Asymptotic Results for Finite Energy Solutions of Semilinear Elliptic Equations

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0. INTRODUCTION

In this paper we will examine the asymptotic behaviour at infinity of solutions of some semilinear elliptic equations. In many radial situations and especially the autonomous case, the possible behaviour of positive (and also oscillatory) solutions has been classified. Let us mention early results by Fowler [FO] and recent results by McLeod, Ni, and Serrin [MNS], and Pucci and Serrin [PS]. Behaviour of nonradial solutions has been studied by Gidas, Spruck, and Caffarelli [GS, CG], and by Aviles [AV].

The studies in the papers mentioned above are focused on determining all possible behaviour of solutions of the equation. In this paper we will only consider "finite energy" solutions and we will consider both radial and nonradial situations. This is a natural class of solutions to consider since solutions obtained by variational methods have "finite energy."

The results obtained in this paper are "designed" for the existence results obtained in [EG2]. Let us mention that [LN1, LN2, NOS1, NOS2] contain existence results of the same type. In [EG2] we considered the following problem:

\[ \Delta u + f(x, u) = 0 \quad \text{in } \Omega, \]
\[ u \in \mathcal{D}^{1,2}_0(\Omega), \quad u > 0 \quad \text{in } \Omega. \]

Here \( \Omega \) is an open connected set in \( \mathbb{R}^n \). The space \( \mathcal{D}^{1,2}_0(\Omega) \) is just the completion of \( C_0^\infty(\Omega) \) with the "energy" \( \| \nabla u \|_2 \) as a norm.

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Unless otherwise stated, we will assume that \( n > 2 \), \( f(x, s) \geq 0 \), for all \( x \in \Omega \) and \( s \geq 0 \), and that \( \Omega^c \) is bounded.

In [EG2] we considered the more general case when the Laplacian is replaced by the \( m \)-Laplacian. In this paper we will prove results for the Laplacian only (except for Theorem 6 and its Corollary). The problem with the method in the present paper is that we can not find a substitute for the Kelvin transform in the general case. However, we believe that it is possible to obtain the results for the \( m \)-Laplacian, using a more direct method not involving the Kelvin transform.

Since \( f > 0 \) in (0.1) the most rapid decay we can expect is \( u(x) \sim C|x|^{2-n} \), this can be shown using a comparison argument (see [LN1]). We will show that for a large class of functions \( f \) it turns out that solutions of (0.1) have this decay. In fact we will also get an estimate of the second term in the asymptotic expansion.

Another equation that has been frequently studied in the literature is

\[
Au - a(x)u + f(x, u) = 0 \quad \text{in } \Omega,
\]

\[
u \in \mathcal{D}^{1,2}_0(\Omega), u > 0 \quad \text{in } \Omega.
\]

Here \( a \) is a non-negative function, and \( f \) and \( \Omega \) satisfy the hypothesis above. If \( a \) is bounded from above and below by positive constants, then the space \( \mathcal{D}^{1,2}_0(\Omega) \) is often replaced by \( H^{1,2}_0(\Omega) \). This is the Sobolev space obtained by taking the completion of \( C^\infty_0(\Omega) \) with the norm \( \|u\| = \|\nabla u\|_2 + \|u\|_2 \).

If we assume that \( f(x, s) = o(s) \) holds uniformly for large \( x \) as \( s \to 0 \) and \( a \) is bounded from below by a positive constant for large \( x \), then it is well known that radial solutions of (0.2) do tend to zero at least exponentially fast as \( x \to \infty \). We will show that under some additional conditions on \( f \) this fast decay holds also for nonradial solutions.

Finally in Section 4 we will prove a result when the Laplacian is replaced by the so-called \( m \)-Laplacian.

In [EG2] the existence of solutions of a class of quasilinear equations is proved using variational methods. This is done in a rather straightforward way. The problem with variational methods is that often it does not give the same qualitative information, as for example, ODE methods. However, in this paper we prove that almost all solutions obtained in [EG2] have the fastest possible decay, at infinity. The methods used in the present paper contain ideas introduced by Moser [MO] and Trudinger [TR].

The formulation of the results might look a little bit complicated. One reason for this is that we treat both superlinear, sublinear, and linear cases. It turns out that the results are different in the different cases. As some examples given in Section 5 show, this difference is not due to the methods applied.

Also in Section 5 we give examples that show that the results obtained are the best possible or close to the best possible.
1. The Main Results

Before we state the results about the asymptotic behaviour of solutions of (0.1) let us recall the two main results contained in [EG2]. The first is only for radial solutions and the second contains results for both the radial and the nonradial cases. Our asymptotic results are "designed" for applications to these existence results. However, the results in the present paper also include linear, sublinear, and critical growth.

Below the following functions will be used repeatedly:

\[ \mathcal{f}(x, s) = \sup_{t < s} f(x, t), \quad \omega(x) = |x|^{(2-n)/2} \quad (1.1) \]

**Theorem A.** Assume that \( f(x, s) \) is radial in the \( x \) variable and that \( \Omega \) is radial (e.g., \( \Omega = \mathbb{R}^n \setminus \overline{B}(0) \) or \( \Omega = \mathbb{R}^n \)). Furthermore, assume that

(i) \[ \mathcal{f}(x, c \omega(x)) \omega(x) \in L^1(\Omega) \text{ for each positive } c, \text{ and} \]

\[ \int_{\Omega} \mathcal{f}(x, c \omega(x)) \omega(x) \, dx = o(\varepsilon^2), \text{ as } \varepsilon \to 0. \]

(ii) There is a radial function \( \varphi \in D_0^{1,2}(\Omega) \) such that

\[ t^{-1} \int f(x, t \varphi) \varphi \, dx \to \infty, \text{ as } t \to \infty. \]

(iii) There are constants \( \theta \in (0, 1/2) \) and \( M < \infty \) such that

\[ \int_0^s f(x, t \omega(x)) \, ds \leq \theta f(x, t \omega(x)) t \text{ holds for all } t \geq M \text{ and } x \in \Omega. \]

(iv) The function \( s \mapsto f(x, s) \) is continuous for a.e. \( x \in \Omega \).

Then (0.1) has a radial solution such that \( u(x) = o(\omega(x)) \) for large \( x \).

**Theorem B.** Let us assume that \( f \) satisfies the following conditions.

(i) The estimate \( f(x, s) \leq \sum_{i=1}^n b_i(x) s^{p_i} \) holds for all \( x \in \Omega \). Here each pair \( (p_i, b_i) \) satisfies the following conditions:

(a) \( b_i \) is locally bounded in \( \Omega \setminus \{0\} \)

(b) \( 1 < p_i < (n+2)/(n-2) \)

(c) there are numbers \( v_1 \) and \( v_2 \) such that \( b_i(x) = o(|x|^{v_1}) \), for \( x \) small, \( b_i(x) = o(|x|^{v_2}) \), for \( x \) large and \( (n+2+2v_2)/(n-2) \leq p_i \leq (n+2+2v_1)/(n-2) \).

(ii) There is a \( \varphi \in D_0^{1,2}(\Omega) \) such that \( t^{-1} \int f(x, t \varphi) \varphi \, dx \to \infty, \text{ as } t \to \infty. \)

(iii) There is a constant \( \theta \in (0, 1/2) \) such that

\[ \int_0^s f(x, s) \, ds \leq \theta f(x, s) t, \text{ for all } x \in \Omega \text{ and positive } t. \]

(iv) The function \( s \mapsto f(x, s) \) is continuous for a.e. \( x \in \Omega \).

Then (0.1) has a solution, which tends to zero uniformly as \( x \to \infty. \)

If \( \Omega \) is radial and \( f \) is radial in the space variable, then the result holds if we remove the condition \( p_i < (n+2)/(n-2) \) in (i)(b).
Let me again mention that similar results can be found in [LN1, LN2, NOS1, NOS2].

The paper [EG2] (see also [LN1, LN2, PS]) also contains the following improved asymptotic result.

**Proposition C.** Let \( \Omega \) be radial and unbounded and let \( f \) be radial in the space variable. Assume that we can find constants \( p > 1 \) and \( v \) satisfying \( p > (n + 2 + 2v)/(n - 2) \) such that \( f(x, s) \leq C \, |x|^p \) holds for large \( x \).

Then each radial solution \( u \) of (0.1) has a finite positive limit \( \lim_{x \to \infty} |x|^{n-2} u(x) = C \). If \( \Omega = \mathbb{R}^n \), then \( C = (1/(n-2) |S_{n-1}|) \int f(x, u(x)) \, dx \). Here \( |S_{n-1}| \) denotes the area of the \( n-1 \) sphere.

It is easy to find examples where Theorems A and B above imply the existence of a radial solution but Proposition C does not apply. Furthermore, Proposition C does not contain any nonradial situations at all.

In this paper we will prove the following two results formulated so that they can be applied to the solutions obtained in Theorems A and B above.

**Theorem 1.** Assume that \( u \) is a non-negative, radial function with finite energy in a neighborhood of infinity. Furthermore assume that \( \Delta u + f(x, u) + h > 0 \) for large \( x \), where \( h \) is radial and integrable and the function \( f \) is radial in the space variable and satisfies the following conditions.

(i) \( \int f(x, C \omega(x)) \omega(x) \) is integrable in a neighborhood of infinity for each positive \( C \).

(ii) There is a constant \( C \), such that the "superlinearity" condition \( \int f(x, s) s^{-\mu} \leq C \int f(x, t) t^{-\mu} \) holds for all \( t > s > 0 \) and large \( x \).

(iii) There is a positive constant \( \alpha \) such that \( \int f(x, |x|^{2-n+\alpha}) \) is integrable in a neighborhood of infinity.

Then \( \lim \sup_{x \to \infty} |x|^{n-2} u(x) < \infty \). If we also have \( -\Delta u \geq 0 \) and \( u \neq 0 \) for large \( x \), then \( \lim_{x \to \infty} |x|^{n-2} u(x) \) exists and is a finite positive number. If \( u \) is a global solution of (0.1) with \( \Omega = \mathbb{R}^n \), then the limit is the number given in Proposition C.

Condition (i) is the best possible in the sense that the theorem is not true if we replace \( \omega \) with \( \omega(x) |x|^{-\delta} \) with \( \delta > 0 \) (see Example 1). Also the restriction that \( f(x, \cdot) \) is superlinear (ii) is needed (see Example 6).

If we only assume that (i) and (ii) above hold we can conclude that for each positive \( \alpha \), \( |x|^{n-2-\alpha} u(x) \) is bounded at infinity.

Condition (iii) in the theorem above is not very restrictive (see Example 5). For example, it is implied by the following more explicit "superlinearity condition."

(iii') There are constants \( M \) and \( \mu > 2 \) such that \( \int_{|x| > M} f(x, \epsilon \omega(x)) \omega(x) \, dx = O(\epsilon^\mu) \), as \( \epsilon \to 0 \).
Thus the extra condition (iii) is only needed if we have linear or almost linear growth. We believe however that the conclusion in the theorem holds without condition (iii). The fact that the linear situation is more delicate is indicated in Examples 3 and 4.

**Theorem 2.** Assume that $u$ is a non-negative function with finite energy in a neighborhood of infinity. Furthermore assume that $\Delta u + f(x, u) \geq 0$ for large $x$, where the function $f$ satisfies the following condition.

The estimate $f(x, s) \leq h(x) + b_0(x)s + \sum_{i=1}^{N} b_i(x)s^{p_i}$ holds for large $x$. Here we assume that there is a $q > n/2$ such that $h(x) |x|^{n+2}, b_0(x) |x|^4 \in L^q(|x|^{-2n} \, dx)$. The exponent $p_i$ is strictly larger than one, and one of the following conditions holds for each pair $(b_i, p_i)$.

1. $h_i(x) = O(\frac{1}{|x|^n})$, where $p_i = (n + 2 + 2\nu_i)/(n - 2)$.
2. $b_i \in L^{q_i}$ and $p_i < (n + 2)/(n - 2)$, where $q_i = 2n/(n + 2 - (n - 2)p_i)$.

If $p_i > (n + 2)/(n - 2)$ for some $i$, then we also need to assume that $u$ is radical.

Then $\lim \sup_{x \to \infty} |x|^{-n-2} u(x) < \infty$. Furthermore, if $-\Delta u \geq 0$ and $u \neq 0$ for large $x$, then $\lim \sup_{x \to \infty} |x|^{-n-2} u(x) - C$ exists and is a finite positive number. If $u$ is a global solution of (0.1) with $\Omega = \mathbb{R}^n$, then the limit is the same as that in Proposition C.

The conditions on $h$ and $b_0$ look strange and artificial. In Example 4, we show that the conditions are essentially the best possible. This example also explains the conditions.

Example 1 shows that conditions (a) and (b) are best possible in the sense that we can not choose $p_i$ any smaller (or $\nu_i$ any larger).

That the restriction to radial functions when $p_i$ is supercritical (i.e., $p_i > (n + 2)/(n - 2)$) is necessary is shown by Example 7.

That a function $u$ has finite energy in a neighborhood of infinity means just that there is a radial non-negative smooth function $\eta$ that is identically one for large $x$ such that $u \eta \in L^2(\mathbb{R}^n)$. To say that $u$ has finite energy in a neighborhood of the origin is defined in the obvious analogous way. The condition that $u$ has finite energy in a neighborhood of infinity in Theorems 1 and 2 above is really necessary (Example 1).

Since we are considering non-smooth functions, we will interpret the differential inequality $-\Delta u \leq g$ in $\Omega$ as $\int \nabla u \cdot \nabla \varphi \, dx \leq \int g \varphi \, dx$, $\forall \varphi \in C_0^\infty(\mathbb{R}^n)$.

In both theorems above we have not included a sublinear term of, for example, the type $b(x) u^p$, with $0 < p < 1$. However, since we are considering differential inequalities, they can be introduced by interpolating between $h$ and a linear term $b_0 u$, as in Example 6.

We have the following consequence of Theorems 1 and 2 above and their proofs.
COROLLARY 3. Let $u$ be a solution obtained in Theorem A or B above. In addition to Theorem A we need to assume that (iii) (or (iii')) in Theorem 1 holds. Then $u(x) = C |x|^2 - n + O(|x|^{2-n})$ as $x \to \infty$, for some positive $a$.

More specifically, $u = \max(1, v)$ or if $u$ is radial $u = \max(2, v)$, where $v$ is either

(i) the best constant such that for each positive constant $B$, there is a constant $D$ such that $f(x, B |x|^{2-n}) \leq D |x|^{-v-n}$ holds for large $x$, or

(ii) $v = 2 - n/q$, where $q$ is the largest number such that $f(x, B |x|^{2-n}) |x|^{n-2} \in L^q$, in a neighborhood of infinity for each positive $B$.

Furthermore, if $\Omega = \mathbb{R}^n$, then $C = (1/(n-2) |S_{n-1}|) \int_{\mathbb{R}^n} f(x, u(x)) \, dx$.

Example 2 in Section 5 shows that the estimate of the error term in the expansion is best possible.

The idea of the proof of Theorems 1 and 2 is to use the classical Kelvin transform $u(x) \rightarrow |x|^{2-n} u(x/|x|^2)$. Then as we will see the new function has finite energy, and satisfies a differential inequality, in a neighborhood of the origin. To prove the theorems we only need to study the transformed problem in a neighborhood of the origin. For this purpose we will use a kind of Moser iterations involving weighted Sobolev inequalities or some other adequate inequalities.

A bonus of the proofs of Theorems 1 and 2 is the following result.

COROLLARY 4. Let $u$ be a solution obtained in Theorem A or B above with $0 \in \Omega$. In addition to the conditions in Theorem A we need to assume that there is an $\alpha > 0$ such that $f(x, C |x|^{-n}) |x|^{2-n}$ is integrable in a neighborhood of the origin.

Then $u$ is bounded in a neighborhood of the origin.

In all the results above we assumed that $n > 2$. If $n \leq 2$ we cannot have solutions tending to zero at infinity. If $n = 2$, $u$ has finite energy in a neighborhood of infinity and $\Delta u \leq 0$, then $\liminf_{x \to \infty} u(x) > 0$. This follows from the maximum principle and the Kelvin transform (which is just a conformal change of coordinates in this case) as below.

To see if solutions are bounded in $\mathbb{R}^2$, we argue as follows. The differential inequality $-\Delta u \leq f(x, u)$ transforms to $-\Delta v \leq f(x/|x|^2, v)$, where $v(x) = u(x/|x|^2)$. As in the proof of Theorems 1 and 2 it follows that this inequality holds also at the origin and that $v$ has finite energy in a neighborhood of the origin. Now it is easy to find sufficient conditions for $v$ to be bounded in a neighborhood of the origin and hence for $u$ to be bounded in a neighborhood of infinity. A sufficient condition is, for example, $f(x, s) \leq b(x)s^p$, where $p > 0$ and $b \in L^q(|x|^{-2} \, dx)$ for some $q > 1$. 
An application of Theorem 2 yields the following result.

**Theorem 5.** Assume that \( u \) is non-negative, has finite energy in a neighborhood of infinity, and assume that \( u \) satisfies \(-\Delta u \leq -\lambda u + f(x, u)\) for large \( x \), where \( \lambda \) is a positive constant and \( f(x, s) \leq o(s) + C_s^{(n+2)/(n-2)} \) holds uniformly for large \( x \) and positive \( s \). Then \( u \) decays at least exponentially fast at infinity, i.e., \( u(x) \leq C \exp(-\delta |x|) \) for some positive constants \( \delta \) and \( C \), as \( x \to \infty \).

In particular, if \( u \) solves (0.2) with \( f(x, u) = b(x)u^p, \ 1 < p \leq (n+2)/(n-2) \), and \( a \) is bounded from below and \( b \) is bounded from above by positive constants, for large \( x \), then \( u \) decays at least exponentially fast at infinity. This includes the equations studied in [BL, BC, LI].

Let \( n = 2 \) and assume that \( f(x, s) \leq o(s) + C_s^a \), for some \( a \in (1, \infty) \), holds for large \( x \) and positive \( s \). If \( u \geq 0 \) has finite energy and is in \( L^2 \) in a neighborhood of infinity and if there is a positive \( \lambda \) such that \(-\Delta u \leq -\lambda u + f(x, u)\), holds for large \( x \), then it follows that \( u \) decays at least exponentially fast at infinity.

Professor W.-M. Ni has pointed out that the conclusion in Theorem 5 follows from Lemma 3.1 in [ZH], provided \( f \) has subcritical growth and \( n \geq 5 \). The last part, with \( n = 2 \), was included after a suggestion from Professor W.-M. Ni.

Above we discussed the case \( n > 2 \) and also \( n = 2 \). The remaining case is just \( n = 1 \), which is in many respects easier. To discuss the problem in this situation we will consider a more general equation where the Laplacian is replaced by the \( m \)-Laplacian. Now \( n < 2 \) is replaced by \( n < m \), which can contain situations where the dimension is larger than one. The new problem is to study solutions of the equation

\[
\Delta_m u + f(x, u) = 0 \quad \text{in } \Omega,
\]
\[
u \in D_0^{1,m}(\Omega), \ u > 0 \quad \text{in } \Omega.
\]

Here \( \Delta_m \) is the \( m \)-Laplace operator defined by \( \Delta_m u = \nabla \cdot (|\nabla u|^{m-2} \nabla u) \), where \( 1 < m < \infty \).

In [EG2] we proved the following existence result.

**Theorem D.** Assume that \( n < m \) and \( \Omega \neq \mathbb{R}^n \). Put \( \omega_m(x) = d(\Omega^c, x)^\alpha \), \( \alpha = (m-n)/m \), here \( d(\Omega^c, x) \) denotes the distance between the complement of \( \Omega \) and the point \( x \). Furthermore, assume that

(i) \( \tilde{f}(x, c\omega_m(x)) \omega_m(x) \in L^1(\Omega) \) for each positive \( c \), and

\[
\int_{\Omega} \tilde{f}(x, \varepsilon \omega_m(x)) \varepsilon \omega_m(x) \, dx = o(\varepsilon^m), \ \text{as } \varepsilon \to 0.
\]
(ii) There is a function $\varphi \in \mathcal{D}_0^{1,m}(\Omega)$ such that $t^{1-m} \int f(x, t\varphi)\varphi \, dx \to \infty$, as $t \to \infty$.

(iii) There are constants $\theta \in (0, 1/m)$ and $M < \infty$ such that $\int_0^1 f(x, s\omega_m(x)) \, ds \leq \theta f(x, t\omega_m(x))$ holds for all $t \geq M$ and $x \in \Omega$.

(iv) The function $s \to f(x, s)$ is continuous for a.e. $x \in \Omega$.

Then (1.2) has a solution such that $u(x) = O(\omega_m(x))$.

In the radial situation Pucci and Serrin [PS] proved that under some extra conditions the solutions obtained above are bounded as $x \to \infty$. We will see that this holds also in the nonradial situation.

**Theorem 6.** Assume that $n < m$, $u$ is non-negative with finite $m$-energy in a neighborhood of infinity. Furthermore, assume that $\Delta_m u + f(x, u) \geq 0$ for large $x \in \Omega$, where the function $f$ satisfies the following conditions.

(i) $\int f(x, C\omega_m(x)) \omega_m(x) \, dx < \infty$ for each positive $C$.

(ii) There is a constant $C$, such that the “superlinearity” condition $f(x, s) s^{1-m} \leq C f(x, t) t^{1-m}$ holds for all $t > s > 0$ and large $x$ in $\Omega$.

(iii) There are constants $\delta > 0$ and $\rho > 1$, such that $\int f(x, C\omega_m(x)^\delta) \omega_m(x)^{\delta + m(\rho - \delta)}$ is integrable for each positive $C$.

Then $u$ is bounded as $x \to \infty$.

Condition (i) is best possible in the sense that the theorem is not true if we replace $\omega$ by $\omega^\alpha$, where $\alpha > 1$.

If we only assume that (i) and (ii) hold we can conclude that $u(x) \omega_m(x)^{-\delta}$ is bounded for each positive $\delta$.

Again, as in Theorem 1, the extra condition (iii) is not very restrictive. It is only required if $f(x, s)$ has growth near $s^{m-1}$.

Let us also mention that the conclusions in Theorems D and 6 hold also if the complement of $\Omega$ is unbounded.

The proof of Theorem 6 will be given in Section 4.

**Corollary 7.** Under the extra assumption (iii) in Theorem 6 above, it follows that the solution obtained in Theorem D is bounded.

2. Proof of Theorems 1 and 2

In this section we will establish some lemmas that will give us Theorems 1 and 2.

**Lemma 8.** If $u \in \mathcal{D}_0^{1,2}(\mathbb{R}^n)$, then the function (Kelvin transform) $v(x) = |x|^{2-n} u(x/|x|^2)$ is also in $\mathcal{D}_0^{1,2}(\mathbb{R}^n)$. As a special case we see that if $u$ has
finite energy in a neighborhood of infinity then \( v \) has finite energy in a neighborhood of the origin.

**Proof.** We have \( v(x) = \frac{1}{|x|^n} u(y) \), where \( y = x/|x|^2 \). A simple calculation yields

\[
\nabla_x v = \frac{1}{|y|^n} \nabla_y u - 2 \frac{1}{|y|^{n-2}} (y \cdot \nabla_y u) y - (n-2) \frac{1}{|y|^{n-2}} y u.
\]

Thus, since \( dx = |y|^{-2n} dy \), we obtain

\[
\int |\nabla_x v|^2 dx \leq C \int (|\nabla_y u|^2 + |y|^{-2} u^2) dy \leq C \int |\nabla_y u|^2 dy.
\]

Here we used the fact that the imbedding \( \mathcal{D}_{0}^{1,2}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n, |y|^{-2} dy) \) is continuous. Of course \( C \) denotes a general constant that does not depend on \( u \) but it might change from line to line, and we will keep this convention throughout the paper.

Let us remark that some elementary but a bit tedious manipulations show that the Kelvin transform is actually an isometry on \( \mathcal{D}_{0}^{1,2}(\mathbb{R}^n) \) (i.e., it preserves the scalar product \( \langle u, w \rangle = \int \nabla u \cdot \nabla w \, dx \)).

Now we will prove two regularity results for some differential inequalities. These will then be used to prove Theorems 1 and 2.

**Lemma 9.** Let \( f \) be a function satisfying the growth condition \( f(x, s) \leq b_0(x)(1 + s) + \sum_{i=1}^{n} b_i(x) s^{p_i} \) for small \( x \) and positive \( s \), where \( b_0 \in L^{q_0} \), for some \( q_0 > n/2 \), \( p_i > 1 \) and each pair \((b_i, p_i) (i \geq 1)\) satisfies one of the conditions

(a) \( b_i(x) = O(|x|^q) \), where \( p_i = (n + 2 + 2q_i)/(n-2) \).

(b) \( b_i \in L^{q_i} \) and \( p_i < (n + 2)/(n-2) \), where \( q_i = 2n/(n+2 - (n-2) p_i) \).

If \( p_i > (n + 2)/(n-2) \) for some \( i \), then we also need to assume that \( u \) below is radial. If \( u \) is a non-negative function with finite energy, satisfying the differential inequality \( \Delta u + f(x, u) \geq 0 \) in a neighborhood of the origin, then it follows that \( u \) is bounded in a neighborhood of the origin.

**Proof.** The proof utilises Moser iterations and a trick due to Trudinger \([TR]\), in a standard way. We will restrict the proof to the case when \( f(x, s) \leq C |x|^s s^p \), with \( p = (n + 2 + 2\beta)/(n-2) > 1 \). The general case will be outlined later.

Take \( G \) to be a \( C^1 \) function such that \( G(s) = s^\beta \) if \( 0 < s < N \), linear if \( s \geq N \), and zero otherwise, and put \( F(u) = \int_0^u |G'(s)|^2 \, ds \) (here \( \beta > 1 \)). It is easy to verify that \( sF(s) \leq s^2 G(s)^2 \leq \beta^2 G(s)^2 \).
Take $\eta$ to be a smooth radial and non-negative function which is identically one in a neighborhood of the origin and has small support. Then if we apply $\eta^2 F(u)$ as a test function in the differential inequality we get

$$\int |\nabla u|^2 G'(u)^2 \eta^2 \, dx + 2 \int \nabla u \cdot \nabla \eta F(u) \eta \, dx \leq \beta^2 \int f(x, u) u^{-1} G(u)^2 \eta^2 \, dx.$$ 

Next we note that

$$2 |\nabla u \cdot \nabla \eta F(u) \eta| \leq \frac{1}{2} |\nabla u|^2 \eta^2 u^{-1} F(u) + 2 |\nabla \eta|^2 F(u) u$$

$$\leq \frac{1}{2} |\nabla u|^2 \eta^2 G'(u)^2 + 2 \beta^2 |\nabla \eta|^2 G(u)^2.$$ 

Thus we finally obtain

$$\|\nabla (G(u) \eta)\|^2 L^2 \leq C \beta^2 \left( \int |\nabla \eta|^2 G(u)^2 \, dx + \int |x|^r u^{-1} G(u)^2 \eta^2 \, dx \right). \quad (2.1)$$

Now using Hölder's inequality and a weighted Sobolev imbedding theorem (e.g., see [EG1, Lemma 7]) we get

$$\|G(u) \eta\|^2 L^{p+1, |x|^r} \leq C \beta^2 \left( \int |\nabla \eta|^2 G(u)^2 \, dx + \|G(u) \eta\|^2 L^{p+1, |x|^r} \|x_{\{\text{supp } \eta\}} u^{-1} G(u)^2 \eta^2 \, dx \right). \quad (2.2)$$

Since $p > 1$, we can choose the support of $\eta$ small so that $C \beta^2 \|x_{\{\text{supp } \eta\}} u^{-1} G(u)^2 \eta^2 \, dx < 1/2$, and bring the last term in (2.2) over to the left side. Then if we let $N$ tend to infinity in $G$ we finally find that

$$\|u^p \eta\|^2 L^{p+1, |x|^r}$$

is finite for $\beta = n/(n - 2) > 1$.

Now we can apply the standard argument in [GT, Proof of Theorem 8.17] (with the standard Sobolev inequality replaced by the weighted one as above).

We will show that for some small positive $r$ we have

$$\sup_{|x| < r} |u(x)| = \lim_{q \to \infty} \left( \int_{B_r(0)} |u|^q \, dx \right)^{1/q} < \infty, \quad (2.3)$$

and thus it follows that $u$ is bounded in a neighborhood of the origin.

Although the proof of (2.3) is standard, we will give it here since it is a crucial ingredient in our main result.

In the argument below we choose $\eta$ to be a radial non-negative function with support in $B_{r_1}(0)$ and such that $\eta \equiv 1$ if $|x| < r_2 < r_1$ and $|\nabla \eta| \leq 2/(r_1 - r_2)$. 

By the above we have \( u \in L^{(p+1)n/(n-2)}(B_{r_1}(0), \ |x|^\gamma \ dx) \), provided \( r_1 \) is small enough, and thus we get using (2.1)

\[
\| (u(x))_{\alpha} \|_{p+1, |x|^\gamma} \leq C \left( \frac{\beta}{r_1 - r_2} \right)^{1/p} \| G(u) \chi_{r_1} \|_{(p+1)\delta, |x|^\gamma},
\]

(2.4)

for some \( \delta < 1 \). Here \( \chi_{r}(x) = 1 \) if \( |x| \leq r \) and zero otherwise.

In the proof of (2.4) we used the fact that there is a \( \delta < 1 \) such that

\[
\| G(u) \chi_{r_1} \|_2 \leq C \| G(u) \chi_{r_1} \|_{(p+1)\delta, |x|^\gamma},
\]

and

\[
\int_{\{ |x| < r_1 \}} u^{p-1} G(u)^2 |x|^\gamma \ dx \leq C \| u \chi_{r_1} \|_{(p+1)n/(n-2), |x|^\gamma}^2 \| G(u) \chi_{r_1} \|_{(p+1)\delta, |x|^\gamma}.
\]

Thus if we define

\[
\Phi_v(q, r) = \left( \int_{\{ |x| < r \}} |u|^q |x|^\gamma \ dx \right)^{1/q}
\]

and let \( N \to \infty \) in (2.4) we get

\[
\Phi_v((p+1)\beta, r_2) \leq \left( \frac{C\beta}{r_1 - r_2} \right)^{1/p} \Phi_v((p+1)\delta, r_1).
\]

(2.5)

Since \( u \in L^{p+1}(B_{r_0}(0), \ |x|^\gamma \ dx) \) we can iterate (2.5) with

\[
\beta = \delta^{-m} > 1, \quad r_m = r_0(2 + 2^{-m})/4 \quad \text{for} \quad m = 1, 2, ..., \]

to obtain

\[
\Phi_v((p+1)\delta^{-m}, r_m) \leq C_m \Phi_v(p+1, r_0),
\]

where \( C_m \leq C d_m(2/\delta)^m, \ d_m = \sum_{k=1}^{m} k^k, \) and \( e_m = \sum_{k=1}^{m} k \delta^k \). Thus it follows that \( C_m \) stays bounded as \( m \) tends to infinity and (2.3) follows.

If we instead have \( f(x, u) \leq b(x)u^p \), where \( p > 1 \) and the pair \((b, p)\) satisfies condition (b), we can modify the argument above as follows.

The last term in (2.1) becomes \( C\beta^2 \int b u^{p-1} G(u)^2 \eta^2 \ dx \). By Hölder's inequality we can estimate this term by \( C\beta^2 \| G(u) \eta \|_{2n(n-2)}^2 \| u \chi_{\{ \text{supp } \eta \}} \|_p \| b \chi_{\{ \text{supp } \eta \}} \|_q \), where \( q \) is the exponent given in condition (b). Again choosing the support of \( \eta \) small we deduce that \( u \) is locally in \( L^{2n\beta/(n-2)} \) for \( \beta = n/(n-2) > 1 \). Now we can perform the Moser iterations as above to arrive at (2.3).

If \( f(x, u) \leq (1 + u) h_0(x) \), then by considering the function \( \bar{u} = u + 1 \), we reduce the study to the case when \( f(x, u) \leq b_0(x)u \). Here we do not need
the initial step above. In fact we get (2.3) if we just perform the Moser
iterations in a standard fashion as in the second step above.

In the general case when \( N > 1 \) we argue almost as before. The only
difference is that we use \( \Phi_{v_1,\ldots,v_N}(q_1, \ldots, q_N, r) = \sum_{i=1}^{N} \Phi_{v_i}(q_i, r) \) and instead of (2.5) we get

\[
\Phi_{v_1,\ldots,v_N}((p_1+1)\beta, \ldots, (p_N+1)\beta, r_2) \\
\leq \left( \frac{C\beta}{r_1-r_2} \right)^{1/\beta} \Phi_{v_1,\ldots,v_N}((p_1+1)\delta\beta, \ldots, (p_N+1)\delta\beta, r_1). \tag{2.5'}
\]

Now the proof follows exactly as before. \( \Box \)

In the proof of Lemma 9 above we used the weighted Sobolev imbedding
\( D_0^{1,2}(\Omega) \subset L^{p+1}(\Omega, |x|^r \, dx), \ p + 1 = 2(n + \nu)/(n - 2) \). In the proof of the
following lemma this imbedding will be replaced by the following element-
ary estimate \( |u(x)| \leq C \|\nabla u\|_2 \|x|^{(2-n)/2} \), which holds for all radial functions \( u \in D_0^{1,2}(\Omega) \). For a proof of this estimate see, for example, Lemma 5
in [EG2].

**Lemma 10.** Assume that \( h \) and \( f \) are radial in the space variable and let
\( \tilde{f}(x, s) \) and \( \omega(x) \) be as defined in (1.1). Assume that \( h(x) \omega(x)^2 \) and
\( \tilde{f}(x, C\omega(x)) \omega(x) \) are locally integrable in a neighborhood of the origin for
each positive \( C \) and assume that there is a constant \( D \) such that \( \tilde{f}(x, s)s^{-1} \leq
Df(x, t)t^{-1} \) holds for all \( t > s \geq 0 \) and small \( x \).

Then if \( u \) is radial, non-negative, has locally finite energy in a
neighborhood of the origin, and satisfies the differential inequality
\( Au + f(x, u) + h \geq 0 \), then \( |x|^\alpha u \) is bounded in a neighborhood of the origin
for each positive \( \alpha \).

If we also assume that there is a positive constant \( \alpha \) such that
\( \tilde{f}(x, |x|^{-\alpha}) |x|^{2-n} \) is integrable in a neighborhood of the origin, then \( u \) is
bounded in a neighborhood of the origin.

**Proof.** First by replacing \( u \) by \( \tilde{u} = u + 1 \), we can replace \( h \) by \( h\tilde{u} \) in the
differential inequality. Thus \( h \) can be considered as part of \( f \). Therefore in
the proof of the first part of the lemma we can assume that \( h \) is zero.

As in the proof of Lemma 9 we get

\[
\|\nabla(G(u)\eta)\|_2 \leq C\beta^2 \left( \int |\nabla\eta|^2 G(u)^2 \, dx + \int \tilde{f}(x, u) u^{-1} G(u)^2 \eta^2 \, dx \right). \tag{2.6}
\]

Since we have the simple estimate \( |u(x)| \leq C \|\nabla u\|_2 \omega(x) \) for all radial
functions with finite energy, we can estimate the left side of (2.6) from
below by sup \( CG(u)^2 \eta^2 \omega^{-2} \).
The last term in (2.6) can be estimated as

\[ C\beta^2 \int \mathcal{F}(x, u) u^{-1} G(u)^2 \eta^2 \, dx \]

\[ \leq C\beta^2 \left( \sup G(u)^2 \eta^2 \omega^{-2} \right) \int_{\{\text{supp } \eta\}} \mathcal{F}(x, C\omega(x)) \omega(x) \, dx. \]

If we let the support of \( \eta \) be small, we can make the integral on the right side small so that we can bring it over on the left side in estimate (2.6) to obtain

\[ G(u)^2 \eta^2 \omega^{-2} \leq C\beta^2 \int |\nabla \eta|^2 G(u)^2 \, dx \]

and thus if we let \( N \to \infty \) in \( G \) we get

\[ u^{2\beta}(x) \eta^2 \leq C\beta^2 \int |x|^{2-n} \int u^{2\beta} |\nabla \eta|^2 \, dx. \]

Using the a priori estimate \( u(x) \leq C |x|^{(2-n)/2} \) we see that we can iterate the inequality above \( N \) times with \( \beta = \delta^m, \ m = 1, 2, \ldots, N, \) where \( \delta \in (1, n/(n-2)) \), to find that

\[ u(x)^{\delta^N} \leq C |x|^{(2-n)/2}, \]

in a neighborhood of the origin. This shows the first part of the lemma. Note that we can not continue this procedure indefinitely, since in each step we need to make the support of \( \eta \) smaller.

To get the second part of the lemma we integrate the differential inequality to obtain for \( 0 < r < r_0 \)

\[ u(r) \leq u(r_0) + \int_{r}^{r_0} \frac{1}{s^{n-1}} \int_0^s (f(t, u(t)) + h(t)) t^{n-1} \, dt \, ds. \]

Here we used the fact that there is a sequence \( s_k \) such that \( s_k \searrow 0 \), and \( u'(s_k)s_k^{n-1} \to 0 \), as \( k \to \infty \). This follows since \( u \) has finite energy.

Now we can change the order of integration to find that

\[ u(r) \leq u(r_0) + \int_{r}^{r_0} \frac{1}{n-2} \int_0^s t^{2-n} (f(t, u(t)) + h(t)) t^{n-1} \, dt \]

\[ \leq u(r_0) + \int_{r}^{r_0} t^{2-n} (f(t, u(t)) + h(t)) t^{n-1} \, dt. \]
From the above, this inequality, and the assumptions, we find that $u$ is bounded in a neighborhood of the origin. □

**Proof of Theorems 1 and 2 and Corollary 3.** Assume that $u$ has locally finite energy in a neighborhood of infinity and satisfies the differential inequality $\Delta u + f(x, u) \geq 0$ for large $x$. Then if $v$ is the Kelvin transform $v(x) = |x|^{2-n} u(x/|x|^2)$, it has locally finite energy in a neighborhood of the origin (Lemma 8), and satisfies the differential inequality $\Delta v + |x|^{-n-2} f(x/|x|^2, |x|^{n-2} v) \geq 0$ in a neighborhood of the origin. The fact that this holds also at the origin follows from the following simple fact. If $\varphi \in C_0^\infty(\mathbb{R}^n)$ is non-negative and $n > 2$, then we can find a sequence of non-negative functions $(\varphi_k \in \mathcal{D}_0^{1,2}(\mathbb{R}^n))$, whose support does not contain the origin and such that $\varphi_k \to \varphi$ in $\mathcal{D}_0^{1,2}(\mathbb{R}^n)$. In fact it is well known that this holds also if $n = 2$; however, in this case the construction of the sequence $\varphi_k$ is a little bit more delicate.

It is easy to see that the conditions in Theorems 1 and 2 transform into those in Lemmas 10 and 9 and thus it follows that $u(x) \leq C |x|^{2-n}$ for large $x$.

The fact that $u(x) \geq C |x|^{2-n}$, if $\Delta u \leq 0$ and $u \not\equiv 0$, is well known. For example, it follows from Proposition 11 below. See also [LN1] for a different and more general argument.

Finally, the fact that $|x|^{n-2} u(x)$ has a limit as $x \to \infty$ follows since the Kelvin transform of $u$ and $\Delta v \in L^q$ in a neighborhood of the origin, for some $q > n/2$. Now standard regularity theory shows that $v$ is Hölder continuous at the origin and the claim follows.

To get the second term in the expansion in Corollary 3, we just calculate the Hölder exponent. For part (i) in the corollary we use the imbedding from Morrey spaces rather than standard Sobolev spaces (cf. [GT, Sect. 7.9]). Part (ii) follows using the standard Sobolev spaces. □

### 3. PROOF OF THEOREM 5 AND SOME REMARKS

Having proved Theorems 1 and 2 we are now able to prove Theorem 5 in a simple and standard way.

**Proof of Theorem 5.** First we consider only the case when $n \geq 3$. We note that Theorem 2 yields $u(x) \leq C |x|^{2-n}$, as $x \to \infty$. The rest of the proof follows from a well known comparison argument (cf. [KA, Sects. 5 and 6] or [GNN, Proposition 4.1]).

If we take $R$ large enough it follows that $\Delta u \geq (\lambda/2) u$ if $|x| \geq R$. We know that there is a positive radial function that decays exponentially fast and satisfies $\Delta v = (\lambda/2) v$ in $|x| > R$. Thus the function $w = v - u$ is positive if $|x| = R$, tends to zero as $x \to \infty$, and satisfies $\Delta w \leq (\lambda/2) w$. 

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Thus an application of the maximum principle shows that \( w \geq 0 \). This proves the claim.

The result that \( v \) decays exponentially fast is well known (see, for example, (2.19) in [BL]). In fact it follows that \( u(x) |x|^{(n-1)/2} \exp(\sqrt{\lambda/2} |x|) \to C > 0 \), as \( x \to \infty \). This more precise result can be used to get more information about the asymptotics of \( u \).

Next we will prove the result when \( n = 2 \). To apply the second argument above we only needed to show that \( u(x) \to 0 \), as \( x \to \infty \) (the stronger conclusion in Theorem 2 was not used). For this we can use a simple argument from [EG2, Theorem 11].

We know that \( -\Delta u \leq cu^2 \) holds in a neighborhood of infinity, for some \( \alpha \in (1, \infty) \).

Thus from standard elliptic estimates we find that

\[
\sup_{x \in B(y)} u(x) \leq C \|u\|_{L^\infty(B_2(y))}
\]

holds for large \( y \), where \( C \) depends only on the energy of \( u \) in a neighborhood of infinity and \( \alpha \). This estimate can be obtained using the same argument as in the proof of Theorem 8.17 in [GT]. It was assumed that \( u \) is \( L^2 \) integrable in a neighborhood of infinity. Furthermore, we know that \( u \) is bounded in a neighborhood of infinity (this fact was pointed out after Theorem 2 in Section 1). Hence combining the last two facts with (3.1), we find that \( u \to 0 \) as \( x \to \infty \). This proves the claim when \( n = 2 \).

Note that the last argument works also for \( n \geq 3 \) above, if we have subcritical growth. In fact, the argument can be modified to include also the limiting case.

The proof above can be extended to more general situations when the nonlinearity \( -\Delta u + f(x,u) \) is replaced by \( g(x,u) \). Now the assumption is that \( g(x,s) \) is negative for small positive \( s \) and positive for large \( s \). In this case we can replace the comparison argument above with that in [MC]. However, we will not pursue this idea any further.

Another question one might ask is the following: Assume that \( u \) is a non-negative and non-trivial function with locally finite energy in a neighborhood of infinity and satisfies \( \Delta u \leq bu \), for large \( x \). When does it follow that \( u(x) \geq C \frac{1}{|x|^{2-n}} \), for large \( x \)? From the above we conclude that we cannot have \( b \equiv 1 \), but if \( b \equiv 0 \) then it is true. The following is a slight generalisation of this observation.

**Proposition 11.** Assume that \( u \) is non-negative and has finite energy in a neighborhood of infinity and that \( \Delta u \leq bu \), where \( b \frac{|x|^a \in L^q(|x|^{-2n} dx) \) for some \( q > n/2 \). Then either there is a constant \( C \) such that \( u \geq C \frac{1}{|x|^{2-n}} \) for large \( x \) or \( u \equiv 0 \) in a neighborhood of infinity.
To see that the conditions are near the best possible we can argue as in Example 4. Another simple "radial" example is the following.

If we take \( b(x) = A |x|^2 \), then \( u(x) = |x|^\alpha \) with \( \alpha = (1/2)(2 - n - \sqrt{(n - 2)^2 + 4A}) \) satisfies \( Au = bu \).

Proof. If \( v \) is the Kelvin transform of \( u \), then \( \Delta v \leq |x|^{-4} b(x/|x|^2)v \). Now the claim follows from [GT, Theorem 8.18], provided \( |x|^{-4} b(x/|x|^2) \in L^p \) holds for some \( p > n/2 \), locally in a neighborhood of the origin (in [GT], it is assumed that the coefficient is bounded, but the proof in this case is exactly the same). In fact the maximum principle in [GT] shows that either \( v \) is identically zero or strictly positive in a neighborhood of the origin.

Thus the result follows if we transform back to the original setting.

4. RESULTS FOR THE \( m \)-LAPLACIAN

The reason we can work with the more general \( m \)-Laplacian here is that we do not need the Kelvin transform.

Proof of Theorem 6. The proof is almost the same as that of Lemma 10. However, since we are considering the \( m \)-Laplacian we need to make some modifications. Define a \( C^1(R) \) function by \( G(s) = s^\beta \) if \( 0 \leq s \leq N \), linear if \( N < s \), and zero otherwise \( (\beta > 1) \). Put \( F(u) = \int_0^N |G'(s)|^m ds \). Now we have the following relations: \( s^m - 1 F(s) \leq s^m G'(s)^m \leq \beta^m G(s)^m \).

Let \( \eta \) be a non-negative smooth radial function that is zero if \( x \) is small and identically one if \( x \) is large.

If we apply \( F(u)\eta^m \) as a test function in the differential inequality and argue as in the proof of Lemma 9, with the obvious modifications, we obtain

\[
\|\nabla(G(u)\eta)\|_m^m \leq C\beta^m \left( \int |\nabla \eta|^m G(u)^m dx + \int \mathcal{F}(x, u) u^{1-m}G(u)^m \eta^m dx \right). \tag{4.1}
\]

Since \( n < m \), Sobolev's inequality yields \( |v(x)| \leq C \|\nabla v\|_m \omega_m(x) \). Thus the left side in (4.1) can be estimated from below by \( C(G(u)\eta \omega_m)^m \).

The second term on the right side in (4.1) can be estimated as

\[
\int \mathcal{F}(x, u) u^{1-m} \omega_m^m \left( \frac{G(u)\eta}{\omega_m} \right)^m dx \leq C \sup \left( \frac{G(u)\eta}{\omega_m} \right)^m \int_{\{\text{supp} \eta\}} \mathcal{F}(x, \omega_m) \omega_m dx.
\]
If we make the support of $\eta$ small enough, we can bring this term over to the left side in (4.1) and arrive at

$$(G(u) \eta \omega_m^{-1})^m \leq C \beta^m \int |\nabla \eta|^m G(u)^m \, dx.$$ 

Finally letting $N \to \infty$ in $G$, yields

$$u^\beta \eta \leq \omega_m C \beta \left( \int |\nabla \eta|^m \omega_m^\beta \, dx \right)^{1/m}.$$ 

Since this holds for any $\beta > 1$, it follows that $u \omega_m^{-\delta}$ is bounded for any $\delta > 0$. Note that we need to make the support of $\eta$ smaller as $\beta$ increases.

To finish the proof note that the last term in (4.1) can be estimated by

$$C \beta^m \left( \frac{G(u) \eta}{\omega_m^{\beta}} \right)^m \int_{\{\text{supp}\eta\}} f(x, C \omega_m^{\delta}) \omega_m^{\delta(1-m)+\rho m} \, dx.$$ 

Here $\delta > 0$, $\rho > 1$ are the numbers given in condition (iii) in the theorem.

Take $\eta$ so that it is one if $|x| > r_2$, zero if $|x| < r_1$, and so that $|\nabla \eta| \leq 2/(r_2 - r_1)$. If we let $N \to \infty$ in $G$, then after some simple estimates (4.1) becomes

$$\sup_{|x| > r_2} (u \omega_m^{-1/\beta}) \leq \left( \frac{C \beta}{r_2 - r_1} \right)^{1/\beta} \sup_{|x| > r_1} (u \omega_m^{-\rho/\beta}).$$

This inequality can be iterated exactly as inequality (2.5) in the proof of Lemma 9. Thus for some positive $r_0$, $\sup_{|x| > r_0} (u \omega_m^{-1/\beta})$ stays bounded as $\beta \to \infty$. This proves the claim in Theorem 6. 

5. SOME COMMENTS, EXAMPLES, AND CONCLUDING REMARKS

In this section we will start with a remark on uniform estimates of the decay in terms of the local energy near infinity. Then a number of examples are given. They show that different conditions in the theorems in the present paper are best possible or near the best possible.

In what follows we will say that $u$ has fast decay if $u(x) = O(|x|^{-n})$, as $x \to \infty$ and slow decay otherwise.

**Uniform Estimates.** Under the assumptions in Theorems 1 and 2 the conclusion was that $u(x) < C |x|^{2-n}$. When can we conclude that the only dependence $C$ has on $u$ is on the energy of $u$ in a neighborhood of infinity, i.e., $C = C(n, f, \|\nabla (u \eta)\|_2)$, where $\eta \equiv 1$ in a neighborhood of infinity?
Going through the proof it turns out that that we have such an estimate in all cases except Theorem 2(a). Example 2 below shows that under the assumptions in Theorem 2(a), we do not always get a uniform estimate.

**Example 1.** Consider the equation

\[ \Delta u + |x|^v u^p = 0, \]  
(5.1)

where we assume that \( p > 1 \) and \( v > -2 \). There is a solution of the form \( u(x) = C |x|^\alpha \), with \( \alpha = -(v + 2)/(p - 1) \), provided \( \alpha \geq 2 - n \). This solution has slow decay if \( \alpha > 2 - n \), i.e., if \( p > (n + 2 + 2v)/(n - 2) - (v + 2)(n - 2) \). Furthermore, our solution has finite energy in a neighborhood of infinity if \( \alpha < (1/2)(2 - n) \), which is the same as \( p < (n + 2 + 2v)/(n - 2) \).

**Conclusion (i).** If \( v > -2 \) and \( p \geq (n + 2 + 2v)/(n - 2) \), then there is a function \( u \) satisfying (5.1) with slow decay, and infinite energy in a neighborhood of infinity.

**Conclusion (ii).** If \( v > -2 \) and \( (n + 2 + 2v)/(n - 2) - (v + 2)/(n - 2) < p < (n + 2 + 2v)/(n - 2) \), then there is a function \( u \) satisfying (5.1) with slow decay, and finite energy in a neighborhood of infinity.

**Example 2.** In Corollary 3 we gave the second term in the asymptotic expansion at infinity. We will see that this result is the best possible.

Equation (5.1) with \( v > -2 \) and \( p = (n + 2 + 2v)/(n - 2) \) has a one parameter family of solutions

\[ u_\lambda(x) = C_{n,v} \lambda^{1/(p - 1)}(1 + \lambda |x|^{v + 1})^{(2 - n)/(v + 2)}, \]

\[ = \lambda^{-(n - 2)/(2 + v)}C_{n,v} |x|^{2 - n} + C_{n,v} |x|^{-n - v} + O(|x|^{-n - 2 - 2v}), \]

where \( \lambda \) is a positive parameter.

**Conclusion (i).** This shows that the expansion in Corollary 3 is best possible in the radial case. Translating the problem above to an artificially nonradial situation shows that the result in this case is also best possible.

**Conclusion (ii).** Since the energy of \( u_\lambda \) is independent of \( \lambda \), we cannot obtain a uniform estimate as described in the beginning of this section.

**Example 3.** The function \( u(x) = |x|^{2 - n} \log(|x|) \) has finite energy in a neighborhood of infinity and satisfies

\[ \Delta u + b(x)u = 0, \]  
(5.2)

with \( b(x) = (n - 2) |x|^{-2}/\log(|x|) \).
Conclusion (i). It is not enough to have \( b(x) = O(|x|^{-2}(\log |x|)^{-1}) = o(|x|^{-2}) \), or \( b \in L'^{1/2} \) in a neighborhood of infinity to conclude that a solution of (5.2) has fast decay. Thus neither condition (a) nor (b) in Theorem 2 extends to the linear situation.

Conclusion (ii). Even in the radial case in Theorem 2, it is not enough to have \( h(x) |x|^n \) and \( b_0(x) |x|^4 \) in \( L^{n/2}(|x|^{-2n} \, dx) \) in a neighborhood of infinity.

Example 4. In this example we will discuss the conditions \( h(x) |x|^{n+2} \), \( b_0(x) |x|^4 \in L^q(|x|^{-2n} \, dx) \) for some \( q > n/2 \), in Theorem 2. The condition looks strange and a little bit artificial. However, in the previous example we saw that the conclusion in Theorem 2 does not hold if we allow \( q = n/2 \). Below we give a stronger example.

Put \( h(x) = \sum_{m=1}^\infty c_m s(x - x_m) \), where \( s(x) = \chi_{|x| < 1/2} |x|^2 \log |x| \), \( c_m > 0 \), \( c_m \to 0 \) fast, and \( x_m \to \infty \), as \( m \to \infty \). Then \( u = (-\Delta)^{-1} h \) has finite energy, \( u(x_k) \geq C \int c_k s(y) |y|^{2-n} \, dy = \infty \). We can choose \( c_m \) decaying fast so that for any given \( \alpha > 0 \), \( h(x) |x|^n \in L^{\alpha} \).

Taking \( b_0 = h/u \) yields a similar conclusion for the linear term \( b_0 u \).

Conclusion (i). The condition that \( h \) or \( b \) is locally in \( L^q \) for some \( q > n/2 \) is necessary to get any control of the asymptotics.

Now take \( h(x) = \sum_{m=1}^\infty \chi_{|x| < r_m} (x - x_m) \), where \( r_m \to 0 \) and \( x_m \to \infty \), as \( m \to \infty \). If we take \( r_m \to 0 \) fast enough, then \( u = (-\Delta)^{-1} h \) has finite energy.

Furthermore, we get

\[
\int h(x)^q |x|^\beta \gamma \, dx \sim \sum_{m=1}^\infty |x_m|^{\beta \gamma} r_m^n
\]

and

\[
u(x_k) > c \int \chi_{|x| < r_k}(y) |y|^{2-n} \, dy \sim r_k^2.
\]

Thus if we can find \( \beta \) and \( \gamma \) such that \( \beta > 0 \) and \( n(n-2) - n\beta + 2\gamma > 0 \), then we get \( h(x) |x|^\beta \in L^q(|x|^{-\gamma} \, dx) \), some \( q > n/2 \), and \( \limsup_{x \to \infty} |x|^n u(x) = \infty \). This shows that the factors \( |x|^n \) and \( |x|^{-2n} \, dx \) cannot be relaxed in Theorem 2.

Assuming that \( u(x) \sim |x|^{2-n} \), we can get similar counterexamples for \( b_0(x) \).

Conclusion (ii). The factors \( |x|^n \), \( |x|^4 \), and the weight \( |x|^{-2n} \, dx \) in the conditions for \( h \) and \( b_0 \) in Theorem 2 are best possible.
Finally let us remark that the condition on $h$ is somewhat related to condition (b) in Theorem 2. To see this, assume that $u$ satisfies (0.1) with $f(x, u) = b_1(x)u^{p_1}$, where the pair $(b_1, p_1)$ satisfies (b) with $q_1$. Then if we assume that $u \sim r^{2-n}$ and take $h = b_1u^{p_1}$ the condition on $h$, with $q = q_1 > n/2$ becomes $\int (h(x) |x|^{n+2})_{q_1} |x|^{-2n} \, dx = \int |b_1|^{q_1} \, dx < \infty$.

**Example 5.** Assume that $u$ is radial and tends to zero at infinity and satisfies $-\Delta u = g$. Integrating the equation yields

$$u(r) = \frac{u'(R) R^{n-1}}{(n-2)r^{n-2}} + \int_0^1 t^{1-n} \int_R^t s^{n-1}g(s) \, ds \, dt$$

$$= \frac{u'(R) R^{n-1}}{(n-2)r^{n-2}} + \frac{1}{n-2} \int_R^\infty (\max(r, s))^{2-n} s^{n-1}g(s) \, ds.$$

Thus if $g$ is non-negative, monotone convergence shows that a necessary and sufficient condition for $r^{n-2}u(r)$ to stay bounded as $r \to \infty$ is that $\int_R^\infty g(s)s^{n-1} \, ds < \infty$.

For a general integrable non-negative function $g$, define the radial function $\tilde{g}(r) = \int_{|x|=r} g(x) \, d\sigma$, where $d\sigma$ is the $n-1$ dimensional surface measure. Then if $-\Delta u = g$, it follows that $-\Delta \tilde{u} = \tilde{g}$. Thus by the argument above, a necessary condition for $|x|^{n-2}u(x)$ to stay bounded as $x \to \infty$ is that $\int_R^\infty \tilde{g}(s)s^{n-1} \, ds < \infty$.

**Conclusion (i).** In the radial situation $h \in L^1$ in a neighborhood of infinity is a necessary condition in Theorem 1.

**Conclusion (ii).** In general, a necessary condition on $h$ and $f$ is that $\tilde{h}(r)$ and $\tilde{f}(\cdot, |\cdot|^{2-n})(r)$ are integrable in a neighborhood of infinity.

**Example 6.** Consider $\Delta u + b(x)u^p = 0$, with $b(x) = O(|x|^\gamma)$, $0 < p < 1$, and $\gamma < -2$. Then if we use Theorem 2 and interpolate $bu^p \leq b_0u + h$, we find that the solution has fast decay if $p > (n + \gamma)/(n - 2)$. Note that this expression is different from the one for the superlinear situation. In the radial case we could have used Theorem 1 instead.

To see that this is sharp, assume that $p \leq (n + \gamma)/(n - 2)$, $b(r) = r^\gamma$, and $u(r) \sim r^{2-n}$. Then $\int_R^\infty b(s) u(s)^p s^{n-1} \, ds = +\infty$, which is a contradiction (see Conclusion (ii) in Example 5).

**Conclusion.** Theorem 1 or 2 yields sharp results also in sublinear situations.

The following example gives a partial answers to a question posed by Professor K. McLeod (personal communication).
Example 7. Let \( p > (n + 2)/(n - 2) \) and \( \varepsilon > 0 \) be given and consider \((0.1)\) with \( f(x, s) = b(x)s^p \) and \( \Omega = \mathbb{R}^n \). Then there is a smooth function \( b \) bounded between zero and one such that \((0.1)\) has a smooth solution \( u(x) \) with energy less than \( \varepsilon \). Furthermore, there are points \( \{ x_n \} \) such that \( x_n \to \infty \) and \( u(x_n) \to \infty \) as \( n \to \infty \).

Proof of Claim. Let \( b_0 \) be a smooth non-negative radial function with support in the unit ball \( B_1(0) \), and such that \( b_0 \) is zero in a neighborhood of the origin and \( b_0 \leq 1 \).

From Theorem A it follows that there is a positive radial function \( v \in \mathcal{D}_{0}^{1,2}(\Omega) \) satisfying \( \Delta v + b_0 v^p = 0 \) in \( \mathbb{R}^n \). Furthermore we know that \( v \) is smooth and that \( v(x) \leq C |x|^{2-n} \) for large \( x \).

If we put \( b_{x_0,\lambda}(x) = b_0((x-x_0)\lambda) \), then \( v_{x_0,\lambda}(x) = \lambda^{2/(p-1)}v((x-x_0)\lambda) \) satisfies

\[
\Delta v_{x_0,\lambda} + b_{x_0,\lambda} v_{x_0,\lambda}^p = 0,
\]

\[
\|\nabla v_{x_0,\lambda}\|_2 = \lambda^{-\alpha} \|\nabla v\|_2,
\]

\[
v_{x_0,\lambda}(x) \leq \lambda^{-\alpha} c_{\alpha} |x-x_0|^{2-n}, \quad \text{for } |x-x_0| \text{ large.}
\]

Here \( \alpha = ((n-2)/2)(p-(n+2)/(n-2)) > 0 \).

Pick \( \lambda_k \geq 2 \) such that it tends to infinity fast as \( k \to \infty \), and points \( x_k, \lambda_k \), such that \( x_k \to \infty \), as \( k \to \infty \) and \( |x_k - x_{k'}| > 4 \) if \( k \neq k' \).

Now if we let \( w(x) = \sum_{k=1}^{\infty} v_{x_k,\lambda_k}(x) \), it follows that \( 0 < w(x) \leq C \sum_{k=1}^{\infty} \lambda_k^{-\alpha} < \infty \), provided we chose \( \lambda_k \) properly above. Furthermore, it is easy to see that \( w \) is smooth and can be differentiated termwise two times.

Now we get

\[
0 \leq -\Delta w = \sum_{k=1}^{\infty} b_{x_k,\lambda_k} v_{x_k,\lambda_k}^p \leq w^p,
\]

where we used the fact that the different \( b_{x_k,\lambda_k} \) have disjoint support. Thus since \( w \) is strictly positive we can take \( b = -\Delta w/w^p \).

The energy of \( w \) can be estimated by

\[
\|\nabla w\|_2 \leq \sum_{k=1}^{\infty} \|\nabla v_{x_k,\lambda_k}\|_2 = \|\nabla v\|_2 \sum_{k=1}^{\infty} \lambda_k^{-\alpha}.
\]

Clearly we can choose the \( \lambda_k \) so that the left side becomes as small as we like.

Finally we note that

\[
u(x_k) \geq v_{x_k,\lambda_k}(x_k) = v(0) \lambda_k^{2/(p-1)} \to \infty, \quad \text{as } k \to \infty.
\]
Conclusion. The restriction to critical or subcritical growth in the nonradial situation is necessary.

Example 8. Consider \( \Delta_m u + |x|^\gamma u^p = 0 \), with \( p > m - 1 \), \( \nu < -m \), and \( n < m \). By Theorem 6, a solution with finite \( m \)-energy in a neighborhood of infinity is bounded at infinity if \( p + 1 < m(\nu - n)/(m - n) \).

There is a solution on the form \( u(r) = Cr^\alpha \), where \( \alpha = (\nu - m)/(p - (m - 1)) \). This solution has finite energy if \( \alpha < (m - n)/m \).

Conclusion (i). There is a solution with finite energy in a neighborhood of infinity which is unbounded at infinity if \( p + 1 > m(\nu - n)/(m - n) \).

Conclusion (ii). There is a solution that has infinite \( m \)-energy in a neighborhood of infinity and is unbounded at infinity provided \( m < p + 1 \leq m(\nu - n)/(m - n) \).

References


