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Global stability of the endemic equilibrium of multigroup SIR models with nonlinear incidence

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ABSTRACT

In this paper, we introduce a basic reproduction number for a multigroup epidemic model with nonlinear incidence. Then, we establish that global dynamics are completely determined by the basic reproduction number R_0. It shows that, the basic reproduction number R_0 is a global threshold parameter in the sense that if it is less than or equal to one, the disease free equilibrium is globally stable and the disease dies out; whereas if it is larger than one, there is a unique endemic equilibrium which is globally stable and thus the disease persists in the population. Finally, a numerical example is also included to illustrate the effectiveness of the proposed result.

1. Introduction

Multigroup models have been introduced in the literature to describe the transmission dynamics of infectious diseases in heterogeneous host populations, such as measles, mumps, gonorrhea, HIV/AIDS, West-Nile virus and vector borne diseases such as Malaria. Many factors can lead to heterogeneity in a host population. Groups can be divided geographically into communities, cities, and countries, or epidemiologically, to incorporate differential infectivity or co-infection of multiple strains of the disease agent. The seminal paper by Lajmanovich and Yorke [1] on a class of SIS multigroup models for the transmission dynamics of Gonorrhea is one of the earliest works on multigroup models. In that paper, a complete analysis of the global dynamics is established, and the proof of the global stability of the unique endemic equilibrium using a global Lyapunov function is given. Much research has been done on multigroup models in recent years as well, see, for example, [2–9] and references therein. It is well known that the global dynamics of multigroup models in high dimensions, especially the global stability of the endemic equilibrium, is a very challenging problem. The question of uniqueness and global stability of the endemic equilibrium, when the basic reproduction number R_0 is greater than one, has largely been open in most cases.

Recently a graph–theoretic approach to the method of global Lyapunov functions was proposed in [10–12] and it was used to establish the global stability of a unique endemic equilibrium of a multigroup SEIR model described by a system of ordinary differential equations. Their results completely solve the open problem on the uniqueness and global stability of endemic equilibrium for this class of multi-group models. By using the results or ideas of the paper [10], the uniqueness and global stability of the endemic equilibrium for several classes of multigroup epidemic models were investigated in [11–15], when the basic reproduction number R_0 is greater than 1, and some previously open problems were resolved.

In general, a multigroup model is formulated by dividing the population of size $N(t)$ into n distinct groups. For $1 \le k \le n$, the k-th group is further partitioned into three compartments: the susceptible, infectious, and recovered, whose numbers of individuals at time t are denoted by $S_k(t)$, $I_k(t)$ and $R_k(t)$, respectively. The nonlinear term $\beta_{kj} f_{kj}(S_k, I_j)$ represents the cross infection from group j to group k. The influx of individuals into the k-th group is given by a constant Λ_k, of which a fraction p_k is assumed to be immune, and the remaining fraction $1 - p_k$ is susceptible. A simple immunization policy is considered

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where a fraction θ_k of the compartment S_k is vaccinated. The matrix $B = (\beta_{ij})_{n\times n}$ is an irreducible contact matrix, where $\beta_{ij} \geq 0$. Within the k-th group, it is assumed that natural death occurs in S_k, I_k and R_k compartments with rate constants d_k^S, d_k^I and d_k^R, respectively. Individuals in I_k have an additional mortality due to the disease with a constant rate ϵ_k. We assume that individuals in I_k recover with a constant rate γ_k, and once recovered they remain permanently immune to the disease. In addition δ_k is the recovery rate of infected individuals in the k-th group R_k. Based on these assumptions, a general multigroup epidemic model with nonlinear incidence is described by the following system of differential equations:

$$\begin{cases} S_k' = (1-p_k)\Lambda_k - (d_k^S + \theta_k)S_k - \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j), \\ I_k' = \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j) - (d_k^I + \epsilon_k + \gamma_k)I_k, \\ R_k' = p_k\Lambda_k + \theta_k S_k + \gamma_k I_k - (d_k^R + \delta_k)R_k \end{cases} \tag{1}$$

for $1 \leq k \leq n$. The model in this form covers many previous ones in the literature, for example, [10,16].

In this paper, we consider the global dynamic behavior of the general multigroup SIR model (1). It is shown that the basic reproduction number R_0 (defined in Section 2) is a global threshold value in the sense that if it is less than or equal to one, the disease free equilibrium is globally asymptotically stable and the disease dies out; whereas if it is larger than one, there is a unique endemic equilibrium which is globally asymptotically stable and thus the disease persists in the population. These main results are proved in Section 2. Finally, a numerical example and simulation is also included in Section 3 to illustrate the effectiveness of the proposed result.

2. Main results

For each k, adding the three equations in (1), we obtain

$$(S_k + I_k + R_k)' = \Lambda_k - d_k^S S_k - (d_k^I + \epsilon_k)I_k - (d_k^R + \delta_k)R_k \leq \Lambda_k - d_k^*(S_k + I_k + R_k),$$

where $d_k^* = \min\{d_k^S, d_k^I + \epsilon_k, d_k^R + \delta_k\}$, then

$$\limsup_{t\to\infty}(S_k + I_k + R_k) \leq \frac{\Lambda_k}{d_k^*}.$$

Similarly it follows from the first equation in (1) that

$$\limsup_{t\to\infty} S_k \leq \frac{(1-p_k)\Lambda_k}{d_k^S + \theta_k}.$$

We assume the basic assumptions on functions $f_{ij}(S_i, I_j)$ as follows:

(H1) Define $C_{ij}(S_i) = \lim_{I_j\to 0^+} \frac{f_{ij}(S_i, I_j)}{I_j}$. Then for all $0 < S_i \leq S_i^0$, $0 < C_{ij}(S_i) \leq \infty$, where $S_i^0 = \frac{(1-p_i)\Lambda_i}{d_i^S + \theta_i}$;
(H2) $f_{ij}(S_i, I_j) \leq C_{ij}(S_i)I_j$ for all $I_j > 0$ and $0 < S_i \leq S_i^0$;
(H3) $C_{ij}(S_i) < C_{ij}(S_i^0)$, for all $0 < S_i \leq S_i^0$.

Note that the class of $f_{ij}(S_i, I_j)$ satisfying (H1)–(H3) include many common incidence functionals such as $f_{ij}(S_i, I_j) = S_i I_j$ [10, 16], $f_{ij}(S_i, I_j) = \frac{\eta S_i I_j}{1+\theta S_i}$ [2], $f_{ij}(S_i, I_j) = \frac{k S_i I_j}{1+\alpha I_j^2}$ [9], $f_{ij}(S_i, I_j) = g(S_i)h(I_j)$ [13], $f_{ij}(S_i, I_j) = S_i g_j(I_j)$ [17], $f_{ij}(S_i, I_j) = S_i^q I_j$ [18], $f_{ij}(S_i, I_j) = \frac{\beta S_i I_j}{\phi(I_j)}$ [19].

Since the variables R_k do not appear in the first two equations of (1), we can work on the reduced system as follows:

$$\begin{cases} S_k' = (1-p_k)\Lambda_k - (d_k^S + \theta_k)S_k - \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j), \\ I_k' = \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j) - (d_k^I + \epsilon_k + \gamma_k)I_k, \end{cases} \tag{2}$$

where $k = 1, 2, \ldots, n$, in the feasible region

$$\Gamma = \left\{(S_1, I_1, S_2, I_2, \ldots, S_n, I_n) \in \mathbb{R}_+^{2n} : S_k \leq \frac{(1-p_k)\Lambda_k}{d_k^S + \theta_k}, S_k + I_k \leq \frac{\Lambda_k}{d_k^*}, k = 1, 2, \ldots, n\right\}. \tag{3}$$

It can be verified that Γ in (3) is positively invariant with respect to (2). In addition, the behavior of R_k can then be determined from the last equation in (1). Also let Γ° denote the interior of Γ.

It is clear that $P_0 = (S_1^0, 0, S_2^0, 0, \ldots, S_n^0, 0)$ is a disease-free equilibrium of the system (2), where $S_i^0 = \frac{(1-p_i)\Lambda_i}{d_i^S + \theta_i}$. An equilibrium $P^* = (S_1^*, I_1^*, S_2^*, I_2^*, \ldots, S_n^*, I_n^*)$ in the interior Γ° of Γ is called an endemic equilibrium, where S_i^*, I_i^* satisfy the following equilibrium equations:

$$(1-p_k)\Lambda_k = (d_k^S+\theta_k)S_k^* + \sum_{j=1}^{n}\beta_{kj}f_{kj}(S_k^*, I_j^*), \tag{4}$$

$$(d_k^I+\epsilon_k+\gamma_k)I_k^* = \sum_{j=1}^{n}\beta_{kj}f_{kj}(S_k^*, I_j^*). \tag{5}$$

Set $R_0 = \rho(M_0)$ to be the spectral radius of the following matrix

$$M_0 = M(S_1^0, S_2^0, \ldots, S_n^0) = \left(\frac{\beta_{ij}C_{ij}(S_i^0)}{d_i^I+\epsilon_i+\gamma_i}\right)_{n\times n}.$$

In case that $C_{ij}(S_i^0) = \infty$ for some i, j, we set $R_0 = \infty$. In the epidemic literature R_0 is referred to as the basic reproduction number, and our definition here is consistent with the standard ones in [20,21].

We have the following result regarding the global stability of the disease-free equilibrium:

Theorem 2.1. *Assume that $B = (\beta_{ij})$ is irreducible and* (H1)–(H3) *hold.*

1. *If $R_0 \le 1$, then P_0 is the unique equilibrium of the system* (2) *and it is globally stable in Γ.*
2. *If $R_0 > 1$, then P_0 is unstable and the system* (2) *is uniformly persistent in Γ°.*

Proof. Similar to the proof of Proposition 3.1 of [10]. Since B is irreducible, then M_0 is also irreducible. From the well-known Perron–Frobenius Theorem, M_0 has a positive principal eigenvector $w = (w_1, w_2, \ldots, w_n)$ such that $w_k > 0$, $k = 1, 2, \ldots, n$, and $w\cdot\rho(M_0) = w\cdot M_0$.

We define a Lyapunov function $V = \sum_{k=1}^{n}\frac{w_k}{d_k^I+\epsilon_k+\gamma_k}I_k$. Then we have

$$\begin{aligned}
V' &= \sum_{k=1}^{n}\frac{w_k}{d_k^I+\epsilon_k+\gamma_k}I_k' \\
&= \sum_{k=1}^{n}\frac{w_k}{d_k^I+\epsilon_k+\gamma_k}\left[\sum_{j=1}^{n}\beta_{kj}f_{kj}(S_k, I_j) - (d_k^I+\epsilon_k+\gamma_k)I_k\right] \\
&\le \sum_{k=1}^{n}\frac{w_k}{d_k^I+\epsilon_k+\gamma_k}\left[\sum_{j=1}^{n}\beta_{kj}C_{kj}(S_k)I_j - (d_k^I+\epsilon_k+\gamma_k)I_k\right] \\
&\le \sum_{k=1}^{n}\frac{w_k}{d_k^I+\epsilon_k+\gamma_k}\left[\sum_{j=1}^{n}\beta_{kj}C_{kj}(S_k^0)I_j - (d_k^I+\epsilon_k+\gamma_k)I_k\right] \\
&= w\cdot(M_0I - I) = [\rho(M_0)-1]w\cdot I \\
&\le 0, \quad \text{if } \rho(M_0)\le 1.
\end{aligned}$$

Here $I = diag(I_1, I_2, \ldots, I_n)$. If $\rho(M_0) < 1$, then $V' = 0$ if and only if $I = 0$. If $\rho(M_0) = 1$, then $V' = 0$ implies

$$\sum_{k=1}^{n}\frac{w_k}{d_k^I+\epsilon_k+\gamma_k}\left[\sum_{j=1}^{n}\beta_{kj}C_{kj}(S_k)I_j\right] = \sum_{k=1}^{n}w_kI_k. \tag{6}$$

If at least for one $k = 1, 2, \ldots, n$, $S_k \ne S_k^0$, then

$$\begin{aligned}
\sum_{k=1}^{n}\frac{w_k}{d_k^I+\epsilon_k+\gamma_k}\left[\sum_{j=1}^{n}\beta_{kj}C_{kj}(S_k)I_j\right] &< \sum_{k=1}^{n}\frac{w_k}{d_k^I+\epsilon_k+\gamma_k}\left[\sum_{j=1}^{n}\beta_{kj}C_{kj}(S_k^0)I_j\right] \\
&= w\cdot M_0I = w\cdot\rho(M_0)I = w\cdot I,
\end{aligned}$$

which implies that (6) has only the trivial solution $I = 0$. Therefore, $V' = 0$ if and only if $I = 0$ or $S_k = S_k^0$ for all $1\le k\le n$ provided that $\rho(M_0)\le 1$. It can be verified that the only compact invariant subset of the set where $V' = 0$ is the singleton $\{P_0\}$. Hence by LaSalle's Invariance Principle [22], P_0 is globally asymptotically stable in Γ if $\rho(M_0)\le 1$.

If $R_0 = \rho(M_0) > 1$ and $I \ne 0$, then

$$w\cdot M_0 - w = [\rho(M_0)-1]\cdot w > 0,$$

and thus by continuity,

$$\sum_{k=1}^{n}\frac{w_k}{d_k^I+\epsilon_k+\gamma_k}\left[\sum_{j=1}^{n}\beta_{kj}f_{kj}(S_k, I_j) - (d_k^I+\epsilon_k+\gamma_k)I_k\right] > 0$$

in a neighborhood of P_0 in Γ°. This implies that P_0 is unstable. With a uniform persistence result from [23] and a similar argument as in the proof of Proposition 3.3 of [5], the instability of P_0 implies the uniform persistence of the system (2) when $R_0 > 1$. This completes the proof of Theorem 2.1. □

Now we show that the endemic equilibrium P_0 of the system (2) is unique and globally asymptotically stable when $R_0 > 1$. Note that the system (2) is uniformly persistent if $R_0 > 1$ from Theorem 2.1, together with the uniform boundedness of the solution of (2) in Γ°, then the system (2) admits at least one endemic equilibrium

$$P^* = (S_1^*, I_1^*, S_2^*, I_2^*, \ldots, S_n^*, I_n^*), \quad S_i^* > 0,\ I_i^* > 0,\ \text{for } 1 \le i \le n.$$

The global stability result about P^* is as follow:

Theorem 2.2. *Assume that $B = (\beta_{ij})$ is irreducible and (H1)–(H3) hold. If $R_0 > 1$, P^* is an arbitrary endemic equilibrium, and $f_{kj}(S_k, I_j)$ satisfies the following conditions: for all $S_k \ne S_k^*$, $1 \le k \le n$,*

$$(S_k - S_k^*)[f_{kk}(S_k, I_k^*) - f_{kk}(S_k^*, I_k^*)] > 0, \tag{7}$$

and for all $S_k, I_j > 0$, $1 \le k, j \le n$,

$$(f_{kk}(S_k^*, I_k^*)f_{kj}(S_k, I_j) - f_{kj}(S_k^*, I_j^*)f_{kk}(S_k, I_k^*)) \cdot \left(\frac{f_{kk}(S_k^*, I_k^*)f_{kj}(S_k, I_j)}{I_j} - \frac{f_{kj}(S_k^*, I_j^*)f_{kk}(S_k, I_k^*)}{I_j^*} \right) \le 0, \tag{8}$$

then there exists a unique endemic equilibrium P^* for the system (2), and P^* is globally asymptotically stable in Γ°.

Proof. We prove that P^* is globally asymptotically stable in Γ°, which implies that the endemic equilibrium is unique. Let

$$V_k = \int_{S_k^*}^{S_k} \frac{f_{kk}(\xi, I_k^*) - f_{kk}(S_k^*, I_k^*)}{f_{kk}(\xi, I_k^*)} \mathrm{d}\xi + (I_k - I_k^* \ln I_k).$$

Then by using the equilibrium equations (4) and (5), one obtains

$$\begin{aligned} V_k' &= \left(1 - \frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k, I_k^*)}\right) \cdot \left[(1-p_k)\Lambda_k - (d_k^S + \theta_k)S_k - \sum_{j=1}^n \beta_{kj} f_{kj}(S_k, I_j)\right] \\ &\quad + \left(1 - \frac{I_k^*}{I_k}\right) \cdot \left[\sum_{j=1}^n \beta_{kj} f_{kj}(S_k, I_j) - (d_k^I + \epsilon_k + \gamma_k)I_k\right] \\ &= \left(1 - \frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k, I_k^*)}\right) \cdot \left[(d_k^S + \theta_k)S_k^* + \sum_{j=1}^n \beta_{kj} f_{kj}(S_k^*, I_j^*) - (d_k^S + \theta_k)S_k - \sum_{j=1}^n \beta_{kj} f_{kj}(S_k, I_j)\right] \\ &\quad + \left(1 - \frac{I_k^*}{I_k}\right) \cdot \sum_{j=1}^n \left[\beta_{kj} f_{kj}(S_k, I_j) - \beta_{kj} f_{kj}(S_k^*, I_j^*)\frac{I_k}{I_k^*}\right] \\ &= -\frac{d_k^S + \theta_k}{f_{kk}(S_k, I_k^*)}(S_k - S_k^*)[f_{kk}(S_k, I_k^*) - f_{kk}(S_k^*, I_k^*)] \\ &\quad + \sum_{j=1}^n \beta_{kj} f_{kj}(S_k^*, I_j^*) \left[2 - \frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k, I_k^*)} - \frac{I_k^* f_{kj}(S_k, I_j)}{I_k f_{kj}(S_k^*, I_j^*)} - \frac{I_k}{I_k^*} + \frac{f_{kk}(S_k^*, I_k^*)f_{kj}(S_k, I_j)}{f_{kj}(S_k^*, I_j^*)f_{kk}(S_k, I_k^*)}\right]. \end{aligned}$$

Let $a_{kj} = \beta_{kj} f_{kj}(S_k^*, I_j^*)$, and

$$F_{kj}(S_k, I_k, I_j) = 2 - \frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k, I_k^*)} - \frac{I_k^* f_{kj}(S_k, I_j)}{I_k f_{kj}(S_k^*, I_j^*)} - \frac{I_k}{I_k^*} + \frac{f_{kk}(S_k^*, I_k^*)f_{kj}(S_k, I_j)}{f_{kj}(S_k^*, I_j^*)f_{kk}(S_k, I_k^*)}.$$

Then by condition (7),

$$V_k' \le \sum_{j=1}^n a_{kj} F_{kj}(S_k, I_k, I_j).$$

Let $\Phi(a) = 1 - a + \ln a$, then it is easy to verify that $\Phi(a) \le 0$ for any $a > 0$ and the equality holds only when $a = 1$. Furthermore, under condition (8),

$$
\begin{aligned}
F_{kj}(S_k, I_k, I_j) &= \Phi\left(\frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k, I_k^*)}\right) - \ln\left(\frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k, I_k^*)}\right) + \Phi\left(\frac{I_k^* f_{kj}(S_k, I_j)}{I_k f_{kj}(S_k^*, I_j^*)}\right) \\
&\quad - \ln\left(\frac{I_k^* f_{kj}(S_k, I_j)}{I_k f_{kj}(S_k^*, I_j^*)}\right) - \frac{I_k}{I_k^*} + 1 + \frac{I_j}{I_j^*} - \frac{I_j}{I_j^*}\cdot\frac{f_{kk}(S_k, I_k^*) f_{kj}(S_k^*, I_j^*)}{f_{kj}(S_k, I_j) f_{kk}(S_k^*, I_k^*)} \\
&\quad + \left(\frac{f_{kk}(S_k^*, I_k^*) f_{kj}(S_k, I_j)}{f_{kj}(S_k^*, I_j^*) f_{kk}(S_k, I_k^*)} - 1\right)\cdot\left(1 - \frac{I_j}{I_j^*}\cdot\frac{f_{kk}(S_k, I_k^*) f_{kj}(S_k^*, I_j^*)}{f_{kj}(S_k, I_j) f_{kk}(S_k^*, I_k^*)}\right) \\
&\le -\ln\left(\frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k, I_k^*)}\right) - \ln\left(\frac{I_k^* f_{kj}(S_k, I_j)}{I_k f_{kj}(S_k^*, I_j^*)}\right) - \frac{I_k}{I_k^*} + \frac{I_j}{I_j^*} \\
&\quad + \Phi\left(\frac{f_{kk}(S_k, I_k^*) f_{kj}(S_k^*, I_j^*) I_j}{f_{kj}(S_k, I_j) f_{kk}(S_k^*, I_k^*) I_j^*}\right) - \ln\left(\frac{f_{kk}(S_k, I_k^*) f_{kj}(S_k^*, I_j^*) I_j}{f_{kj}(S_k, I_j) f_{kk}(S_k^*, I_k^*) I_j^*}\right) \\
&\quad + \left(\frac{f_{kk}(S_k^*, I_k^*) f_{kj}(S_k, I_j)}{f_{kj}(S_k^*, I_j^*) f_{kk}(S_k, I_k^*)} - 1\right)\cdot\left(1 - \frac{I_j}{I_j^*}\cdot\frac{f_{kk}(S_k, I_k^*) f_{kj}(S_k^*, I_j^*)}{f_{kj}(S_k, I_j) f_{kk}(S_k^*, I_k^*)}\right) \\
&\le -\ln\left(\frac{I_k^*}{I_k}\cdot\frac{I_j}{I_j^*}\right) - \frac{I_k}{I_k^*} + \frac{I_j}{I_j^*} = \left(-\frac{I_k}{I_k^*} + \ln\frac{I_k}{I_k^*}\right) - \left(-\frac{I_j}{I_j^*} + \ln\frac{I_j}{I_j^*}\right).
\end{aligned}
$$

Taking $G_k(I_k) = -\frac{I_k}{I_k^*} + \ln\frac{I_k}{I_k^*}$, then we can show that V_k, F_{kj}, G_k, a_{kj} satisfy the assumptions of Theorem 3.1 and Corollary 3.3 in [12]. Therefore, the function $V = \sum_{k=1}^n c_k V_k$ as defined in the Theorem 3.1 of [12] is a Lyapunov function for the system (2), namely, $V' \le 0$ for all $(S_1, I_1, S_2, I_2, \ldots, S_n, I_n) \in \Gamma$. One can only show that the largest invariant subset where $V' = 0$ is the singleton P^* using the same argument as in [11,12]. By LaSalle's Invariance Principle, P^* is globally asymptotically stable in Γ°. This completes the proof of Theorem 2.2. □

We remark that Lyapunov functions for similar models have been used in [11,13] and others. Here we take advantage of the new general result proved in [12] to prove the global stability of endemic equilibrium for a more general class of multigroup SIR models.

3. A numerical example

Consider the system (2) when $k = 2$, one has the two-group model as follows:

$$
\begin{cases}
S_1' = (1 - p_1)\Lambda_1 - (d_1^S + \theta_1)S_1 - \left[\beta_{11}\dfrac{S_1 I_1}{1 + I_1^2} + \beta_{12}\dfrac{S_1 I_2}{1 + I_2^2}\right], \\
I_1' = \left[\beta_{11}\dfrac{S_1 I_1}{1 + I_1^2} + \beta_{12}\dfrac{S_1 I_2}{1 + I_2^2}\right] - (d_1^I + \epsilon_1 + \gamma_1)I_1, \\
S_2' = (1 - p_2)\Lambda_2 - (d_2^S + \theta_2)S_2 - \left[\beta_{21}\dfrac{S_2 I_1}{1 + I_1^2} + \beta_{22}\dfrac{S_2 I_2}{1 + I_2^2}\right], \\
I_2' = \left[\beta_{21}\dfrac{S_2 I_1}{1 + I_1^2} + \beta_{22}\dfrac{S_2 I_2}{1 + I_2^2}\right] - (d_2^I + \epsilon_2 + \gamma_2)I_2.
\end{cases}
\tag{9}
$$

Here $f_{kj}(S_k, I_j) = \frac{S_k I_j}{1+I_j^2}$, for $k, j = 1, 2$. We use the parameter values as in Table 1.

Table 1
Sample values of parameters.

Parameter	p_1	p_2	Λ_1	Λ_2	d_1^S	d_1^I	d_2^S	d_2^I	θ_1	ϵ_1	γ_1	θ_2	ϵ_2	γ_2
Value	1/2	1/3	2	3/2	1/4	1/4	1/8	1/8	1/4	3/4	1	3/4	3	7/8

If $B = \begin{pmatrix}\beta_{11} & \beta_{12}\\ \beta_{21} & \beta_{22}\end{pmatrix} = \begin{pmatrix}5/12 & 1/12\\ 1/12 & 5/12\end{pmatrix}$, we have $M_0 = \begin{pmatrix}5/12 & 1/12\\ 1/12 & 5/12\end{pmatrix}$, $R_0 = 0.5 < 1$. Hence the disease-free equilibrium $P_0 = (2, 0, 4, 0)$ is the unique equilibrium of the system (9) and it is globally stable in Γ from Theorem 2.1.

If $B = \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$, we have $M_0 = \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$, $R_0 = 3 > 1$. Then $P^* = (0.901, 0.275, 1.065, 0.183)$ is a unique endemic equilibrium for the system (9) and it is globally asymptotically stable in Γ° from Theorem 2.2.

The numerical simulations for these two examples are shown in Fig. 1.

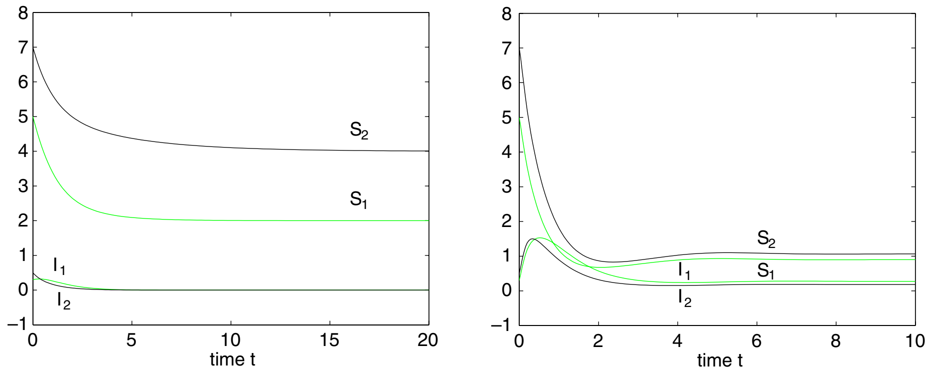

Fig. 1. Numerical simulations for (9) with parameter values in Table 1.

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References

[1] A. Lajmanovich, J.A. York, A deterministic model for gonorrhea in a nonhomogeneous population, Math. Biosci. 28 (1976) 221–236.
[2] R.M. Anderson, R.M. May, Population biology of infectious diseases I, Nature 280 (1979) 361–367.
[3] E. Beretta, V. Capasso, Global stability results for a multigroup SIR epidemic model, in: T.G. Hallam, L.J. Gross, S.A. Levin (Eds.), Mathematical Ecology, World Scientific, Teaneck, NJ, 1988, pp. 317–342.
[4] W. Huang, K.L. Cooke, C. Castillo-Chavez, Stability and bifurcation for a multiple-group model for the dynamics of HIV/AIDS transmission, SIAM J. Appl. Math. 52 (1992) 835–854.
[5] M.Y. Li, J.R. Graef, L. Wang, J. Karsai, Global dynamics of a SEIR model with varying total population size, Math. Biosci. 160 (1999) 191–213.
[6] H.W. Hethcote, The mathematics of infectious diseases, SIAM Rev. 42 (2000) 599–653.
[7] H.R. Thieme, Mathematics in Population Biology, Princeton University Press, Princeton, 2003.
[8] I.G. Laukó, Stability of disease free sets in epidemic models, Math. Comput. Modelling 43 (2006) 1357–1366.
[9] D. Xiao, S. Ruan, Global analysis of an epidemic model with nonmonotone incidence rate, Math. Biosci. 208 (2007) 419–429.
[10] H. Guo, M.Y. Li, Z. Shuai, Global stability of the endemic equilibrium of multigroup SIR epidemic models, Can. Appl. Math. Q. 14 (2006) 259–284.
[11] H. Guo, M.Y. Li, Z. Shuai, A graph–theoretic approach to the method of global Lyapunov functions, Proc. Amer. Math. Soc. 136 (2008) 2793–2802.
[12] M.Y. Li, Z. Shuai, Global-stability problem for coupled systems of differential equations on networks, J. Differential Equations 248 (2010) 1–20.
[13] Z.H. Yuan, L. Wang, Global stability of epidemiological models with group mixing and nonlinear incidence rates, Nonlinear Anal. Real World Appl. 11 (2010) 995–1004.
[14] M.Y. Li, Z. Shuai, C. Wang, Global stability of multi-group epidemic models with distributed delays, J. Math. Anal. Appl. 361 (2010) 38–47.
[15] Z.H. Yuan, X.F. Zou, Global threshold property in an epidemic model for disease with latency spreading in a heterogeneous host population, Nonlinear Anal. Real World Appl. 11 (2010) 3479–3490.
[16] A. Korobeinikov, Global properties of SIR and SEIR epidemic models with multiple parallel infectious stages, Bull. Math. Biol. 71 (2009) 75–83.
[17] Y.N. Kyrychko, K.B. Blyuss, Global properties of a delayed SIR model with temporary immunity and nonlinear incidence rate, Nonlinear Anal. Real World Appl. 6 (2005) 495–507.
[18] Z. Zhao, L.S. Chen, X.Y. Song, Impulsive vaccination of SEIR epidemic model with time delay and nonlinear incidence rate, Math. Comput. Simulation 79 (2008) 500–510.
[19] L.M. Cai, X.Z. Li, Analysis of a SEIV epidemic model with a nonlinear incidence rate, Appl. Math. Modelling 33 (2009) 2919–2926.
[20] O. Diekmann, J.A.P. Heesterbeek, Mathematical Epidemiology of Infectious Diseases: Model Building, Analysis and Interpretation, Wiley, New York, 1999.
[21] P. van den Driessche, J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of diseases transmission, Math. Biosci. 180 (2002) 29–48.
[22] J.P. LaSalle, The Stability of Dynamical Systems, in: SIAM Regional Conference Series in Applied Mathematics. No. 25, Society for Industrial and Applied Mathematics, Philadelphia, 1977.
[23] H.I. Freedman, M.X. Tang, S.G. Ruan, Uniform persistence and flows near a closed positively invariant set, J. Dynam. Differential Equations 6 (1994) 583–600.