perfect square. They contribute

$$-\omega(n)\, n^{k/2-1}\bar{\omega}(\sqrt{n})\,\phi(\sqrt{n})\,\frac{\sqrt{n}}{2}\,\mathrm{par}(p)\prod_{l\mid M}\mathrm{par}(l)$$

to the L.H.S. of (3.1). This equals their contribution to the R.H.S. of (3.1) since $e=e(\omega)=e(\bar{\omega})$ and $\mathrm{par}(p)$ depends only on p, v, and e while $\mathrm{par}(l)$ for $l\mid M$ depends only on M and ϕ.

Now consider the remaining terms, those classified into the hyperbolic (h) and elliptic (e) cases in Theorem 2.2. Note that the formulas for $\mathrm{tr}_{p^v M,\bar{\omega}\phi} T_k(n)$ and $\mathrm{tr}_{p^v M,\omega\phi} T_k(n)$ involve summations over the same index set. We will show that equality in (3.1) holds term by term, i.e., for each fixed element (s, f) of the index set. Note that for fixed s and f, $a(s)$ and $b(s, f)$ are independent of which side of (3.1) they occur in. Further since $c'_\chi(s, f, l)$ depends only on χ_l, $c'_{\bar{\omega}\phi}(s, f, l)=c'_{\omega\phi}(s, f, l)$ for all $l\mid M$. Thus for fixed s and f, to show the corresponding contributions to the L.H.S. and R.H.S. of (3.1) are equal, we need only show that

$$\omega(n)\, c'_{\bar{\omega}\phi}(s, f, p)=c'_{\omega\phi}(s, f, p). \tag{3.2}$$

For a character ψ mod p^v let

$$c'_A(s, f, p;\psi)=\sum_x \psi(x) \qquad \text{and} \qquad c'_B(s, f, p;\psi)=\sum_y \psi(y),$$

where x runs over all elements of $A(s, f, p)$ and y runs over all elements of $B'(s, f, p)$ where the notation is as in Theorem 2.2. Hence

$$c'_\psi(s, f, p)=c'_A(s, f, p;\psi)+c'_B(s, f, p;\psi), \tag{3.3}$$

where we take $B'(s, f, p)=\varnothing$ if $s^2-4n/f^2 \not\equiv 0 \pmod{p}$. From Lemmas 2.1 and 2.5 we find that

$$c'_A(s, f, p;\psi)=\begin{cases} p^{c_1}(\psi((s+p^a d)/2)+\psi((s-p^a d)/2)) & (1)\\ p^{c_2}\psi(s/2) & (2)\\ -2^{c_3}\psi(s/2) & (3)\\ 0 & (4)\end{cases}$$

and that

$$c'_B(s, f, p;\psi)=\begin{cases} p^{c_5}(\psi((s+p^a d)/2)+\psi((s-p^a d)/2)) & (5)\\ p^{c_6}\psi(s/2) & (6)\\ -2^{c_7}\psi(s/2) & (7)\\ 0 & (8),\end{cases}$$

where the cases and the constants c_i depend only on p, v, n, s, f, and $e(\psi)$. For example, assume $s^2-4n=p^{2a}d^2$ with p odd and d a unit of Z_p and let $b=\operatorname{ord}_p(f)$. If v is odd, $v=2\rho+1$, and $a-b\leqslant\rho$, then by Case A of Lemma 2.5, $A(s,f,p)=\{(s\pm p^a d)/2+zp^{2\rho+2b-a+1}\mid z\in Z/p^{a-b}Z\}$. Then Lemma 2.1 shows that $c'_A(s,f,p;\psi)$ equals either $p^{a-b}(\psi((s+p^a d)/2)+\psi((s-p^a d)/2))$ (case (1)) or 0 (case (4)) depending on whether $e(\psi)\leqslant 2\rho+2b-a+1$ or $e(\psi)>2\rho+2b-a+1$. Further (see Lemma 2.5) cases (1) and (5) occur only if $s^2-4n=p^{2a}d^2$ for some d of Z_p. In these cases we have $n=((s+p^a d)/2)((s-p^a d)/2)$ and

$$\psi(n)\left(\bar\psi\left(\frac{s+p^a d}{2}\right)+\bar\psi\left(\frac{s-p^a d}{2}\right)\right)=\psi\left(\frac{s+p^a d}{2}\right)+\psi\left(\frac{s-p^a d}{2}\right).$$

Cases (3) and (7) occur only if $p=2$ (for example, when $p=2$, $v=2\rho+1$, $s^2-4n=t^2d^2$ (d a unit of Z_2), $a=\operatorname{ord}_2(t)=\rho+1+\operatorname{ord}_2(f)$, and $e(\psi)=a$ in which case we find that $c'_A(s,f,2;\psi)=2^\rho\psi(s/2+2^{e(\psi)-1})=-2^\rho\psi(s/2)$). Also in cases (2), (3), (6), and (7) we always have $(s/2)^2\equiv n \pmod{p^{e(\psi)}}$ so that $\psi(n)\,\bar\psi(s/2)=\psi(s/2)$ (for $p\neq 2$, $s^2-4n\equiv 0 \pmod{p^v}$ in these cases while for $p=2$ we sometimes need use the fact that $e(\psi)\geqslant 2$). Since $e(\bar\omega)=e(\omega)$, it follows that $\omega(n)\,c'_A(s,f,p;\bar\omega)=c'_A(s,f,p;\omega)$ and $\omega(n)\,c'_B(s,f,p;\bar\omega)=c'_B(s,f,p;\omega)$ in all cases. This establishes (3.2) and completes the proof of the lemma. ∎

THEOREM 3.2. *Let p be a prime and M a positive integer prime to p. Assume ω is a character* mod p^v *and ϕ is a character* mod M. *Then*

$$S_k^0(p^v M,\bar\omega\phi)^\omega\cong S_k^0(p^v M,\omega\phi). \tag{3.4}$$

Proof. Let $N=p^vM$ and put $f=f(\omega\phi)=f(\bar\omega\phi)$. From (1.4) it follows that the R.H.S. of (3.1) equals the trace of $T_k(n)$ acting on $\bigoplus_{f\mid a\mid N}\delta(N/a)\,S_k^0(a,\omega\phi)$ while by Lemma 1.2 and (1.4) the L.H.S. of (3.1) equals the trace of $T_k(n)$ acting on $\bigoplus_{f\mid a\mid N}\delta(N/a)\,S_k^0(a,\bar\omega\phi)^\omega$. As H is a commutative semi-simple ring, we see that (see, e.g., [18, p. 174]) Eq. (3.1) implies that

$$\bigoplus_{f\mid a\mid N}\delta(N/a)\,S_k^0(a,\omega\phi)\cong\bigoplus_{f\mid a\mid N}\delta(N/a)\,S_k^0(a,\bar\omega\phi)^\omega. \tag{3.5}$$

The theorem now follows directly from (3.5) by induction on N/f. ∎

COROLLARY 3.3. *Let the hypotheses be as in Theorem* 3.2. *Assume ω_1 and ω_2 are characters mod p^v. Then*

$$S_k^0(p^vM,\bar\omega_1\omega_2\phi)^{\omega_1}\cong S_k^0(p^vM,\omega_1\bar\omega_2\phi)^{\omega_2}.$$

Proof. Let $\omega = \omega_1 \bar{\omega}_2$ in Theorem 3.2. Then

$$S_k^0(p^\nu M, \bar{\omega}_1 \omega_2 \phi)^{\omega_1 \bar{\omega}_2} \cong S_k^0(p^\nu M, \omega_1 \bar{\omega}_2 \phi)$$

and the corollary follows from Lemma 1.2. ∎

COROLLARY 3.4. *Let the hypotheses be as in Theorem* 3.2. *If* $e(\omega) < \nu$, *then*

$$S_k^0(p^\nu M, \bar{\omega}\phi)^\omega = S_k^0(p^\nu M, \omega\phi).$$

If $e(\omega) = \nu$ and $F(\tau)$ *is a (normalized) newform in* $S_k^0(p^\nu M, \bar{\omega}\phi)$, *then* $F_\omega(\tau) = G(\tau) - a_G(p)\, G(p\tau)$ *for some (normalized) newform* $G(\tau)$ *in* $S_k^0(p^\nu M, \omega\phi)$ *where* $a_G(p)$ *is the* pth *Fourier coefficient of* $G(\tau)$.

Proof. Let $F(\tau)$ be a newform in $S_k^0(p^\nu M, \bar{\omega}\phi)$. If $e(\omega) < \nu$, then by Proposition 3.6 of [2], $F_\omega(\tau) \in S_k(p^\nu M, \omega\phi)$. However, by Theorem 3.2, $F_\omega(\tau)$ is equivalent to some newform in $S_k^0(p^\nu M, \omega\phi)$ and thus it follows from the theory of newforms that $F_\omega(\tau)$ is a newform in $S_k^0(p^\nu M, \omega\phi)$ (see the last paragraph of Section 1 above). Since newforms form a basis of $S_k^0(p^\nu M, \bar{\omega}\phi)$, we see that $S_k^0(p^\nu M, \bar{\omega}\phi)^\omega = S_k^0(p^\nu M, \omega\phi)$. If $e(\omega) = \nu$, again by Proposition 3.6 of [2], $F_\omega(\tau) \in S_k(p^{\nu+1} M, \omega\phi)$ and by Theorem 3.2, $F_\omega(\tau)$ is equivalent to some newform $G(\tau)$ in $S_k^0(p^\nu M, \omega\phi)$. By [11] (see, e.g. , [2, p. 231]), $F_\omega(\tau) = aG(\tau) + bG(p\tau)$. Comparing the first Fourier coefficients shows that $a = 1$ and then comparing the pth Fourier coefficients shows that $b = -a_G(p)$. ∎

Remark 3.5. In the case $e(\omega) = \nu$, $a_G(p)$ is never zero by Theorem 3 of [11] so that $F_\omega(\tau)$ is never a newform in this case.

LEMMA 3.6. *Let p be a prime and let M be a positive integer prime to p. Assume* ω *and* χ *are characters* mod p^ν *with* $e(\chi) \leqslant \nu/2$ *and* $e(\omega) \leqslant \nu/2$ *and let* ϕ *be a character* mod M. *Then*

$$\begin{aligned}\chi(n)(\operatorname{tr}_{p^\nu M, \omega\phi} T_k(n) - \operatorname{tr}_{p^{\nu-1} M, \omega\phi} T_k(n)) \\ = \operatorname{tr}_{p^\nu M, \omega\chi^2\phi} T_k(n) - \operatorname{tr}_{p^{\nu-1} M, \omega\chi^2\phi} T_k(n) \qquad (3.6)\end{aligned}$$

for all n with $(n, pM) = 1$.

Proof. Assume χ is non-trivial as there is nothing to prove otherwise. In particular we assume $\nu \geqslant 2$ if $p \neq 2$ and $\nu \geqslant 4$ if $p = 2$. As in the proof of Lemma 3.1 we show that the trace identity (3.6) holds term by term. Hence, again as in the proof of Lemma 3.1, it suffices to restrict to the case $M = 1$ (hence $\phi = 1$) which we do. The degree terms, if they occur, contribute 0 to both sides of (3.6). If n is a perfect square, the mass terms contribute

$$\chi(n)\, n^{k/2-1}\omega(\sqrt{n})\,\frac{k-1}{12}\,(1+1/p)(p^{\nu}-p^{\nu-1})$$

to both sides of (3.6) while the contribution of the parabolic terms to both sides of (3.6) is

$$-\chi(n)\, n^{k/2-1}\omega(\sqrt{n})\,\frac{n}{2}\,(p^{\rho}-p^{\rho-1}),$$

where $\rho=[\nu/2]$. This follows from Theorem 2.2 since $\mathrm{par}_{\nu}(p)-\mathrm{par}_{\nu-1}(p)=p^{\rho}-p^{\rho-1}$ when $e\leqslant \nu/2$ (where $e=e(\omega)$ or $e(\omega\chi^2)$). Here $\mathrm{par}_{\sigma}(p)$ refers to the term $\mathrm{par}(p)$ which occurs in Theorem 2.2 when $\mathrm{ord}_p(N)=\sigma$. Now we consider the elliptic and hyperbolic terms. We use the notation introduced in the proof of Lemma 3.1. As in the proof of Lemma 3.1 it suffices to show that

$$c'_{\omega\chi^2}(s,f,p)_{\nu}-c'_{\omega\chi^2}(s,f,p)_{\nu-1}=\chi(n)(c'_{\omega}(s,f,p)_{\nu}-c'_{\omega}(s,f,p)_{\nu-1}). \quad (3.7)$$

Here $c'_{\psi}(s,f,p)_{\sigma}$ refers to the term $c'_{\psi}(s,f,p)$ which occurs in Theorem 2.2 when $\mathrm{ord}_p(N)=\sigma$. As in (3.3) we write

$$c'_{\psi}(s,f,p)_{\sigma}=c'_A(s,f,p;\psi)_{\sigma}+c'_B(s,f,p;\psi)_{\sigma}. \quad (3.8)$$

We will show that (3.7) holds by considering all the possible cases which are given in Lemma 2.5. Assume ψ is a character (such as ω or $\omega\chi^2$) modulo a power of p with $e(\psi)\leqslant \nu/2$. Consider Cases A and D when $a-b\leqslant \nu/2-1$. We have

$$c'_A(s,f,p;\psi)_{\sigma}=p^{a-b}\left(\psi\left(\frac{s+p^a d}{2}\right)+\psi\left(\frac{s-p^a d}{2}\right)\right)$$

$$c'_B(s,f,p;\psi)_{\sigma}=p^{a-b-1}\left(\psi\left(\frac{s+p^a d}{2}\right)+\psi\left(\frac{s-p^a d}{2}\right)\right)$$

for $\sigma=\nu$ or $\nu-1$. Hence by (3.8) both sides of (3.7) are zero in this case. Consider Case D when $a-b=[\nu/2]$. If $\nu=2\rho+1$, then

$$c'_A(s,f,2;\psi)_{\nu}=2^{\rho}\left(\psi\left(\frac{s+2^a d}{2}\right)+\psi\left(\frac{s-2^a d}{2}\right)\right)$$

$$c'_A(s,f,2;\psi)_{\nu-1}=2^{\rho}\psi\left(\frac{s+2^a d}{2}\right)$$

$$c'_B(s,f,2;\psi)_{\nu}=c'_B(s,f,2;\psi)_{\nu-1}=2^{\rho-1}\left(\psi\left(\frac{s+2^a d}{2}\right)+\psi\left(\frac{s-2^a d}{2}\right)\right).$$

Hence

$$c'_\psi(s, f, 2)_\nu - c'_\psi(s, f, 2)_{\nu-1} = 2^\rho \psi\left(\frac{s-2^a d}{2}\right). \tag{3.9}$$

We claim that

$$\omega\chi^2\left(\frac{s-2^a d}{2}\right) = \chi(n)\,\omega\left(\frac{s-2^a d}{2}\right). \tag{3.10}$$

First $e(\chi^2) \leqslant e(\chi) - 1 \leqslant \rho - 1 \leqslant a - 1$ so that $\chi^2((s-2^a d)/2) = \chi^2(s/2)$. Second, $s^2/4 \equiv n \pmod{2^{2a-2}}$ and $2a-2 \geqslant 2\rho - 2 \geqslant \rho \geqslant e(\chi)$ since $\rho \geqslant 2$ as $p=2$. Thus $\chi(n) = \chi(s^2/4)$. Now (3.9) and (3.10) establish equality in (3.7). If $\nu = 2\rho$ then we find that

$$c'_\psi(s, f, 2)_\nu - c'_\psi(s, f, 2)_{\nu-1} = 2^{\rho-1}\psi\left(\frac{s+2^a d}{2}\right) + 2^{\rho-1}\psi\left(\frac{s-2^a d}{2}\right).$$

But $e(\psi) \leqslant \rho \leqslant a$ so that $\psi((s+2^a d)/2) = \psi((s-2^a d)/2)$ and as above we see that $\omega\chi^2((s-2^a d)/2) = \chi(n)\,\omega((s-2^a d)/2)$ which gives equality in (3.7). Consider Case E when $a-b=[\nu/2]$. If $\nu = 2\rho+1$, then $c'_A(s, f, 2; \psi)_\nu = c'_B(s, f, 2; \psi)_\nu = c'_B(s, f, 2; \psi)_{\nu-1} = 0$ and $c'_A(s, f, 2; \psi)_{\nu-1} = 2^\rho \psi((s+2^a)/2)$. Thus $c'_\psi(s, f, 2)_\nu - c'_\psi(s, f, 2)_{\nu-1} = -2^\rho\psi((s+2^a)/2)$. Again as above $\omega\chi^2((s+2^a)/2) = \chi(n)\,\omega((s+2^a)/2)$ which establishes equality in (3.7). If $\nu = 2\rho$ then

$$c'_A(s, f, 2; \psi)_\nu = 2^\rho\psi\left(\frac{s+2^a}{2}\right)$$

$$c'_B(s, f, 2; \psi)_\nu = 0$$

$$c'_A(s, f, 2; \psi)_{\nu-1} = c'_B(s, f, 2; \psi)_{\nu-1} = 2^{\rho-1}\psi\left(\frac{s+2^a}{2}\right).$$

Hence both sides of (3.7) are zero in this case. Consider Case F when $\nu = 2\rho$ and $a-b=\rho-1$. Then $c'_A(s, f, 2; \psi)_\nu = c'_B(s, f, 2; \psi)_\nu = c'_B(s, f, 2; \psi)_{\nu-1} = 0$ and $c'_A(s, f, 2; \psi)_{\nu-1} = 2^{\rho-1}\psi((s+2^{a+1})/2)$. Then $c'_\psi(s, f, 2)_\nu - c'_\psi(s, f, 2)_{\nu-1} = -2^{\rho-1}\psi((s+2^{a+1})/2)$ and equality in (3.7) follows as above. Finally in all other cases we have

$$c'_D(s, f, p; \psi)_\sigma = \psi\left(\frac{s}{2}\right) c'_D(s, f, p; 1)_\sigma$$

and

$$\omega\chi^2(s/2) = \chi(n)\,\omega(s/2)$$

for $D=A$ or B and $\sigma=\nu$ or $\nu-1$ which by (3.8) establishes equality in (3.7). This completes the proof of Lemma 3.6. ∎

THEOREM 3.7. *Let p be a prime and let M be a positive integer prime to p. Assume ω and χ are characters* mod p^ν *with $e(\omega)\leqslant \nu/2$ and $e(\chi)\leqslant \nu/2$ and let ϕ be a character* mod M. *Then*

$$\bigoplus_{i=e(\omega)}^{\nu} S_k^0(p^iM, \omega\phi)^\chi \cong \bigoplus_{i=e(\omega\chi^2)}^{\nu} S_k^0(p^iM, \omega\chi^2\phi).$$

Proof. Let $f=f(\phi)$. By (1.5) the R.H.S. of (3.6) equals the trace of $T_k(n)$ on

$$\bigoplus_{i=e(\omega\chi^2)}^{\nu} \bigoplus_{f|a|M} \delta(M/a)\, S_k^0(p^ia, \omega\phi) \tag{3.11}$$

while by (1.5) and Lemma 1.2 the L.H.S. of (3.6) equals the trace of $T_k(n)$ on

$$\bigoplus_{i=e(\omega)}^{\nu} \bigoplus_{f|a|M} \delta(M/a)\, S_k^0(p^ia, \omega\phi)^\chi. \tag{3.12}$$

Hence (3.11) and (3.12) are isomorphic as H-modules and the theorem follows by induction on M/f. ∎

LEMMA 3.8. *Let p be a prime and M be a positive integer prime to p. Assume ω is a character* mod p^ν *with $\nu/2<e(\omega)<\nu$ and let ϕ be a character* mod M. *Then*

$$\begin{aligned}\sum_{e(\chi)=\nu-e(\omega)} &\operatorname{tr}_{p^{e(\omega)}M,\omega\chi^2\phi} T_k(1)\\ &= \operatorname{tr}_{p^\nu M,\omega\phi} T_k(1) - 2\operatorname{tr}_{p^{\nu-1}M,\omega\phi} T_k(1)\\ &\quad + \begin{cases}\operatorname{tr}_{p^{\nu-2}M,\omega\phi} T_k(1) & \text{if } e(\omega)<\nu-1\\ 0 & \text{if } e(\omega)=\nu-1,\end{cases}\end{aligned} \tag{3.13}$$

where the sum $\sum_{e(\chi)=\nu-e(\omega)}$ is over all primitive characters χ modulo $p^{\nu-e(\omega)}$.

Proof. Note that our assumptions imply that $\nu\geqslant 3$ and that $\nu\geqslant 5$ if $e(\omega)<\nu-1$. As in the proofs of Lemmas 3.1 and 3.6 we show that the trace identity (3.13) holds term by term. Hence, as in the proof of Lemma 3.6, it suffices to restrict to the case $M=1$ which we do. Since ω is non-trivial, there are no degree terms. The contribution of the mass terms to the R.H.S. of (3.13) is

$$\frac{k-1}{12}(1+1/p)(p^\nu-2p^{\nu-1}+\{p^{\nu-2}\}),$$

where we have used the convention that for an expression q

$$\{q\} = \begin{cases} q & \text{if } e(\omega) < v-1 \\ 0 & \text{if } e(\omega) = v-1. \end{cases}$$

We will continue to use this convention for the remainder of the proof. The contribution of the mass terms to the L.H.S. of (3.13) is $((k-1)/12)(1+1/p)\, p^{e(\omega)}\alpha$ where α equals the number of primitive characters modulo $p^{v-e(\omega)}$. But the number of primitive characters modulo p^s is $(p^s - p^{s-1}) - (p^{s-1} - p^{s-2}) = p^s - 2p^{s-1} + p^{s-2}$ if $s \geqslant 2$ and is equal to $p-2$ if $s=1$. Hence $p^{e(\omega)}\alpha = p^v - 2p^{v-1} + \{p^{v-2}\}$ and the mass terms contribute the same amount to the L.H.S. and R.H.S. of (3.13). Since $e(\omega\chi^2) = e(\omega)$, the parabolic terms contribute $(p^{v-e(\omega)} - 2p^{v-e(\omega)-1} + \{p^{v-e(\omega)-2}\})(-(1/2)\cdot 2p^{e(\omega)-e(\omega)}) = -p^{v-e(\omega)} + 2p^{v-e(\omega)-1} - \{p^{v-e(\omega)-2}\}$ to the L.H.S. of (3.13). On the other hand since $e(\omega) > v/2$, they contribute $(-1/2)\, 2p^{v-e(\omega)} + (1/2)\, 4p^{v-1-e(\omega)} - (1/2)\{2p^{v-2-e(\omega)}\}$ to the R.H.S. of (3.13).

Now we consider the remaining terms. Since $n=1$, s takes only the values 0 and ± 1 and $f=1$ in all cases. As in Lemmas 3.1 and 3.6 it suffices to show that

$$\sum_{e(\omega)=v-e(\omega)} c'_{\omega\chi^2}(s, 1, p)_{e(\omega)}$$
$$= c'_\omega(s, 1, p)_v - 2c'_\omega(s, 1, p)_{v-1} + \{c'_\omega(s, 1, p)_{v-2}\}, \tag{3.14}$$

where the notation is as in (3.7). First consider the case $s=0$, i.e., $s^2 - 4n = -4$. If p is odd and $(-1/p) = -1$ then $c'_A(0, 1, p; \psi)_\sigma = c'_B(0, 1, p; \psi)_\sigma = 0$ for all σ and ψ by Case B of Lemma 2.5 as $a=b=0$. Hence the L.H.S. and R.H.S. of (3.14) are both zero in this case. If $p=2$ the same is true by Case F of Lemma 2.5 (since $e(\omega) \geqslant 2$ and $v-2 \geqslant 3$ if $e(\omega) < v-1$). If p is odd and $(-1/p)=1$, letting $d^2 = -4$ with $d \in Z_p$, we see by Case A of Lemma 2.5 that $c'_A(0, 1, p; \psi)_\sigma = \psi(d/2) + \psi(-d/2)$ and $c'_B(0, 1, p; \psi)_\sigma = 0$ for all σ and ψ as $A_0 = \{d/2, -d/2\}$ and $B'_0 = \phi$ in all cases. Thus $c'_\psi(0, 1, p; \psi)_\sigma = \psi(d/2) + \psi(-d/2)$ in all cases. Hence the R.H.S. of (3.14) equals 0 if $e(\omega) < v-1$ and equals $-(\omega(d/2) + \omega(-d/2))$ if $e(\omega) = v-1$. On the other hand the L.H.S. of (3.14) equals

$$(\omega(d/2) + \omega(-d/2))\left(\sum_{\chi_1} \chi_1(-1) - \sum_{\chi_2} \chi_2(-1)\right),$$

where $\sum_{\chi_1}$ is over all characters χ_1 on the group $G_1 = (Z/p^{v-e(\omega)}Z)^\times$ and $\sum_{\chi_2}$ is over all characters χ_2 on the group $G_2 = (Z/p^{v-e(\omega)-1}Z)^\times$. If $e(\omega) < v-1$, both G_1 and G_2 are non-trivial so $\sum_{\chi_1} \chi_1(-1) =$

$\sum_{\chi_2} \chi_2(-1)=0$ while if $e(\omega)=v-1$, then $\sum_{\chi_1} \chi_1(-1)=0$ and $\sum_{\chi_2} \chi_2(-1)=\mathrm{id}(-1)=1$. Now assume $s=\pm 1$, i.e., $s^2-4n=-3$. If $p=2$ or 3 or $p>3$ and $(-3/p)=-1$ then by Cases E, C, or B of Lemma 2.5, $c'_A(\pm 1, 1, p; \psi)_\sigma = c'_B(\pm 1, 1, p; \psi)_\sigma = 0$ for all σ and ψ so that both sides of (3.14) are zero in these cases. So assume $p>3$ and $d^2=-3$ for some $d \in Z_p$. Then by Case A of Lemma 2.5, $c'_A(s, 1, p; \psi)_\sigma = \psi((s+d)/2)+\psi((s-d)/2)$ and $c'_B(s, 1, p; \psi)_\sigma = 0$ for all ψ and σ where $s=\pm 1$. Thus the R.H.S. of (3.14) equals zero if $e(\omega)<v-1$ and equals $-\psi((s+d)/2)-\psi((s-d)/2)$ if $e(\omega)=v-1$. But then just as in the case when $s=0$, this equals the L.H.S. of (3.14) since $((s+d)/2) \not\equiv 1 \pmod{p}$ and $((s-d)/2)^2 \not\equiv 1 \pmod{p}$. This completes the proof of Lemma 3.8. ∎

Theorem 3.9 below relates to several theorems in Atkin and Li [2]. Let $F(\tau)$ be a newform in $S_k^0(p^{e(\varepsilon)}M, \varepsilon\phi)$ with ε a character mod $p^{e(\varepsilon)}$ and ϕ a character mod M. If ψ is a character mod a power of p with $e(\psi)<e(\varepsilon)$, then letting $\omega=\varepsilon\psi^2$ and $v=e(\varepsilon)+e(\psi)$ in Theorem 3.9, we see that $F_\psi(\tau)$ is a newform in $S_k^0(p^v M, \varepsilon\psi^2\phi)$. This is Theorem 4.2 of [2]. If $F(\tau)$ is a newform in $S_k^0(p^v M, \omega\phi)$ with $e(\omega)>v/2$, then by Theorem 3.9 there is a character χ with $e(\chi)=v-e(\omega)$ and a newform $G(\tau)$ in $S_k^0(p^{e(\omega)}M, \omega\bar{\chi}^2\phi)$ such that $F(\tau)=G_\chi(\tau)$. This is Theorem 4.3 of [2]. Finally, Theorem 3.1 of [2] provides information on the exact level of twists of newforms. Assume ω and χ are characters mod powers of p with $e(\omega)=\alpha \geq 0$ and $e(\chi)=\beta \geq 1$. According to Theorem 3.1 of [2], if $F(\tau)$ is a newform in $S_k^0(p^v M, \omega\phi)$, the exact level of $F_\chi(\tau)$ is $p^{v'}M$ where $v'=\max\{v, \alpha+\beta, 2\beta\}$ provided that (a) $\alpha+\beta<v$ and $2\beta<v$ if $v'=v$ or (b) $e(\omega\chi)=\max\{\alpha, \beta\}$ if $v'>v$. In case (b) assume that $\alpha>\beta$. Note that this implies $\alpha>v/2$. By Theorem 3.9, $F(\tau)=G_\psi(\tau)$ form some newform $G(\tau)$ in $S_k^0(p^\alpha M, \omega\bar{\psi}^2\phi)$ and some primitive character ψ mod $p^{v-\alpha}$. Since $v'=\alpha+\beta>v$, $e(\psi\chi)=\beta$ and as above $F_\chi(\tau)=G_{\psi\chi}(\tau)$ is a newform in $S_k^0(p^{v'}M, \omega\chi^2\phi)$ so in particular $F_\chi(\tau)$ has exact level $p^{v'}M$ in agreement with Theorem 3.1 of [2].

Part (a) of Theorem 3.1 of [2] follows from Theorem 3.12 below. For the remaining case in part (b) of Theorem 3.1 of [2], see Section 4 (especially Remark 4.11) below.

THEOREM 3.9. *Let p be a prime and let M be a positive integer prime to p. Let ω be a character* mod p^v *with $e(\omega)>v/2$ and let ϕ be a character* mod M. *Then*

$$S_k^0(p^v M, \omega\phi) = \bigoplus_{e(\chi)=v-e(\omega)} S_k^0(p^{e(\omega)}M, \omega\bar{\chi}^2\phi)^\chi,$$

where the sum $\bigoplus_{e(\chi)=v-e(\omega)}$ is over all primitive characters χ modulo $p^{v-e(\omega)}$.

Proof. If $e(\omega)=\nu$ there is nothing to prove so assume $e(\omega)<\nu$. By Theorem 4.3 of [2] and Theorem 3 of [11] we have

$$S_k^0(p^\nu M, \omega\phi) \subseteq \sum_{e(\chi)=\nu-e(\omega)} S_k^0(p^{e(\omega)}M, \omega\chi^2\phi)^{\bar{\chi}},$$

where the $\sum$ on the right is not necessarily a direct sum. To complete the proof we show that

$$\dim S_k^0(p^\nu M, \omega\phi) = \sum_{e(\chi)=\nu-e(\omega)} \dim S_k^0(p^{e(\omega)}M, \omega\chi^2\phi)^{\bar{\chi}}. \tag{3.15}$$

Let $f=f(\phi)$. By (1.5)

$$S_k(p^\nu M, \omega\phi) \oplus \{S_k(p^{\nu-2}M, \omega\phi)\}$$
$$\cong 2S_k(p^{\nu-1}M, \omega\phi) \oplus \bigoplus_{f|a|M} \delta(M/a)\, S_k^0(p^\nu a, \omega\phi),$$

where

$$\{S_k(p^{\nu-2}M, \omega\phi)\} = \begin{cases} S_k(p^{\nu-2}M, \omega\phi) & \text{if } \; e(\omega) \leqslant \nu-2 \\ 0 & \text{if } \; e(\omega)=\nu-1. \end{cases}$$

Hence the R.H.S. of (3.13) equals

$$\sum_{f|a|M} \delta(M/a) \dim S_k^0(p^\nu a, \omega\phi). \tag{3.16}$$

On the other hand $e(\omega)>\nu/2$ and $e(\chi)=\nu-e(\omega)$ imply $e(\omega\chi^2)=e(\omega)$ so that by (1.5)

$$S_k(p^{e(\omega)}M, \omega\chi^2\phi) \cong \bigoplus_{f|a|M} \delta(M/a)\, S_k^0(p^{e(\omega)}a, \omega\chi^2\phi).$$

Now $\dim S_k^0(p^{e(\omega)}a, \omega\chi^2\phi) = \dim S_k^0(p^{e(\omega)}a, \omega\chi^2\phi)^{\bar{\chi}}$ (for example, by Lemma 1.2) so that the L.H.S. of (3.13) equals

$$\sum_{e(\omega)=\nu-e(\omega)} \sum_{f|a|M} \delta(M/a) \dim S_k^0(p^{e(\omega)}a, \omega\chi^2\phi)^{\bar{\chi}}. \tag{3.17}$$

Lemma 3.8 shows the equality of (3.16) and (3.17). Using induction on M/f then establishes (3.15) and completes the proof of the theorem. ∎

Remark 3.10. One can give an alternate proof of Theorem 3.9 which is independent of [2] and [11] by proving the trace identity (3.13) of Lemma 3.8 holds for all $T_k(n)$ with $(n, pM)=1$, not just for the identity operator $T_k(1)$. In fact this is how we originally proved Theorem 3.9.

However, proving that the trace identity (3.13) holds for general $T_k(n)$ is much more complicated than for the case $T_k(1)$ of Lemma 3.8. Also one should note that as in Corollary 4.5 of [2], Theorem 3.9 implies that $S_k^0(2^\nu M, \omega\phi)=0$ if $\nu \geqslant 3$ and $e(\omega)=\nu-1$.

COROLLARY 3.11. *Let N be a positive integer, ψ a character* mod N, *and $k \geqslant 2$ an integer satisfying $\psi(-1)=(-1)^k$. Assume $e(\psi_l) > \frac{1}{2}\operatorname{ord}_l(N)$ for all primes l dividing N. Then*

$$S_k^0(N, \psi) = \bigoplus_{f(\chi)=N/f(\psi)} S_k^0(f(\psi), \psi\chi^2)^{\bar{\chi}}.$$

Here the sum is over all primitive characters χ mod $N/f(\psi)$. *Recall that $f(\chi)$ denotes the conductor of χ and note that $f(\psi\chi^2)=f(\psi)$ for all χ.*

Proof. This follows immediately from Theorem 3.9. ∎

THEOREM 3.12. *Let p be a prime and let M be a positive integer prime to p. Assume ω and χ are characters* mod p^ν *with $e(\chi) < \nu/2$ and $e(\chi)+e(\omega) < \nu$. Then*

$$S_k^0(p^\nu M, \omega\phi)^\chi = S_k^0(p^\nu M, \omega\chi^2\phi).$$

Proof. First assume that $e(\omega) < \nu/2$. It then follows immediately from Theorem 3.7 that $S_k^0(p^\nu M, \omega\phi)^\chi \cong S_k^0(p^\nu M, \omega\chi^2\phi)$. By Proposition 1.2, $S_k^0(p^\nu M, \omega\phi)^\chi \subseteq S_k(p^\nu M, \omega\chi^2\phi)$. The theory of newforms then implies that $S_k^0(p^\nu M, \omega\phi)^\chi = S_k^0(p^\nu M, \omega\chi^2\phi)$. Next assume $e(\omega)=\nu/2$. By Theorem 3.7 we have

$$\bigoplus_{i=\nu/2}^{\nu} S_k^0(p^i M, \omega\phi)^\chi \cong \bigoplus_{i=\nu/2}^{\nu} S_k^0(p^i M, \omega\chi^2\phi) \tag{3.18}$$

since $e(\omega)=e(\omega\chi^2)=\nu/2$. Now by Theorem 3.9,

$$\begin{aligned}\bigoplus_{i=\nu/2}^{\nu-1} S_k^0(p^i M, \omega\phi)^\chi &= \bigoplus_{i=\nu/2}^{\nu-1} \bigoplus_{e(\psi)=i-\nu/2} S_k^0(p^{\nu/2} M, \omega\chi^2\phi)^{\bar{\psi}\chi} \\ &= \bigoplus_{\psi \in G} S_k^0(p^{\nu/2} M, \omega\psi^2\phi)^{\bar{\psi}\chi},\end{aligned}$$

where G is the character group of $(Z/p^{\nu/2-1}Z)^\times$. On the other hand, again by Theorem 3.9,

$$\begin{aligned}\bigoplus_{i=\nu/2}^{\nu-1} S_k^0(p^i M, \omega\chi^2\phi) &= \bigoplus_{\psi \in G} S_k^0(p^{\nu/2} M, \omega\chi^2\psi^2\phi)^{\bar{\psi}} \\ &= \bigoplus_{\psi \in G} S_k^0(p^{\nu/2} M, \omega\psi^2\phi)^{\psi\bar{\chi}}\end{aligned}$$

since $e(\bar{\chi}) = e(\chi) < v/2$ and v even implies that $\bar{\chi} \in G$. Thus from (3.18) we conclude that $S_k^0(p^v M, \omega\phi)^\chi \cong S_k^0(p^v M, \omega\chi^2\phi)$ and equality follows as above. Finally we assume that $e(\omega) > v/2$. Then by Theorem 3.9,

$$S_k^0(p^v M, \omega\phi)^\chi = \bigoplus_{e(\psi) = v - e(\omega)} S_k^0(p^{e(\omega)} M, \omega\psi^2\phi)^{\bar{\psi}\chi}$$

and

$$S_k^0(p^v M, \omega\chi^2\phi) = \bigoplus_{e(\psi) = v - e(\omega)} S_k^0(p^{e(\omega)} M, \omega\chi^2\psi^2\phi)^{\bar{\psi}}.$$

Now since $e(\bar{\chi}) = e(\chi) < v - e(\omega)$, as ψ runs over all characters mod p^v with $e(\psi) = v - e(\omega)$, so does $\psi\bar{\chi}$ and the theorem follows. ∎

THEOREM 3.13. *Let M be a positive integer prime to 2. Let v be even, $v = 2\rho$ and assume that ω and χ are characters* mod 2^v *with $e(\omega) = \rho$ and $e(\chi) \leqslant \rho$. Let ϕ be a character* mod M. *Then*

$$S_k^0(2^v M, \omega\phi)^\chi = S_k^0(2^v M, \omega\chi^2\phi).$$

Proof. Note that if $e(\chi) < \rho$, Theorem 3.13 is a special case of Theorem 3.12 so we assume $e(\chi) = \rho$. Let $\chi' = \chi\omega$ so that $e(\chi') < \rho$. Then $S_k^0(2^v M, \omega\phi)^\chi = S_k^0(2^v M, \omega\phi)^{\bar{\omega}\chi'} \cong S_k^0(2^v M, \bar{\omega}\phi)^{\chi'}$ (by Theorem 3.2) $\cong$ $S_k^0(2^v M, \bar{\omega}\chi'^2\phi)$ (by Theorem 3.12) $= S_k^0(2^v M, \omega\chi^2\phi)$. Since $S_k^0(2^v M, \omega\phi)^\chi$ $\subseteq S_k(2^v M, \omega\chi^2\phi)$ by Proposition 1.1, the theorem follows from the theory of newforms. ∎

THEOREM 3.14 (Shemanske [15] if $\omega\phi = 1$). *Let M be a positive integer prime to 2. Let $v \geqslant 4$ be even, $v = 2p$ and assume ω and χ are characters* mod 2^v with $e(\omega) \leqslant \rho - 1$ *and $e(\chi) = \rho$. Let ϕ be a character* mod M. *Then*

$$S_k^0(2^v M, \omega\phi) = \bigoplus_{i = e(\omega\chi^2)}^{v-1} S_k^0(2^i M, \omega\chi^2\phi)^{\bar{\chi}}.$$

Proof. Theorem 4.4iii of [2] implies

$$S_k^0(2^v M, \omega\phi) \subseteq \sum_{i = e(\omega\chi^2)}^{v-1} S_k^0(2^i M, \omega\chi^2\phi)^{\bar{\chi}}. \tag{3.19}$$

On the other hand if $F(\tau)$ is an element of $S_k^0(2^i M, \omega\chi^2\phi)$ for some i, $e(\omega\chi^2) \leqslant i \leqslant v - 1$, then by Theorems 3.1 and 3.2 and Corollary 3.1 of [2], we see that $F_{\bar{\chi}}(\tau)$ is a newform of level $2^v M$. Hence we have equality in (3.19). Now the multiplicity one theorem (see, e.g., Theorem 5 of [11]) shows that the sum on the right hand side of (3.19) is a direct sum which establishes the theorem. ∎

Remark 3.15. Theorem 3.14 is a rather far reaching generalization of results stated (implicitly) by Atkin and Lehner in Theorem 7 of [1] for the cases $S_k^0(2^4M, 1)$ and $S_k^0(2^6M, 1)$. Note that for these cases $\omega\phi$ and $\omega\chi^2\phi$ are trivial. In particular on p. 158 of [1] it is stated that, "It is interesting to note that all newforms on $\Gamma_0(16M)$ and $\Gamma_0(64M)$ [with trivial character] can be inferred from a knowledge of the oldforms." Theorem 3.14 shows that in essence this statement remains true for $S_k^0(2^{2\rho}M, \omega\phi)$ if $\rho \geqslant 2$ and $e(\omega) \leqslant \rho - 1$. In particular there are no "2-primitive" (see [2, p. 236]) forms in these cases. Theorem 3.14 can be proved independently of [2] and [11] by establishing the trace identity

$$\begin{aligned}\operatorname{tr}_{2^\nu M,\omega\phi} T_k(n) - 2\operatorname{tr}_{2^{\nu-1}M,\omega\phi} T_k(n) + \operatorname{tr}_{2^{\nu-2}M,\omega\phi} T_k(n)\\ = \bar{\chi}(n)(\operatorname{tr}_{2^{\nu-1}M,\omega\chi^2\phi} T_k(n) - \operatorname{tr}_{2^{\nu-2}M,\omega\chi^2\phi} T_k(n)).\end{aligned}$$

In fact Theorem 3.14 was first proved (in the case $\omega\phi = 1$) by Shemanske in Theorem 7.7 of [15] by proving such a trace identity.

Our next theorem explains the vanishing of pth Fourier coefficients of newforms where p is a prime dividing the level.

THEOREM 3.16. *Let $F(\tau)$ be a newform in $S_k^0(N, \psi)$ with $F(\tau) = \sum a(n)\, x^n$. Let p be a prime dividing N. Then the following are equivalent*:

(i) $a(p) = 0$

(ii) $p^2 \mid N$ *and* $e(\psi_p) < \operatorname{ord}_p(N)$

(iii) *$F = G_\chi$ for some newform G in $S_k^0(N', \psi\bar{\chi}^2)$ for some N' and some character χ modulo a power of p where N' differs from N by a power of p.*

Further, assuming (i), *N' in* (iii) *can be chosen so that $N' \leqslant N$ if $p \neq 2$ or if $p = 2$ and* $\operatorname{ord}_2(N) \geqslant 4$. *If $p = 2$ and* $\operatorname{ord}_2(N) = 2$ (*resp.* 3), then *N' can be chosen to equal* $4N$ (*resp.* $2N$).

Proof. Let $N = p^\nu M$ with $p \nmid M$. (i) $\to$ (ii) follows from Theorem 3 of [1] and Theorem 3 of [11]. Now assume (ii) holds. If $e = e(\psi_p) > \nu/2$, then by Theorem 3.9, $F = G_\chi$ for some newform G in $S_k^0(p^e M, \psi\bar{\chi}^2)$ where χ is a primitive character modulo $p^{\nu-e}$. Thus we assume $e = e(\psi_p) \leqslant \nu/2$. Let $\chi = (\ /p)$, the Legendre symbol if $p \neq 2$ and let $\chi(n) = (-1)^{(n-1)/2}$ for odd n if $p = 2$. Thus $e(\chi) = 1$ (2 if $p = 2$). By Theorem 3.7 if p is odd or if $p = 2$ and $\nu \geqslant 4$ we have

$$\bigoplus_{i=e}^{\nu} S_k^0(p^i M, \psi) \cong \bigoplus_{i=e}^{\nu} S_k^0(p^i M, \psi)^\chi.$$

Thus $F \sim G_\chi$ where G is some newform in $S_k^0(p^i M, \psi)$ for some i, $e \leqslant i \leqslant \nu$. But by Proposition 1.1, $G_\chi \in S_k(N, \psi)$ so that from the theory of nexforms we see that $G_\chi = F$. Finally assume $p=2$ and $\nu \leqslant 3$. Then since $e(\psi_2) \leqslant \nu/2$ we have $\psi_2 = 1$. By Theorem 3.15 we have

$$\bigoplus_{i=0}^{3} S_k(2^i M, \psi) \cong S_k^0(16M, \psi)^\chi.$$

If $F \in S_k^0(8M, \psi)$, then $F \sim G_\chi$ for some newform G in $S_k^0(16M, \psi)$. By Proposition 1.1, $G_\chi \in S_k(16M, \psi)$ so that $G_\chi = aF(\tau) + bF(2\tau)$ for some $a, b \in \mathbb{C}$. As shown above the 2nd and hence all even Fourier coefficients of G are zero so that $G = (G_\chi)_\chi = aF(\tau)_\chi + bF(2\tau)_\chi = aF(\tau)_\chi$. But all even Fourier coefficients of F are zero so that $aF = G_\chi$. Comparing the first Fourier coefficients gives $a=1$, If $F \in S_k^0(4M, \psi)$, the proof is the same except that $G_\chi = aF(\tau) + bF(2\tau) + cF(4\tau)$. The fact that (iii) $\rightarrow$ (i) is clear. Also the conditions on N' are clear from the proof of (ii) $\rightarrow$ (iii). ∎

Remark 3.17. If $e(\psi_p) > \nu/2$, then F is the twist of some $G = \sum a(n)\, x^n$ with $a(p) \neq 0$. This is not true in general as the example $S_k^0(p^3, 1)$, p odd demonstrates. One can give an easy alternate proof of the fact that (ii) $\rightarrow$ (iii) by using Theorem 3.2 of [2]. However, in doing so one does not obtain the information about N'.

4. Decompositions Involving Theta Series

If $F(\tau)$ is a newform in $S_k^0(M, \phi)$ and χ is a primitive character mod p^r with p a prime not dividing M, it is well known (see [2, p. 228]) that $F_\chi(\tau)$ is a newform in $S_k^0(p^{2r}M, \chi^2\phi)$. More generally, if $F(\tau)$ is a newform in $S_k^0(p^\nu M, \omega\phi)$ and χ is a character mod a power of p with $r = e(\chi)$ "large" (e.g., $r > \nu/2$ and $r > e(\omega)$) we also expect $F_\chi(\tau)$ to be a newform in $S_k^0(p^{2r}M, \chi^2\omega\phi)$. Thus we are led to investigate spaces of newforms of the general type $S_k^0(p^{2r}M, \omega\phi)$ where $e(\omega) = r$ if p is odd and $e(\omega) = r-1$ if $p=2$. Decompositions of these and similar spaces involve both twists of newforms and theta series. This is the topic of this section. The relevant material on theta series is contained in [8] and especially [9]. With the exception of the proofs of Lemmas 4.1 and 4.2 (which are given in [9]), this section can be read independently of [9].

We begin by stating two results from [9]. Let p be a prime and let M be a positive integer prime to p. Let s be a positive integer, ω a character mod p^s, ϕ a character mod M, and set $\psi = \omega\phi$. Let k be an integer $\geqslant 2$ satisfying $\psi(-1) = (-1)^k$. Assume $e(\omega) \leqslant s/2$ and let $L(p)$ be a quadratic field extension of Q_p. Then for any positive integer n we can define the Brandt matrix $B(n) = B_{k-2}(n; M; L(p), s; \tilde{\psi})$. The Brandt matrix $B(n)$ gives an explicit matrix representation of the action of the Hecke operator $T(n)$ acting on a space of theta series. These theta series are modular forms of weight k and character ψ on $\Gamma_0(p^s M)$. The precise definition of the Brandt matrices can be found in [8, Sect. 7.3]. The definition is complicated and will not be needed for this paper.

LEMMA 4.1. *Let p be an odd prime and M a positive integer prime to p. Let r be a positive integer, ω a character* mod p^r, *ϕ a character* mod M, *and set $\psi = \omega\phi$. Let k be an integer $\geqslant 2$ satisfying $\psi(-1) = (-1)^k$. Assume $e(\omega) \leqslant r-1$ and let $L(p)$ denote either of the ramified quadratic extensions of Q_p. Then*

$$\begin{aligned} &2(\mathrm{tr}_{p^{2r}M,\psi} T_k(n) - 2\,\mathrm{tr}_{p^{2r-1}M,\psi} T_k(n) + \mathrm{tr}_{p^{2r-2}M,\psi} T_k(n)) \\ &\quad = \mathrm{tr}\, B_{k-2}(n; M; L(p), 2r; \tilde{\psi}) - \mathrm{tr}\, B_{k-2}(n; M; L(p), 2r-1; \tilde{\psi}) \\ &\qquad + \sum_{\chi} \bar{\chi}(n)\, \mathrm{tr}_{p^r M, \chi^2\psi} T_k(n) \\ &\qquad - \begin{cases} (n/p) \deg T_2(n) & \text{if } r=1, \text{ and } k=2, \text{ and } \psi \text{ is trivial} \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{4.1}$$

for all n with $(n, pM) = 1$. Here the sum $\sum_\chi$ is over all the ($p^r - 2p^{r-1} + p^{r-2}$ if $r \geqslant 2$ and $p-2$ if $r=1$) primitive characters χ of $(Z/p^r Z)^\times$.

Proof. If $r=1$ and $\psi = 1$, Lemma 4.1 reduces to Theorem 7.1 of [14]. In the general case Lemma 4.1 is the same as Lemma 6.5 of [9]. ∎

LEMMA 4.2. *Let the notation and hypotheses be as in Lemma* 4.1 *except that we now assume $r=1$ and ω is a character of $(Z/pZ)^\times$. Then*

$$\begin{aligned} &2\,\mathrm{tr}_{p^2M,\psi} T_k(n) - 3\,\mathrm{tr}_{pM,\psi} T_k(n) = \mathrm{tr}\, B_{k-2}(n; M; L(p), 2; \tilde{\psi}) \\ &\quad + \sum_{\chi} \bar{\chi}(n)\, \mathrm{tr}_{pM,\chi^2\psi} T_k(n) \\ &\quad - \begin{cases} \left(1 + \left(\dfrac{n}{p}\right)\right) \deg T_2(n)\, \xi(n) & \text{if } k=2, \phi \text{ is trivial, and } \omega = \xi^2 \\ & \text{where } \xi \text{ is a character of } (Z/pZ)^\times \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{4.2}$$

for all n with $(n, pM) = 1$ where the sum is over all the $p-2$ primitive characters of $(Z/pZ)^\times$.

Proof. This is the same as Lemma 6.7 of [9]. ∎

THEOREM 4.3. *Let p be an odd prime and M a positive integer prime to p. Let r be a positive integer, ω a character* mod p^r, *ϕ a character* mod M, *and set $\psi = \omega\phi$. Let k be an integer $\geqslant 2$ satisfying $\psi(-1) = (-1)^k$. Assume*

$e(\omega) \leqslant r-1$. Then there exist spaces Θ' and Θ'' of theta series such that the following isomorphisms hold. If $r=1$ we have

$$2S_k^0(p^2M, \phi) \cong \Theta' \oplus S_k^0(p, M, \phi)^\gamma \oplus 2S_k^0(M, \phi)^\gamma \oplus \bigoplus_{\chi/\sim} 2S_k^0(pM, \chi^2\phi)^{\bar{\chi}} \tag{4.3'}$$

as H-modules where $\gamma = (\ /p)$ and the sum $\bigoplus_{\chi/\sim}$ is over all the $\frac{1}{2}(p-3)$ classes of primitive characters mod p *excepting γ modulo the equivalence $\chi \sim \bar{\chi}$. If $r \geqslant 2$, we have*

$$2S_k^0(p^{2r}M, \psi) \cong \Theta'' \oplus \bigoplus_{\chi/\sim} 2S_k^0(p^rM, \chi^2\psi)^{\bar{\chi}} \tag{4.3''}$$

as H-modules where the sum $\bigoplus_{\chi/\sim}$ is over all the $\frac{1}{2}(p^r - 2p^{r-1} + p^{r-2})$ classes of primitive characters mod p^r *modulo the equivalence $\chi \sim \overline{\chi\omega}$.*

Remark 4.4. Note that if $\chi_1 = \overline{\chi\omega}$, then $S_k^0(p^rM, \chi^2\psi)^{\bar{\chi}} \cong S_k^0(p^rM, \chi_1^2\psi)^{\bar{\chi}_1}$ by Corollary 3.3. Also, if ω is either the trivial character or an odd character mod p, it is shown in Theorem 7.16 of [9] that Θ' and Θ'' are the subspace of "newforms" Θ^0 in a space of theta series. Θ^0 is defined in an manner exactly analogous to the definition of the space of newforms $S_k^0(N, \psi)$ as a subspace of $S_k(N, \psi)$—see [9, Sect. 7.1]. In our more general case it is almost certain that Θ' and Θ'' can also be identified with a space of "newforms."

Proof of Theorem 4.3. By Proposition 5.1 of [9], the Brandt matrix $B_{k-2}(n; M; L(p), v; \tilde{\psi})$ gives the action of the Hecke operator $T_k(n)$ on a space $\mathscr{M}_k(\mathcal{O}_v, \tilde{\psi})$ of theta series (see [9, Sect. 5]). Since $\mathscr{M}_k(\mathcal{O}_{2r-1}, \tilde{\psi}) \subseteq \mathscr{M}_k(\mathcal{O}_{2r}, \tilde{\psi})$,

$$\operatorname{tr} B_{k-2}(n; M; L(p), 2r; \tilde{\psi}) - \operatorname{tr} B_{k-2}(n; M; L(p), 2r-1; \tilde{\psi})$$

is the trace of $T_k(n)$ acting on the orthogonal complement of $\mathscr{M}_k(\mathcal{O}_{2r-1}, \tilde{\psi})$ in $\mathscr{M}_k(\mathcal{O}_{2r}, \tilde{\psi})$ which we denote by $\mathscr{M}^1$. If $r=1$, $k=2$, and ψ is trivial then by Theorem 5.6 of [9], we see that $(n/p) \deg T_2(n)$ is the trace of $T_k(n)$ on the "Eisenstein series part" of $\mathscr{M}^1$. Otherwise by Propositions 5.2 and 5.3 and Theorem 5.6 of [9], the "Eisenstein series part" of $\mathscr{M}^1$ is trivial. Hence letting Θ^1 denote the orthogonal complement of the Eisenstein series part in $\mathscr{M}^1$, we see that (4.1) yields

$$2(\operatorname{tr}_{p^{2r}M,\psi} T_k(n) - 2\operatorname{tr}_{p^{2r-1}M,\psi} T_k(n) + \operatorname{tr}_{p^{2r-2}M,\psi} T_k(n)) = \operatorname{tr}_{\Theta^1} T_k(n) + \sum \bar{\chi}(n) \operatorname{tr}_{p^rM,\chi^2\psi} T_k(n) \tag{4.4}$$

for all n with $(n, pM)=1$. Here $\mathrm{tr}_{\Theta^1} T_k(n)$ denotes the trace of $T_k(n)$ acting on the space Θ^1 and the sum $\sum_\chi$ is over all the ($p^r - 2p^{r-1} + p^{r-2}$ if $r \geqslant 2$ and $p-2$ if $r=1$) primitive characters χ of $(Z/p^rZ)^\times$.

By (1.5) the left hand side of (4.4) equals the trace of $T_k(n)$ on

$$\bigoplus_{f(\phi)\,|\,m\,|\,M} 2\delta(M/m)\, S_k^0(p^{2r}m, \psi). \tag{4.5}$$

Now we consider the right hand side of (4.4). Let χ be a primitive character mod p^r and consider $S_k(p^rM, \chi^2\psi) = S_k(p^rM, \chi^2\omega\phi)$. If $r=1$, then $\omega=1$ and $e(\chi^2)=1$ except when $\chi=\gamma$. If $r\geqslant 2$, then $e(\chi^2\omega)=r$ for all χ. Hence by (1.5)

$$S_k(pM, \gamma^2\psi) \cong \bigoplus_{f(\phi)\,|\,m\,|\,M} 2\delta(M/m)\, S_k^0(m, \psi) \oplus \bigoplus_{f(\phi)\,|\,m\,|\,M} \delta(M/m)\, S_k^0(pm, \psi)$$

and for $r \geqslant 2$ or $r=1$ and $\chi \neq \gamma$,

$$S_k(p^rM, \chi^2\psi) \cong \bigoplus_{f(\phi)\,|\,m\,|\,M} \delta(M/m)\, S_k^0(p^rm, \chi^2\psi).$$

Then by Lemma 1.2 if $r=1$ the right hand side of (4.4) equals the trace of $T_k(n)$ on

$$\begin{aligned} &\Theta^1 \oplus \bigoplus_{f(\phi)\,|\,m\,|\,M} 2\delta(M/m)\, S_k^0(m, \phi)^\gamma \oplus \bigoplus_{f(\phi)\,|\,m\,|\,M} \delta(M/m)\, S_k^0(pm, \phi)^\gamma \\ &\oplus \bigoplus_{\chi} \bigoplus_{f(\phi)\,|\,m\,|\,M} \delta(M/m)\, S_k^0(pm, \chi^2\phi)^{\bar\chi}, \end{aligned} \tag{4.6$'$}$$

where the sum $\bigoplus_\chi$ is over all the $p-3$ primitive characters mod p excepting γ. If $r\geqslant 2$, then the right hand side of (4.4) equals the trace of $T_k(n)$ on

$$\Theta^1 \oplus \bigoplus_{\chi} \bigoplus_{f(\phi)\,|\,m\,|\,M} \delta(M/m)\, S_k^0(p^rm, \chi^2\psi)^{\bar\chi}, \tag{4.6$''$}$$

where the sum $\bigoplus_\chi$ is over all the primitive characters mod p^r.

From (4.4), (4.5), (4.6′), and (4.6″) we find if $r=1$, then

$$\begin{aligned} &\bigoplus_{f(\phi)\,|\,m\,|\,M} 2\delta(M/m)\, S_k^0(p^{2r}m, \phi) \\ &\quad\cong \Theta^1 \oplus \bigoplus_{f(\phi)\,|\,m\,|\,M} 2\delta(M/m)\, S_k^0(m, \phi)^\gamma \oplus \bigoplus_{f(\phi)\,|\,m\,|\,M} \delta(M/m)\, S_k^0(pm, \phi)^\gamma \\ &\quad\oplus \bigoplus_{\chi} \bigoplus_{f(\phi)\,|\,m\,|\,M} \delta(M/m)\, S_k^0(pm, \chi^2\phi)^{\bar\chi}, \end{aligned} \tag{4.7$'$}$$

where the sum $\bigoplus_\chi$ is over all the $p-3$ primitive characters mod p excepting γ; and if $r \geqslant 2$, then

$$\bigoplus_{f(\phi)|m|M} 2\delta(M/m)\, S_k^0(p^{2r}m, \psi)$$
$$\cong \Theta^1 \oplus \bigoplus_\chi \bigoplus_{f(\phi)|m|M} \delta(M/m)\, S_k^0(p^r m, \chi^2\psi)^{\bar\chi}, \qquad (4.7'')$$

where the sum $\bigoplus_\chi$ is over all the primitive characters mod p^r.

If $F(\tau)$ is any newform in any space $S_k^0(p^\nu m, \xi)$ where p is a prime not dividing m and ξ is a character mod $p^\nu m$ and χ is any character mod a power of p, then by Theorem 3.2 of [2] there is a newform $G(\tau)$ in some space $S_k^0(p^{\nu'}m, \chi^2\xi)$ such that $F_\chi(\tau) \sim G(\tau)$. The important thing for us is that m, the part of the level prime to p, is the same for $F(\tau)$ and $G(\tau)$. Now consider (4.7″). If $F(\tau)$ is a newform in $S_k^0(p^r m, \chi^2\psi)$ then by (4.7″), $F_{\bar\chi}(\tau)$ is equivalent to some newform $G(\tau)$ in some space $S_k^0(p^{2r}m', \psi)$. But by the above argument, $m = m'$. Noting that if $\chi_1 = \overline{\chi\omega}$, then $S_k^0(p^r M, \chi^2\psi)^{\bar\chi} \cong S_k^0(p^r M, \chi_1^2\psi)^{\bar\chi_1}$ by Corollary 3.3, we obtain (4.3″) for some subspace Θ'' of Θ^1. The isomorphism (4.3′) is obtained similarly from (4.7′). Note that we do not need to use Theorem 3.2 of [2]. We could have used induction on $M/f(\phi)$ instead. In fact, to identify Θ' as a space of "newforms" in a space of theta series, one must use induction. ∎

COROLLARY 4.5. *Let the notation and hypotheses be as in Theorem* 4.3. *Then if* $r = 1$ *we have*

$$S_k^0(p^2M, \phi) \supseteq S_k^0(pM, \phi)^\gamma \oplus S_k^0(M, \phi)^\gamma \oplus \bigoplus_{\chi/\sim} S_k^0(pM, \chi^2\phi)^{\bar\chi}, \qquad (4.8')$$

where $\gamma = (\ /p)$ *and the sum* $\bigoplus_{\chi/\sim}$ *is over all the* $\frac{1}{2}(p-3)$ *classes of primitive characters* mod *p excepting* γ *modulo the equivalence* $\chi \sim \bar\chi$. *If* $r \geqslant 2$, *we have*

$$S_k^0(p^{2r}M, \psi) \supseteq \bigoplus_{\chi/\sim} S_k^0(p^r M, \chi^2\psi)^{\bar\chi}, \qquad (4.8'')$$

where the sum $\bigoplus_{\chi/\sim}$ *is over all the* $\frac{1}{2}(p^r - 2p^{r-1} + p^{r-2})$ *classes of primitive characters mod* p^r *modulo the equivalence* $\chi \sim \overline{\chi\omega}$.

Proof. By Proposition 1.1 every newform in the R.H.S. of (4.8′) (resp. (4.8″)) is contained in $S_k(p^2M, \phi)$ (resp. $S_k(p^{2r}M, \psi)$); (4.8′) (resp. (4.8″)) now follows from (4.3′) (resp. (4.3″)) and the theory of newforms. ∎

We now consider $S_k^0(p^2M, \omega\phi)$ with $e(\omega) = 1$.

PROPOSITION 4.6. *Let the notation and hypotheses be as in Theorem* 4.3

except that we now assume $r=1$ and ω is an odd character mod p. *Then we have*

$$2S_k^0(p^2M, \psi) \cong \Theta' \oplus \bigoplus_{\chi/\sim} 2S_k^0(pM, \chi^2\psi)^{\bar{\chi}} \tag{4.9}$$

as H-modules where the sum $\bigoplus_{\chi/\sim}$ *is over all the* $\frac{1}{2}(p-3)$ *classes of primitive characters* mod p *excepting* $\bar{\omega}$ *modulo the equivalence* $\chi \sim \overline{\chi\omega}$.

Proof. This is Theorem 7.17 of [9] where $\Theta' = \Theta_k^0(M; L, 2; \tilde{\psi})$ is a space of "newforms." For completeness we give a (different) proof here. As in the proof of Theorem 4.3 we see that (4.2) yields

$$2\,\mathrm{tr}_{p^2M,\psi} T_k(n) - 3\,\mathrm{tr}_{pM,\psi} T_k(n) = \mathrm{tr}_{\Theta^1} T_k(n) + \sum_{\chi} \bar{\chi}(n)\,\mathrm{tr}_{pM,\chi^2\psi} T_k(n). \tag{4.10}$$

Using (1.5) and Lemma 1.2 we obtain from (4.10)

$$\begin{aligned} &\bigoplus_{f(\phi)|m|M} 2\delta(M/m)\, S_k^0(p^2m, \psi) \oplus \bigoplus_{f(\phi)|m|M} \delta(M/m)\, S_k^0(pm, \psi) \\ &\qquad \cong \Theta^1 \oplus \bigoplus_{\chi} \bigoplus_{f(\phi)|m|M} \delta(M/m)\, S_k^0(pm, \chi^2\psi)^{\bar{\chi}}, \end{aligned}$$

where the sum $\bigoplus_\chi$ is over all the $p-2$ primitive characters mod p. Note that if $\chi = \bar{\omega}$, then by Theorem 3.2 we have $S_k^0(pm, \chi^2\psi)^{\bar{\chi}} \cong S_k^0(pm, \psi)$ and we obtain

$$\begin{aligned} &\bigoplus_{f(\phi)|m|M} 2\delta(M/m)\, S_k^0(p^2m, \psi) \\ &\qquad \cong \Theta^1 \oplus \bigoplus_{\chi} \bigoplus_{f(\phi)|m|M} \delta(M/m)\, S_k^0(pm, \chi^2\psi)^{\bar{\chi}}, \end{aligned} \tag{4.11}$$

where the sum $\bigoplus_\chi$ is over all the $p-3$ primitive characters mod p excepting $\bar{\omega}$. Now as in the proof of Theorem 4.3 the isomorphism in (4.9) follows from (4.11) using Theorem 3.2 of [2] (or induction). ∎

If ω is an even character mod p we could again use (4.2) together with Theorem 3.2 to obtain (4.12) below. However, it is just as simple to proceed as follows.

PROPOSITION 4.7. *Let the notation and hypotheses be as in Theorem* 4.3 *except that we now assume $r=1$ and ω is a non-trivial even character mod p with $\omega = \lambda^2$. Then we have*

$$\begin{aligned} S_k^0(p^2M, \omega\phi) \cong{}& \Theta''^{\lambda} \oplus S_k^0(M, \phi)^{\lambda} \oplus S_k^0(pM, \phi)^{\lambda} \\ &\oplus S_k^0(M, \phi)^{\gamma\lambda} \oplus S_k^0(pM, \phi)^{\gamma\lambda} \oplus \bigoplus_{\chi/\sim} S_k^0(pM, \chi^2\phi)^{\bar{\chi}\lambda}, \end{aligned} \tag{4.12}$$

where the sum $\bigoplus_{\chi/\sim}$ *is over all the* $\frac{1}{2}(p-5)$ *classes of primitive characters* mod p *excepting* $\chi=\gamma$, $\chi=\lambda$, *and* $\chi=\bar{\lambda}$ *modulo the equivalence* $\chi\sim\bar{\chi}$.

Proof. Note that by assumption $p\geqslant 5$. By Theorem 3.7,

$$S_k^0(pM,\lambda^2\phi)\oplus S_k^0(p^2M,\lambda^2\phi)$$
$$\cong S_k^0(M,\phi)^{\lambda}\oplus S_k^0(pM,\phi)^{\lambda}\oplus S_k^0(p^2M,\phi)^{\lambda}. \tag{4.13}$$

By (4.8′),

$$S_k^0(p^2M,\phi)^{\lambda}\cong\Theta''^{\lambda}\oplus S_k^0(pM,\phi)^{\gamma\lambda}\oplus S_k^0(M,\phi)^{\gamma\lambda}$$
$$\oplus\bigoplus_{\chi/\sim} S_k^0(pM,\chi^2\phi)^{\bar{\chi}\lambda}, \tag{4.14}$$

where $\gamma=(\ /p)$ and the sum $\bigoplus_{\chi/\sim}$ is over all the $\frac{1}{2}(p-3)$ classes of primitive characters mod p excepting γ modulo the equivalence $\chi\sim\bar{\chi}$. The summand corresponding to $\chi=\lambda$ in (4.14) gives the first summand of (4.13). Hence we obtain (4.12). ∎

COROLLARY 4.8. *Let p be an odd prime and M a positive integer prime to p. Let ω be a non-trivial character* mod p, *ϕ a character* mod M, *and set $\psi=\omega\phi$. Let k be an integer $\geqslant 2$ satisfying $\psi(-1)=(-1)^k$. If ω is odd we have*

$$S_k^0(p^2M,\psi)\supseteq\bigoplus_{\chi/\sim} S_k^0(pM,\chi^2\psi)^{\bar{\chi}}, \tag{4.15}$$

where the sum $\bigoplus_{\chi/\sim}$ *is over all the* $\frac{1}{2}(p-3)$ *classes of primitive characters* mod p *excepting* $\bar{\omega}$ *modulo the equivalence* $\chi\sim\overline{\chi\omega}$. *If ω is even then letting* $\omega=\lambda^2$ *we have*

$$S_k^0(p^2M,\omega\phi)\supseteq S_k^0(M,\phi)^{\lambda}\oplus S_k^0(pM,\phi)^{\lambda}\oplus S_k^0(M,\phi)^{\gamma\lambda}$$
$$\oplus S_k^0(pM,\phi)^{\gamma\lambda}\oplus\bigoplus_{\chi/\sim} S_k^0(pM,\chi^2\phi)^{\bar{\chi}\lambda}, \tag{4.16}$$

where the sum $\bigoplus_{\chi/\sim}$ *is over all the* $\frac{1}{2}(p-5)$ *classes of primitive characters* mod p *excepting* $\chi=\gamma$, $\chi=\lambda$, *and* $\chi=\bar{\lambda}$ *modulo the equivalence* $\chi\sim\bar{\chi}$.

Proof. By Proposition 1.1 every newform in the R.H.S. of (4.15) or (4.16) is contained in $S_k(p^2M,\psi)$. Equations (4.15) and (4.16) now follow from (4.9), (4.12), and the theory of newforms. ∎

THEOREM 4.9. *Let p be an odd prime and M a positive integer prime to p. Let r be a positive integer, ω a character* mod p^r, *ϕ a character* mod M, *and set $\psi=\omega\phi$. Let k be an integer $\geqslant 2$ satisfying $\psi(-1)=(-1)^k$. Assume $r\geqslant 2$,*

$e(\omega)=r$, and let $\omega=\varepsilon\lambda^2$ for any character ε with $e(\varepsilon)\leqslant r-1$ and $e(\lambda)=r$. Then

$$S_k^0(p^{2r}M, \varepsilon\lambda^2\phi) \cong \Theta''^{\lambda} \oplus \bigoplus_{i=e(\varepsilon)}^{2r-1} S_k^0(p^iM, \varepsilon\phi)^{\lambda} \oplus \bigoplus_{\chi/\sim} S_k^0(p^rM, \varepsilon\chi^2\phi)^{\bar{\chi}\lambda}, \tag{4.17}$$

where the sum $\bigoplus_{\chi/\sim}$ is over all the equivalence classes of primitive characters χ mod p^r such that $\chi\bar{\lambda}$ and $\chi\lambda\varepsilon$ remain primitive mod p^r, modulo the equivalence $\chi\sim\overline{\chi\varepsilon}$. Here Θ'' is "half" of the Θ'' in (4.3″).

Proof. By Theorem 3.7

$$\bigoplus_{i=r}^{2r} S_k^0(p^iM, \varepsilon\lambda^2\phi) \cong \bigoplus_{i=e(\varepsilon)}^{2r} S_k^0(p^iM, \varepsilon\phi)^{\lambda}. \tag{4.18}$$

By (4.3″)

$$S_k^0(p^{2r}M, \varepsilon\phi)^{\lambda} \cong \Theta''^{\lambda} \oplus \bigoplus_{\chi/\sim} S_k^0(p^rM, \chi^2\psi)^{\bar{\chi}\lambda}, \tag{4.19}$$

where the sum $\bigoplus_{\chi/\sim}$ is over all the $\frac{1}{2}(p^r-2p^{r-1}+p^{r-2})$ classes of primitive characters mod p^r modulo the equivalence $\chi\sim\overline{\chi\varepsilon}$. Now by Theorem 3.9 we can write the L.H.S. of (4.18) as

$$\begin{aligned} &S_k^0(p^{2r}M, \varepsilon\lambda^2\phi) \oplus \bigoplus_{i=r}^{2r-1} \left(\bigoplus_{e(\rho)=i-r} S_k^0(p^r, \varepsilon\lambda^2\rho^2\phi)^{\bar{\rho}} \right) \\ &\quad = S_k^0(p^{2r}M, \varepsilon\lambda^2\phi) \oplus \bigoplus_{e(\rho)<r} S_k^0(p^r, \varepsilon\lambda^2\rho^2\phi)^{\bar{\rho}}, \end{aligned} \tag{4.20}$$

where $\bigoplus_{e(\rho)=i-r}$ denotes the sum over all primitive characters mod p^{i-r} and $\bigoplus_{e(\rho)<r}$ denotes the sum over all characters mod p^{r-1}. Each space $S_k^0(p^r, \varepsilon\lambda^2\rho^2\phi)^{\bar{\rho}}$ occurring in (4.20) appears as a summand (when $\chi=\lambda\rho$) in the R.H.S. of (4.19). Hence from (4.18), (4.19), and (4.20) we obtain (4.17). ∎

COROLLARY 4.10. *Let the notation and hypotheses be as in Theorem 4.9. Then*

$$S_k^0(p^{2r}M, \varepsilon\lambda^2\phi) \supseteq \bigoplus_{i=e(\varepsilon)}^{2r-1} S_k^0(p^iM, \varepsilon\phi)^{\lambda} \oplus \bigoplus_{\chi/\sim} S_k^0(p^rM, \varepsilon\chi^2\phi)^{\bar{\chi}\lambda}. \tag{4.21}$$

Proof. By Proposition 1.1 every newform in the R.H.S. of (4.21) is contained in $S_k(p^{2r}M, \varepsilon\lambda^2\phi)$; (4.21) now follows from (4.17) and the theory of newforms. ∎

Remark 4.11. We now explain how Corollary 4.10 relates to Theorems 3.1 and 4.1 in Atkin and Li [2]. First consider Theorem 3.1 of [2]. Recall (see the paragraph preceding Theorem 3.9 above) that the only case of Theorem 3.1 of [2] remaining to be considered is base (b) when $\alpha \leqslant \beta$. Let $F(\tau)$ be a newform in $S_k^0(p^v M, \omega\phi)$ and let χ be a primitive character mod p^r. Let $\alpha = e(\omega)$ and $\beta = r$. Theorem 3.1 of [2] states that if $\alpha \leqslant r$, $2r > v$, and $e(\chi\omega) = r$, then $F_\chi(\tau)$ has exact level $p^{2r}M$. Since we are considering Corollary 4.10 we make the additional assumptions that p is odd and $r \geqslant 2$. Now if $\alpha < r$, letting $\omega = \varepsilon$ and $\chi = \lambda$, $F_\chi(\tau)$ appears in the first summand on the R.H.S. of (4.21), hence is a newform of level $p^{2r}M$, and hence has exact level $p^{2r}M$. If $\alpha = r$, then $\omega = \varepsilon\xi^2$ for some ε with $e(\varepsilon) < r$. Let $\lambda = \chi\xi$. Then $\chi = \bar{\xi}\lambda$. Also $\xi\bar{\lambda} = \bar{\chi}$ and $\xi\lambda\varepsilon = \chi\omega$ are primitive mod p^r. Thus $F_\chi(\tau)$ appears in the second summand on the R.H.S. of (4.21), hence is a newform of level $p^{2r}M$, and hence has exact level $p^{2r}M$. Theorem 4.1 of [2] states that if $F(\tau)$ is a newform in $S_k^0(p^v M, \omega\phi)$ and χ is a character mod a power of p with $e(\chi) = r \geqslant v$ and $e(\omega\chi) = r$, then $F_\chi(\tau)$ is a newform in $S_k^0(p^{2r}M, \omega\chi^2\phi)$. This follows by the same argument as above. If p is odd and $r = 2$, we leave it to the reader to check that Theorem 4.1 of [2] and the above case of Theorem 3.1 of [2] follow from Corollary 4.8. If $p = 2$, Theorem 4.1 of [2] and the above case of Theorem 3.1 of [2] follow from Theorem 3.14.

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