Lie Superalgebras

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INTRODUCTION

"Graded Lie algebras have recently become a topic of interest in physics in the context of 'supersymmetries' relating particles of different statistics" (see the survey [22], from which this quotation is taken and which contains an extensive bibliography).

In this paper, we attempt to construct a theory of Lie superalgebras or, as the physicists call them, Z-graded Lie algebras. We prefer the term "superalgebra," which is also inspired by physicists, because speaking generally, Lie superalgebras are not Lie algebras.

A superalgebra is a $\mathbb{Z}_2$-graded algebra $A = A_0 \oplus A_1$ (that is, if $a \in A_\alpha$, $b \in A_\beta$, $\alpha, \beta \in \mathbb{Z}_2 = \{0, 1\}$, then $ab \in A_{\alpha+\beta}$). A Lie superalgebra is a superalgebra $G = G_0 \oplus G_1$ with an operation $[,]$ satisfying the following axioms:

[Insert mathematical axioms here]
\[ [a, b] = -(-1)^{ad}[b, a] \quad \text{for } a \in G_a, \ b \in G_b, \]
\[ [a, [b, c]] = [[a, b], c] + (-1)^{ad}[b, [a, c]] \quad \text{for } a \in G_a, \ b \in G_b. \]

We mention that the Whitehead operation in homotopy groups satisfies these axioms; Lie superalgebras also occur in several cohomology theories, for example, in deformation theory (see [22, 24]).

Lie superalgebras appear in [4] as Lie algebras of certain generalized groups, nowadays called Lie supergroups, whose function algebras are algebras with commuting and anticommuting variables. Recently, a satisfactory theory, similar to Lie's theory, has been developed on the connection between Lie supergroups and Lie superalgebras [5].

We now give a brief account of the main features of the theory of finite-dimensional Lie superalgebras. Let \( G \) be a finite-dimensional Lie superalgebra. Then \( G \) contains a unique maximal solvable ideal \( R \) (the solvable radical). The Lie superalgebra \( G/R \) is semisimple (that is, has no solvable ideals). Therefore, the theory of finite-dimensional Lie superalgebras is reduced in a certain sense to the theories of semisimple and of solvable Lie superalgebras. (But note that Levi's theorem on \( G \) being a semidirect sum of \( R \) and \( G/R \) is not true, in general, for Lie superalgebras.)

The main fact in the theory of solvable Lie algebras is Lie's theorem, which asserts that every finite-dimensional irreducible representation of a solvable Lie algebra over \( \mathbb{C} \) is one-dimensional. For Lie superalgebras this is not true, in general. In the paper we obtain a classification of finite-dimensional irreducible representations of solvable Lie superalgebras (Section 5.2.2, Theorem 7). In particular, we derive a necessary and sufficient condition for any finite-dimensional irreducible representation to be one-dimensional (Section 5.2.2, Proposition 5.2.4).

Next, it is well known that a semisimple Lie algebra is a direct sum of simple ones. This is by no means true for Lie superalgebras. However, there is a construction that allows us to describe finite-dimensional semisimple Lie superalgebras in terms of simple ones (Section 5.1.3, Theorem 6). It is similar to the construction in [21].

So we come to the fundamental problem of classifying the finite-dimensional simple Lie superalgebras. A solution of this problem in the case of an algebraically closed field of characteristic 0 is the main aim of the paper and occupies the major part of it (Chapters 2–4). The principal difficulty lies in the fact that the Killing form (see the definition in Section 2.3.1) may be degenerate, which cannot happen in the case of simple Lie algebras. Therefore, the classical technique Killing–Cartan is not applicable here. The classification is divided into two main parts (presented in Chapters 2 and 4, respectively).

In the first part we give a classification of the classical Lie superalgebras. A Lie superalgebra \( G = G_0 \oplus G_1 \) is called \textit{classical} if it is simple and the representation of the Lie algebra \( G_0 \) on \( G_1 \) is completely reducible. This clas-
sification is divided into two parts, corresponding to the cases of a nondegenerate and a zero Killing form. In Section 2.3 we give a classification of all finite dimensional Lie superalgebras with a nondegenerate Killing form (Theorem 1). This is the first key point of the classification. Here the usual technique is applicable. In Section 2.4 we consider the second key point: the case of a zero Killing form (Proposition 2.4.1). The fact that the Killing form is zero is used to obtain severe restrictions on the index of the representations of $G_0$ on $G_1$ (for the definition of the index see Section 1.4.3). Each of the parts corresponding to Sections 2.3 and 2.4 is, in its turn, divided into two parts according to whether or not the representation of $G_0$ on $G_1$ is irreducible (see Section 2.2). The resulting classification of the classical Lie superalgebras that are not Lie algebras is as follows (Theorem 2): (a) four series $A(m, n)$, $B(m, n)$, $C(n)$, and $D(m, n)$, in many respects similar to the Cartan series $A_n$, $B_n$, $C_n$, and $D_n$; (b) two exceptional Lie superalgebras: a 40-dimensional $F(4)$ and a 31-dimensional $G(3)$, and a family of 17 dimensional exceptional Lie superalgebras $D(2, 1; \alpha)$, which are deformations of $D(2, 1)$; (c) two "strange" series $P(n)$ and $Q(n)$. The construction of all these classical Lie superalgebras is carried out in Section 2.1.

In the second part we give a classification of the nonclassical simple Lie superalgebras. For this purpose we construct a filtration $G = L_{-1} \supset L_0 \supset L_1 \supset \cdots$, where $L_0$ is a maximal subalgebra containing $G_0$, and $L_i = \{a \in L_{i-1} \mid [a, L] \subseteq L_{i-1}\}$ for $i > 0$. Then we classify $\mathbb{Z}$-graded Lie superalgebras with the properties that the associated graded Lie superalgebra $\text{Gr} G = \bigoplus_{i > 1} \text{Gr}_i G$ necessarily has (Section 4.1.1, Theorem 4). This is the third key point. In the proof we make essential use of the method developed in our paper [11] for the classification of infinite-dimensional Lie algebras. After this it only remains to reconstruct the Lie superalgebra $G$ with filtration from the $\mathbb{Z}$-graded Lie superalgebra $\text{Gr} G$.

The final classification of simple finite-dimensional Lie superalgebras is as follows (Section 4.2.1, Theorem 5): (a) the classical Lie superalgebras (listed above); (b) the Lie superalgebras of Cartan type $W(n)$, $S(n)$, $H(n)$, $\bar{S}(n)$, where the first three series are analogous to the corresponding series of simple infinite-dimensional Lie algebras of Cartan type and $\bar{S}(n)$ is a deformation of $S(n)$. The construction of the Lie superalgebras of Cartan type is carried out in Sections 3.1 and 3.3.

The finite-dimensional irreducible representations of the simple Lie algebras are described by the theorem on the highest weight. A similar result holds for simple Lie superalgebras (Section 5.2.3, Theorem 8). Full reducibility of finite-dimensional representations is lacking, in general.

It is not hard to reduce the classification of simple Lie superalgebras over nonclosed fields for the classical Lie superalgebras to the same problem for simple Lie algebras and for Lie superalgebras of the Cartan type a complete list can be made. This is done in Section 5.3, where we list all finite-dimensional simple real Lie superalgebras (Theorem 9).
Finally, in Section 5.4 we attempt to extend Cartan's results on the classification of complete infinite-dimensional primitive Lie algebras to Lie superalgebras. In this direction we have only obtained a partial result (Theorem 10). This also makes clear the reason for the appearance of finite-dimensional Lie superalgebras of Cartan type: Lie superalgebras of Cartan type are Lie superalgebras of vector fields in commuting and anticommuting variables, and also their subalgebras defined by the action on the volume, Hamiltonian, and contact forms. If there are no commuting variables, then the superalgebra is finite-dimensional, and so there is no finite-dimensional analog for the contact Lie algebra.

Here is a brief account of the contents of the paper.

Chapter 1 is introductory. In it we give the basic definitions (Section 1.1), establish the simplest properties of gradings and filtrations (Sections 1.2 and 1.3), and quote the necessary information on finite-dimensional representations of semisimple Lie algebras (Section 1.4).

Chapter 2 is devoted to a description (Section 2.1) and classification (Sections 2.2–2.4) of Lie superalgebras with a nondegenerate Killing form (Theorem 1) and of the classical Lie superalgebras (Theorem 2). In Section 2.5 we describe the root systems of the classical Lie superalgebras and find all up to equivalence systems of simple roots. We classify the simple finite-dimensional contragredient Lie superalgebras (Theorem 3); their properties are very close to those of simple Lie algebras.

In Chapter 3 we introduce and study two algebras of differential forms (Section 3.2) with anticommuting and commuting differentials; it is curious that the second algebra has all the properties that one would naturally expect of an algebra of differential forms. In Sections 3.1 and 3.3 we construct the finite-dimensional Lie superalgebras of Cartan type and study their properties.

In Chapter 4 we classify $\mathbb{Z}$-graded Lie superalgebras that arise in the construction of filtrations in simple Lie superalgebras for which the representation of $G_0$ on $G_1$ is reducible (Section 4.1, Theorem 4), and then, on the basis of this classification, we complete the classification of simple Lie superalgebras (Section 4.2, Theorem 5).

Theorems 1, 2, 4, and 5, and also partially Theorems 6 and 7, were announced by the author in the note [16] (Theorem 4 even earlier in [13]).

In Chapter 5 we discuss the following problems. In Section 5.1 we give a description of the finite-dimensional semisimple Lie superalgebras in terms of the simple ones (Theorem 6) and we find the Lie superalgebras of derivations of all simple Lie superalgebras. As in [21], Theorem 6 is a consequence of a general result on differentially simple superalgebras (Proposition 5.1.1). Section 5.2 is concerned with the theory of finite-dimensional irreducible representations of solvable and simple Lie superalgebras (Theorem 7 and 8). In Section 5.3 we treat the classification of simple finite-dimensional Lie superalgebras over nonclosed fields (Propositions 5.3.1–5.3.3). We also give a
classification of the simple real Lie superalgebras (Theorem 9). In Section 5.4 we introduce infinite-dimensional Lie superalgebras of Cartan type and formulate the theorem on \( \mathbb{Z} \)-graded Lie superalgebras that arise in the classification of infinite-dimensional complete primitive Lie superalgebras (Theorem 10). Finally, in Section 5.5 we discuss some unsolved problems.

All spaces and algebras are regarded over a ground field \( k \), which is assumed to be algebraically closed and of characteristic 0 unless the contrary is stated. The symbol \( \langle M \rangle \) denotes the linear span over \( k \) of a subset \( M \) of a linear space, the symbol \( \oplus \) the direct sum of \( k \)-spaces, and \( \otimes \) the tensor product of \( k \)-spaces.

Here, I would also like to express my deep indebtedness to F. A. Berezin, E. B. Vinberg, and D. A. Leites for numerous conversations and constructive help. I also thank Professor I. Kaplansky for his interest in my work; having become acquainted with his preprint on root systems of simple Lie superalgebras with a nondegenerate invariant form I could remove some errors that had slipped into the original version of the article.

Remark. The history of this article began in 1969 when, impressed by Stavraky's example of a simple Lie superalgebra \( A(1, 0) \) [19], the author was led to employ the technique of [11] to prove the present Theorem 4. Two years later, having read [4], I decided to publish this result [13]. The classification of classical Lie superalgebras (Theorem 2) was obtained in 1974 under the stimulation of the physicists' interest in the subject. At the beginning of 1975 the key to the complete solution of the classification problem of simple Lie superalgebras was found (filtration!) and Theorem 5 was proved. The results were announced in [16]. Then the results of the Chapter 5 were obtained and by September 1975 the work was completed. In October 1975 the manuscript was submitted to the Soviet journal Uspehi Matematicheskikh Nauk, but later was withdrawn and resubmitted to the present journal. In the beginning of 1976 the paper was translated into English. I am grateful to Professor Sternberg for his genuine interest in my work and for making the translation of it possible. I am obliged to Professor Hirsch who translated the text.

In the English version some remarks on further results in the field have been added.

1. Basic Definitions and Preliminary Remarks

1.1. Superalgebras and Lie Superalgebras—Supertrace

1.1.1. Superalgebras. We recall that if \( A \) is an algebra and \( M \) an Abelian group, then an \( M \)-grading of \( A \) is a decomposition of \( A \) into a direct sum of subspaces \( A = \oplus_{x \in M} A_x \) for which \( A_A A_B \subseteq A_{A+B} \). An algebra \( A \) equipped
with an \( M \)-grading is called \( M \)-graded. If \( a \in A_\alpha \), then we say that \( a \) is homogeneous of degree \( \alpha \) and we write \( \deg a = \alpha \). A subspace \( B \) of an \( M \)-graded algebra \( A \) is called \( M \)-graded if \( B = \bigoplus_{\alpha \in M} (B \cap A_\alpha) \). A subalgebra (or ideal) of an \( M \)-graded algebra is an \( M \)-graded subalgebra (or ideal). A homomorphism \( \Phi : A \rightarrow A' \) of \( M \)-graded algebras preserves the grading in the sense that \( \Phi(A_\alpha) \subseteq A'_{\varphi(\alpha)} \), where \( \varphi \) is an automorphism of \( M \).

Now let \( \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} \) be the residue class ring mod 2, with the elements \( \bar{0} \) and \( \bar{1} \).

A superalgebra is a \( \mathbb{Z}_2 \)-graded algebra \( A = A_0 \oplus A_1 \). The elements of \( A_0 \) are called even, those of \( A_1 \) odd. Throughout what follows, if \( \deg a \) occurs in an expression, then it is assumed that \( a \) is homogeneous, and that the expression extends to the other elements by linearity.

The direct and semidirect sum of superalgebras are defined in the usual way. With the definition of the tensor product things are different. Let \( A \) and \( B \) be superalgebras. Their tensor product \( A \otimes B \) is the superalgebra whose space is the tensor product of the spaces of \( A \) and \( B \), with the induced \( \mathbb{Z}_2 \)-grading and the operation defined by

\[
(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{(\deg a_1)(\deg b_1)} a_1 a_2 \otimes b_1 b_2, \quad a_i \in A, \ b_i \in B.
\]

There is a natural way of defining a bracket \([ , ]\) in a superalgebra \( A \), i.e., by the equality,

\[
[a, b] = ab - (-1)^{(\deg a)(\deg b)} ba. \quad (1.1.1)
\]

A superalgebra is called commutative if \([a, b] = 0\) for all \( a, b \in A \). Quite generally, permutability in a superalgebra is understood in the sense of the bracket (1.1.1). Associativity of superalgebras is defined as for algebras.

For an associative superalgebra \( A \) we have the following important identity:

\[
[a, bc] = [a, b]c + (-1)^{(\deg a)(\deg b)} b[a, c]. \quad (1.1.2)
\]

**Example 1.** Let \( M \) be an Abelian group and \( V = \bigoplus_{\alpha \in M} V_\alpha \) an \( M \)-graded space. Then the associative algebra \( \text{End} V \) is equipped with the induced \( M \)-grading \( \text{End} V = \bigoplus_{\alpha \in M} \text{End}_\alpha V \), where

\[
\text{End}_\alpha V = \{ a \in \text{End} V \mid a(V_\beta) \subseteq V_{\alpha + \beta} \}.
\]

In particular, for \( M = \mathbb{Z}_2 \) we obtain the associative superalgebra \( \text{End} V = \text{End}_0 V \oplus \text{End}_1 V \).

**Example 2.** Let \( \Lambda(n) \) be the Grassmann algebra in \( n \) variables \( \xi_1, \ldots, \xi_n \). Then \( \Lambda(n) \) becomes \( \mathbb{Z}_2 \)-graded if we set \( \deg \xi_i = \bar{1}, \ i = 1, \ldots, n \). The result is called a Grassmann superalgebra. It is commutative and associative. Evidently \( \Lambda(m) \otimes \Lambda(n) = \Lambda(m + n) \).
A generalization of this example is the commutative superalgebra \( \Lambda(m, n) = k[x_1, \ldots, x_m] \otimes \Lambda(n) \), where the polynomial algebra \( k[x_1, \ldots, x_m] \) is regarded as a superalgebra with trivial \( \mathbb{Z}_2 \)-grading.

1.1.2. Definition of a Lie superalgebra. A Lie superalgebra is a superalgebra \( G = G_0 \oplus G_1 \) with an operation \([\ , \ ]\) satisfying the following axiom:

\[
[a, b] = -(-1)^{(\deg a)(\deg b)} [b, a] \quad \text{(anticommutativity)},
\]

\[
[a, [b, c]] = [[a, b], c] + (-1)^{(\deg a)(\deg b)} [b, [a, c]] \quad \text{(Jacobi identity)}.
\]

Observe that \( G_0 \) is an ordinary Lie algebra, that multiplication on the left by elements of \( G_0 \) determines a structure of a \( G_0 \)-module on \( G_1 \), and that multiplication of elements of \( G_1 \) determines a homomorphism of \( G_0 \)-modules \( \varphi: S^2 G_1 \to G_0 \). Thus, every Lie superalgebra can be specified by three objects: the Lie algebra \( G_0 \), the \( G_0 \)-module \( G_1 \), and the homomorphism of \( G_0 \)-modules \( \varphi: S^2 G_1 \to G_0 \), with the sole condition

\[
\varphi(a, b)c + \varphi(b, c)a + \varphi(c, a)b = 0 \quad \text{for } a, b, c \in G_1. \tag{1.1.3}
\]

Example 1. If \( A \) is an associative superalgebra, then the bracket (1.1.1) turns \( A \) into a Lie superalgebra. (The Jacobi identity follows from (1.1.2).) We denote the resulting Lie superalgebra by \( A_L \).

Example 2. Let \( G \) be a Lie superalgebra and \( \Lambda(n) \) a Grassmann superalgebra. Then \( G \otimes \Lambda(n) \) is also a Lie superalgebra.

The definitions of a solvable and a nilpotent Lie superalgebra are the same as for Lie algebras. A Lie superalgebra is called simple (semisimple) if it contains no nontrivial (no solvable) ideals.

1.1.3. The universal enveloping superalgebra. Let \( G = G_0 \oplus G_1 \) be a Lie superalgebra. As usual, a pair \((U(G), i)\), where \( U(G) \) is an associative superalgebra and \( i: G \to U(G)_L \) is a homomorphism of Lie superalgebras, is called the universal enveloping superalgebra of \( G \) if for any other pair \((U', i')\) there is a unique homomorphism \( \theta: U \to U' \) for which \( i' = \theta \circ i \).

The universal enveloping superalgebra of \( G = G_0 \oplus G_1 \) is constructed as follows [24]. Let \( T(G) \) be the tensor superalgebra over the space \( G \) with the induced \( \mathbb{Z}_2 \)-grading, and \( R \) the ideal of \( T(G) \) generated by the elements of the form:

\[
[a, b] - a \otimes b + (-1)^{(\deg a)(\deg b)} b \otimes a.
\]

We set \( U(G) = T(G)/R \). The natural map \( G \to U(G) \) evidently induces a homomorphism \( i: G \to U(G)_L \), and the pair \((U(G), i)\) is the required enveloping superalgebra.

In [24] the following theorem is verified.
**The Poincaré-Birkhoff-Witt Theorem.** Let \( G = G_0 \oplus G_1 \) be a Lie superalgebra, \( a_1, ..., a_m \) be a basis of \( G_0 \), and \( b_1, ..., b_n \) be a basis of \( G_1 \). Then the elements of the form

\[
a_1^{k_1} ... a_m^{k_m} b_1^{i_1} ... b_n^{i_n}, \quad \text{where} \quad k_i \geq 0 \quad \text{and} \quad 1 \leq i_1 < ... < i_s \leq n,
\]

form a basis of \( U(G) \).

Finally, we define the diagonal homomorphism. As it is easy to see, the map

\[
a \mapsto i(a) \otimes 1 + (-1)^{\deg a} 1 \otimes i(a), \quad a \in G,
\]

is a homomorphism of Lie superalgebras \( G \rightarrow (U(G) \otimes U(G))_L \) and therefore determines a homomorphism of associative superalgebras:

\[
\Delta : U(G) \rightarrow U(G) \otimes U(G),
\]

which is called the diagonal homomorphism.

1.1.4. Derivations and automorphisms of a superalgebra. A derivation of degree \( s, s \geq 2 \), of a superalgebra \( A \) is an endomorphism \( D \in \text{End}_s A \) with the property

\[
D(ab) = D(a)b + (-1)^{\deg a} aD(b).
\]

We denote by \( \text{der}_s A \subset \text{End}_s A \) the space of all derivations of degree \( s \), and we set \( \text{der} A = \text{der}_0 A \oplus \text{der}_1 A \). The space \( \text{der} A \subset \text{End} A \) is easily seen to be closed under the bracket (1.1.1), in other words, it is a subalgebra of \( (\text{End} A)_L \); it is called the superalgebra of derivations of \( A \). Every element of \( \text{der} A \) is called a derivation of \( A \).

**Example 1.** Let \( G \) be a Lie superalgebra. It follows from the Jacobi identity that \( \text{ad} a : b \mapsto [a, b] \) is a derivation of \( G \). These derivations are called inner; they form an ideal inder \( G \) of \( \text{der} G \), because \([D, \text{ad} a] = \text{ad} Da \) for \( D \in \text{der} G \).

**Example 2.** Let \( A(n) = A_0(n) \oplus A_1(n) \) be a Grassmann superalgebra. Let us find \( \text{der} A(n) \). For this purpose it is convenient to represent \( A(n) \) in the form \( \tilde{A}(n)/I \), where \( \tilde{A}(n) \) is the free associative superalgebra with the generators \( \xi_1, ..., \xi_n \) whose \( Z_2 \)-grading is given by \( \deg \xi_i = 1 \), \( i = 1, ..., n \), and \( I \) is the ideal generated by all the elements \( \xi_i \xi_j + \xi_j \xi_i \). Note that if \( P \) and \( Q \) are homogeneous elements of \( \tilde{A}(n) \), then \( PQ - (-1)^{\deg P \deg Q} QP \in I \).

Let \( D \) be a derivation of degree \( s \) of \( \tilde{A}(n) \). Then

\[
D(\xi_i \xi_j + \xi_j \xi_i) = D(\xi_i)\xi_j + (-1)^s \xi_j D(\xi_i) + D(\xi_j)\xi_i + (-1)^s \xi_i D(\xi_j) = (D(\xi_i)\xi_j + (-1)^s \xi_j D(\xi_i)) + (D(\xi_j)\xi_i + (-1)^s \xi_i D(\xi_j)) \in I,
\]

from which it follows that \( I \) is invariant under \( D \). Since, obviously, there is one and only one derivation of \( \tilde{A}(n) \) with prescribed values \( D(\xi_i) \in \tilde{A}(n) \), we
see that for any $P_1, \ldots, P_n \in \Lambda(n)$ there is one and only one derivation $D \in \text{der } \Lambda(n)$ for which $D(\xi_i) = P_i \in \Lambda(n)$.

In particular, the relations $\partial/\partial \xi_i(\xi_j) = \delta_{ij}$ define the derivation $\partial/\partial \xi_i$, $i = 1, \ldots, n$. The derivation $D \in \text{der } \Lambda(n)$ for which $D(\xi_i) = P_i$ can now be written as a linear differential operator:

$$D = \sum_{i=1}^{n} P_i \frac{\partial}{\partial \xi_i}, \quad P_i \in \Lambda(n).$$

It is equally easy to find the automorphism group of $\Lambda(n)$ [3]. Observe that there is a unique homomorphism $\varphi: \Lambda(n) \to k$; we agree to write $f(0)$ instead of $\varphi(f)$, $f \in \Lambda(n)$. If now $\Phi$ is an automorphism of $\Lambda(n)$, then $\deg \Phi(\xi_i) = \bar{1}$ and $\det(\partial/\partial \xi_i(\Phi(\xi_i))(0)) \neq 0$; any map $\xi_i \mapsto \Phi(\xi_i) \in \Lambda(n)$, $i = 1, \ldots, n$, having these two properties extends uniquely to an automorphism of $\Lambda(n)$.

Any automorphism $\Phi$ of $\Lambda(n)$ induces an automorphism of $\text{der } \Lambda(n)$ according to the formula

$$(\Phi D)f = \Phi(D(\Phi^{-1}f)), \quad f \in \Lambda(n).$$

Note that if $D$ is an even derivation of a superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$, then $\exp tD$, $t \in k$, is a one-parameter group of automorphisms. In particular, if $A$ is a Lie superalgebra, then $\exp(\text{ad } a)$ for $a \in A_{\bar{0}}$ is an automorphism of $A$; the group generated by these automorphisms is called the group of inner automorphisms. The preceding remark leads to the following result.

**Proposition 1.1.1.** Let $\mathcal{G}$ and $\mathcal{G}_0$ be the connected components of the identity in the automorphism groups of superalgebras $A$ and $A_0$, and let $\mathcal{K}$ be the subgroup of $\mathcal{G}$ consisting of the automorphisms that act identically on $A_0$. Then the restriction induces an epimorphism $\mathcal{G} \twoheadrightarrow \mathcal{G}_0$ with kernel $\mathcal{K}$. In particular, if $A$ is a Lie superalgebra, then every inner automorphism of $A_0$ extends to an inner automorphism of $A$.

1.1.5. The superalgebra $l(V)$ and the supertrace. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded space. Then (Section 1.1.1) the algebra $\text{End } V$ is endowed with a $\mathbb{Z}_2$-grading and so becomes an associate superalgebra. Now $(\text{End } V)_L$ is a Lie superalgebra (Section 1.1.2) which we denote by $l(V)$ or $l(m, n)$, where $m = \dim V_{\bar{0}}$, $n = \dim V_{\bar{1}}$. In the theory of Lie superalgebras $l(V) = l(V)_0 \oplus l(V)_1$ plays the same role as the general linear Lie algebra in the theory of Lie algebras. If we regard the same decomposition $V = V_{\bar{0}} \oplus V_{\bar{1}}$ as a $\mathbb{Z}$-grading of $V$, then it corresponds to a $\mathbb{Z}$-grading of $l(V)$, which is compatible with the $\mathbb{Z}_2$-grading: $l(V) = L_{\bar{0}} \oplus L_{\bar{1}}$.

Let $e_1, \ldots, e_m$, $e_{m+1}, \ldots, e_{m+n}$ be a basis of $V$, formed from bases of $V_{\bar{0}}$ and $V_{\bar{1}}$. It is natural to call such a basis homogeneous. In this basis the matrix of an operator $a$ from $l(V)$ can be written in the form $[a_{ij}^{\alpha \beta \gamma}]$, where $\alpha$ is an $(m \times m)$-, $\beta$ an $(n \times n)$-, and $\gamma$ an $(n \times m)$-matrix. The matrices of even
elements have the form $[0, \delta]$, and those of odd ones $[\gamma, \delta]$. Here $l_1$ consists of the matrices of the form $[\gamma, \delta]$ and $l_{-1}$ of the form $[0, \delta]$. Hence it is clear that the $l_0$-modules $l_1$ and $l_{-1}$ are contragredient and the $l_0$-module $l_1$ isomorphic to $gl_m \otimes gl_n$.

Now we come to the definition of the supertrace. For the matrix $a = [\gamma, \delta] \in l_{(m,n)}$ this is the number

$$
\text{str}(a) = \text{tr} \alpha - \text{tr} \delta.
$$

Observe that the supertrace of the matrix of an operator $a \in l(V)$ docs not depend on the choice of a homogeneous basis. Therefore, we have the right to speak of the supertrace of $a$, meaning the supertrace of this operator in any homogeneous basis.

To state properties of the supertrace (and for other purposes) it is useful to introduce the following definitions. Let $G = G_0 \oplus G_1$ be a $\mathbb{Z}_2$-graded space and $f$ be a bilinear form on $G$. Then $f$ is called consistent if $f(a, b) = 0$ for $a \in G_0$, $b \in G_1$, and supersymmetric if $f(a, b) = (-1)^{(\deg a)(\deg b)}f(b, a)$. If $G$ is a Lie superalgebra, $f$ is called invariant if $f([a, b], c) = f(a, [b, c])$.

**Proposition 1.1.2.** (a) The bilinear form $(a, b) = \text{str}(ab)$ on $l(V)$ is consistent, supersymmetric, and invariant.

(b) $\text{str}([a, b]) = 0$ for any $a, b \in l(V)$.

**Proof.** The consistency follows from the fact that $ab \in l(V)_1$ for $a \in l(V)_0$, $b \in l(V)_1$.

Supersymmetry for $a, b \in l(V)_0$ follows from the corresponding property of the trace, and for $a \in l(V)_0$, $b \in l(V)_1$ from consistency. It remains to consider the case $a, b \in l(V)_1$. Let $a = (a_\alpha \delta)$ and $b = (b_\gamma \delta)$ be the matrices of $a$ and $b$ in a homogeneous basis. Then $(a, b) = \text{tr} \alpha \delta - \text{tr} \beta \gamma$, $(b, a) = \text{tr} \gamma \beta - \text{tr} \delta \alpha$, from which it follows that $(a, b) = -(b, a)$, as required.

(b) is simply another way of writing down supersymmetry.

We still have to verify invariance. By (1.1.2) we have $[b, ac] = [b, a]c + (-1)^{(\deg a)(\deg b)}a[b, c]$. Therefore, by (b):

$$
0 = \text{str}([b, ac]) = ([b, a], c) + (-1)^{(\deg a)(\deg b)}(a, [b, c]),
$$

as required.

**1.1.6. Linear representations of Lie superalgebras.** Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded linear space. A linear representation $\rho$ of a Lie superalgebra $G = G_0 \oplus G_1$ in $V$ is a homomorphism $\rho: G \rightarrow l(V)$. 

For brevity we often say in this case that $V$ is a $G$-module, and instead of $\rho(g)(v)$ we write $g(v), g \in G, v \in V$. Note that, by definition, $G_i(V_j) \subseteq V_{i+j}$, $i, j \in \mathbb{Z}_2$, and $[g_1, g_2](v) = g_1(g_2(v)) - (-1)^{(\deg g_1)(\deg g_2)}g_2(g_1(v))$. Note also that the map $\text{ad}: G \to \mathfrak{I}(G)$ for which $(\text{ad}g)(a) = [g, a]$ is a linear representation of $G$. It is called the adjoint representation.

A submodule of a $G$-module $V$ is assumed to be $\mathbb{Z}_2$-graded; a $G$-module $V$ is said to be irreducible if it has no nontrivial submodules. By a homomorphism of $G$-modules $\Phi: V \to V'$ we mean one that preserves the $\mathbb{Z}_2$-grading in the sense that $\Phi(V_i) = V_{\varphi(i)}$, where $\varphi$ is a bijection $\mathbb{Z}_2 \to \mathbb{Z}_2$.

**Schur’s Lemma.** Let $V = V_0 \oplus V_1$, $\mathcal{M}$ an irreducible family of operators from $l(V)$, and $\mathcal{C}(\mathcal{M}) = \{a \in l(V) \mid [a, m] = 0, m \in \mathcal{M}\}$. Then either $\mathcal{C}(\mathcal{M}) = \langle 1 \rangle$ or $\dim V_0 = \dim V_1$ and $\mathcal{C}(\mathcal{M}) = \langle 1, A \rangle$, where $A$ is a nondegenerate operator in $V$ permuting $V_0$ and $V_1$, and $A^2 = 1$.

**Example.** We consider the Lie superalgebra $N = N_0 \oplus N_1$, where $N_0 = \langle e \rangle, N_1 = \langle a_1, ..., a_n, b_1, ..., b_n \rangle$ and $[a_i, b_i] = e$, $i = 1, ..., n$, the remaining brackets being zero. We construct a family of representations $\rho_\alpha, \alpha \in k^*$, of $N$ in $\Lambda(n)$ by setting: $\rho_\alpha(a_i)u = \partial u/\partial \xi_i, \rho_\alpha(b_i)u = \alpha \xi_i u, \rho_\alpha(e)u = au$. Clearly, $\rho_\alpha$ is a $2^n$-dimensional irreducible representation of $N$.

We now consider the Lie superalgebra $N' = N \oplus \langle e \rangle$, where $[N, e] = 0$, $[e, c] = e$, and the superalgebra $\Lambda'(n) = \Lambda(n) \otimes k[e]$, where $\deg e = 1, e^2 = \alpha/2, \alpha \in k$. We define a representation $\rho_\alpha'$ of $N'$ in $\Lambda'(n)$ by setting $\rho_\alpha'(h)(u \otimes v) = \rho_\alpha(h)u \otimes v, \rho_\alpha'(e)(u \otimes v) = (1 \otimes e)(u \otimes v), u \otimes v \in \Lambda'(n)$. Clearly, $\rho_\alpha'$ is a $2^{n+1}$-dimensional irreducible representation of $N'$.

Both $N$ and $N'$ are nilpotent. They are called Heisenberg superalgebras. Note that $\rho_\alpha$ and $\rho_\alpha'$ fall under the two cases of Schur’s lemma.

This example shows that Lie’s theorem need not be true for Lie superalgebras. However, Engel’s theorem remains valid, and the proof is the same as for Lie algebras [10].

**Engel’s Theorem.** Let $G$ be a subalgebra of $l(V)$ and suppose that all the operators of $G$ are nilpotent. Then there is a vector $v \in V, v \neq 0$, that is annihilated by all the operators of $G$.

Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded space. By the symmetric (respectively, exterior) algebra over $V$ we mean the $\mathbb{Z}$-graded superalgebra $S(V) = S(V_0) \otimes \Lambda(V_1) = \oplus S^k(V)$ (respectively, $\Lambda(V) = \Lambda(V_0) \otimes S(V_1) = \oplus \Lambda^k(V)$).

If $V$ is a module of the Lie superalgebra $G$, then we have homomorphisms $G \to \text{der } S(V)$ and $G \to \text{der } \Lambda(V)$, so $S(V)$ and $\Lambda(V)$ become $G$-modules. The submodules $S^k(V)$ (respectively, $\Lambda^k(V)$) are called the symmetric (respectively, exterior) powers of the $G$-module $V$. 
1.2. Z-Graded Lie Superalgebras

1.2.1. Z-gradings. A Z-grading of a superalgebra $G$ is a decomposition of it into a direct sum of finite-dimensional $\mathbb{Z}$-graded subspaces $G = \bigoplus_{i \in \mathbb{Z}} G_i$ for which $G_i G_j \subseteq G_{i+j}$. A Z-grading is said to be consistent if $G_0 = \bigoplus G_{2i}$, $G_1 = \bigoplus G_{2i+1}$.

By definition, if $G$ is a Z-graded Lie superalgebra, then $G_0$ is a subalgebra and $[G_0, G_1] \subseteq G_1$; therefore, the restriction of the adjoint representation to $G_0$ induces linear representations of it on the subspaces $G_i$.

A Z-graded Lie superalgebra $G = \bigoplus_{i \in \mathbb{Z}} G_i$ is called irreducible if the representation of $G_0$ on $G_1$ is irreducible.

A Z-graded Lie superalgebra $G = G_0 \oplus G_1$ is called transitive if for $a \in G_i$, $i > 0$, it follows from $[a, G_{-1}] = 0$ that $a = 0$, and bitransitive if in addition for $a \in G_i$, $i < 0$, it follows from $[a, G_{1}] = 0$ that $a = 0$.

These properties are closely connected with $G$ being simple, as is shown by the following proposition.

**Proposition 1.2.1.** If in a simple Z-graded Lie superalgebra $G = \bigoplus_{i \in \mathbb{Z}} G_i$ the subspace $G_{-1} \oplus G_0 \oplus G_1$ generates $G$, then it is bitransitive.

The proof is the same as that of [11, Proposition 1].

1.2.2. Local Lie superalgebras. Let $\hat{G}$ be a $\mathbb{Z}$-graded space, decomposed into a direct sum of $\mathbb{Z}$-graded subspaces, $\hat{G} = G_{-1} \oplus G_0 \oplus G_1$. Suppose that whenever $|i + j| \leq 1$ a bilinear operation is defined $G_i \times G_j \rightarrow G_{i+j}([x, y])$, satisfying the axiom of anticommutativity and the Jacobi identity for Lie superalgebras, provided that all the commutators in this identity are defined. Then $\hat{G}$ is called a local Lie superalgebra.

To a Z-graded Lie superalgebra $G = \bigoplus G_i$ there corresponds a local Lie superalgebra $G_{-1} \oplus G_0 \oplus G_1$, which we call the local part of $G$.

Homomorphisms, transitivity, bitransitivity, etc., for local Lie superalgebras are defined as for Z-graded Lie superalgebras.

In this subsection we consider only Z-graded Lie superalgebras $G = \bigoplus G_i$ for which the subspace $G_{-1} \oplus G_0 \oplus G_1$ generates $G$.

A Z-graded Lie superalgebra $G = \bigoplus G_i$ with local part $\hat{G}$ is said to be maximal (respectively, minimal) if for any other Z-graded superalgebra $G'$ an isomorphism of the local parts $\hat{G}$ and $\hat{G}'$ extends to an epimorphism of $G$ onto $G'$ (respectively, $G'$ onto $G$).

**Proposition 1.2.2.** Let $\hat{G} = G_{-1} \oplus G_0 \oplus G_1$ be a local Lie superalgebra. Then there is a maximal and a minimal Z-graded Lie superalgebra whose local parts are isomorphic to $\hat{G}$. 
Proposition 1.2.3. (a) A bitransitive $\mathbb{Z}$-graded Lie superalgebra is minimal.

(b) A minimal $\mathbb{Z}$-graded Lie superalgebra with bitransitive local part is bitransitive.

(c) Two bitransitive $\mathbb{Z}$-graded Lie superalgebras are isomorphic if and only if their local parts are isomorphic.

These two propositions are proved just as the corresponding assertions for Lie algebras (see [11, Propositions 4 and 5]).

1.2.3. Invariant bilinear forms. The following proposition is proved in the same manner as [11, Proposition 7].

Proposition 1.2.4. Suppose that on the local part of a $\mathbb{Z}$-graded Lie superalgebra $G = \bigoplus G_i$ a consistent supersymmetric invariant bilinear form $(\ , \ )$ is given (see Section 1.1.5) for which $(G_i , G_j) = 0$ when $i + j \neq 0$. If $G_{-1} \oplus G_0 \oplus G_1$ generates $G$, then the form can be extended uniquely to a consistent supersymmetric invariant bilinear form with the same property on the whole $G$.

The following assertion is proved in the same manner as [7, Corollary 2 to Theorem 4].

Proposition 1.2.5. Let $G = \bigoplus_{i=-d}^{t} G_i$ be a simple finite-dimensional $\mathbb{Z}$-graded Lie superalgebra, with $G_{-k} = G_{k}^* , k \geq 0$, $G_{-d} \neq 0$, $G_i \neq 0$. On $G$ there exists a nondegenerate consistent supersymmetric invariant bilinear form $(\ , \ )$ if and only if the representations of $G_0$ on $G_{-d}$ and $(G_i)^*$ are equivalent. For this form $(G_i , G_j) = 0$ when $i + j \neq t - d$.

Clearly, the kernel of an invariant form is an ideal. Therefore, we have the following result.

Proposition 1.2.6. If $G$ is a simple Lie superalgebra, then an invariant form on it is either nondegenerate or identically zero, and any two invariant forms on $G$ are proportional.

1.2.4. Conditions for simplicity. In this subsection we state some conditions for the simplicity of Lie superalgebras. The proofs are standard.

Proposition 1.2.7. The following conditions are necessary for a Lie superalgebra $G = G_{\overline{0}} \oplus G_{\overline{1}}$ to be simple:

1. The representation of $G_{\overline{0}}$ on $G_{\overline{1}}$ is faithful and $[G_{\overline{1}} , G_{\overline{1}}] = G_{\overline{0}}$.

If, in addition,

2. the representation of $G_{\overline{0}}$ on $G_{\overline{1}}$ is irreducible,

then $G$ is simple.
1.2.8. The following conditions are necessary for a \( \mathbb{Z} \)-graded Lie superalgebra \( G = \bigoplus_{i \in \mathbb{Z}} G_i \) to be simple:

1. \( G \) is transitive and irreducible; \([G_{-1} , G_1] = G_0\).

If, in addition,

2. the kernel of the \( G_0 \)-module \( G_1 \) is 0 and \( G_i = G_i^* \) for \( i > 0 \),

then \( G \) is simple.

1.2.5. Some properties of \( \mathbb{Z} \)-graded Lie superalgebras of the form \( \bigoplus_{i \in \mathbb{Z}} G_i \).

The following assertion facilitates the work with \( \mathbb{Z} \)-graded Lie superalgebras.

**Proposition 1.2.9** (cf. [23]). Let \( G = \bigoplus_{i \in \mathbb{Z}} G_i \) be a transitive irreducible Lie superalgebra with a consistent \( \mathbb{Z} \)-grading, and \( G_1 \neq 0 \). Then \([G_{-1}, G_1] \subseteq [G_{-1}, G_1] \).

**Proof.** Observe, first of all, that \([G_{-1}, [G_{-1} , G_1]] = G_{-1}\), because \([G_{-1}, [G_{-1} , G_1]]\) is a nontrivial \( G_0 \)-submodule of \( G_{-1}\). Let \( C \) be the centralizer in \( G_0 \) of \([G_{-1}, G_1]\). Since the Lie algebra \( G_0 \) is reductive, it is sufficient to show that the Lie algebra \( C \) of linear transformations of the space \([G_{-1}, [G_{-1} , G_1]]\) is Abelian. To do this we have to verify that for \( x, y \in G_{-1} \), \( a, b \in C \) the expression \( d = [[[t, x], y], a] \) is symmetric in \( a \) and \( b \). Now \( d = [[[t, x], y], a] \), \( b = -[[[t, x], [y, a]], b] = [[[t, x], [y, a]], b] = -[[[t, x], [y, b]], [a, b]] \), which proves the assertion.

**Proposition 1.2.10.** Let \( G = \bigoplus_{i \in \mathbb{Z}} G_i \) be a \( \mathbb{Z} \)-graded Lie superalgebra satisfying the conditions of Proposition 1.2.9 and suppose that, in addition, the representation of \( G_0 \) on \( G_1 \) is irreducible. Let \( H \) be a Cartan subalgebra of \( G_{-1} \), \( F_A \) the highest weight vector of the representation of \( G_0 \) on \( G_{-1} \), and \( E_M \) the lowest weight vector of the representation of \( G_0 \) on \( G_1 \).

(a) If the representations of \( G_0 \) on \( G_{-1} \) and \( G_1 \) are contragredient, then

1. \( M = -A \),
2. \([F_A, E_M] = h \neq 0\), where \( h \in H \),
3. \([G_1, G_1] = 0\),
4. the Lie superalgebra \( G_{-1} \oplus [G_{-1} , G_1] \oplus G_1 \) is simple.

(b) If the representations of \( G_0 \) on \( G_{-1} \) and \( G_1 \) are not contragredient, then

1. \([F_A, E_M] = e_\alpha \neq 0\), where \( \alpha = A + M \) is a non-zero root of the Lie algebra \([G_0, G_0] \),
2. \([G_{-1}, G_1] = [G_0, G_0] \),
3. \([G_0, G_0] \) is simple.
Proof. Since $G_{-1} = \langle [\cdots [F_A, e_{-\gamma_1}], \cdots, e_{-\gamma_k}] \rangle$ and $G_1 = \langle [\cdots [E_M, e_{\gamma_1}], \cdots, e_{\gamma_k}] \rangle$, where $\gamma_1, \ldots, \gamma_k > 0$, we evidently have

$$[G_{-1}, G_1] = \langle [\cdots [F_A, E_M], e_{\gamma_1}, \cdots, e_{\gamma_k}] \rangle. \quad (1.2.1)$$

Since, by transitivity, $[G_{-1}, G_1] \neq 0$, we obtain from (1.2.1) that $[F_A, E_M] \neq 0$. But $[t, [F_A, E_M]] = (A + M)(t)[F_A, E_M]$ for any $t \in H$. Since contragredience of the representations of $G_0$ on $G_{-1}$ and $G_1$ means that $A + M = 0$, we have now established (1) and (2) in (a) and (1) in (b).

Let us prove (3) in (a). We consider the graded subalgebra $\mathcal{G}$ of $G$ generated by the subspace $G_{-1} \oplus G_0 \oplus G_1$. Clearly $\mathcal{G}$ is bitransitive. There is an obvious automorphism $\varphi$ of its local part carrying the positive roots of $G_0$ into the negative ones and interchanging $G_{-1}$ with $G_1$. Since, according to Proposition 1.2.3, $\mathcal{G}$ is minimal, $\varphi$ extends to an automorphism of $\mathcal{G}$. Therefore, it follows from $[G_{-1}, G_1] = 0$ that $[G_1, G_1] = 0$, as required.

$(4)$ in (a) follows from Proposition 1.2.8.

Let us now prove (2) and (3) in (b). From (1.2.1) we see that $[G_{-1}, G_1] = \langle [\cdots [e_{\gamma_1}, e_{\gamma_1}], \cdots, e_{\gamma_1}] \rangle \subseteq \mathcal{H}$, where $\mathcal{H}$ is the simple component of the semisimple Lie algebra $[G_0, G_0]$ the root of which is $\alpha$. From Proposition 1.2.9 we see that $[G_{-1}, G_{-1}] = \mathcal{H} = [G_0, G_0]$, which proves (2) and (3).

**Proposition 1.2.11.** Let $G = G_{-1} \oplus G_0 \oplus G_1$ be a transitive $\mathbb{Z}$-graded Lie superalgebra satisfying the conditions of Proposition 1.2.9. Then either the representation of $G_0$ on $G_1$ is faithful and irreducible, or dim $G_1 = 1$.

**Proof.** Let $G_1 = G_1' \oplus G_1''$ be some nontrivial decomposition of $G_1$ into a direct sum of $G_0$-submodules. Applying Proposition 1.2.9 to $G_{-1} \oplus G_0 \oplus G_1'$ we see that $[G_0, G_0] \subseteq [G_{-1}, G_1']$. On the other hand, clearly $[[G_{-1}, G_1'], G_{-1}] \oplus [G_1', G_1']$ is an ideal in $G$. Hence it follows, in particular, that $[[G_0, G_0], G_1'] = 0$. So we find that if the $G_0$-module $G_1$ is reducible, then $[G_0, G_0]$ acts trivially on $G_1$. But in that case it clearly follows from transitivity that dim $G_1 = 1$.

It remains to show that the following situation is impossible: $H \subseteq G_0$, $H \neq [G_0, G_0]$ is a simple subalgebra of $G_0$ whose representation on $G_1$ is trivial. In that case $[x, G_{-1}] \subseteq H$ for every $x \in G_1$, which contradicts Proposition 1.2.9. This completes the proof of Proposition 1.2.11.

**Proposition 1.2.12.** Let $G = \bigoplus_{i \geq 1} G_i$ be a $\mathbb{Z}$-graded irreducible transitive Lie superalgebra. If the even part of the center $C$ of $G_0$ is nontrivial, then $C_{\overline{0}} = \langle z \rangle$, and $[z, g] = sg$ for $g \in G_s$. 

Proof. By Schur's lemma, \( C_0 = \langle x \rangle \), where \([x, g] = -g \) for \( g \in G_{-1} \). Now let \( g_{k+1} \in G_{k+1} \), \( x \in G_{-1} \). Then we have by induction:

\[
[x, [x, g_{k+1}]] = -[x, g_{k+1}] + [x, [x, g_{k+1}]] - k[x, g_{k+1}],
\]

from which we see that \([x, [x, g_{k+1}]] - (k + 1)g_{k+1} = 0\). By the transitivity of \( G \), what we need now follows.

**Proposition 1.2.13.** Let \( G = \bigoplus_{i \geq -1} G_i \) be a \( \mathbb{Z} \)-graded irreducible transitive Lie superalgebra for which the representation of \( G_0 \) on \( G_1 \) is faithful. Then \( G \) is bitransitive.

**Proof.** Clearly, \( V = \{a \in G_{-1} \mid [a, G_1] = 0\} \) is a submodule of the \( G_0 \)-module \( G_{-1} \). By the transitivity of \( G \) we have \([G_{-1}, G_1] \neq 0\); therefore, \( V \neq G_{-1} \); consequently, \( V = 0 \).

### 1.3. Lie Superalgebras with Filtrations

**1.3.1. Filtrations.** A sequence of embedded \( \mathbb{Z}_g \)-graded subspaces in a superalgebra \( L: L = L_{-1} \supset L_0 \supset L_1 \supset \cdots \) is called a filtration if

\[
L_i L_j \subseteq L_{i+j} \quad \text{and} \quad \bigcap L_i = 0, \quad i, j \in \mathbb{Z}.
\]

A Lie algebra \( L \) with a filtration is called transitive if for any \( a \in L_i \setminus L_{i+1}, i > 0 \), there is an element \( b \in L \) for which \([a, b] \notin L_i \). This condition can also be written as follows:

\[
L_i = \{a \in L_{i-1} \mid [a, L] \subseteq L_{i-1}\}, \quad i > 0. \tag{1.3.1}
\]

Let \( L \) be a Lie superalgebra and \( L_0 \) be a subalgebra of \( L \) that contains no nonzero ideals of the whole algebra \( L \). Then (1.3.1) defines a filtration in \( L \).

The first property of a filtration is easily verified by induction, using the Jacobi identity, and the second follows from the fact that \( \bigcap L_i \), clearly, is an ideal of \( L \) and so, by hypothesis, \( \bigcap L_i = 0 \).

The filtration constructed in this way is called the transitive filtration of the pair \((L, L_0)\).

With a Lie superalgebra \( L \) we can associate, in the usual way, the \( \mathbb{Z} \)-graded Lie superalgebra

\[
\text{Gr} L = \bigoplus_{i \geq -1} \text{Gr}_i L, \quad \text{where} \quad \text{Gr}_i L = L_i/L_{i+1}.
\]

Owing to the grading of the subspace \( L_i \), the algebra \( \text{Gr} L \) is equipped with a natural \( \mathbb{Z}_g \)-grading; however, the \( \mathbb{Z} \)-grading of \( \text{Gr} L \) is not, in general, consistent.

A \( \mathbb{Z} \)-graded superalgebra \( G = \bigoplus_{i \geq -1} G_i \) is canonically equipped with a filtration: \( L_i = \bigoplus_{i \geq i} G_i \).
A superalgebra $L$ with filtration is transitive if and only if $\text{Gr} L$ is transitive. If $\text{Gr} L$ is simple, then so is $L$.

1.3.2. Connection between $L$ and $\text{Gr} L$. If in a Lie superalgebra with filtration $L = L_{-1} \supset L_0 \supset L_1 \supset \cdots$ subspaces $G_i$ are given such that $L_s = G_s \oplus L_{s+1}$ and $[G_i, G_j] \subseteq G_{i+j}$, then we say that $L$ is equipped with a grading that is consistent with its filtration. In that case, clearly, $L \simeq \text{Gr} L$, provided that $L$ is finite-dimensional.

**Proposition 1.3.1.** Let $L = L_{-1} \supset L_0 \supset L_1 \supset \cdots$ be a transitive finite-dimensional Lie superalgebra with a filtration for which the representation of $\text{Gr}_0 L$ on $\text{Gr}_{-1} L$ is irreducible and even part of the center of $\text{Gr}_0 L$ is nontrivial. Then $L \simeq \text{Gr} L$.

**Proof.** According to Proposition 1.2.12 there is an element $z \in \text{Gr}_0 L$ such that $[z, g] = sg$ for $g \in \text{Gr}_s L$. Let $\tilde{z}$ be some inverse image of $z$ under the map $L_0 \to \text{Gr}_0 L = I_0/I_1$. As is easy to see, $\tilde{z}$ is diagonalizable in $L$, so that $L_s = G_s \oplus L_{s+1}$, where $G_s$ is the eigenspace of $\tilde{z}$ for the eigenvalue $s$. This gives us the required grading, consistent with the filtration of $L$.

1.3.3. Properties of filtrations. Let $L = L_0 \oplus L_1$ be a Lie superalgebra and $L_0$ be a maximal proper subalgebra containing $L_1$. Suppose that $L_0$ does not contain nonzero ideals of $L$. Let us construct the transitive filtration of the pair $(L, L_0)$ (see 1.1.1):

$$L_i = \{a \in L_{i-1} \mid [a, L] \subseteq L_{i-1}\}, \quad i > 0.$$  

Let $\text{Gr} L = \bigoplus_{i>-1} \text{Gr}_i L$ be the associated $\mathbb{Z}$-graded Lie superalgebra.

**Proposition 1.3.2.** $\text{Gr} L$ has the following properties:

(a) $\text{Gr} L$ is transitive;

(b) the $\mathbb{Z}$-grading of $\text{Gr} L$ is consistent with the $\mathbb{Z}_2$-grading;

(c) $\text{Gr} L$ is irreducible;

(d) if the representation of $L_0$ on $L_1$ is reducible, then $\text{Gr}_1 L \neq 0$.

**Proof.** (a) follows from the transitivity of $L$. The fact that $L_0$ contains $L_0$ implies that $\text{Gr}_{-1} L \subseteq (\text{Gr} L)_1$. By the transitivity of $\text{Gr} L$, we obtain (b) by induction.

Let us now prove (c). Suppose the contrary; then there exists a $\mathbb{Z}_2$-graded subspace $\tilde{L}$ of $L$ containing $L_0$ but different from $L$ and $L_0$ for which $[L_0, \tilde{L}] \subseteq \tilde{L}$. Then $\tilde{L} = L_0 \oplus V$, where $V \subseteq L_1$ and $[V, V] \subseteq L_0$ because $L_0 \supseteq L_0$. Therefore, $[\tilde{L}, \tilde{L}] = [L_0 \oplus V, L_0 \oplus V] = [L_0, L_0] + [L_0, V] + [V, V] \subseteq \tilde{L}$. But this contradicts the maximality of $L_0$.

Now we prove (d). If $\text{Gr}_1 L = 0$; then clearly $\text{Gr}_0 L = L_0$. Consequently, by (c), the representation of $L_0$ on $L_1$ is then irreducible.
Proposition 1.3.3. Lie superalgebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$ is solvable iff Lie algebra $G_{\bar{0}}$ is solvable.

Proof. Let $G_{\bar{0}}$ be solvable, $L_{\bar{0}}$ be a maximal proper subalgebra containing $G_{\bar{0}}$, and $J$ be a maximal ideal among ideals of $G$, containing in $L_{\bar{0}}$. We have a filtered Lie superalgebra $G/J = L \supseteq L_{\bar{0}} \supseteq L_{\bar{1}} \supseteq \cdots$, which satisfies all the conditions of Proposition 1.3.2. In particular, $\text{Gr} L$ is irreducible. But $\text{Gr}_{\bar{0}} L$ is solvable since $G_{\bar{0}}$ is solvable and so $\dim \text{Gr}_{\bar{0}} L = 0$ or $1$. Therefore $\dim G/J = 1$ or $2$ and $G/J$ is a solvable Lie superalgebra. By induction, $J$ is a solvable superalgebra too. So $G$ is a solvable Lie superalgebra.

1.4. Information from the Theory of Representations of Semisimple Lie Algebras

1.4.1. The theorem on the highest weight. Let $G$ be a semisimple Lie algebra and $H$ be a Cartan subalgebra of it. We consider a representation $\rho$ of $G$ in a finite-dimensional space $V$ (or, as we usually say, a $G$-module $V$). For $\lambda \in H^*$ we set $V_\lambda = \{v \in V \mid h(v) = \lambda(h)v\}$. If $V_\lambda \neq 0$, then $\lambda$ is called a weight of $\rho$ and a nonzero vector $v_\lambda$ in $V_\lambda$ is called a weight vector. Let $\mathcal{L}_\rho$ be the set of all weights; then $V = \bigoplus_{\lambda \in \mathcal{L}_\rho} V_\lambda$.

A weight $\alpha$ of the adjoint representation of $G$ is called a root of $G$. Then $G = \bigoplus_{\alpha} G_{\alpha}$, where $G_\alpha = H$ and $\dim G_{\alpha} = 1$ for $\alpha \neq 0$. A nonzero vector $e_\alpha$ in $G_{\alpha}$ is called a root vector. $G_{\alpha} V_\lambda \neq 0$ is contained in $V_{\lambda + \alpha}$ if $\lambda + \alpha \in \mathcal{L}_\rho$, and $G_{\alpha} V_\lambda = 0$ if $\lambda + \alpha \notin \mathcal{L}_\rho$.

Let $(a, b) = \text{tr}(\text{ad} a)(\text{ad} b)$ be the Killing form on $G$. Both the Killing form and its restriction to $H$ are nondegenerate; therefore, it induces on $H^*$ a nondegenerate form. If $\alpha \neq 0$, then $[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha}) h_\alpha \neq 0$, where the vector $h_\alpha \in H$ is determined by the relation $\alpha(h) = (h_\alpha, h)$.

Let $\Delta'$ be the set of all nonzero roots, $\Delta^+$ be the set of positive roots (in some fixed lexicographical ordering), and $\Sigma = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots of $G$. Then $\Delta' = \Delta^+ \cup -\Delta^+$, the system $\Sigma$ forms a basis of the space $H^*$, and every root $\alpha \in \Delta^+$ is of the form $\alpha = \sum k_i \alpha_i$, where the $k_i$ are nonnegative integers.

Let $H_0^*$ be the linear span of $\Delta'$ over $\mathbb{Q}$. The Killing form is positive definite on $H_0^*$, and $\mathcal{L}_\rho \subseteq H_0^*$. Let $\alpha \in \Delta'$, $\lambda \in \mathcal{L}_\rho$; then the set of weights of the form $\lambda + \alpha \beta$ forms a progression: $\lambda - p\alpha, \lambda - (p - 1)\alpha, \ldots, \lambda - \alpha, \lambda, \lambda + \alpha, \ldots, \lambda + q\alpha$, where $p$ and $q$ are nonnegative integers and $p - q = 2(\lambda, \alpha)/(\alpha, \alpha)$.

The numbers $2(\lambda, \alpha_i)/(\alpha_i, \alpha_i)$ are called the numerical marks of the linear function $\lambda \in H^*$.

If $\lambda \in \mathcal{L}_\rho$, then its numerical marks are integers. A function $\lambda \in H_0^*$ is said to be dominant if its numerical marks are nonnegative integers.

Now let $\rho$ be an irreducible finite-dimensional representation of a Lie algebra $G$. A highest (respectively, lowest) weight of $\rho$ is a weight $\Lambda \in \mathcal{L}_\rho$ (respectively, $M \in \mathcal{L}_\rho$) for which $\Lambda + \alpha \notin \mathcal{L}_\rho$ (respectively, $M + \alpha \notin \mathcal{L}_\rho$) for $\alpha \in \Delta^+$. The highest and lowest weights are unique, and $\dim V_\Lambda = \dim V_M = 1$. Every
nonzero vector in \( V_A \) (respectively, \( V_{\lambda'} \)) is called a highest (respectively, lowest) vector of \( \rho \). The theorem on the highest weight asserts that the function \( \Lambda \) is dominant and that for any dominant linear function \( \Lambda \) there is a unique irreducible finite-dimensional representation with the highest weight \( \Lambda \).

A representation \( \rho \) of a Lie algebra \( G \) in a space \( V \) induces a representation \( \rho^* \) of it in the dual space \( V^* \); \( \rho \) and \( \rho^* \) are said to be contragredient. \( \lambda \in \mathcal{L}_\rho \) if and only if \( -\lambda \in \mathcal{L}_{\rho^*} \). In particular, if \( \Lambda \) is the highest weight of \( \rho \), then \( (\Lambda) \) is the lowest weight of \( \rho^* \).

If a Lie algebra \( G \) has a faithful irreducible finite-dimensional representation \( \rho \), then \( G = G' \oplus C \), where \( G' \) is a semisimple Lie algebra, \( C \) is the center of \( G \), and \( \rho(C) \) are scalar operators. The restriction of \( \rho \) to \( G' \) is also irreducible and its highest weight (vector) is defined as the highest weight (vector) of \( \rho \).

1.4.2. Diagrams of highest and dominant roots. A semisimple Lie algebra can be represented by a Dynkin diagram. Let \( \Sigma = \{\alpha_1, \ldots, \alpha_r\} \) be the system of simple roots; then \( a_{ij} = -2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i) \) are nonnegative integers. The Dynkin diagram of \( G \) consists of \( r \) circles corresponding to the simple roots, and the \( i \)th circle is joined to the \( j \)th by an \( a_{ij} \alpha_{ij} \) segment with arrows pointing to the \( i \)th circle when \( a_{ij} < a_{ji} \). An irreducible representation of \( G \) is represented by a Dynkin diagram equipped with the nonzero numerical marks \( 2(\Lambda, \alpha_i)/(\alpha_i, \alpha_i) \) of the highest weight \( \Lambda \) standing against the corresponding circles.

If \( G \) is simple, then its adjoint representation is irreducible; its highest weight \( \theta \) is the highest root of \( G \). The diagrams of the highest roots of all simple Lie algebras are given in Table I. Apart from the highest roots, the simple Lie algebras also have the dominant roots given in Table II.

1.4.3. The index. Let \( \rho \) be a finite-dimensional faithful linear representation of a semisimple Lie algebra \( G \) in a space \( V \). Then the bilinear form \( (a, b)_V = \text{tr} \rho(a) \rho(b) \) on \( G \) is nondegenerate. If \( G \) is simple, then
\[
(a, b)_V = l_V(a, b), \quad a, b \in G,
\]
### TABLE II

<table>
<thead>
<tr>
<th>Type</th>
<th>Diagram</th>
<th>dim $\hat{V}$</th>
<th>$l_V$</th>
</tr>
</thead>
</table>
| $B_n(n > 3)$ | \[ \begin{array}{c}
         1 \\
         \circ \circ \circ \circ \circ \circ
       \end{array} \] $\Rightarrow$ \[ \begin{array}{c}
         1 \\
         \circ
       \end{array} \] | \[ \begin{array}{c}
         n \\
         \frac{n(n-1)}{2}
       \end{array} \] | $\frac{n-2}{2n}$ |
| $C_n(n > 2)$ | \[ \begin{array}{c}
         1 \\
         \circ \circ \circ \circ \circ \circ
       \end{array} \] $\Leftarrow$ \[ \begin{array}{c}
         1 \\
         \circ
       \end{array} \] | \[ \begin{array}{c}
         n \\
         \frac{n(n+1)}{2}
       \end{array} \] | $\frac{n+2}{2n}$ |

### TABLE III

<table>
<thead>
<tr>
<th>Type</th>
<th>$r$</th>
<th>Diagram</th>
<th>$\dim \hat{V}$</th>
<th>$l_V$</th>
</tr>
</thead>
</table>
| **A** $sl_n$ | $n-1$ | \[ \begin{array}{c}
         1 \\
         \circ \circ \circ \circ \circ \circ
       \end{array} \] $\Rightarrow$ \[ \begin{array}{c}
         1 \\
         \circ
       \end{array} \] | \[ n \] | $\frac{1}{2n}$ |
| $A^2 sl_n$ | $n-1$ | \[ \begin{array}{c}
         1 \\
         \circ \circ \circ \circ \circ \circ
       \end{array} \] $\Leftarrow$ \[ \begin{array}{c}
         1 \\
         \circ
       \end{array} \] | \[ \frac{n(n-1)}{2} \] | $\frac{n-2}{2n}$ |
| $S^2 sl_n$ | $n-1$ | \[ \begin{array}{c}
         1 \\
         \circ \circ \circ \circ \circ \circ
       \end{array} \] $\Rightarrow$ \[ \begin{array}{c}
         1 \\
         \circ
       \end{array} \] | \[ \frac{n(n+1)}{2} \] | $\frac{n+2}{2n}$ |
| $A^3 sl_n$ | $5, 6, 7$ | \[ \begin{array}{c}
         1 \\
         \circ \circ \circ \circ \circ \circ
       \end{array} \] $\Leftarrow$ \[ \begin{array}{c}
         1 \\
         \circ
       \end{array} \] | \[ 20, 35, 56 \] | $\frac{1}{2}, \frac{5}{6}, \frac{15}{8}$ |
| $B$ $so_n$ | \[ \frac{n-1}{2} \] | \[ \begin{array}{c}
         1 \\
         \circ \circ \circ \circ \circ \circ
       \end{array} \] $\Rightarrow$ \[ \begin{array}{c}
         1 \\
         \circ
       \end{array} \] | \[ n \] | $\frac{1}{n-2}$ |
| $spin_n$ | $3, 4, 5, 6$ | \[ \begin{array}{c}
         1 \\
         \circ \circ \circ \circ \circ \circ
       \end{array} \] $\Rightarrow$ \[ \begin{array}{c}
         1 \\
         \circ
       \end{array} \] | \[ 8, 16, 32, 64 \] | $\frac{1}{2}, \frac{5}{6}, \frac{11}{6}, \frac{11}{4}$ |
| $C$ $sp_n$ | \[ \frac{n}{2} \] | \[ \begin{array}{c}
         1 \\
         \circ \circ \circ \circ \circ \circ
       \end{array} \] $\Leftarrow$ \[ \begin{array}{c}
         1 \\
         \circ
       \end{array} \] | \[ n \] | $\frac{1}{n+2}$ |
| $A^2 sp_n$ | \[ \frac{n}{2} \] | \[ \begin{array}{c}
         1 \\
         \circ \circ \circ \circ \circ \circ
       \end{array} \] $\Leftarrow$ \[ \begin{array}{c}
         1 \\
         \circ
       \end{array} \] | \[ \frac{n(n-1)}{2} - 1 \] | $\frac{n-2}{n+2}$ |
| $A^3 sp_n$ | $3$ | \[ \begin{array}{c}
         1 \\
         \circ \circ \circ \circ \circ \circ
       \end{array} \] $\Leftarrow$ \[ \begin{array}{c}
         1 \\
         \circ
       \end{array} \] | \[ 14 \] | $\frac{5}{8}$ |
| $D$ $so_n$ | \[ \frac{n}{2} \] | \[ \begin{array}{c}
         1 \\
         \circ \circ \circ \circ \circ \circ
       \end{array} \] $\Rightarrow$ \[ \begin{array}{c}
         1 \\
         \circ
       \end{array} \] | \[ n \] | $\frac{1}{n-2}$ |
| $spin_n$ | $5, 6, 7$ | \[ \begin{array}{c}
         1 \\
         \circ \circ \circ \circ \circ \circ
       \end{array} \] $\Rightarrow$ \[ \begin{array}{c}
         1 \\
         \circ
       \end{array} \] | \[ 16, 32, 64 \] | $\frac{1}{2}, \frac{5}{6}, \frac{11}{6}$ |
| **E** $E_6$ | $6$ | \[ \begin{array}{c}
         1 \\
         \circ \circ \circ \circ \circ \circ
       \end{array} \] $\Rightarrow$ \[ \begin{array}{c}
         1 \\
         \circ
       \end{array} \] | \[ 27 \] | $\frac{1}{4}$ |
| $E_7$ | $7$ | \[ \begin{array}{c}
         1 \\
         \circ \circ \circ \circ \circ \circ \circ
       \end{array} \] $\Rightarrow$ \[ \begin{array}{c}
         1 \\
         \circ
       \end{array} \] | \[ 56 \] | $\frac{1}{3}$ |
| **F** $F_4$ | $4$ | \[ \begin{array}{c}
         1 \\
         \circ \circ \circ \circ \circ \circ
       \end{array} \] $\Rightarrow$ \[ \begin{array}{c}
         1 \\
         \circ
       \end{array} \] | \[ 26 \] | $\frac{1}{3}$ |
| **G** $G_2$ | $2$ | \[ \begin{array}{c}
         1 \\
         \circ \circ \circ \circ \circ \circ
       \end{array} \] $\Rightarrow$ \[ \begin{array}{c}
         1 \\
         \circ
       \end{array} \] | \[ 7 \] | $\frac{1}{4}$ |
where \( l_\nu \) is a positive rational number, independent of \( a \) and \( b \). It is called the index of \( \rho \). The index of a direct sum of representations is the sum of the indices of the representations.

The index of a one-dimensional representation of a simple Lie algebra \( G \) is 0, and that for irreducible representation is 1 only of the adjoint representation. A list of the irreducible representations of the simple Lie algebras for which \( 0 < l_\nu < 1 \) is given, to within transition to a contragredient representation, in Table III, which is taken from [1].

Here \( \mathfrak{sl}_n \), \( \mathfrak{sp}_n \), and \( \mathfrak{so}_n \) stand for standard representations of these Lie algebras; \( S^k \) and \( \Lambda^k \) denote \( k \)th symmetrical and exterior degrees, respectively, \( S^{0^k} \) and \( \Lambda_0^k \) their highest component, \( \text{spin}_k \) stands for irreducible spinor representation of \( \mathfrak{so}_k \); \( G_2 \), \( F_4 \), \( E_6 \), \( E_7 \) denote also the simplest representations of the corresponding Lie algebras.

1.4.4. A technical lemma. We now prove a lemma on representations that is used in an essential way in the classification of the classical Lie superalgebras.

**Lemma 1.4.1.** Let \( \rho \) be a faithful irreducible finite-dimensional representation of a semisimple Lie algebra \( G \) in a space \( V \). Let \( \Delta \) be the system of all roots of \( G \), \( \mathcal{L} \) the system of weights of \( \rho \), and \( \Lambda \) the highest weight.

(a) If \( 2\Lambda \in \Delta \), then the \( G \)-module \( V \) is isomorphic to \( \mathfrak{sp}_n \); 
(b) if \( \Lambda - \mu \in \Delta \) for any \( \mu \in \mathcal{L} \), then \( G \)-module \( V \) is isomorphic to \( \mathfrak{sl}_n \) or \( \mathfrak{sp}_n \); 
(c) if \( \Lambda - \mu \in \Delta \) for any \( \mu \in \mathcal{L} \), \( \mu \neq -\Lambda \), then \( G \)-module \( V \) is isomorphic to \( \mathfrak{sl}_n \), \( \mathfrak{sp}_n \), \( \mathfrak{so}_n \), \( \text{spin}_r \), or \( G_2 \).

**Proof.** Let \( M \) be the lowest weight of \( \rho \). Observe that \( 2\Lambda \) and \( \Lambda - M \) can lie in \( \Delta \) only when \( G \) is simple. This is, therefore, the case in (a) and (b).

(a) By hypothesis, \( 2\Lambda \in \Delta \). However, from Tables I and II it is clear that only \( \mathfrak{C}_r \) has a dominant root with mark 2. Half this root is the highest weight of the \( \mathfrak{C}_r \)-module \( \mathfrak{sp}_{2r} \). This proves (a).

(b) By hypothesis, \( \Lambda - M \in \Delta \); this root is dominant, and the sum of its marks is not less than 2. From Tables I and II it is clear that \( \Lambda - M = \theta \) is the highest root of one of the Lie algebras \( \mathfrak{A}_r \) or \( \mathfrak{C}_r \); therefore, the \( G \)-module \( V \) is isomorphic to \( \mathfrak{sl}_{r+1} \) or \( \mathfrak{sp}_{2r} \), respectively.

(c) If \( \Lambda \neq -M \), then by hypothesis \( \Lambda - M \in \Delta \), and from the proof of (b) it is clear that \( V \) is isomorphic to \( \mathfrak{sl}_n \). Now let

\[
\Lambda = -M
\]  

(1.4.1)

If \( \theta \) is the highest root of one of the simple components of \( G \), then clearly \( \Lambda - \theta \in \mathcal{L} \). If \( \Lambda - \theta = M \), then from (1.4.1) and the proof of (b) it follows that \( V \) is isomorphic to \( \mathfrak{sp}_n \). But if \( \Lambda - \theta \neq M \), then there is a simple root \( \alpha \)
for which $A - \theta - \alpha \in \mathcal{L}$. Therefore, $A - \theta - \alpha = M$, because otherwise, by hypothesis and (1.4.1), $A - (A - \theta - \alpha) = \theta + \alpha \in A$, which is impossible. Hence,
\[2A - \theta = \alpha \text{ is a simple root.} \quad (1.4.2)\]

From (1.4.2) it follows that if $G$ is not simple, then the number of simple components is 2 and the highest root of each simple component is a simple root. Therefore, $G = A_1 \oplus A_1$, and $\mathcal{V}$ is isomorphic to $\mathfrak{so}_4 = \mathfrak{sl}_2 \otimes \mathfrak{sl}_2$.

Now suppose that $G$ is simple. Now (1.4.2) means that $A = \frac{1}{2}(\theta + \alpha)$ is a dominant linear function. If the circle corresponding to $\alpha$ is not at an end of the diagram, then it has at least two negative marks, so that $\theta$ has at least two positive marks. From Table I we see that in this case $G$ is of type $A_r$, moreover, that $r = 3$, $\alpha = \alpha_3$, and $\mathcal{V}$ is isomorphic to $A^2\mathfrak{sl}_4 = \mathfrak{so}_6$. But if the circle is at an end of the diagram and $r > 2$, then the mark of $\alpha$ is negative, therefore, the positive mark of $\theta$ is not at an end. From Table I we see that in this case $G$ is of type $B_r$ or $D_r$, $\alpha = \alpha_1$, and $\mathcal{V}$ is isomorphic to $\mathfrak{so}_n$ with $n > 6$, or $G$ is of type $B_3$, $\alpha = \alpha_3$, and $\mathcal{V}$ is isomorphic to $\text{spin}_7$. Finally, the case $r \leq 2$, as is easy to see, gives the $G$-modules $\mathfrak{so}_n$ with $n = 3, 5$ and $G_2$.

The proof of the lemma is now complete.

2. Classical Lie Superalgebras

A finite-dimensional Lie superalgebra $G = G_0 \oplus G_\mathfrak{t}$ is called classical if it is simple and the representation of $G_0$ on $G_\mathfrak{t}$ is completely reducible.

The aim of this chapter is the description and classification of the classical Lie superalgebras.

2.1. Examples of Classical Lie Superalgebras

2.1.1. The Lie superalgebras $\mathfrak{A}(m, n)$. We recall some facts from Section 1.1. Let $\mathcal{V} = V_0 \oplus V_\mathfrak{t}$ be a $\mathbb{Z}_2$-graded space, $\dim V_0 = m$, $\dim V_\mathfrak{t} = n$. The associative algebra $\text{End} \mathcal{V}$ becomes an associative superalgebra if we let $\text{End}_i \mathcal{V} = \{a \in \text{End} \mathcal{V} \mid aV_{s, i} \subseteq V_{i+s, i}\}$, for $i, s \in \mathbb{Z}_2$.

The bracket $[a, b] = ab - (-1)^{\deg a \deg b} ba$ makes $\text{End} \mathcal{V}$ into a Lie superalgebra, denoted by $l(\mathcal{V})$ or $l(m, n)$. If we regard the same decomposition $\mathcal{V} = V_0 \oplus V_\mathfrak{t}$ as a $\mathbb{Z}$-grading of $\mathcal{V}$, then the same construction gives a consistent $\mathbb{Z}$-grading: $l(\mathcal{V}) = G_{-1} \oplus l(V_0) \oplus G_1$. On $l(\mathcal{V})$ we define the supertrace, a linear function $\text{str}: l(\mathcal{V}) \to k$. Its basic property is $\text{str}([a, b]) = 0$, $a, b \in l(\mathcal{V})$.

From this it follows that the subspace

$$sl(m, n) = \{a \in l(m, n) \mid \text{str} a = 0\}$$
is an ideal in \( l(m, n) \) of codimension 1. The resulting \( \mathbb{Z} \)-grading \( sl(m, n) = G_{-1} \oplus sl(m, n)_\mathbb{Z} \oplus G_1 \) looks in some homogeneous basis of \( V \) as follows: 

\[
sl(m, n)_\mathbb{Z} \text{ is the set of matrices of the form } (\begin{array}{cc} a & 0 \\ 0 & b \end{array}), \quad \text{where } tr a = tr b, \quad G_1 \text{ is the set of matrices of the form } (\begin{array}{cc} a & 0 \\ 0 & b \end{array}) \text{ and } G_{-1} \text{ of the form } (\begin{array}{cc} a & 0 \\ 0 & b \end{array}) \text{ (where } a \text{ is an } (m \times m)\text{-, } b \text{ an } (n \times n)\text{-, } / \text{ an } (m \times n)\text{-, and } y \text{ an } (n \times m)\text{-matrix}).
\]

Now \( sl(n, n) \) contains the one-dimensional ideal consisting of the scalar matrices \( \lambda 1_{2n} \). The Lie superalgebra \( sl(1, 1) \) is three-dimensional and nilpotent.

We set

\[
A(m, n) = sl(m + 1, n + 1) \quad \text{for } m \neq n, \quad m, n \geq 0,
A(n, n) = sl(n + 1, n + 1)/\langle 1_{2n+2} \rangle, \quad n > 0.
\]

The \( \mathbb{Z} \)-grading of \( sl(m + 1, n + 1) \) induces a \( \mathbb{Z} \)-grading of \( A(m, n) \) of the form 

\[
A(m, n) = G_{-1} \oplus G_0 \oplus G_1.
\]

2.1.2. The Lie superalgebras \( B(m, n) \), \( D(m, n) \) and \( C(n) \). Again, let \( V = V_0 \oplus V_1 \) be a \( \mathbb{Z}_2 \)-graded space, \( \dim V_0 = m \), \( \dim V_1 = n \). Let \( F \) be a non-degenerate consistent supersymmetric bilinear form on \( V \), so that \( V_0 \) and \( V_1 \) are orthogonal and the restriction of \( F \) to \( V_0 \) is a symmetric and to \( V_1 \) a skew-symmetric form (in particular, \( n = 2r \) is even).

We define in \( l(m, n) \) the subalgebra \( osp(m, n) = osp(m, n)_0 \oplus osp(m, n)_1 \) by setting

\[
osp(m, n)_s = \{ a \in l(m, n), \mid F(a(x), y) - -(-1)^{s(deg x)}F(x, a(y)) \}, \quad s \in \mathbb{Z}_2.
\]

We call \( osp(m, n) \) an orthogonal-symplectic superalgebra (for \( n = 0 \) or \( m = 0 \) it turns into an orthogonal or symplectic Lie algebra, respectively).

Let us find the explicit matrix form of the elements of \( osp(m, n) \). We treat two cases separately.

\( m = 2l + 1 \). In some basis the matrix of the form \( F \) can be written as

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

from which we see that a matrix in \( osp(m, n) \) is of the form

\[
\begin{bmatrix}
a & b & u & x & x_1 \\
c & -a^T & v & y & y_1 \\
-c^T & -u^T & 0 & z & z_1 \\
-x_1^T & -x_1^T & z_1^T & d & e \\
y^T & -x^T & -z^T & f & -d^T
\end{bmatrix}.
\]
here $a$ is any $(l \times l)$-matrix; $b$ and $c$ are skew-symmetric $(l \times l)$-matrices; $d$ is any $(r \times r)$-matrix; $e$ and $f$ are symmetric $(r \times r)$-matrices; $n$ and $v$ are $(l \times 1)$-matrices; $x$ and $y$ are $(l \times r)$-matrices, and $z$ is an $(r \times 1)$-matrix.

In particular, we see that $\text{osp}(m, n)_0$ is a Lie algebra of type $B_\ell \oplus C_r$, and the $\text{osp}(m, n)_0$-module $\text{osp}(m, n)_1$ is isomorphic to $\text{so}_m \otimes \text{sp}_n$.

$m = 2l$. In some basis the matrix of $F$ can be written as

$$
\begin{bmatrix}
0 & 1 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 1 & \cdots & 0 \\
-1 & 0 & \cdots & 0
\end{bmatrix},
$$

from which we see that a matrix in $\text{osp}(m, n)$ has the same form as in the first case, with the middle row and column deleted.

In particular, we find that $\text{osp}(m, n)$ for $l \geq 2$ is a Lie algebra of type $D_\ell \oplus C_r$, and that the $\text{osp}(m, n)_0$-module $\text{osp}(m, n)_1$ is isomorphic to $\text{so}_m \otimes \text{sp}_n$.

By analogy with Cartan's notation we set:

- $B(m, n) = \text{osp}(2m + 1, 2n)$, $m \geq 0$, $n > 0$;
- $D(m, n) = \text{osp}(2m, 2n)$, $m \geq 2$, $n > 0$;
- $C(n) = \text{osp}(2, 2n - 2)$, $n \geq 2$.

We now examine the Lie superalgebra $C(n)$. Subalgebra $C(n)_0$ consists of matrices of the form

$$
\begin{bmatrix}
\alpha & 0 \\
0 & -\alpha \\
\end{bmatrix}
\begin{bmatrix}
a & b \\
c & -a^T \\
\end{bmatrix},
$$

where $a$, $b$, and $c$ are $(n - 1 \times n - 1)$-matrices, $b$ and $c$ being symmetric, and $\alpha \in k$. Furthermore, $C(n)$ has the consistent $\mathbb{Z}$-grading: $C(n) = G_{-1} \oplus C(n)_0 \oplus G_1$, where $G_{-1}$ and $G_1$ consist of the matrices of the form (respectively):

$$
\begin{bmatrix}
0 & 0 & 0 \\
y_1 & 0 & \cdots \\
\end{bmatrix}
\begin{bmatrix}
x_1 & x_1 & \cdots \\
0 & 0 & \cdots \\
\end{bmatrix}.
$$

The representations of $C(n)_0$ on $G_{-1}$ and $G_1$ are contragradient, and the $C(n)_0$-module $G_1$ is isomorphic to $\text{csp}_{2n-2}$.
Supplement. We consider another realization of \(\text{osp}(m, n)\). Let \(V_0\) be an \(m\)-dimensional space with a nondegenerate symmetric bilinear form \((\cdot, \cdot)_0\) and \(V_1\) an \(n\)-dimensional space with a nondegenerate skew-symmetric bilinear form \((\cdot, \cdot)_1\), \(n = 2r\). Then \(\text{osp}(m, n)\) can be realized as follows:

\[
\text{osp}(m, n)_0 = \Lambda^2 V_0 \oplus S^2 V_1, \\
\text{osp}(m, n)_1 = V_0 \otimes V_1.
\]

The definitions of the operations are

\[
[a \wedge b, c] = (a, c)_0 b - (b, c)_0 a, \\
[a \circ b, c] = (a, c)_1 b + (b, c)_1 a,
\]

where \(a \wedge b \in \Lambda^2 V_0\), \(c \in V_0\); \(a \circ b \in S^2 V_1\), \(c \in V_1\). These brackets define brackets on \(\Lambda^2 V_0\) and \(S^2 V_1\) in the usual way:

\[
[ab, cd] = [ab, c]d + [ab, d].
\]

Finally, for \(a \otimes c, b \otimes d \in V_0 \otimes V_1\) we set

\[
[a \otimes c, b \otimes d] = (a, b)_0 c \circ d + (c, d)_1 a \wedge b.
\]

In this realization there is a natural way of defining an interesting \(\mathbb{Z}\)-grading of \(\text{osp}(m, n)\):

\[
\text{osp}(m, n) = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_1 \oplus G_2.
\]

To obtain this, we represent \(V_1\) as a direct sum of isotropic subspaces \(V_1 = V_1' \oplus V_1^*\). The following decomposition is then a \(\mathbb{Z}\)-grading:

\[
\text{osp}(m, n) = S^2 V_1' \oplus (V_0 \otimes V_1') \oplus (V_1' \otimes V_1^* \oplus \Lambda^2 V_0) \oplus (V_0 \otimes V_1^*) + S^2 V_1^*.
\]

Clearly, \(G_0 \simeq gl_r \oplus so_n\), the representations of \(G_0\) on \(G_i\) and \(G_{-i}\) are contragredient, the \(G_0\)-module \(G_1\) is isomorphic to \(gl_r \otimes so_n\), and \(G_2\) to \(S^2 gl_r\).

2.1.3. The Lie superalgebra \(\mathbf{P}(n), n \geq 2\). This is a subalgebra of \(sl(n+1, n+1)\), consisted of the matrices of the form:

\[
\left(\begin{array}{c|c}
  a & b \\
  \hline
  c & -a^t
\end{array}\right)
\]

where \(\text{tr } a = 0\), \(b\) is a symmetric matrix, and \(c\) is a skew-symmetric matrix.

2.1.4. The Lie superalgebra \(\mathbf{Q}(n), n \geq 2\). First we denote \(\overline{\mathbf{Q}}(n)\) a subalgebra of \(sl(n + 1, n + 1)\), consisting of the matrices of the form

\[
\left(\begin{array}{c|c}
  a & b \\
  \hline
  b & a
\end{array}\right),
\]
where tr $b = 0$. Lie superalgebra $Q(n)$ has one-dimensional center $C = \langle 1_{2n+2} \rangle$. We put $Q(n) = Q(n)/C$.

2.1.5. The Lie superalgebras $F(4)$, $G(3)$, and $D(2, 1; \alpha)$.

**Proposition 2.1.1.** (a) There is one and only one 40-dimensional classical Lie superalgebra $F(4)$ for which $F(4)_0$ is a Lie algebra of type $B_3 \oplus A_1$ and its representation on $F(4)_1$ is $\text{spin} \otimes sl_2$.

(b) There is one and only one 31-dimensional classical Lie superalgebra $G(3)$ for which $G(3)_0$ is a Lie algebra of type $G_2 \oplus A_1$ and its representation on $G(3)_1$ is $G_2 \otimes sl_2$.

(c) There is a one-parameter family of 17-dimensional Lie superalgebras $D(2, 1; \alpha)$, $\alpha \in k^* \setminus \{0, -1\}$, consisting of all simple Lie superalgebras for which $D(2, 1; \alpha)_0$ is a Lie algebra of type $A_1 \oplus A_1 \oplus A_1$ and its representation on $D(2, 1; \alpha)_1$ is $sl_2 \otimes sl_2 \otimes sl_2$.

The proof can be obtained by a direct construction of epimorphisms of $G_\alpha$-modules $S^\alpha G_1 \to G_\beta$ satisfying (1.1.3). However, in Section 2.5 (Proposition 2.5.4) we give an alternative proof, by means of contragredient Lie superalgebras (cf. [8, 11]).

2.1.6. Properties and uniqueness. From the description of the classical Lie superalgebras in Sections 2.1.1-2.1.5 above and Propositions 1.2.7 and 1.2.8, we derive the following result.

**Proposition 2.1.2.** (a) All the Lie superalgebras $A(m, n)$, $B(m, n)$, $C(n)$, $D(m, n)$, $D(2, 1; \alpha)$, $F(4)$, $G(3)$, $P(n)$, $Q(n)$ are classical.

(b) For the Lie superalgebras $B(m, n)$, $D(m, n)$, $D(2, 1; \alpha)$, $F(4)$, $G(3)$, and $Q(n)$ the $G_\alpha$-module $G_1$ is irreducible and isomorphic to the modules in the following list:

\[
\begin{array}{ccc}
G & G_0 & G_0 \mid G_1 \\
B(m, n) & B_m \oplus C_n & so_{2m+1} \oplus sp_{2n} \\
D(m, n) & D_m \oplus C_n & so_{2m} \oplus sp_{2n} \\
D(2, 1; \alpha) & A_1 \oplus A_1 \oplus A_1 & sl_2 \otimes sl_2 \otimes sl_2 \\
F(4) & B_3 \oplus A_1 & \text{spin} \otimes sl_2 \\
G(3) & G_2 \oplus A_1 & G_2 \otimes sl_2 \\
Q(n) & A_n & ad sl_{n+1}
\end{array}
\]

(c) The Lie superalgebras $A(m, n)$, $C(n)$, and $P(n)$ admit a unique consistent $\mathbb{Z}$-grading of the form $G_{-1} \oplus G_0 \oplus G_1$. Here the $G_\alpha$-modules $G_1$ and $G_{-1}$ are irreducible and for $A(m, n)$ and $C(n)$ contragredient; they are isomorphic to the modules in the following list:
Proposition 2.1.3. Let $G = G_0 \oplus G_1$ be one of the Lie superalgebras $A(m, n)$, $B(m, n)$, $C(n)$, $D(m, n)$, $D(2, 1; \omega)$, $F(4)$, $G(3)$, $P(n)$, or $Q(n)$. Then the $G_0$-module $S^2 G_1$ contains $G_0$ with multiplicity 1.

This is not hard to prove, by using the table in [9]. Here we can also exploit the fact that in the tensor product of two irreducible $G_0$-modules, of which one has a simple spectrum, the multiplicity of any simple submodule is at most 1.

Proposition 2.1.4. Let $G = G_0 \oplus G_1$ be a simple Lie superalgebra for which the representation of $G_1$ on $G_0$ is the same as for one of the Lie superalgebras $A(m, n)$, $B(m, n)$, $C(n)$, $D(m, n)$ where $(m, n) \neq (2, 1)$, $F(4)$, $G(3)$, $P(n)$, or $Q(n)$. Then $G$ is isomorphic to this algebra.

Proof. Let $\Phi : S^2 G_1 \to G_0$ be the homomorphism of $G_0$-modules defined by the bracket on $G_1$, and let $\Phi'$ be the same map for the corresponding superalgebra, as listed in the proposition. On account of simplicity, $\Phi$ and $\Phi'$ are epimorphisms. By Proposition 2.1.3, $\Phi$ and $\Phi'$ are projections of $S^2 G_1$ onto the same subspace. Since the homomorphism $\Phi$, to within a constant factor, determines the superalgebra uniquely, we have to show that $\Phi$ and $\Phi'$ are proportional projections onto $G_0 \subseteq S^2 G_1$. If $G_0$ is simple, this is clear; therefore, we assume that $G_0$ is not simple. The projections $\phi$ and $\phi'$ can be decomposed: $\Phi = \Phi_0 + \Phi_1 + \cdots$, $\Phi' = \Phi_0' + \Phi_1' + \cdots$, where $\Phi_0$ and $\Phi_0'$ are projections onto the center, and $\Phi_i$ and $\Phi_i'$, $i > 0$, are projections onto the simple components; $\Phi_1, \Phi_1' \neq 0$; by Schur's lemma, $\Phi_i = c_i \Phi_i'$, $c_i \in \mathbb{C}^\times$.

Now we observe that the kernel of each bilinear map $\Phi_i$ (similarly, $\Phi_i'$): $S^2 G_1 \to G_0$ is trivial. For $\ker \Phi_i \subseteq G_1$ is a $G_0$-submodule; therefore, if $G_0$ is irreducible on $G_1$, then $\ker \Phi_i = 0$; but if $G_0$ is reducible on $G_1$, then $G_1 = G_{1 \ominus} \oplus G_1$ is a direct sum of irreducible $G_0$-modules, and if $G_{1 \ominus} \subseteq \ker \Phi_i$; then we see again that $\ker \Phi_i = 0$ because $\Phi_i(G_1, G_1) = 0$.

Now suppose that $\Phi$ and $\Phi'$ are not proportional. Taking $\Phi'' = \Phi + c \Phi'$ for a suitable $c \in \mathbb{C}$, we may assume that $\Phi''(G_1, G_1) = H_0$ is a nonzero ideal in $G_0$ that does not contain simple component $G_0^{(s)}$ of $G_0$ of maximal dimension. From what we have said above it follows that $H = H_0 \oplus G_1$ is a simple Lie superalgebra. Evidently, in all cases listed in the proposition the $H_0$-module $G_1$ contains more than two irreducible components. This contradicts Proposition 2.2.2 below, and Proposition 2.1.4 is proved.
2.2. Splitting of the Classification of Classical Lie Superalgebras into Two Cases

Let $G = G_0 \oplus G_1$ be a classical Lie superalgebra. Then $G_0 = G_0' \oplus C$, where $G_0'$ is a semisimple Lie algebra and $C$ is the center of $G$. We treat two cases separately:

Case I. The representation of $G_0$ on $G_1$ is irreducible. Then $G_0$ is a semisimple Lie algebra. If this is not so, then there exists a center element $z \in G_0$ for which $[z, g] = 2g$ for a $g \in G_0$, a contradiction.

Case II. The representation of $G_0$ on $G_1$ is reducible. Then we consider in $G$ a proper maximal subalgebra $L_0$ containing $G_0$. We construct the appropriate transitive filtration (see Section 1.3.1) $G = L_{-1} \supset L_0 \supset L_1 \supset \cdots$, where

$$L_i = \{ a \in L_{i-1} | [a, L] \subseteq L_{i-1} \} , \quad i > 0 .$$

Let $Gr L = \bigoplus_{i>1} Gr_i L$, where $Gr_i L = L_i / L_{i+1}$ is an associated $\mathbb{Z}$-graded Lie superalgebra. This filtration induces one on $G_0$: $G_0 = (L_0 \cap G_0) \supset (L_1 \cap G_0) \supset \cdots$. Since $G_0$ is a reductive Lie algebra, we see that $L_1 \cap G_0 = 0$.

Since the $\mathbb{Z}$-grading of $Gr L$ is consistent with the $\mathbb{Z}_2$-grading (Proposition 1.3.2), we have $Gr G_0 = \bigoplus_{i>0} Gr_{2i} L$, from which it follows that $Gr_0 L = 0$, because $L_1 \cap G_0 = 0$.

Thus, $Gr L = Gr_{-1} L \oplus Gr_0 L \oplus Gr_1 L$. From Proposition 1.3.2 and Proposition 1.2.11, we now obtain:

**Lemma 2.2.1.** Let $G = G_0 \oplus G_1$ be a classical Lie superalgebra for which the representation of $G_0$ on $G_1$ is reducible. Then $G$ has a filtration $G = L_{-1} \supset L_0 \supset L_1$ with the following properties:

(a) transitivity;

(b) $L_0 = G_0 \oplus L_1$;

(c) the representations of $G_0$ on $L_{-1} / L_0$ and on $L_1$ are irreducible;

(d) either the representation of $G_0$ on $L_1$ is faithful or $\dim L_1 = 1$.

We now derive from Lemma 2.2.1 the following result:

**Proposition 2.2.2.** Let $G = G_0 \oplus G_1$ be a classical Lie superalgebra for which the representation of $G_0$ on $G_1$ is reducible. Then $G$ has a filtration $G = L_{-1} \supset L_0 \supset L_1$ for which $Gr L = Gr_{-1} L \oplus Gr_0 L \oplus Gr_1 L$ is a simple $\mathbb{Z}$-graded Lie superalgebra; the representations of $Gr_0 L$ on $Gr L$ and $Gr_{-1} L$ are faithful and irreducible, $(Gr L)_0 = Gr_0 L \simeq G_0$, and the representation of $G_0$ on $G_1$ is equivalent to that of $Gr_0 L$ on $Gr_{-1} L \oplus Gr_1 L$.

**Proof.** If the center of $Gr_0 L$ is nontrivial, then $L \simeq Gr L$, according to Proposition 1.3.1, and there is nothing to prove. But if it is trivial, then $[Gr_{-1} L, Gr_1 L] = Gr_0 L$, by Proposition 1.2.9. If, in addition, the representation
of $\text{Gr}_0 L$ on $\text{Gr}_1 L$ is faithful, then all the conditions of Proposition 1.2.8 are satisfied, and therefore, $\text{Gr} L$ is simple.

According to Lemma 2.2.1 it only remains to show that the case $\dim L_1 = 1$ is impossible. Now $G = V \oplus G_0 \oplus L_1$ is a decomposition into the direct sum of $G_0$-invariant subspaces, $[V, L_1] = G_0$ and $[V, V] \subseteq G_0$, from parity arguments, and $[G_0, L_1] = 0$. Hence, it follows that $V \oplus G_0$ is an ideal in $G$, which contradicts simplicity.

Thus, Case II leads us to a classification of the simple $\mathbb{Z}$-graded Lie superalgebras $G = G_{-1} \oplus G_0 \oplus G_1$, with $G_0 = G_0$, where the representations of $G_0$ on $G_1$ and $C_{-1}$ are faithful and irreducible.

2.3. Classification of Lie Superalgebras with Nondegenerate Killing Form

2.3.1. Definition and properties of the Killing form. The Killing form on a Lie superalgebra $G$ is the bilinear form

$$(a, b) = \text{str}((\text{ad } a)(\text{ad } b)).$$

From the properties of the supertrace (see Proposition 1.1.2) we obtain corresponding properties of the Killing form.

**Proposition 2.3.1.** The Killing form on a Lie superalgebra $G_0 \oplus G_1$ has the following properties:

- $(a, b) = 0$ for $a \in G_0$, $b \in G_1$ (consistency),
- $(a, b) = (-1)^{\deg a \deg b}(b, a)$ (supersymmetry),
- $([a, b], c) = (a, [b, c])$ (invariance).

From Proposition 2.3.1 we derive, in particular, the next result.

**Proposition 2.3.2.** If the Killing form on $G = G_0 \oplus G_1$ is nondegenerate, then its restriction to $G_0$ is nondegenerate, and its restriction to $G_1$ gives a nondegenerate bilinear skew-symmetric form that is invariant under the representation of $G_0$ on $G_1$.

Just as for Lie algebras (see [10], for example), we can prove the following two propositions.

**Proposition 2.3.3.** A Lie superalgebra with a nondegenerate Killing form splits into an orthogonal direct sum of Lie superalgebras (with nondegenerate Killing forms).

**Proposition 2.3.4.** Every derivation of a Lie superalgebra with nondegenerate Killing form is inner.
Proposition 2.3.5. Let $G = \bigoplus G_i$ be a $\mathbb{Z}$-graded Lie superalgebra with nondegenerate Killing form. Then

(a) $(G_i, G_j) = 0$ for $i \neq -j$,

(b) $(a, G_i) \neq 0$ for $a \in G_{-i}$, $a \neq 0$,

(c) the representations of $G_0$ on $G_i$ and $G_{-i}$ are contragredient,

(d) there is an element $z \in G_0$ for which $[z, g] = sg$ for $g \in G_s$.

Proof. If $a \in G_i$, $b \in G_j$, then $(\text{ad} a)(\text{ad} b)$ for $i + j \neq 0$ is clearly a nilpotent operator on $G$; therefore, $(a, b) = 0$, which gives (a). Now (b) follows from (a) and the fact that the Killing form is nondegenerate; (c) follows from (b).

Let us prove (d). The endomorphism $D$ for which $D(g) = sg$ for $g \in G_s$ evidently gives rise to a derivation of degree 0. According to Proposition 2.3.4, this derivation is inner.

Now let $G = G_0 \oplus G_1$ be a Lie superalgebra. On $G_0$ we can define two bilinear forms:

$$(a, b)_0 = \text{tr}(\text{ad} a)(\text{ad} b)|_{G_0} \quad \text{and} \quad (a, b)_1 = \text{tr}(\text{ad} a)(\text{ad} b)|_{G_1}. \quad (2.3.1)$$

By the definition of the Killing form:

$$(a, b) = (a, b)_0 - (a, b)_1 \quad \text{for} \quad a, b \in G_0. \quad (2.3.2)$$

From (2.3.1) and (2.3.2) it follows that if $G_0 = G_0' \oplus G_1'$ is a direct sum of Lie algebras and $G_0'$ is simple, then

$$(a, b) = (1 - l)(a, b)_0 \quad \text{for} \quad a, b \in G_0', \quad (2.3.3)$$

where $l$ is the index of the representation of $G_0'$ on $G_1$ (see Section 1.4.3).

Proposition 2.3.6. A simple Lie superalgebra $G = G_0 \oplus G_1$ with non-degenerate Killing form is classical.

Proof. The unipotent radical $N$ of $G_0$ is known (see [9], for example) to lie in a kernel of the form $(\ , \ )_\nu$ of the representation of $G_0$ in $V$. Therefore, if $a \in N$, $b \in G_0$, then by (2.3.2): $(a, b)_0 = (a, b)_1 = 0$. It follows that $a$ lies in the kernel of the Killing form of $G$. Hence $N = 0$ and $G$ is a classical Lie superalgebra, because of Proposition 1.3.2.

By means of the Killing form we can write the Jacobi identity for three odd elements in a very convenient way. Let $G = G_0 \oplus G_1$ be a Lie superalgebra with nondegenerate Killing form. We choose in $G_0$ any basis $u_i$ and the dual basis $v_i$ relative to the restriction of the Killing form to $G_0$. Let $a, b, c \in G_1$. Then: $[a, b] = \sum \alpha_i v_i$. Multiplying both sides scalarly by $u_i$, we obtain $\alpha_i = \langle [a, b], u_i \rangle$. Making use of the invariance of the Killing form, we have: $\alpha_i = \langle [a, [b, u_i]], - (a, [u_i, b]) \rangle$. Thus, $[a, b] = -\sum (a, [u_i, b])v_i$. 


Therefore, the Jacobi identity \([[[a, b], c] + [[b, c], a] + [[c, a], b] = 0\) gives us:

\[
\sum_i (\langle a, [u_i, b] \rangle [v_i, c] + \langle b, [u_i, c] \rangle [v_i, a] + \langle c, [u_i, a] \rangle [v_i, b]) = 0.
\] (*)

2.3.2. Classification of the simple Lie superalgebras with nondegenerate Killing form in Case I. In this section we prove the following proposition:

**Proposition 2.3.7.** Let \(G = G_0 \oplus G_1\) be a simple Lie superalgebra with nondegenerate Killing form for which the representation of \(G_0\) on \(G_1\) is irreducible and \(G_1 \neq 0\). Then \(G\) is isomorphic to \(B(m, n), D(m, n)\) with \(m - n \neq 1\), \(F(4)\), or \(G(3)\).

Before the proof we give a lemma.

As we have shown in Section 2.2, \(G_0\) is semisimple. Let \(H\) be a Cartan subalgebra of \(G_0\), \(\Delta\) be the system of all roots, and \(\Delta'\) be that of nonzero roots. Let \(\mathcal{L}\) be the system of weights of the representation of \(G_0\) on \(G_1\), and \(G_1 = \bigoplus V_\lambda\) the weight decomposition. From Proposition 2.3.2 it follows that

\[
\langle V_\lambda, V_\mu \rangle = 0 \quad \text{for } \lambda \neq -\mu; \tag{2.3.4}
\]

if \(\lambda \in \mathcal{L}\), then \(-\lambda \in \mathcal{L}\) and \(\langle v_\lambda, v_{-\lambda} \rangle \neq 0\). \(\tag{2.3.5}\)

Let \(G_0 = \bigoplus G_0^{(s)}\) be the decomposition of \(G_0\) into a direct sum of simple components. Evidently, this decomposition is orthogonal relative to the bilinear forms \(\langle , \rangle\) and \(\langle , \rangle_0\). We denote by \(\langle , \rangle_0^{(s)}\) the restriction of \(\langle , \rangle_0\) to \(G_0^{(s)}\).

Let \(l_s\) be the index of the representation of \(G_0^{(s)}\) in \(G_1\). Let \(h_1, \ldots, h_r\) be a basis of \(H\) formed from bases of the Cartan subalgebras \(H \cap G_0^{(s)}\) of \(G_0^{(s)}\). Let \(\hat{h}_1, \ldots, \hat{h}_r\) be the dual basis with respect to \(\langle , \rangle\) and \(\hat{h}_1, \ldots, \hat{h}_r\) with respect to \(\langle , \rangle_0\). From (2.3.3) it clearly follows that

\[
\hat{h}_i = (1 - l_s) h_i \quad \text{for } h_i \in G_0^{(s)}. \tag{2.3.6}
\]

In particular,

Killing form is nondegenerate iff \(l_s \neq 1\) for some \(s\). \(\tag{2.3.7}\)

**Lemma 2.3.8.** (a) If \(\lambda \in \mathcal{L}\), \(2\lambda \notin \Delta\), then

\[
\langle \lambda, \lambda \rangle = \sum_s \frac{\langle \lambda, \lambda \rangle_0^{(s)}}{1 - l_s} = 0.
\]

(b) If \(\lambda, \mu \in \mathcal{L}\) and \(\lambda \pm \mu \notin \Delta\), then

\[
\langle \lambda, \mu \rangle = \sum_s \frac{\langle \lambda, \mu \rangle_0^{(s)}}{1 - l_s} = 0.
\]
Proof. We consider the following basis of $G_0$:

$$\{u_i\} = \{e_\alpha, \alpha \in \Lambda'; h_i, i = 1, \ldots, r\}.$$  

The dual basis with respect to $(\ , \ )$ is

$$\{v_i\} = \{e_{-\alpha}, \alpha \in \Lambda'; h_i, i = 1, \ldots, r\}.$$  

Let us prove (a). Let $h \in \mathcal{G}$. We set $a = c = e_\lambda$, $b = e_{-\lambda}$, where $(e_\lambda, e_{-\lambda}) = 1$ (by (2.3.5) such a vector exists). Now we write the identity (*) for the chosen bases $\{u_i\}$ and $\{v_i\}$ and the vectors $a, b, c$. Taking (2.3.4) into account, we have

$$(e_\lambda, [e_{-2\lambda}, v_i])[e_{2\lambda}, v_{-\lambda}] = 2e_\lambda \sum_i \lambda(h_i) \lambda(h_i).$$

Therefore, if $2\lambda \notin \Delta$, then

$$\sum_i \lambda(h_i) \lambda(h_i) = 0. \quad (2.3.8)$$

Since $(\lambda, \mu) = \sum_i \lambda(h_i) \mu(h_i)$ and $(\lambda, \mu)_0 = \sum_i \lambda(h_i) \mu(h_i)$ for any $\lambda, \mu \in H^*$, when (2.3.6) is taken into account, it can be rewritten in the form

$$(\lambda, \lambda) = \sum_s \frac{(\lambda, \lambda)_0^{(s)}}{1 - l_s} = 0,$$

as required.

(b) is proved similarly, but in (*) we must put $a = v_\lambda$, $b = v_{-\lambda}$, $c = e_\lambda$. This proves Lemma 2.3.8.

Proof of Proposition 2.3.7. If $G_0$ is simple, and $\Lambda$ is the highest weight, then

$$\sum_s \frac{(\Lambda, \Lambda)_0^{(s)}}{1 - l_s} = \frac{(\Lambda, \Lambda)_0}{1 - l_1} \neq 0,$$

and by Lemma 2.3.8(a) it follows that $2\Lambda \in \Delta$. From Lemma 1.4.1(a) it therefore follows that $G_0$-module $G_\Pi$ is isomorphic to $sp_n$. By Proposition 2.1.4, we now see that $G$ is isomorphic to $B(0, n/2)$. Since $l_1 = 1/(n + 2)$ (see Table III), by virtue of (2.3.7) the Killing form on $B(0, n/2)$ is nondegenerate.

Suppose now that $G_0$ is semisimple, but not simple. We represent $G_0$ in the form $G_0 = G_0^{\mathfrak{I}} \oplus G_0^{\mathfrak{II}}$, where $G_0^{\mathfrak{I}}$ and $G_0^{\mathfrak{II}}$ consist of all simple components of $G_0$ for which $1 - l_i$ is positive and negative, respectively. As is clear from Lemma 2.3.8(a), both these subalgebras are nontrivial. Let $\Lambda = \Lambda^{\mathfrak{I}} + \Lambda^{\mathfrak{II}}$ be the highest weight of the representation of $G_0$ on $G_\Pi$ (where I and II indicate that the weight is restricted to the relevant direct summand). We consider a weight of the form $\mu = \mu^{\mathfrak{I}} + \Lambda^{\mathfrak{II}}$, where $\mu^{\mathfrak{I}} \neq \pm \Lambda^{\mathfrak{I}}$. Observe that, clearly,

$$\Lambda + \mu \notin \Delta. \quad (2.3.9)$$
Next, \((A, \mu) = (A^I, \mu^I) + (A^I, A^I) = (A, A) - (A^I, A^I)\). Since \(2A \notin A\), by Lemma 2.3.8(a),

\[(A, \mu) - (A^I, \mu^I) - (A^I, A^I) \quad (2.3.10)\]

This relation can be rewritten (see the proof of Lemma 2.3.8) in the form

\[(A, \mu) = \sum_s \frac{(A, \mu)^{(s)}_0 - (A, A)^{(s)}_0}{1 - \lambda_s}, \quad (2.3.11)\]

where the summand is over the simple components occurring in \(G^1\). Since \(A\) is the highest weight, \((A, A)^{(s)}_0 \geq \mu^0, \mu^0)^{(s)}_0\) for all \(s\). Therefore, all the terms in (2.3.11) are negative, by the Cauchy–Bunjakowskii inequality. Consequently,

\[(A, \mu) \neq 0. \quad (2.3.12)\]

Now we can use Lemma 2.3.8(b), according to which it follows from (2.3.9) and (2.3.12) that \(A - \mu \in A\). Thus, if \(\mu^I \neq -A^I\), then \(A^I - \mu^I \in A\). Of course, the same is true for \(G^1\). Therefore, we find from Lemma 1.4.1(c) that the linear representation of \(G^1\) on \(G\) can only be equivalent to the tensor product of two of the following linear Lie algebras: \(sp, n > 2; sl, n > 3; so, n > 3\).

We recall now (Proposition 2.3.2) that the representation of \(G^1\) on \(G\) admits a nondegenerate skew-symmetric invariant bilinear form. This can only be the case when one of the factors of the tensor product has a skew-symmetric invariant and bilinear form and the other an invariant symmetric form. Therefore, only the following possibilities remain:

1. \(so \otimes sp_n\),
2. \(sp_n \otimes spin_7\),
3. \(sp_n \otimes G_2\).

In case (1) we obtain from Proposition 2.1.4 that \(G\) is isomorphic to \(B(m - 1/2, n/2)\) for odd \(m > 1\), or \(D(m/2, n/2)\) for even \(m > 2\). Since \(l_1 = n/m - 2\) and \(l_2 = m/n + 2\), (Table III), by (2.3.7) the Killing form is nondegenerate on \(B(m, n)\) and also on \(D(m, n)\) when \(m - n \neq 1\).

In cases (2) and (3) we use Lemma 2.3.8(a) again:

\[\frac{(A, A)^{(1)}_0}{1 - l_1} + \frac{(A, A)^{(a)}_0}{1 - l_2} = 0. \quad (2.3.13)\]

In cases (2) and (3), (2.3.13) yields that \(n = 2\) and therefore we see from Proposition 2.1.4 that \(G\) is isomorphic to \(F(4)\) and \(G(3)\), respectively.

This completes the proof of Proposition 2.3.7.
2.3.3. Classification of the simple Lie superalgebras with nondegenerate Killing form in Case II. In this section we prove the following proposition.

**Proposition 2.3.9.** Let \( G = G_- \oplus G_0 \oplus G_1 \) be a simple Lie superalgebra with a consistent \( \mathbb{Z} \)-grading for which the representations of \( G_0 \) on \( G_1 \) and \( G_{-1} \) are faithful and irreducible and the Killing form is nondegenerate. Then \( G \) is isomorphic (even as a \( \mathbb{Z} \)-graded superalgebra) to one of \( A(m, n) \), \( m \neq n \), or \( C(n) \).

The proof of this proposition is based on the same arguments as that of Proposition 2.3.7.

It follows from Proposition 2.3.5 that \( G_0 \) is the direct sum of the one-dimensional center \( C \) and the semisimple Lie algebra \( G'_0 \). Here \( C = \langle z \rangle \), where \([z, g] = \pm g \) for \( g \in G_{\pm 1} \), and the representations of \( G_0 \) on \( G_{-1} \) and \( G_1 \) are contragredient. Let \( H \) be a Cartan subalgebra of \( G'_0 \), \( \Delta \) its root system, \( \mathcal{L}_{\pm 1} \) the systems of weights of the representations of \( G'_0 \) on \( G_{\pm 1} \). From Proposition 2.3.5 it follows that

\[
\mathcal{L}_{-1} = -\mathcal{L}_1, \\
(G_1, G_1) = (G_{-1}, G_{-1}) = 0, \\
(v_\lambda, V_{-\lambda}) \neq 0 \quad \text{for} \quad \lambda \in \mathcal{L}_1, \quad -\lambda \in \mathcal{L}_{-1}.
\]

Let \( G'_0 = \bigoplus G_0^{(s)} \) be the decomposition of \( G'_0 \) into the direct sum of simple components. We denote by \( (\ , \)_0 \) the restriction of \( (\ , \) \) to \( G_0^{(s)} \) and by \( l_s \) the index of the representation of \( G_0^{(s)} \) in \( G_1 \). Note that it is also the index of the representation of \( G_0^{(s)} \) in \( G_{-1} \). Just as in Section 2.2, we choose a basis \( h_1, \ldots, h_r \) of \( H \), its dual basis \( \alpha_1, \ldots, \alpha_r \) with respect to \( (\ , \) \) and \( \tilde{h}_1, \ldots, \tilde{h}_r \) with respect to \( (\ , \) \).

From (2.3.3) it follows that

\[
h_i = (1 - 2l_s) \tilde{h}_i \quad \text{for} \quad h_i \in G_0^{(s)}.
\]

In particular,

Killing form is nondegenerate, iff \( l_s \neq \frac{1}{2} \) for some \( s \).

**Lemma 2.3.10.** (a) If \( \lambda \in \mathcal{L}_1 \), then

\[
(\lambda, \lambda) = \sum_s \frac{(\lambda, \lambda)_0^{(s)}}{1 - 2l_s} - \frac{1}{2 \dim G_1} = 0.
\]

(b) If \( \lambda, \mu \in \mathcal{L}_1 \) and \( \lambda - \mu \notin \Delta \), then

\[
(\lambda, \mu) = \sum_s \frac{(\lambda, \mu)_0^{(s)}}{1 - 2l_s} - \frac{1}{2 \dim G_1} = 0.
\]
Proof. We consider the following basis of $G_0$:

$$\{u_i\} = \{e_\alpha, \alpha \in \Delta'; h_i, i = 1, \ldots, r; x\}.$$

The dual basis with respect to $(,)$ is

$$\{v_j\} = \{e_{-\alpha}, \alpha \in \Delta'; h_i, i = 1, \ldots, r; -1/(2 \dim G_1) x\}.$$

Let us prove (a). Let $h \in G_0$. We set $a = c = v_{-\alpha} \in G_{-1}, \ b = v_\alpha \in G_1,$

where $(v_\alpha, v_{-\alpha}) = 1.$ (According to (2.3.16) such vectors exist.) We now write
down the identity (*) in the chosen bases $\{u_i\}$ and $\{v_j\}$ and the vectors $a, b, c.$

Taking (2.3.15) and (2.3.4) into account, we have

$$(\lambda, \lambda) = \sum_i \lambda(h_i) \lambda(h_i) - \frac{1}{2 \dim G_1} = 0,$$

from which, using (2.3.17), we obtain (a).

(b) is proved similarly, only in (*) we must put $a = v_{-\alpha} \in G_{-1}, \ b = v_\alpha \in G_1,$

c = v_\mu \in G_1.$

Proof of Proposition 2.3.9. We represent $G_0'$ in the form $G_0' = G_0^{\prime I} \oplus G_0^{\prime II},$

where $G_0^{\prime I}$ and $G_0^{\prime II}$ consist of those simple components for which $1 - 2l_i$
is positive and negative, respectively. For definiteness, let $G_0^{\prime I} \neq 0.$ Let

$A = A^{\prime I} + A^{\prime II}$ be the highest weight of the representation of $G_0'$ in $G_1.$ We

consider a weight of the form $\mu = \mu^{\prime I} + A^{\prime II}.$ Just as in the proof of Proposition

2.3.7 we find that

$$(A, \mu) \neq 0.$$

From Lemma 2.3.10(b) it now follows that $A^{\prime I} - \mu^{\prime I} \in \Delta^I.$ So we see that if

$\mu \in \mathcal{L}^I,$ then $A^{\prime I} - \mu \in \Delta^I.$ Therefore, we obtain from Lemma 1.4.1(b) that the

$G_0'$-module $G_1$ can only be isomorphic to a linear Lie algebra $sl_n$ or $sp_n$ or to any
tensor product of them. So we have the following possibilities for the representation

of $G_0$ on $G_1$:

1. $gl_m \otimes sl_n,$

2. $csp_n,$

3. $gl_m \otimes sp_n, m \geq 2, n \geq 4,$

4. $csp_m \otimes sp_n, m \geq 2, n \geq 4.$

In case (1), $l_1 = n/(2m), l_2 = m/(2n)$ (Table III); hence, by (2.3.18), $m \neq n.$

But then we see from Proposition 2.1.4 that $G$ is isomorphic to $A(m - 1, n - 1).$

In case (2), we derive from the same Proposition 2.1.4 that $G$ is isomorphic to $C((n/2) + 1).$ From (2.3.18) it is clear that the Killing form for these Lie
superalgebras is nondegenerate.
That cases (3) and (4) are impossible we deduce from Lemma 2.3.10(a):

$$
\frac{(A, A)_{\mathfrak{g}}}{1 - 2\mathfrak{g}} + \frac{(A, A)_{\mathfrak{h}}}{1 - 2\mathfrak{h}} = \frac{1}{2 \dim G_1}.
$$

(2.3.19)

In case (3), (2.3.19) yields

$$\frac{m - 1}{m(2m - 2n)} + \frac{1}{2(n + 2 - 2m)} = \frac{1}{2mn},$$

so that either $m = 1$ or $n = 2$.

In case (4), (2.3.19) yields

$$\frac{1}{2(n + 2 - 2m)} + \frac{1}{2(m + 2 - 2n)} = \frac{1}{2mn},$$

which is impossible.

Thus, cases (3) and (4) cannot occur. This completes the proof of Proposition 2.3.9.

2.3.4. Conclusion of the classification of simple Lie superalgebras with nondegenerate Killing form.

**Theorem 1.** A simple finite-dimensional Lie superalgebra $G = G_0 \oplus G_1$ with nondegenerate Killing form is isomorphic to one of the simple Lie algebras or to one of the following classical Lie superalgebras:

- $A(m, n)$ with $m \neq n$,
- $B(m, n)$,
- $C(n)$,
- $D(m, n)$ with $m - n \neq 1$,
- $F(4)$,
- $G(3)$.

**Proof.** From Proposition 2.3.6 it follows that $G$ is classical. In accordance with Section 2.2 we have to discuss two cases.

**Case I.** The representation of $G_0$ on $G_1$ is irreducible. Then Theorem 1 follows from Proposition 2.3.7.

**Case II.** The representation of $G_0$ on $G_1$ is reducible. In that case, according to Lemma 2.2.1, $G$ has a filtration $G = L_{-1} \supset L_0 \supset L_1$ for which $\text{Gr} G$ is a $\mathbb{Z}$-graded Lie superalgebra satisfying all the conditions of Proposition 2.3.9, $(\text{Gr} G)_0 \cong G_0$, and the $(\text{Gr} G)_1$-module $(\text{Gr} G)_1$ is isomorphic to the $G_0$-module $G_1$. By Proposition 2.3.9, $\text{Gr} G$ is one of $A(m, n)$, $m \neq n$, or $C(n)$. From Proposition 2.1.4 it now follows that $G \cong \text{Gr} G$, and the theorem is proved.

2.4. Completion of the Classification of the Classical Lie Superalgebras

The classification of the classical Lie superalgebras is given by the following theorem.
THEOREM 2. A classical Lie superalgebra is isomorphic either to one of the simple Lie algebras \( A_n, B_n, ..., E_8 \), or to one of \( A(m, n), B(m, n), C(n), D(m, n), D(2, 1; \alpha), F(4), G(3), P(n), \) or \( Q(n) \).

By virtue of Theorem 1 and of Proposition 1.2.6, what remains to be proved is the following proposition.

PROPOSITION 2.4.1. A classical Lie superalgebra \( G = G_0 \oplus G_1 \) with zero Killing form is isomorphic to one of \( A(n, n), D(n + 1, n), P(n), Q(n), \) or \( D(2, 1; \alpha) \).

2.4.1. Beginning of the proof of Proposition 4.1.

LEMMA 2.4.2. Let \( G = G_0 \oplus G_1 \) be a Lie superalgebra with zero Killing form for which the representation of \( G_0 \) on \( G_1 \) is faithful and completely reducible. Then the index \( l_i \) of the representation of any simple component \( G_0^{(i)} \) of \( G_0 \) on \( G_1 \) is 1. In particular, the index of the representation of \( G_0^{(i)} \) on any irreducible component of \( G_1 \) does not exceed 1.

Proof. By (2.3.3), \( (a, b) = (1 - \lambda)(a, b) \) for \( a, b \in G_1 \). Since \( (a, b) = 0 \) and \( (a, b) \) is nondegenerate on \( G_1 \), we see that \( l_i = 1 \).

2.4.2. Classification of the classical Lie superalgebras with zero Killing form in Case I.

PROPOSITION 2.4.3. Let \( G = G_0 \oplus G_1 \) be a simple Lie superalgebra with zero Killing form for which the representation of \( G_0 \) on \( G_1 \) is irreducible. Then \( G \) is isomorphic to one of \( Q(n), D(n + 1, n), \) or \( D(2, 1; \alpha) \).

Proof. As we remarked in Section 2.2, \( G_0 \) is semisimple. Therefore, the representation of \( G_0 \) on \( G_1 \) is equivalent to the tensor product of some simple irreducible linear Lie algebras:

\[
G_0(G_1) = G_0^{(1)}(V_1) \otimes \cdots \otimes G_0^{(k)}(V_k).
\]  

(2.4.1)

Let \( l_1, ..., l_k \) be their indices. Then, clearly, the index \( l_i \) of the representation of \( G_0^{(i)} \) on \( G_1 \) is equal to

\[
l_i = l_i \prod_{s \neq i} \dim V_s = 1,
\]

(2.4.2)

according to Lemma 2.4.2.

If \( G_0 \) is simple, then \( l_1 = l_i = 1 \), and so (see Section 1.4.3) the representation of \( G_0 \) on \( G_1 \) is the adjoint one, and an epimorphism of \( G_0 \)-modules, \( S^2G_1 \rightarrow G_0 \), exists only when \( G_0 \) is of type \( A_n \) (see [9], for example). From Proposition 2.1.4, it follows that this case leads to \( Q(n) \).

Now suppose that \( G_0 \) is not simple. Then it follows from (2.4.2) that \( l_i = t_i^{-1} \),
where \( t_i \geq 2 \) is an integer. If \( t_i \leq 4 \), then \( \prod_{i \neq i} \dim V_i \leq 4 \) by (2.4.2); hence, \( \dim V_s \leq 4 \) for \( s \neq i \). From Table III (see Section 1.4.3) it is clear that if \( \dim V_s \leq 4 \), then \( t_s \leq 8 \), and by (2.4.2), then \( \dim V_s \leq 8 \). Thus, from \( t_i \leq 4 \) it follows that \( \dim V_i \leq 8 \). From Table III we can now see that only \( sl_n \), \( sp_n \), \( so_n \), and \( \text{spin}_r \) can occur in (2.4.1). We claim that the last two cases are impossible. If \( G_0^{(1)}(V_1) = \text{spin}_n \) in (2.4.1), then \( s = 2 \) and \( \dim V_2 = 5 \). But then \( G_0^{(2)}(V_2) = sl_n \) or \( so_n \). In the first case \( l_2 = \frac{8}{5} \), and in the second \( l_2 = \frac{8}{8} \). If \( G_0^{(1)}(V_2) = G_2 \), then \( G_0(G_1) = G_2 \otimes sl_4 \) or \( G_2 \otimes sl_2 \otimes sl_2 \) or \( G_2 \otimes sp_4 \). In the first case \( l_2 = \frac{7}{8} \), in the second \( l_2 = \frac{2}{8} \), and in the third \( l_2 = \frac{7}{8} \). Hence, only \( sl_n \), \( sp_n \), and \( so_n \) can occur in (2.4.1):

\[
G_0(G_1) = sl_{n_1} \otimes \cdots \otimes sl_{n_s} \otimes sp_{r_1} \otimes \cdots \otimes sp_{r_2} \otimes so_{m_1} \otimes \cdots \otimes so_{m_l},
\]

where \( 2 \leq n_1 \leq \cdots \leq n_s, \ 4 \leq r_1 \leq \cdots \leq r_2, \ 5 \leq m_1 \leq \cdots \leq m_l \).

Relations (2.4.2) can be rewritten in the form

\[
\begin{align*}
\left( \prod_{i \neq i} n_i \right) \left( \prod_{i \neq i} r_i \right) \left( \prod_{i \neq i} m_i \right) &= 2n_i, \\
\left( \prod_{i \neq i} n_i \right) \left( \prod_{i \neq i} r_i \right) \left( \prod_{i \neq i} m_i \right) &= r_i + 2, \\
\left( \prod_{i \neq i} n_i \right) \left( \prod_{i \neq i} r_i \right) \left( \prod_{i \neq i} m_i \right) &= m_i - 2.
\end{align*}
\]

From these relations it is evident that \( \alpha \leq 3, \ \beta \leq 2, \ \gamma \leq 1 \). If \( \alpha = 3 \), then it is clear from (2.4.3) that \( G_0(G_1) = sl_2 \otimes sl_2 \otimes sl_2 \), and we have \( D(2,1;\delta) \). If \( \alpha = 2 \), the only possibility is \( G_0(G_1) = sl_n \otimes sl_n \); but then \( n_1 = 2n_2, \ n_2 = 2n_1 \), which is impossible. If \( \beta = 2 \), then by (2.4.4) the only possibility is \( G_0(G_1) = sp_r \otimes sp_{r+2} \), which clearly cannot be realized. If \( \alpha = \beta = \gamma = 1 \), then we have by multiplying (2.4.3), (2.4.4), and (2.4.5), \( n_1 r_1^2 m_1^2 = 2(r_1 + 2)(m_1 - 2) \), which is impossible. The cases \( \alpha = \beta = 1, \ \gamma = 0 \) and \( \alpha = \gamma = 1, \ \beta = 0 \), are also impossible. There remains the case \( G_0(G_1) = sp_r \otimes so_n \). Then \( n = r + 2 \) (see Table III).

Thus, the only remaining possibility for the representation of \( G_0 \) on \( G_1 \) is \( so_n \otimes sp_n, n > 2 \). This is, in fact, realized for \( D(n/2 + 1, n/2) \). By Proposition 2.1.4, there can only be one superalgebra with this representation of \( G_0 \) on \( G_1 \). This proves the proposition.

2.4.3. Classification of the classical Lie superalgebras with zero Killing form in Case II.

**Proposition 2.4.4.** Let \( G = G_{-1} \oplus G_0 \oplus G_1 \) be a simple Lie superalgebra with a consistent \( \mathbb{Z} \)-grading for which the representations of \( G_0 \) on \( G_1 \) and \( G_{-1} \)
are faithful and irreducible, and the Killing form is zero. Then $G$ is isomorphic (even as a $\mathbb{Z}$-graded superalgebra) to one of $A(n, n)$ or $P(n)$.

Before the proof we give two lemmas.

**Lemma 2.4.5.** Under the conditions of Proposition 2.4.4, $G_0$ is semisimple.

**Proof.** If the center $C$ of $G_0$ is nontrivial, then by Proposition 1.2.12, there exists a $z \in C$ such that $[z, g] = \pm g$ for $g \in G_{\pm 1}$. But then, clearly, $(z, z) = -\dim G_{-1} - \dim G_1$, which is impossible, because the Killing form is zero.

**Lemma 2.4.6.** If under the conditions of Proposition 2.4.4 the representations of $G_0$ on $G_1$ and $G_{-1}$ are contragradient, then the highest weight of the representation of $G_0$ on $G_1$ must have more than one nonzero numerical mark.

**Proof.** Let $H$ be a Cartan subalgebra of $G_0$, $\alpha_1, \ldots, \alpha_s$ its system of simple roots, $h_1, \ldots, h_s$ a basis of $H$, where $h_i = [e_{\alpha_i}, e_{-\alpha_i}]$, $\alpha_i(h_i) = 2$. If $E_{a_0}$ is the lowest weight vector of the representation of $G_0$ on $G_{-1}$, and $F_{-a_0}$ the highest weight vector of that of $G_0$ on $G_1$, then $[E_{a_0}, F_{-a_0}] = h_0 \in H$, $h_0 \neq 0$ (see Proposition 1.2.10(a)), then

$$\alpha_0(h_0) = 0,$$  \hspace{1cm} \text{(2.4.6)}

$$\alpha_i(h_0) = 0 \quad \text{for} \quad \alpha_0(h_i) = 0,$$  \hspace{1cm} \text{(2.4.7)}

$$\det(\alpha_i(h_j))_{i,j=0} = 0.$$  \hspace{1cm} \text{(2.4.8)}

(2.4.6) follows from 0 = $[E_{a_0}, [E_{a_0}, F_{-a_0}]] = -2\alpha_0(h_0)F_{-a_0}$, (2.4.7) is obtained by multiplying both sides of $[E_{a_0}, F_{-a_0}] = h_0$ by $e_{\alpha_i}$, and (2.4.8) follows from the linear dependence of the vectors $h_0, h_1, \ldots, h_r$.

Suppose now that $\alpha_0(h_s) \neq 0$ for one $s$ only. Then $\alpha_s(h_0) \neq 0$; otherwise, $h_0 = 0$ by (2.4.6) and (2.4.7), and the remaining elements of the first column are zeros. By hypothesis, $\alpha_0(h_s) \neq 0$, but the remaining elements of the first row are zeros. It then follows that $\det(\alpha_i(h_j)) = \alpha_0(h_s)\alpha_s(h_0)\det A$, where $A$ is the Cartan matrix of the Dynkin diagram of $G_0$, with the $s$th circle omitted. Since $\det A \neq 0$, we have reached a contradiction to (2.4.8).

We divide the proof of Proposition 2.4.4 into two cases corresponding to the following two lemmas.

**Lemma 2.4.7.** If under the conditions of Proposition 2.4.4 the representations of $G_0$ on $G_{-1}$ are contragradient, then $G$ is isomorphic to $A(n, n)$.

**Proof.** By Lemma 2.4.5, the representation of $G_0$ on $G_1$ is equivalent to the tensor product of some simple irreducible linear Lie algebras:

$$G_0(G_1) = G_0^{(1)}(V_1) \otimes \cdots \otimes G_0^{(k)}(V_k).$$
Let $l_1, \ldots, l_k$ be their indices. Since contragredient representations have equal indices, the index of the representation of $G^{(0)}$ on $G_1$ is $\frac{1}{2}$ (by Lemma 2.4.2). Therefore,

$$l_i \prod_{s \in t} \dim V_s = \frac{1}{2}. \tag{2.4.9}$$

If $G_0$ is simple, then we see from (2.4.9) that $l_i = \frac{1}{2}$. From Table III it is clear that there are only the two possibilities for $G_0(G_2)$: $\Lambda^2s_6$ and $\Lambda_0^2s_6$. Both cannot occur according to Lemma 2.4.6.

Suppose now that $G_0$ is not simple. From Table III it is clear that $2(\dim V_i)l_i \geq 1$ and that equality holds for $s_i$ only. Therefore, we see from (2.4.9) at once that the only possibility for $G_0(G_2)$ is $s_i \otimes s_i$. By Proposition 2.1.4, $G$ is then isomorphic to $A(n - 1, n - 1)$, and the lemma is proved.

**Lemma 2.4.8.** Under the conditions of Proposition 2.4.4, if the representations of $G_0$ on $G_{-1}$ and $G_1$ are not contragredient, then $G$ is isomorphic to $P(n)$.

**Proof.** It follows from Lemma 2.4.5 that $G_0$ is semisimple, and from Proposition 1.2.10 (and Proposition 1.2.1) that $G_0$ is simple. Thus, the relevant pair of representations of $G_0$ on $G_{-1}$ and $G_1$ can only be one from Table III, for which the sum of the indices is 1. This leads to the following cases:

1. $s_i$ and $\Lambda^2s_6$;
2. $s_i^*$ and $\Lambda_0^2s_6$;
3. $\Lambda^2s_n$ and $S^2s_n$, $n > 4$;
4. $\Lambda_0^2s_n^*$ and $S^2s_n$.

Proposition 1.2.10(b) imposes yet another restriction: If $\Lambda$ is the highest weight of the representation of $G_0$ on $G_{-1}$ and $M$ is the lowest weight of that on $G_1$, then $\Lambda + M$ is a root of $G_0$. This rules out cases (1)--(3) at once. The fourth case corresponds (on the basis of Proposition 2.1.4) only to $P(n)$, and the lemma is proved.

The conclusion of the proof of Proposition 2.4.1 proceeds verbatim on the same lines as that of Theorem 1 (see Section 2.3.4), on the basis of Propositions 2.4.3 and 2.4.4.

This completes the proof of Theorem 2.

**Additional Remark.** The following authors independently obtained classification results on classical Lie superalgebras under the following restrictions:


2.5. Contragredient Lie Superalgebras

2.5.1. Definition of the superalgebras $G(A, \tau)$. Let $A = (a_{ij})$ be an $(r \times r)$-matrix with elements from a field $k$ and $\tau$ be a subset of $I = \{1, 2, \ldots, r\}$. Let $G_-, G_0$, and $G_1$ be vector spaces over $k$ with bases $\{f_i\}$, $\{h_i\}$, and $\{e_i\}$, $i \in I$, respectively. As is easy to see, the following relations determine the structure of a local Lie superalgebra $\tilde{G}(A, \tau)$ on the space $G_- \oplus G_0 \oplus G_1$:

\begin{align*}
[e_i, f_j] &= \delta_{ij} h_i, \\
[h_i, f_j] &= 0, \\
[h_i, e_j] &= a_{ij} e_j, \\
[h_i, h_j] &= 0
\end{align*}

\begin{align*}
deg h_i &= 0, \\
deg e_i &= deg(\tau) = 0 \\
deg f_i &= 1 \\
\text{for } i \notin \tau,
\end{align*}

\begin{align*}
\text{for } i \in \tau.
\end{align*}

According to Proposition 1.2.2, there exists a minimal $\mathbb{Z}$-graded Lie superalgebra $G(A, \tau)$ with local part $\tilde{G}(A, \tau)$. We call $G(A, \tau)$ contragredient Lie superalgebra, $A$ its Cartan matrix, and $r$ its rank. Note that when $\tau = \emptyset$, we have contragredient Lie algebras whose theory is developed in [11].

When $h_i$ is replaced by $c h_i$ and $f_i$ by $c f_i$, $c \in k^*$, then the $i$th row of $A$ is multiplied by $c$. Therefore, we may (and will) assume that if $a_{ii} \neq 0$, then $a_{ii} = 2$. If we can obtain the pair $(\tilde{A}, \tilde{\tau})$ from $(A, \tau)$ by multiplying several rows by nonzero constants and renumbering the indices, then we regard $(A, \tau)$ and $(\tilde{A}, \tilde{\tau})$ as equivalent; the corresponding contragredient Lie superalgebras are isomorphic.

Observe that if $I_1 \subseteq I$, $A_1$ is the corresponding principal minor of $A$, and $\tau_1 = I_1 \cap \tau$, then the subalgebra of $G(A, \tau)$ generated by the elements $e_i$, $f_i$, and $h_i$, $i \in I_1$, is isomorphic to $G(A_1, \tau_1)$. Note also that if $A$ is decomposable,

\[ A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \]

then $G(A, \tau)$ splits into the direct sum of the algebras with the Cartan matrices $A_1$ and $A_2$.

Let $H = \langle h_1, \ldots, h_r \rangle$, let $\alpha_1, \ldots, \alpha_r$ be the linear functions on $H$ defined by the relations $\alpha_j(h_i) = a_{ij}$, $j = 1, \ldots, r$, and let $M$ be the free Abelian group with the generators $\alpha_1, \ldots, \alpha_r$. We set $G_\alpha = \langle [\ldots[e_i, e_j], \ldots, e_i] \rangle$, $G_{-\alpha} = \langle [\ldots[f_i, f_j], \ldots, f_i] \rangle$, $\alpha = \sum a_{ij} \alpha_j$.

As in [11], it is easy to show that $G(A, \tau) = H \oplus (\oplus G_\alpha)$.

Many assertions about contragredient Lie algebras in [8, 11] remain valid for Lie superalgebras (with the same proofs). Here we state only those that are needed in what follows.
Proposition 2.5.1. The center $C$ of $G(A, \tau)$ consists of the elements of the form $\sum x_i \gamma_i$, where $\sum a_i x_i = 0$.

Proposition 2.5.2. Let $G(A, \tau)$ be finite-dimensional and $C$ be its center. Then $G(A, \tau)/C$ is simple if and only if

\[ a_{i_1} a_{i_2} \ldots a_{i_t} \neq 0 \]

for any $i, j \in I$ there exists a sequence $i_1, \ldots, i_t \in I$ (m) for which

Proposition 2.5.3. Suppose that $G(A, \tau)$ is finite-dimensional and satisfies (m). Then on $G(A, \tau)/C$ there is a nondegenerate consistent supersymmetric invariant bilinear form. This induces a form $(, )$ on $G(A, \tau)$ having the following properties:

1. The kernel of the form $( , )$ is $C$;
2. $(G_{\alpha}, G_{\beta}) = 0$ when $\alpha \neq -\beta$;
3. the form $( , )$ determines a nondegenerate pairing of $G_{-\alpha}$ with $G_{\alpha}$;
4. $[e_{\alpha}, e_{-\alpha}] = (e_{\alpha}, e_{-\alpha}) h_{\alpha}$, where $h_{\alpha}$ is a nonzero vector in $H$ for which $(h_{\alpha}, h) = \alpha(h)$, $h \in H$.

Proof. The mapping $\theta: e_i \mapsto -f_i$, $h_i \mapsto -h_i$, $f_i \mapsto (-1)^{\deg(e_i)} e_i$ evidently induces an automorphism of $G(A, \tau)/C$. Hence, all the conditions for Proposition 1.2.5 are satisfied and the required form exists. The remaining properties are proved as in [11].

2.5.2. Existence of the exceptional Lie superalgebras $D(2, 1; a)$, $F(4)$, and $G(3)$.

Proposition 2.5.4. We consider the following matrices ($\alpha \in k/\{0, -1\}$):

\[
D_\alpha = \begin{bmatrix} 0 & 1 & \alpha \\ -1 & 2 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}, \quad F_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix}.
\]

(a) For $G(D_\alpha, \{1\})$, $G(F_4, \{1\})$, and $G(G_3, \{1\})$ the $G_8$-module $G_1$ is isomorphic to $sl_2 \otimes sl_2 \otimes sl_2$, spin$_3 \otimes sl_2$, and $G_3 \otimes sl_2$, respectively.

(b) The $G(D_\alpha, \{1\})$ exhaust all simple Lie superalgebras $G = G_8 \oplus G_1$ for which the $G_8$-module $G_1$ is isomorphic to $sl_2 \otimes sl_2 \otimes sl_2$. Two members $D(2, 1; \alpha)$ and $D(2, 1; \beta)$ of this family are isomorphic if and only if $\alpha$ and $\beta$ lie in the same orbit of the group $V$ of order 6 generated by $\alpha \mapsto -1 - \alpha$, $\alpha \mapsto 1/\alpha$.

Proof. (a) and the condition for isomorphy of members of the family $D(2, 1; \alpha)$ are established exactly as for [8, Proposition 3.6].

Now let $G = G_8 \oplus G_1$ be a simple Lie superalgebra for which the $G_8$-
module $G_i$ is isomorphic to $sl_2 \otimes sl_2 \otimes sl_2$. Then $G_0 = A^{(1)}_1 \oplus A^{(q)}_1 \oplus A^{(a)}_1$, where $A^{(i)}_1 = \langle e_{a_i}, h_{a_i}, f_{-a_i} \rangle$. Let $e_1$ and $f_1$ be the lowest and highest weight vectors. Since $G$ is simple, $[e_1, f_1] = h_1 \neq 0$. We define a $\mathbb{Z}$-grading $G = \bigoplus G_t$ on $G$ by setting $G_{-1} = \langle f_1, f_2, f_3 \rangle$, $G_0 = \langle h_1, h_2, h_3 \rangle$, and $G_1 = \langle e_1, e_2, e_3 \rangle$. Then $G$ is equipped with the structure of a contragredient Lie superalgebra $G(A, \tau)$, where

$$A = \begin{pmatrix} 0 & \alpha_1(h_1) & \alpha_3(h_1) \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}, \quad \tau = \{1\}.$$ 

Since $G$ is simple, $\alpha_1(h_1), \alpha_3(h_1) \neq 0$. So we have $G \simeq D(2, 1; \alpha)$. 

Remark. The family $D(2, 1; \alpha)$ becomes a family of Lie algebras if the characteristic of the field is 2. This family of Lie algebras is studied in [8]. Note that group $V$ has three exceptional orbits: \{0, -1, \infty\}, \{1, -2, -\frac{1}{2}\}, and \{-\frac{1}{2} \pm i3^{1/2}/2\}. $D(2, 1)$ corresponds to the second orbit. Superalgebra, corresponding to the third orbit, admits an outer automorphism of order 3.

2.5.3. The root decomposition of the classical Lie superalgebras. Let $G = G_0 \oplus G_1$ be a Lie superalgebra and $H$ be a Cartan subalgebra of $G_0$. We call $H$ a Cartan subalgebra of $G$. Since every inner automorphism of $G_0$ extends to one of $G$ (see Proposition 1.1.1) and Cartan subalgebras of a Lie algebra are conjugate, so are Cartan subalgebras of a Lie superalgebra.

A Cartan subalgebra of a classical Lie superalgebra is diagonalizable. Therefore, we have the root decomposition:

$$G = \bigoplus_{\alpha \in H^*} G_{\alpha}, \quad \text{where } G_{\alpha} = \{a \in G \mid [h, a] = \alpha(h) a \text{ for } h \in H\}. \quad (2.5.1)$$

The set $\Delta = \{\alpha \in H^* \mid G_{\alpha} \neq 0\}$ is called the root system. Clearly, $\Delta = \Delta_0 \cup \Delta_1$, where $\Delta_0$ is the root system of $G_0$ and $\Delta_1$ is the system of weights of the representation of $G_0$ on $G_1$; $\Delta_0$ is called the system of even and $\Delta_1$ that of odd roots.

A straightforward inspection of examples of classical Lie superalgebras together with standard arguments from the theory of Lie algebras yields the following information on their root decompositions.

**Proposition 2.5.5.** Let $G$ be a classical Lie superalgebra and let $G = \bigoplus G_{\alpha}$ be its root decomposition relative to a Cartan subalgebra $H$.

(a) $G_0 = H$ in all cases except $Q(n)$;

(b) $\dim G_\alpha = 1$ for $\alpha \neq 0$, except for $A(1, 1)$, $P(2)$, $P(3)$, and $Q(n)$;

(c) On $G$ there is one and, up to a constant factor, only one nondegenerate invariant supersymmetric bilinear form $(\ , \ )$, except for $P(n)$ and $Q(n)$.

(d) $\Delta_0$ and $\Delta_1$ are invariant under the Weil group $W$ of $G_0$. 
If $G$ is $A(m, n)$, $(m, n) \neq (1, 1)$, $B(m, n)$, $C(n)$, $D(m, n)$, $D(2, 1; \alpha)$, $F(4)$, or $G(3)$, then the following properties hold:

1. $[G_\alpha, G_\beta] \neq 0$ if and only if $\alpha, \beta, \alpha + \beta \in \Delta$;
2. $(G_\alpha, G_\beta) = 0$ for $\alpha \neq -\beta$;
3. the form $(\ , \ )$ determines a nondegenerate pairing of $G_\alpha$ with $G_{-\alpha}$;
4. $[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})h_\alpha$, where $h_\alpha$ is the nonzero vector determined by $(h_\alpha, h) = \alpha(h)$, $h \in H$;
5. if $\alpha$ is in $\Delta$ (or $\Delta_0$, or $\Delta_1$), then so is $-\alpha$;
6. $k\alpha \in \Delta$ for $\alpha \neq 0$, $k \neq \pm 1$, if and only if $\alpha \in \Delta_1$ and $(\alpha, \alpha) \neq 0$; here $k = \pm 2$.

2.5. Explicit description of the system of roots and of simple roots. A system of roots $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subseteq \Delta$ is said to be simple if there are vectors $e_i \in G_{\alpha_i}$, $f_i \in G_{-\alpha_i}$, for which $[e_i, f_i] = \delta_i h_i \in H$, the vectors $e_i$ and $f_i$, $i = 1, \ldots, r$, generate $G$, and $\Pi$ is minimal with these properties. Below we describe for $A(m, n)$, $B(m, n)$, $C(n)$, $D(m, n)$, $D(2, 1; \alpha)$, $F(4)$, and $G(3)$ the systems of even nonzero roots $\Delta_0'$ and of odd roots $\Delta_1$, and all systems of simple roots, up to $W$-equivalence.

In all the examples the Cartan subalgebra $H$ is a subspace of the space $D$ of diagonal matrices; the roots are expressed in terms of the standard basis $e_i$ of $D^*$ (more accurately, the restrictions of the $e_i$ to $H$).

$A(m, n)$. The roots are expressed in terms of linear functions $e_1, \ldots, e_{m+1}$, $\delta_1 = e_{m+2}, \ldots, \delta_{n+1} = e_{m+n+2}$.

$$\Delta_0' = \{e_i - e_j; \delta_i - \delta_j\}, \quad i \neq j; \quad \Delta_1 = \{\pm (e_i - \delta_j)\}.$$ 

Up to $W$-equivalence, all the systems of simple roots are determined by two increasing sequences $S = \{s_1 < s_2 < \cdots\}$ and $T = \{t_1 < t_2 < \cdots\}$ and a sign:

$$\Pi_{S,T} = \pm\{e_1 - e_2, e_2 - e_3, \ldots, e_{s_1} - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{t_1} - e_{s_1+1}, \ldots\}.$$ 

The simplest such system is

$$\{e_1 - e_2, e_2 - e_3, \ldots, e_{m+1} - \delta_1, \delta_1 - \delta_2, \ldots, \delta_n - \delta_{n+1}\}.$$ 

$B(m, n)$. The roots are expressed in terms of linear functions $e_1, \ldots, e_m$, $\delta_1 = e_{m+1}, \ldots, \delta_n = e_{2m+n}$.

$$\Delta_0' = \{\pm e_i \pm e_j; \pm 2\delta_i; \pm e_i; \pm \delta_i \pm \delta_j\}, \quad i \neq j; \quad \Delta_1 = \{\pm \delta_i; \pm e_i; \pm \delta_j\}.$$ 

Up to $W$-equivalence, all the systems of simple roots are determined by two increasing sequences $S$ and $T$:

$$\Pi_{S,T} = \{e_1 - e_2, \ldots, e_{t_1} - \delta_1, \delta_1 - \delta_2, \ldots, t_1 - e_{s_1+1}, \ldots, \pm \delta_n (or \pm e_m)\}.$$
The simplest such system is

\[ \{ \delta_1 - \delta_2, \ldots, \delta_n - \epsilon_1, \epsilon_1 - \epsilon_2, \ldots, \epsilon_{m-1} - \epsilon_m, \epsilon_m \} \]

if \( m > 0 \), and

\[ \{ \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n \} \]

if \( m = 0 \).

**C(n).** The roots are expressed in terms of linear functions \( \epsilon_1, \delta_1 = \epsilon_3, \ldots, \delta_{n-1} = \epsilon_{n+1} \).

\[ \Delta_0' = \{ \pm 2\delta_i; \pm \delta_i \pm \delta_j \}; \quad \Delta_1 = \{ \pm \epsilon_1 \pm \delta_i \}. \]

Up to \( W \)-equivalence there are the following systems of simple roots:

\[ \pm \{ \epsilon_1 - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-2} - \delta_{n-1}, 2\delta_{n-1} \}; \]
\[ \pm \{ \delta_1 - \delta_2, \ldots, \delta_i - \epsilon_1, \epsilon_1 - \delta_{i+1}, \ldots, \delta_{n-2} - \delta_{n-1}, 2\delta_{n-1} \}; \]
\[ \pm \{ \delta_1 - \delta_2, \ldots, \delta_{n-2} - \delta_{n-1}, \delta_{n-1} - \epsilon_1, \delta_{n-1} + \epsilon_1 \}. \]

**D(m, n).** The roots are expressed in terms of linear functions \( \epsilon_1, \ldots, \epsilon_m, \delta_1 = \epsilon_{2m+1}, \ldots, \delta_n = \epsilon_{2m+n} \).

\[ \Delta_0' = \{ \pm \epsilon_i \pm \epsilon_j; \pm 2\delta_i; \pm \delta_i \pm \delta_j \}, \quad i \neq j, \quad \Delta_1 = \{ \pm \epsilon_i \pm \delta_i \}. \]

Up to \( W \)-equivalence, all the systems of simple roots are determined by two increasing sequences \( S \) and \( T \), and a number:

\[ \Pi_{S,T}^1 = \{ \epsilon_1 - \epsilon_2, \ldots, \epsilon_{t_1} - \delta_1, \delta - \delta_2, \ldots, \delta_{t_2} - \epsilon_{t_1+1}, \ldots, \epsilon_{m-1} - \epsilon_m, \epsilon_m + \epsilon_m \}
\]

(or \( \delta_n = \epsilon_m, \delta_n + \epsilon_m \));

\[ \Pi_{S,T}^2 = \{ \epsilon_1 - \epsilon_2, \ldots, \epsilon_{t_1} - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{t_2} - \epsilon_{t_1+1}, \ldots, \delta_{n-1} - \delta_n, 2\delta_n \}. \]

The simplest such systems are

\[ \{ \epsilon_1 - \epsilon_2, \ldots, \epsilon_{m-1} - \epsilon_m, \epsilon_m + \epsilon_m \}; \]
\[ \{ \epsilon_1 - \epsilon_2, \ldots, \epsilon_{m-1} - \epsilon_m, \epsilon_m - \epsilon_m \}. \]

**D(2, 1; \alpha).** The roots are expressed in terms of linear functions \( \epsilon_1, \epsilon_2, \) and \( \epsilon_3 \).

\[ \Delta_0' = \{ \pm 2\epsilon_i \}; \quad \Delta_1 = \{ \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \}. \]

Up to \( W \)-equivalence there are four systems of simple roots:

\[ \{ \epsilon_1 + \epsilon_2 + \epsilon_3, -2\epsilon_i, -2\epsilon_j \}, \quad i \neq j, \quad i, j = 1, 2, 3; \]
\[ \{ \epsilon_1 + \epsilon_2 + \epsilon_3, \epsilon_1 - \epsilon_2 - \epsilon_3, -\epsilon_1 - \epsilon_2 + \epsilon_3 \}. \]
F(4). The roots are expressed in terms of linear functions $\varepsilon_1, \varepsilon_2, \varepsilon_3$, corresponding to $B_3$, and $\delta$, corresponding to $A_1$.

$$\Delta'_0 = \{\pm \varepsilon_i \pm \varepsilon_j \pm \varepsilon_k \pm \varepsilon_l \mid i \neq j\}$$

$$\Delta_1 = \{\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}$$

Up to $W$-equivalence there are four systems of simple roots:

$$\{\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \delta), -\varepsilon_1, -\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\}$$

$$\{-\delta, \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \delta), -\varepsilon_1, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\}$$

$$\{\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \delta), \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \delta), \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \delta), \varepsilon_2 - \varepsilon_1, \varepsilon_2 - \varepsilon_3\}$$

$$\{\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \delta), \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \delta), \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\}$$

G(3). The roots are expressed in terms of linear functions $\varepsilon_1, \varepsilon_2, \varepsilon_3$, corresponding to $G_2$, $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$, and $\delta$, corresponding to $A_1$.

$$\Delta'_0 = \{\varepsilon_i - \varepsilon_j \pm \varepsilon_k \pm 2\delta\}$$

$$\Delta_1 = \{\varepsilon_i \pm \delta \mid i \neq j\}$$

Up to $W$-equivalence there is a unique system of simple roots:

$$\{\delta + \varepsilon_1, \varepsilon_2, \varepsilon_3 \pm \varepsilon_2\}$$

### 2.5.5. Examples of finite-dimensional contragredient Lie superalgebras.

Examples can be obtained in the following manner. Let $G$ be one of the Lie superalgebras $sl(m + 1, n + 1), B(m, n), C(n), D(m, n), D(2, 1; \alpha), F(4), \text{or} G(3)$. Let $H$ be a Cartan subalgebra, $\Pi$ be one of its systems of simple roots (listed in Section 2.5.4 above), and $e_i$ and $f_i$ be the corresponding nonzero vectors in $G_{-\alpha_i}$ and $G_{-\alpha_i}$, respectively, $\alpha_i \in \Pi$. Then the vectors $[e_i, f_i] = h_i$ form a basis of $H$. Setting $\deg e_i = -\deg f_i = 1, \deg h_i = 0$, we define a $\mathbb{Z}$-grading on $G$. Since $G$ is simple modulo its center, $G$ is the minimal $\mathbb{Z}$-graded Lie superalgebra with the local part $G_{-1} \oplus G_0 \oplus G_1$. In this way the structure of a contragredient Lie superalgebra is introduced in $G$. Its Cartan matrix is $A = (\alpha_i(h_j)),$ and $\tau = \{i \in I \mid \alpha_i \in \Delta_1\}$.

In the next section we show that these examples exhaust all simple modulo center finite-dimensional contragredient Lie superalgebras.

We now list all the resulting pairs $(A, \tau)$, up to equivalence. For this it is convenient, as usual, to introduce Dynkin diagrams. To begin with we extract from our examples in Tables IV and V all the contragredient Lie superalgebras of rank 1 and 2 with indecomposable Cartan matrices, the corresponding pairs $(A, \tau)$, and Dynkin diagrams. The circles $\bigcirc$, $\otimes$, and $\bullet$ are called, respectively, **white**, **gray**, and **black**. Contragredient Lie superalgebras of rank $r$ are depicted by a diagram consisting of $r$ white, gray, or black circles; the $i$th circle is white if $i \notin \tau$ and gray or black if $i \in \tau$ and $\alpha_{ii} = 0$ or $2$, respectively. The $i$th and the $j$th circles are not joined if $\alpha_{ij} = \alpha_{ji} = 0$; otherwise, they are joined as shown in Table V (note, that isomorphic Lie superalgebras may correspond to different Dynkin diagrams).
Matrix $A$ also satisfies the following restrictions. If $a_{ii} = 0$, then in every submatrix of order 3 with $a_{ii}$ in the center, the diagram of which is not a cycle and does not contain two arrows, the sum of the elements of the second row is 0. For the diagram $\otimes \leftarrow \otimes \rightarrow \bigcirc$, it is always $a_{31} = -2a_{33}$, and for the cycle,

\[\begin{array}{c}
\otimes \\
\bigcirc \rightarrow \otimes,
\end{array}\]
it is always $a_{23} = -2a_{21}$. We introduce matrices

$$D_a = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & \alpha \\ 0 & -1 & 2 \end{pmatrix}, \quad D_a' = \begin{pmatrix} 0 & 1 & -1 - \alpha \\ 1 & 0 & \alpha \\ 1 + \alpha & \alpha & 0 \end{pmatrix}.$$ 

The diagrams

$$\circ \rightarrow \circ \leftarrow \circ \quad \text{and} \quad \circ \leftarrow \circ \rightarrow \circ$$

always correspond to matrices $D_a$ and $D_a'$, respectively, and $\alpha = 1$ unless the contrary is stated.

The following proposition is a consequence of the results in Section 2.4.

**Proposition 2.5.6.** Let $G$ be one of $A(m, n), B(m, n), C(n), D(m, n), D(2, 1; \alpha), F(4)$, or $G(3)$ and let $G \simeq G(A, \tau)/C$, where $C$ is the center. Then $C \neq 0$ only for $A(n, n)$ and in this case $\dim C = 1$. The diagrams of the pairs $(A, \tau)$ can be described as follows (each point can be a white or a gray circle):

- **A**
  $$\begin{array}{cccccc}
  \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  \end{array}$$

- **B**
  $$\begin{array}{cccccc}
  \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  \end{array} \Rightarrow \circ, \quad \begin{array}{cccccc}
  \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  \end{array} \Rightarrow \bullet$$

- **C, D**
  $$\begin{array}{cccccc}
  \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  \end{array} \Rightarrow \circ, \quad \begin{array}{cccccc}
  \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  \end{array} \leftarrow \circ \rightarrow \circ,$$

- **D(2, 1; \alpha)**
  $$\begin{array}{cccccc}
  \circ & \rightarrow \circ & \rightarrow \circ, \quad \begin{array}{cccccc}
  \circ & \rightarrow \circ & \rightarrow \circ \\
  \end{array} \leftarrow \circ \rightarrow \circ, \quad \begin{array}{cccccc}
  \circ & \circ & \circ & \circ & \circ & \circ \\
  \end{array} \leftarrow \circ \rightarrow \circ, \quad \begin{array}{cccccc}
  \circ & \circ & \circ & \circ & \circ & \circ \\
  \end{array} \leftarrow \circ \rightarrow \circ, \quad \begin{array}{cccccc}
  \circ & \circ & \circ & \circ & \circ & \circ \\
  \end{array} \leftarrow \circ \rightarrow \circ,$$

where for the last two diagrams, subdiagrams without the first circle correspond to matrices $D_{-3}$ and $D'_{-3}$, respectively.

- **G(3)**
  $$\begin{array}{cccccc}
  \circ & \rightarrow \circ & \rightarrow \circ \\
  \end{array} \leftarrow \circ.$$
Below (Table VI) we list the "simplest" diagrams, the coefficients of the decomposition of the highest root into simple roots, the index $s$ of the only nonwhite circle, and the number $r$ of the circles.

2.5.6. The classification of finite-dimensional contragredient Lie superalgebras.

**Theorem 3.** Let $G(A, \tau)$ be a finite-dimensional contragredient Lie superalgebra whose Cartan matrix satisfies condition (m) of Proposition 2.5.2, and let $C$ be its center. Then $G' = G(A, \tau)/C$ is classical, and $(A, \tau)$ is equivalent either to one of the pairs listed in Proposition 2.5.6 or to $(A, \phi)$, where $A$ is the Cartan matrix of a simple Lie algebra.

**Proof.** According to Propositions 2.5.1 and 2.5.2, $C \subseteq H$ and $G'$ is simple. Therefore, by Theorem 2 and Proposition 2.5.6, it is sufficient to show that the linear Lie algebra $G_0'$ acting on $G_1'$ is reductive.

According to Proposition 2.5.3, on $G'$ there exists a nondegenerate invariant bilinear form $( , )$. We consider a new form on $G'$: $f(x, y) = (x, \theta y)$, where $\theta$ is the automorphism from the proof of Proposition 2.5.3 ($\theta(G_a) = G_{-a}$). Clearly, the restriction of $f$ to $G_\alpha$ is nondegenerate, and the operators $ad e_\alpha$ and $-ad \theta e_\alpha$ are dual with respect to this form. Hence, it follows that $[e_\alpha, \theta e_\alpha] = h$ is a nonzero element of $H$. For otherwise we would have two dual nonzero commuting nilpotent operators, which is impossible.

Now let $R$ be the radical of $G_0'$. Evidently, $R$ is graded relative to the root

<table>
<thead>
<tr>
<th>$G$</th>
<th>Diagram</th>
<th>$s$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A(m, n)$</td>
<td><img src="image1" alt="Diagram" /></td>
<td>$m + 1$</td>
<td>$m + n + 1$</td>
</tr>
<tr>
<td>$B(m, n)$</td>
<td><img src="image2" alt="Diagram" /></td>
<td>$n$</td>
<td>$m + n$</td>
</tr>
<tr>
<td>$B(0, n)$</td>
<td><img src="image3" alt="Diagram" /></td>
<td>$n$</td>
<td>$n$</td>
</tr>
<tr>
<td>$C(n)$</td>
<td><img src="image4" alt="Diagram" /></td>
<td>$1$</td>
<td>$n$</td>
</tr>
<tr>
<td>$D(m, n)$</td>
<td><img src="image5" alt="Diagram" /></td>
<td>$n$</td>
<td>$m + n$</td>
</tr>
<tr>
<td>$F(4)$</td>
<td><img src="image6" alt="Diagram" /></td>
<td>$1$</td>
<td>$4$</td>
</tr>
<tr>
<td>$G(3)$</td>
<td><img src="image7" alt="Diagram" /></td>
<td>$1$</td>
<td>$3$</td>
</tr>
<tr>
<td>$D(2, 1; \alpha)$</td>
<td><img src="image8" alt="Diagram" /></td>
<td>$1$</td>
<td>$3$</td>
</tr>
</tbody>
</table>
decomposition. We claim that $R \subseteq H$, which will prove the theorem. If this were not so, then $e_\alpha \in R$ for some $\alpha \neq 0$, and $[e_\alpha, \theta e_\alpha] = h \neq 0$, $h \in R$. Since $R$ is solvable, $[h, e_\alpha] = 0$; hence $\alpha(h) = 0$. Now we look at the adjoint representation of the subalgebra $\langle \theta e_\alpha, h, e_\alpha \rangle$. By the Lie theorem, in some basis the matrices of $\text{ad} \theta e_\alpha$ and $\text{ad} e_\alpha$ are triangular. But then the matrix of $\text{ad} h = [\text{ad} e_\alpha, \text{ad} \theta e_\alpha]$ is also triangular with zeros along the main diagonal. Since $\text{ad} h$ is diagonalizable, we see that $h = 0$. This is a contradiction and proves the theorem.

2.5.7. \textbf{Z-gradings.} It is not hard to show, just as in [12], that the relations $\deg e_i = -\deg f_i = k_i$, $\deg h_i = 0$, $k_i \in \mathbb{Z}$, $i = 1, \ldots, r$, determine all possible \textbf{Z}-gradings of finite-dimensional contragredient Lie superalgebras. In particular, if $(A, \tau)$ is a pair from Table VI, then for $k_i = 0$, $i \neq s$, $k_s = 1$ we obtain the \textbf{Z}-gradings of $A(m, n)$, $B(m, n)$, $C(n)$, and $D(m, n)$, as described in Section 2.1, and for $D(2, 1; \alpha)$, $F(4)$, and $G(3)$ we obtain consistent \textbf{Z}-gradings of the form $G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_1 \oplus G_2$, where $\dim G_{\pm 2} = 1$ and the $G_0$-modules $G_{\pm 1}$ are isomorphic, respectively, to $so_4 \otimes k$, $spin_7 \otimes k$, and $G_2 \otimes k$. In the same way, the \textbf{Z}-gradings are defined for $Q(n)$ (as $Q(n)\theta = A_n$ and we can naturally identify spaces $Q(n)k$ and $Q(n)\theta : \xi_i \leftrightarrow e_i$, $h_i \leftrightarrow h_i$).

3. Cartan Lie Superalgebras

3.1. The Lie Superalgebras $\mathbf{W(n)}$

3.1.1. \textit{Definition of $\mathbf{W(n)}$.} Let $A(n)$ be the Grassmann superalgebra with the generators $\xi_1, \ldots, \xi_n$. We denote $\text{der} A(n)$ by $\mathbf{W(n)}$. We recall (see Section 1.1.4) that every derivation $D \in \mathbf{W(n)}$ can be written in the form

$$D = \sum_i P_i \frac{\partial}{\partial \xi_i}, \quad P_i \in A(n),$$

where $\partial/\partial \xi_i$ is the derivation defined by

$$\partial/\partial \xi_i(\xi_j) = \delta_{ij}.$$ 

Letting $\deg \xi_i = 1$, $i = 1, \ldots, n$, we obtain a consistent \textbf{Z}-grading of $A(n)$, which induces one of $\mathbf{W(n)} = \oplus_{k \geq -1} \mathbf{W(n)}_k$, where

$$\mathbf{W(n)}_k = \{D \in \mathbf{W(n)} \mid D(A(n)_s) \subseteq A(n)_{s+k}\}$$

$$= \left\{\sum_i P_i \frac{\partial}{\partial \xi_i} \mid \deg P_i = k + 1, i = 1, \ldots, n\right\}.$$
In particular, \( W(n)_{-1} = \langle \partial/\partial \xi_i, ..., \partial/\partial \xi_n \rangle \). Hence it follows that 
\[
[\partial/\partial \xi_i, \partial/\partial \xi_j] = 0,
\]
that is,
\[
\frac{\partial}{\partial \xi_i} \frac{\partial}{\partial \xi_j} = - \frac{\partial}{\partial \xi_j} \frac{\partial}{\partial \xi_i}.
\]

Formula (3.1.1) is one of the standard facts of analysis on a Grassmann algebra, as developed in [2].

We now list some properties of \( W(n) \).

**Proposition 3.1.1.**

(a) \( W(n) = \bigoplus_{i=1}^{n-1} W(n)_i \) is transitive.

(b) The \( W(n)_0 \)-module \( W(n)_{-1} \) is isomorphic to \( gl_n \).

(c) \( W(n)_k = W(n)_k^1, k \geq 1 \).

(d) \( W(n) \) is simple for \( n \geq 2 \).

(e) If \( G = \bigoplus_{i \in \mathbb{Z}} G_i \) is a transitive \( \mathbb{Z} \)-graded Lie superalgebra for which the \( G_0 \)-modules \( G_{\pm 1} \) are isomorphic to the \( W(n)_0 \)-modules \( W(n)_{\pm 1} \), then \( G \cong W(n) \).

**Proof.** Properties (a)–(c) are easily verified directly. (d) follows from Proposition 1.2.8 and (e) follows from (c) and Proposition 3.1.2 below.

3.1.2. The universality of \( W(n) \) as a \( \mathbb{Z} \)-graded Lie superalgebra. Let \( G = \bigoplus_{i \in \mathbb{Z}} G_i \) be a transitive Lie superalgebra with a consistent \( \mathbb{Z} \)-grading, and \( \dim G_{-1} = n \). Then the map \( \psi_k: G_k \to \text{Hom}(A^{k+1}(G_{-1}), G_{-1}) \), defined by

\[
\psi_k(g)(a_1, ..., a_{k+1}) = \cdots [[[g, a_1], a_2], ..., a_{k+1}],
\]
determines a monomorphism of \( G_0 \)-modules \( \psi_k: G_k \to A^{k+1}(G_{-1}) \otimes G_{-1} \). The \( \psi_k \)'s yield a canonical monomorphism of \( G_0 \)-modules

\[
\psi: G \to \bigoplus_{k \geq 1} (A^{k+1}(G_{-1}) \otimes G_{-1}).
\]

From this we see by dimension arguments that if \( G = W(n) \), then \( \psi \) is an isomorphism of \( W(n)_0 \)-modules.

If we are now given a monomorphism of the \( G_0 \)-module \( G_{-1} \) into \( W(n)_0 \)-module \( W(n)_{-1} \), we obtain a chain of maps:

\[
G \to \bigoplus_k (A^{k+1}(G_{-1}) \otimes G_{-1}) \to W(n).
\]

It is easy to verify that the composite map is a monomorphism of \( \mathbb{Z} \)-graded Lie superalgebras. So we have the following result.
**Proposition 3.1.2.** Let $G = \bigoplus_{i \geq -1} G_i$ be a transitive superalgebra with a consistent $\mathbb{Z}$-grading, and $\dim G_{-1} = n$. Then there is an embedding of $G$ in $\mathcal{W}(n)$ preserving the $\mathbb{Z}$-grading.

3.1.3. The universality of $\mathcal{W}(n)$ as a Lie superalgebra with a filtration. $\mathcal{W}(n)$ is canonically equipped with a filtration. Now it turns out that there is also an embedding theorem for Lie superalgebras with filtration.

**Proposition 3.1.3.** Let $L = L_{-1} \supset L_0 \supset L_1 \supset \cdots$ be a transitive Lie superalgebra with a filtration, $\dim L/L_0 = n$, and suppose that $L_0$ contains $L_0$. Then there is an embedding $\alpha: L \to \mathcal{W}(n)$ preserving the filtration. If $\beta$ is any other such embedding, then there is one and only one automorphism $\Phi$ of $\mathcal{W}(n)$ that is induced by an automorphism of $\Lambda(n)$ for which $\alpha = \Phi \circ \beta$.

The proof carries over almost verbatim from [20], with the definitions replaced by the relevant definitions in Section 1.1.3. True, the proof in [20] only gives the existence of $\Phi$ under the assumption that $(\alpha - \beta)(L) \subseteq L_0$. However, this assumption is easily dispensed with, by modifying $\beta$ to an automorphism of $\mathcal{W}(n)$ induced by a linear automorphism of $\Lambda(n)$.

As a corollary to Proposition 3.1.3 we have the next result.

**Proposition 3.1.4.** Let $L = L_{-1} \supset L_0 \supset L_1 \supset \cdots$ be a subalgebra of $\mathcal{W}(n)$ with the induced filtration, and $\dim L/L_0 = n$. Then every automorphism of $L$ preserving the filtration is induced by an automorphism of $\Lambda(n)$.

Clearly, in $\mathcal{W}(n)$ with $n \geq 3$ there is a unique subalgebra containing $\mathcal{W}(n)_k$, namely, $\bigoplus_{k=0}^{n-1} \mathcal{W}(n)_k$. Hence for $n \geq 3$ this subalgebra, and therefore, the filtration in $\mathcal{W}(n)$, are invariant under all automorphisms. So we obtain the next result from Proposition 3.1.4.

**Proposition 3.1.5.** Every automorphism of $\mathcal{W}(n)$ with $n \geq 3$ is induced by an automorphism of $\Lambda(n)$.

3.2. Two Algebras of Differential Forms

3.2.1. The superalgebra $\Omega(n)$. Let $\Lambda(n)$ be the Grassmann superalgebra on $\xi_1, \ldots, \xi_n$. We denote by $\Omega(n)$ the associative superalgebra over $\Lambda(n)$ with the generators $d\xi_1, \ldots, d\xi_n$ and the defining relations $d\xi_i \circ d\xi_j = d\xi_j \circ d\xi_i$, $\deg d\xi_i = 0$, $i, j = 1, \ldots, n$. Note that $\Omega(n)$ is not commutative (in the sense of the bracket); in particular, $\xi_i d\xi_j = (d\xi_j)\xi_i$.

Every element $\psi \in \Omega(n)$ can be written uniquely as a sum of elements of the form

$$\psi = \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k} d\xi_{i_1} \circ \cdots \circ d\xi_{i_k}, \quad \text{where } a_{i_1 \cdots i_k} \in \Lambda(n).$$
We define on $\Omega(n)$ the differential $d$ as the derivation of degree $\bar{1}$ for which

$$d(\xi_i) = d\xi_i, \quad d(d\xi_i) = 0, \quad i = 1, \ldots, n.$$ 

It is easy to verify, as in Section 1.1.4, that this derivation exists and is unique.

$\Omega(n)$ is called the superalgebra of differential forms with commuting differentials.

**Proposition 3.2.1.** The differential $d$ has the following properties:

(a) $d(\varphi \circ \psi) = d\varphi \circ \psi + (-1)^{\deg \varphi} \circ d\psi$, $\varphi, \psi \in \Omega(n)$.

(b) $df = \sum_i \frac{\partial f}{\partial \xi_i} d\xi_i$, $f \in \Lambda(n)$.

(c) $d^2 = 0$.

(d) Every derivation $D$ of $\Lambda(n)$ extends uniquely to a derivation $\tilde{D}$ of $\Omega(n)$ for which $[\tilde{D}, d] = 0$.

(e) Every automorphism of $\Lambda(n)$ extends uniquely to an automorphism of $\Omega(n)$ commuting with $d$.

**Proof.** (a) is true, by definition.

(b) is proved by induction on the $\mathbb{Z}$-grading of $\Lambda(n) = \oplus A_i$. Suppose that $f \in A_k$; it is enough to prove (b) for $f = f_1 \xi_1$, where $f_1 \in A_{k-1}$. By the inductive hypothesis we have

$$df = (df_1) \xi_1 + (-1)^{k-1} f_1 d\xi_1 = \sum_i \left( \frac{\partial f_1}{\partial \xi_i} \right) d\xi_i + (-1)^{k-1} f_1 d\xi_i$$

$$= \sum_i \frac{\partial}{\partial \xi_i} \left( f_1 \xi_i \right) d\xi_i,$$

as required.

(c) We now define a height $h$ on $\Omega(n)$ by putting $h(d\xi_i) = 1$, $h(\xi_i) = 0$, $i = 1, \ldots, n$. We conduct the proof of (c) by induction on $h$.

If $h(f) = 0$, then $f \in \Lambda(n)$ and

$$d(df) = d \left( \sum_i \frac{\partial f}{\partial \xi_i} d\xi_i \right) = \sum_{i,j} \frac{\partial^2 f}{\partial \xi_i \partial \xi_j} d\xi_i \circ d\xi_j = 0,$$

by (3.1.1). Suppose, next, that $h(\varphi) = k$; it is enough to prove (c) for $\varphi = \varphi_1 \circ d\xi_i$. By induction we have $d^2(\varphi_1 \circ d\xi_i) = d(d\varphi_1 \circ d\xi_i) = 0$.

(d) Let $D = \sum_i P_i(\partial/\partial \xi_i)$ be a derivation of $\Lambda(n)$ of degree $k$. It is easy to verify, as in Section 1.1.4, that there is one (and only one) extension of $D$
to a derivation $D$ of $\Omega(n)$ for which $D(d\xi_i) = (-1)^k d(D\xi_i)$, $i = 1, \ldots, n$. We claim that $Dd = (-1)^k dD$. Let $f \in \Lambda$; then we have

$$D(df) = D \left( \sum_{i,j} \frac{\partial f}{\partial \xi_i} d\xi_j \right) = \sum_i P_i \frac{\partial f}{\partial \xi_i} \partial \xi_i + (-1)^k \sum_i \frac{\partial f}{\partial \xi_i} dP_i$$

$$= \sum_i \left( \sum_j P_j \frac{\partial f}{\partial \xi_i} \partial \xi_j + (-1)^k \frac{\partial f}{\partial \xi_i} \partial \xi_i \right) \partial \xi_i$$

$$= (-1)^k \sum_{i,j} \frac{\partial f}{\partial \xi_i} \left( P_j \frac{\partial f}{\partial \xi_j} \right) \partial \xi_i = (-1)^k d(Df).$$

Now clearly we have

$$Dd(f \, d\xi_{i_1} \circ \cdots \circ d\xi_{i_s}) = (-1)^k dD(f \, d\xi_{i_1} \circ \cdots \circ d\xi_{i_s})$$

as required.

(e) is proved rather like (d) (besides, as is easy to see, it follows from (d)).

The proposition is now proved.

Note that (d) gives us an action of $\mathfrak{w}(n)$ on $\Omega(n)$, commuting with $d$ in the sense of the bracket. There is also an analog to Poincaré's lemma.

**Proposition 3.2.2.** If a differential form $\psi \in \Omega(n)$ is closed, that is, $d\psi = 0$, then $\psi = d\psi$ for some $\psi \in \Omega(n)$.

**Proof.** We consider the linear map (homotopy operator) $K: \Omega(n) \rightarrow \Omega(n)$ and the homomorphism $\epsilon: \Omega(n) \rightarrow \Omega(n)$, defined by the formulas,

$$K(f \, d\xi_{i_1} \circ \cdots \circ d\xi_{i_{s-1}} \circ d\xi_n) = \xi_{n} f \, d\xi_{i_1} \circ \cdots \circ d\xi_{i_{s-1}},$$

$$K(f \, d\xi_{i_1} \circ \cdots \circ d\xi_{i_s}) = 0 \quad \text{when} \quad i_t \neq n, \quad t = 1, \ldots, s, \quad f \in \Lambda,$$

$$\epsilon(\xi_i) = \xi_i \quad \text{and} \quad \epsilon(d\xi_i) = d\xi_i \quad \text{if} \quad i \neq n, \quad \epsilon(\xi_n) = \epsilon(d\xi_n) = 0.$$

It is easy to check that $Kd + dK = 1 + \epsilon$. Therefore, if $d\psi = 0$, then $\psi = d(K(\psi)) - \epsilon(\psi)$, where $\epsilon(\psi)$ is closed and does not depend on $\xi_n$. Now we can use the induction by $n$.

3.2.2. The superalgebra $\Theta(n)$. We denote by $\Theta(n)$ the associative superalgebra over $\Lambda(n)$ with the generators $\theta \xi_1, \ldots, \theta \xi_n$ and defining relations

$$\theta \xi_i \wedge \theta \xi_j = -\theta \xi_j \wedge \theta \xi_i, \quad \deg \theta \xi_i = 1, \quad i, j = 1, \ldots, n.$$

Note that $\Theta(n)$ is commutative (in the sense of the bracket); in particular, $\xi_i \theta \xi_j = -\theta \xi_j \xi_i$.
Every element \( \omega \in \Theta(n) \) can be written uniquely as a sum of elements of the form
\[
\omega_k = \sum_{i_1 < \cdots < i_k} a_{i_1, \ldots, i_k} \theta_1 \xi_{i_1} \wedge \cdots \wedge \theta_k \xi_{i_k}.
\]

We define a differential \( \theta \) on \( \Theta(n) \) as the derivation of degree 0 for which
\[
\theta(\xi_i) = \theta \xi_i, \quad \theta(0) = 0, \quad i = 1, \ldots, n.
\]

It is easy to verify that this derivation exists and is unique.

**Proposition 3.2.3.** The differential \( \theta \) has the following properties:

(a) \( \theta(\omega_1 \wedge \omega_2) = \theta(\omega_1) \wedge \omega_2 + \omega_1 \wedge \theta(\omega_2) \).

(b) \( \theta(f) = \sum_i (\theta \xi_i)(\partial f/\partial \xi_i), f \in \Lambda(n) \).

(c) Every derivation \( D \) of \( \Lambda(n) \) extends uniquely to a derivation \( \bar{D} \) of \( \Theta(n) \) for which \( \bar{D}\theta = \theta \bar{D}, f \in \Lambda(n) \); if \( \theta(\xi_i) = 0, i = 1, \ldots, n \), then \( \bar{D}\theta = \theta \bar{D} \).

(d) Every automorphism \( \Phi \) of \( \Lambda(n) \) extends uniquely to an automorphism \( \Phi \) of \( \Theta(n) \) for which \( \Phi\theta = \theta \Phi, f \in \Lambda(n) \).

The proof is similar to that of Proposition 3.2.1. Observe that \( \theta^a \neq 0 \). For example, \( \theta^a(\xi_1 \xi_2) = 2\xi_1 \wedge \xi_2 \). Also, it is not true that \( [D, \theta] = 0 \) for \( D \in W(n) \). Nevertheless, (c) provides us with some action of \( W(n) \) on \( \Theta(n) \).

### 3.3. Special and Hamiltonian Lie Superalgebras

**3.3.1. Volume forms and the Lie superalgebras \( S(n) \) and \( \bar{S}(n) \).** A volume form is a differential form in \( \Theta(n) \) like
\[
\omega = f \theta \xi_1 \wedge \cdots \wedge \theta \xi_n, \quad f \in \Lambda(n) \cap f(0) \neq 0.
\]

To a volume form \( \omega \) there corresponds in \( W(n) \) the subalgebra
\[
S(\omega) = \{ D \in W(n) | D\omega = 0 \}.
\]

Among these subalgebras we single out two: \( S(n) = S(\theta \xi_1 \wedge \cdots \wedge \theta \xi_n) \) and \( \bar{S}(n) = S((1 + \xi_1 \cdots \xi_n) \theta \xi_1 \wedge \cdots \wedge \theta \xi_n) \) for \( n = 2k \).

The condition for an operator \( \sum P_i(\partial/\partial \xi_i) \) to belong to \( S(\omega) \) can be written as follows:
\[
\sum_i \frac{\partial}{\partial \xi_i} (f P_i) = 0.
\]

Hence it is easy to see that \( S(\omega) \) is the linear span of the elements like
\[
f^{-1} \frac{\partial a}{\partial \xi_i} \frac{\partial}{\partial \xi_j} + f^{-1} \frac{\partial a}{\partial \xi_j} \frac{\partial}{\partial \xi_i}, \quad a \in \Lambda(n).
\]

On \( S(\omega) \) a filtration is induced from \( W(n) \), and on \( S(n) \) clearly even a \( \mathbf{Z} \)-grading.
PROPOSITION 3.3.1.

(a) \( \text{Gr } S(\omega) \cong S(n) = \bigoplus_{k=0}^{\infty} S(n)_k \).

(b) The semisimple part of \( S(\omega)_0 \) is isomorphic to \( \mathfrak{sl}_n \).

(c) The \( S(n)_0 \)-module \( S(n)_{-1} \) is isomorphic to \( \mathfrak{sl}_n \).

(d) The \( S(n)_0 \)-modules \( S(n)_k \) are irreducible and isomorphic to the highest component of the module \( \mathfrak{sl}_n \otimes \Lambda^{n-k-1} \mathfrak{sl}_n \).

(e) \( S(n)_k = S(n)_1^k \), \( k \geq 1 \).

(f) The \( \mathbb{Z} \)-graded Lie superalgebra \( S(n) \) is transitive. The \( S(\omega) \) are simple for \( n \geq 3 \).

(g) Every automorphism of \( S(\omega) \), \( n \geq 3 \), is induced by an automorphism of \( A(n) \) under which the differentiation form \( \omega \) is multiplied by an element of \( k \).

(h) If \( G = \bigoplus_{i=-1}^{\infty} G_i \) is a transitive \( \mathbb{Z} \)-graded Lie superalgebra for which the \( G_0 \)-module \( G_{-1} \) is isomorphic to \( \mathfrak{sl}_n \), then \( G \cong S(n) \).

Proof. Properties (a)-(e) are easily derived from the description of the elements of \( S(\omega) \). The fact that \( S(\omega) \) is simple now follows from Proposition 1.2.8; therefore, (f) is true by (a).

To prove (g), we note (as was done in [15]) that if \( \omega_1 \) and \( \omega_2 \) are not proportional, then \( S(\omega_1) \neq S(\omega_2) \) and that the filtration in \( S(\omega) \) for \( n \geq 3 \) is invariant under automorphisms. (For \( n > 3 \) this is proved as in Section 3.1.3, and for \( n = 3 \) it is obvious.)

Finally, (h) is obtained by embedding \( G \) in \( W(n) \); clearly, then \( G_i = S(n)_i \) for \( i = -1, 0 \), and we can then use Proposition 3.3.2 below.

We set \( D_0 = \sum \xi_i(\partial/\partial \xi_i) \), \( T_k(n) = \{ fD_0 \mid f \in A(n)_k \} \subset W(n)_k \), \( k = 0, \ldots, n-1 \).

The next proposition is easy to obtain, for example, by dimension arguments.

PROPOSITION 3.3.2. \( W(n)_k = S(n)_k \oplus T_k(n) \), \( k \geq 0 \), is a direct sum of irreducible \( W(n)_0 \)-modules, and the \( W(n)_0 \)-module \( T_k(n) \) is isomorphic to \( \Lambda^{n-k} \mathfrak{gl}_n \).

We now obtain a classification of volume forms.

PROPOSITION 3.3.3. Every volume form \( \omega = f\theta \xi_1 \wedge \cdots \wedge \theta \xi_n \) can be reduced by an automorphisms of \( A(n) \) to the shape

\[
(\alpha + \beta \xi_1 \cdots \xi_n) \theta \xi_1 \wedge \cdots \wedge \theta \xi_n, \quad \alpha \neq 0; \quad \beta = 0 \quad \text{when } n \text{ is odd}.
\]

Proof. We may assume that \( n \geq 3 \). The semisimple part of the Lie algebra \( S(\omega)_0 \) (which exists by Levi's theorem) can be carried, by Mal'tsev's theorem, into \( S(n)_0 \subset W(n)_0 \) by an inner automorphism \( \Phi \) of \( W(n)_0 \). According to Proposition 1.1.1, \( \Phi \) extends to an automorphism of \( W(n) \) and is, therefore, induced by an automorphism \( \varphi \) of \( A(n) \) (see Proposition 3.1.5). Replacing \( \omega \)
by \( \varphi(\omega) \), we may assume \( D\omega = 0 \) for \( D \in S(n)_0 \). But \( S(n)_0 \supset \{ \xi_i(\partial/\partial \xi_j), i \neq j \} \); therefore, \( \xi_i(\partial/\partial \xi_j) = 0 \) for \( i \neq j \), and hence, \( f = \alpha + \beta \xi_1 \cdots \xi_n \), and \( \beta = 0 \) for odd \( n \).

Taking Proposition 3.1.3 into account, we can derive the following result from Proposition 3.3.3.

**Proposition 3.3.4.** Every superalgebra \( S(\omega) \) is isomorphic to one of \( S(n) \) or \( \tilde{S}(n) \). These two Lie superalgebras are not isomorphic.

**Proposition 3.3.5.** Let \( L = L^- \supset L_0 \supset L_1 \supset \cdots \) be a Lie superalgebra with a filtration, and \( \text{Gr} \ L \simeq S(n) \). Then \( L \simeq \text{Gr} L \simeq S(n) \) for odd \( n \), and \( L \simeq S(n) \) or \( \tilde{S}(n) \) for even \( n \).

**Proof.** Using Proposition 3.1.3, we embed \( L \) in \( W(n) \). Then \( L_0 \subseteq W(n)_0 \), and by applying Maltsev's theorem to this pair, we may assume that the semisimple part of \( L_0 \) (which is isomorphic to \( sl_n \)) lies in \( W(n)_0 \), and hence, that \( L \supseteq S(n)_0 \).

We now observe that \( W(n) \) and \( L \) are \( S(n)_0 \)-modules, that the \( S(n)_0 \)-modules \( L \) and \( S(n) \) are isomorphic and, by Proposition 3.3.2, that \( W(n) = S(n) \oplus T(n) \) is a direct sum of \( S(n)_0 \)-modules, where \( T(n) = \oplus_k T_k(n) \).

From Propositions 3.3.1(d) and 3.3.2 it is clear that the \( S(n)_0 \)-modules \( T(n) \) and \( L \) can only contain a unique common simple component:

\[ \text{Gr}_{-1} L \simeq T_{n-1}(n). \]

Therefore, \( L = V \oplus (\oplus_{k \geq 0} S(n)_k) \), where \( V \subseteq S(n)_-1 \oplus T_{n-1}(n) \). If \( V = S(n)_-1 \), then \( L = S(n) \). For odd \( n \) there is no other possibility because \( V \subseteq L_1 \).

But if \( V \neq S(n)_-1 \) for even \( n \), then \( L \simeq \tilde{S}(n) \), as is easy to see.

**3.3.2. Hamiltonian forms and the Lie superalgebras \( H(n) \) and \( \tilde{H}(n) \).** A Hamiltonian form is a closed differential form in \( \Omega(n) \) of the kind

\[ \omega = \sum_{i,j=1}^{n} \omega_{ij} d\xi_i \circ d\xi_j, \quad \omega_{ij} \in \Lambda(n)^{\cdot}, \quad \omega_{ij} = -\omega_{ji}, \quad \det(\omega_{ij}(0)) \neq 0. \]

To a Hamiltonian form \( \omega \) there corresponds a subalgebra of \( W(n) \):

\[ \tilde{H}(\omega) = \{ D \in W(n) \mid D\omega = 0 \}. \]

We set \( H(\omega) = [\tilde{H}(\omega), \tilde{H}(\omega)] \), \( \tilde{H}(n) = \tilde{H}((d\xi_1)^{\text{\circ}} + \cdots + (d\xi_n)^{\text{\circ}}) \), and \( H(n) = [\tilde{H}(n), \tilde{H}(n)] \).

It is not difficult to see that the condition for \( \sum P_i(\partial/\partial \xi_i) \) to belong to \( \tilde{H}(\omega) \) can be written as follows:

\[ \frac{\partial}{\partial \xi_i} \left( \sum P_i \omega_{ij} \right) + \frac{\partial}{\partial \xi_i} \left( \sum \omega_{ij} P_i \right) = 0. \]
By the analog to Poincaré's lemma, this condition shows that there is an element \( f \in \Lambda(n) \) (depending on \( D \)) for which \( \sum_t \omega_t P_t = \partial f / \partial \xi_t \). Therefore, if \( (\omega_t) \) is the inverse matrix to \( (\omega_{ij}) \), we see that \( \mathfrak{H}(\omega) \) consists of all the elements of the form

\[
D_f = \sum_{t,j} \left( \omega_{ij} \frac{\partial f}{\partial \xi_i} \right) \frac{\partial}{\partial \xi_j}, \quad f \in \Lambda(n), \quad f(0) = 0,
\]

and that \([D_f, D_g] = D_{\{f,g\}}\), where

\[
\{f, g\} = (-1)^{\deg f} \sum_{i,j} \omega_{ij} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_j}.
\]

In particular, \( \mathfrak{H}(n) \) consists of the elements of the form

\[
D_f = \sum_i \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial \xi_i}, \quad f \in \Lambda(n), \quad f(0) = 0,
\]

and the bracket looks as follows:

\[
\{f, g\} = (-1)^{\deg f} \sum \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_i}.
\]

A filtration is induced on \( \mathfrak{H}(\omega) \) from \( \mathfrak{W}(n) \), and on \( \mathfrak{H}(n) \) and \( \mathfrak{H}(n) \) even \( \mathbb{Z} \)-gradings.

**Proposition 3.3.6.**

(a) \( \text{Gr } \mathfrak{H}(\omega) \cong \mathfrak{H}(n) = \bigoplus_{k=0}^{n-2} \mathfrak{H}(n)_k \).

(b) The semisimple part of \( \mathfrak{H}(\omega)_0 \) is isomorphic to \( \mathfrak{so}_n \).

(c) \( \mathfrak{H}(n) = \bigoplus_{k=0}^{n-2} \mathfrak{H}(n)_k \), that is \( \mathfrak{H}(n) = \mathfrak{H}(n) \oplus \langle D_{\xi_1 \ldots \xi_n} \rangle \).

(d) \( \mathfrak{H}(n)_k = \mathfrak{H}(n)_k \), \( k \geq 1 \).

(e) \( [\mathfrak{H}(n)_k, \mathfrak{H}(n)_{-1}] = \mathfrak{H}(n)_{k-2} \) for \( k \leq n-2 \); \( [\mathfrak{H}(n)_k, \mathfrak{H}(n)_{-1}] = \mathfrak{H}(n)_{k-1} \), \( k \leq n \).

(f) The \( \mathfrak{H}(n)_0 \)-modules \( \mathfrak{H}(n)_k \) are isomorphic to \( \Lambda^{k+2} \mathfrak{so}_n \), \( -1 \leq k \leq n-2 \).

(g) The \( \mathbb{Z} \)-graded Lie superalgebras \( \mathfrak{H}(n) \) and \( \mathfrak{H}(n) \) are transitive.

(h) \( \mathfrak{H}(n) \) is simple for \( n \geq 4 \).

(i) Every automorphism of \( \mathfrak{H}(\omega) \) for \( n \geq 4 \) and of \( \mathfrak{H}(\omega) \) for \( n \geq 3 \) is induced by an automorphism of \( \Lambda(n) \) under which \( \omega \) is multiplied by an element of \( k \).

(j) If \( G = \bigoplus_{i \geq 1} G_i \) is a transitive \( \mathbb{Z} \)-graded Lie superalgebra for which the \( G_0 \)-module \( G_1 \) is isomorphic to \( \mathfrak{so}_n \), then \( G \cong \mathfrak{H}(n) \) or \( \mathfrak{H}(n) \) or \( \mathfrak{P}(3) \).

**Proof.** Properties (a)–(i) are established just like the corresponding assertions in Proposition 3.3.1.
(j) is now a consequence of the following assertion: If $G = \bigoplus_{i \geq -1} G_i$ is a transitive Lie superalgebra with a consistent $\mathbb{Z}$-grading and if the $G_0$-module $G_{-1}$ has an invariant symmetric bilinear form $(\ ,\ )$, then there is an embedding of $G$ in $\mathfrak{H}(n)$ preserving the $\mathbb{Z}$-grading. To prove this we construct embeddings $i_k: G_k \to \Lambda^{k+2}(G^*_1)$ by the formula $i_k(g)(a_1, \ldots, a_{k+2}) = ([\ldots[g, a_1], \ldots, a_{k+1}], a_{k+2})$. The subsequent arguments are the same as in Section 3.1.2.

Remark (cf. [22]). Let $\mathfrak{C}_n$ be a Clifford superalgebra with the natural $\mathbb{Z}_2$-grading. The bracket turns this into a Lie superalgebra $(\mathfrak{C}_n)_L$. The factor algebra $(\mathfrak{C}_n)_L/\langle 1 \rangle$ is isomorphic to $\mathfrak{H}(n)$.

**Proposition 3.3.7.** Let $L = L_{-1} \supset L_0 \supset L_1 \supset \cdots$ be a Lie superalgebra with filtration, and $\text{Gr} L \cong \mathfrak{H}(n)$ or $\mathfrak{H}(n)$. Then $L \cong \text{Gr} L$.

**Proof.** By Levi's theorem, $L_0$ contains a subalgebra $G_0$ isomorphic to $so_n$ (see Proposition 3.3.6(b)). Now $L$ splits into a $\mathbb{Z}_2$-graded direct sum of irreducible $G_0$-submodules, and the $G_0$-module $L$ is isomorphic to the $\text{Gr} G_0 L$-module $\text{Gr} L$. Hence we see that $L = G_{-1} \oplus L_0$ is a direct sum of $G_0$-modules, and that the $G_0$-module $G_{-1}$ is isomorphic to $so_n$. In particular, $[G_{-1}, G_{-1}] \subseteq S^2 so_n$; hence, and from Proposition 3.3.6(f), it is clear that $[G_{-1}, G_{-1}] \subseteq L_{-2}$. It follows that $[G_{-1}, G_{-1}]$ can differ from zero only when $n$ is even and $\text{Gr} L \cong \mathfrak{H}(n)$. However, in that case $[G_{-1}, G_{-1}] = L_{n-2} = \langle D_{\epsilon_1, \ldots, \epsilon_n} \rangle$, and for $a \in G_{-1}$ we have $[[a, a], a] = (a, a) \cdot [D_{\epsilon_1, \ldots, \epsilon_n}, a] = 0$ (by the Jacobi identity); this contradicts the transitivity of $\mathfrak{H}(n)$.

Thus, $[G_{-1}, G_{-1}] = 0$. We now embed $L$ in $W(n)$ with preservation of the filtration $\alpha: L \to W(n)$, using Proposition 3.1.3. Since $[G_{-1}, G_{-1}] = 0$, there is an embedding $\beta: G_{-1} \to W(n)$ for which $\beta(G_{-1}) = W(n)_{-1}$. Therefore, $\alpha$ can be modified (by Proposition 3.1.3) to an automorphism of $W(n)$ for which $\alpha(G_{-1}) = \langle \partial/\partial \xi_1, \ldots, \partial/\partial \xi_n \rangle$.

Now let $\text{Gr} L = \mathfrak{H}(n)$. Then $W(n) \supset L = L_{-1} \supset L_0 \supset \cdots \supset L_{n-3} \supset 0$, where $G_{-1} = W(n)_{-1}$ and the $G_0$-module $L_{n-3}$ is isomorphic to $so_n$, and $L_{n-3} \subseteq W(n)_{n-3} \oplus W(n)_{n-3}$; hence, it follows evidently that $L_{n-3} \subseteq W(n)_{n-3}$. Therefore, $(\text{ad } G_{-1})^{n-3} L_{n-3} \subseteq W(n)_{n-3}$.

In this way, in accordance with Proposition 3.3.6(e), a $\mathbb{Z}$-grading is introduced on $L$ that is consistent with the filtration; therefore, $L \cong \text{Gr} L$.

The arguments for the case $\text{Gr} L \cong \mathfrak{H}(n)$ are similar.

It is now easy to prove the analog to Darboux' lemma.

**Proposition 3.3.8.** Every Hamiltonian form $\omega = \sum \omega_{ij} d\xi_i \wedge d\xi_j$ can be brought by an automorphism of $\Lambda(n)$ to the shape

$$\sum_{i=1}^n (d\xi_i)^2.$$
Proof. We consider $\mathfrak{H}(\omega)$. According to Proposition 3.3.6(a), $\text{Gr } \mathfrak{H}(\omega) \cong \mathfrak{H}(\mathfrak{n})$. Therefore, by Proposition 3.3.7, $\mathfrak{H}(\omega) \cong \mathfrak{H}(\mathfrak{n})$, and from Proposition 3.1.3 it follows that there is an automorphism $\Phi$ of $\mathfrak{A}(\mathfrak{n})$ carrying $\mathfrak{H}(\omega)$ into $\mathfrak{H}(\mathfrak{n})$. Therefore $\partial/\partial\xi_i(\Phi(\omega)) = 0$ (because $\partial/\partial\xi_i \in \mathfrak{A}(\mathfrak{n})$) and $\Phi(\omega) = \sum c_{ij} d\xi_i o d\xi_j$, $c_{ij} \in k$. It remains to apply a linear automorphism.

3.3.3. Definition. The Lie superalgebras $\mathfrak{W}(n)$ for $n \geq 3$, $\mathfrak{S}(n)$ for $n \geq 4$, $\mathfrak{S}(n)$ for even $n \geq 4$, and $\mathfrak{H}(n)$ for $n \geq 5$ are called Curti Lie superalgebras. (For other values of $n$ they are either not simple or isomorphic to classical Lie superalgebras: $\mathfrak{W}(2) \cong \mathfrak{A}(1, 0) \cong \mathfrak{C}(2)$, $\mathfrak{S}(3) \cong \mathfrak{P}(2)$, $\mathfrak{H}(4) \cong \mathfrak{A}(1, 1)$.)

4. The Classification Theorem

In this chapter we complete the classification of the simple Lie superalgebras over an algebraically closed field of characteristic 0.

4.1. Classification of Certain $\mathbb{Z}$-Graded Lie Superalgebras

4.1.1. The noncontragredient case. Here we prove the following important proposition, which is the most complicated technically.

Proposition 4.1.1. Let $G = \bigoplus_{i=-d}^t G_i$ be a bitransitive Lie superalgebra with a consistent $\mathbb{Z}$-grading, for which $d$ or $t = 1$, and suppose that

(a) $G_0$ is semisimple;
(b) the representations of $G_0$ on $G_{-1}$ and $G_1$ are irreducible;
(c) the representations of $G_0$ on $G_{-1}$ and $G_1$ are not contragredient;
(d) $G_{-1} \oplus G_0 \oplus G_1$ generates $G$.

Then $G$ is isomorphic as a $\mathbb{Z}$-graded superalgebra to one of $\mathfrak{S}(n)$, $\mathfrak{H}(n)$ with $n \geq 4$, or $\mathfrak{P}(n)$.

The proof of the proposition is based on an analysis of the relations between the highest vector $F_A$ of the representation of $G_0$ on $G_{-1}$ and the lowest vector $E_M$ of the representation of $G_0$ on $G_1$. We know (see the proof of Proposition 1.2.10) that

$$[F_A, E_M] = e_{-\alpha},$$

(4.1.1)

where $\alpha = -(\Lambda + M)$ is a nonzero root of $G_0$. Interchanging if necessary $G_k$ with $G_{-k}$, we may assume that $\alpha > 0$, and hence, in particular, that

$$[F_A, e_{\alpha}] = [E_M, e_{-\alpha}] = 0.$$

(4.1.2)

In the proof of the proposition we need a number of lemmas.
**Lemma 4.1.2.** Let $G$ be a finite-dimensional $\mathbb{Z}$-graded Lie superalgebra. Let $E_i, F_i \in G$, $i = 1, 2$, be $\mathbb{Z}$-homogeneous of nonzero degree and $\mathbb{Z}_2$-homogeneous elements and $H \in G$ a nonzero element such that

$$[H, E_i] = a_i E_i, \quad [E_i, F_j] = \delta_{ij} H, \quad [H, F_i] = -a_i F_i. \quad (4.1.3)$$

Then $a_1 = a_2 = 0$.

**Proof.** The subalgebra $P$ of $G$ generated by the $E_i$ and $F_i$ is clearly endowed with a $\mathbb{Z}$-grading when we set $\deg E_i = 1$, $\deg F_i = -1$, $\deg H = 0$. Suppose that one of the $a_i$ is not zero. We consider the matrix

$$A = \begin{bmatrix} a_1 & a_3 \\ a_1 & a_2 \end{bmatrix}$$

and set $\tau = \{i \in \{1, 2\} \mid E_i \text{ is odd}\}$. Then the contragredient Lie superalgebra $G(A, \tau)$ is infinite-dimensional. For if both $a_i \neq 0$, this follows, for example, from Theorem 3 (all the matrices of Table V are nondegenerate). But if $a_1 \neq 0$, $a_2 = 0$, then we replace $E_2$ by $E_2' = [E_1, E_2]$ and $F_2$ by $F_2' = [F_1, F_2]$ and arrive at the preceding case. The factor algebra $P_1 = G(A, \tau)/C$, where $C$ is the (one-dimensional) center, is also infinite-dimensional.

Now, evidently, the map $E_i \mapsto e_i, F_i \mapsto f_i$ induces an epimorphism of $\mathbb{Z}$-graded Lie superalgebras $P \twoheadrightarrow P_1$. Therefore, $\dim P = \infty$, which contradicts the fact that $G$ is finite-dimensional.

**Lemma 4.1.3.** Let $G = \bigoplus G_i$ be a finite-dimensional $\mathbb{Z}$-graded Lie superalgebra, with $G_0$ semisimple. Suppose that there exist odd elements $x_{\lambda}$ and $x_{\mu}$, $\mathbb{Z}$-homogeneous of nonzero degree, that are weight vectors of the adjoint representation of $G_0$ on $G$, and a root vector $e_\delta$ of $G_0$, linked by the relations

$$[x_{\lambda}, x_{\mu}] = e_{-\delta}, \quad (4.1.4)$$

$$[x_{\lambda}, e_\delta] = [x_{\mu}, e_{-\delta}] = 0. \quad (4.1.5)$$

Then $(\lambda, \delta) = 0$.

**Proof.** Suppose that $(\lambda, \delta) \neq 0$. The same argument as in the proof of Lemma 3.1 in [12] gives that $2(\lambda, \delta) = (\delta, \delta)$. We choose a root vector $e_{-\delta}$ such that $[e_{-\delta}, e_\delta] = h_\delta$. We consider the elements

$$E_1 = x_{\lambda}^2, \quad F_1 = -(4(\lambda, \delta))^{-1}[x_{\mu}, e_\delta]^3, \quad H = h_\delta, \quad E_2 = [[x_{\lambda}^2, e_{-\delta}], e_{-\delta}], \quad F_2 = (8(\lambda, \delta)(\delta - \lambda, \delta)(4\lambda - \delta, \delta))^{-1}[[[x_{\mu}, e_\delta]^3, e_\delta], e_\delta].$$

A direct calculation shows that these elements satisfy (4.1.3). Therefore, we find from Lemma 4.1.2 that $(\lambda, \delta) = 0$, and we have a contradiction.

Henceforth we assume that all the conditions of Proposition 4.1.1 are satisfied.
and, using Lemma 4.1.3, we look for restrictions on the weight $\Lambda$ and the root $\alpha$.

From (4.1.1), (4.1.3), and the lemma we deduce at once the next result.

**Lemma 4.1.4.** $(\Lambda, \alpha) = 0$.

**Lemma 4.1.5.** Let $\beta$ and $\gamma$ be positive roots of $G_\alpha$.

(a) If $\alpha + \beta$ is a root, then $(\Lambda - \beta, \alpha + \beta) = 0$.

(b) If $\alpha + \beta$ is a root and $\alpha - \beta$ is not, then

$$(\Lambda + \alpha, \beta) = 0 \quad \text{and} \quad \frac{2(\Lambda, \beta)}{(\beta, \beta)} = -\frac{2(\alpha, \beta)}{(\beta, \beta)} = 1.$$ 

(c) If $\alpha + \beta$ is a root and $\alpha - \beta, \alpha - \gamma, \beta - \gamma$ are not, then $(\Lambda, \gamma) = 0$.

**Proof.** (a) We set $x_\lambda = [e_{-\beta}, F_\Lambda], x_\mu = E_M, \delta = \alpha + \beta$. Then it follows from (4.1.1) and (4.1.2) that relations (4.1.4) and (4.1.5) hold. By Lemma 4.1.3 and 4.1.4, we now see that $(\Lambda - \beta, \alpha + \beta) = 0$.

(b) Suppose that $(\Lambda + \alpha, \beta) \neq 0$. We set $x_\lambda = [e_{-\beta}, F_\Lambda], x_\mu = [e_{\beta}, E_M], \delta = \alpha$. Then it follows from (4.1.1) and (4.1.2) that relations (4.1.4) and (4.1.5) hold. By Lemmas 4.1.3 and 4.1.4, $(\Lambda - \beta, \alpha) = -(\alpha, \beta) = 0$. However, by hypothesis, $(\alpha, \beta) < 0$, which is a contradiction.

We have yet to show that $2(\Lambda, \beta)/(\beta, \beta) = 1$. We know that $(\Lambda, \beta) - (\alpha, \beta) - (\beta, \beta) = 0$ and $(\Lambda, \beta) + (\alpha, \beta) = 0$. Adding, we see that $2(\Lambda, \beta) = (\beta, \beta)$, as required.

(c) Suppose that $(\Lambda, \gamma) \neq 0$. Then it is easy to see that $[[[E_M, e_\alpha], e_\beta], e_\gamma], [E_M, e_\alpha]) \neq 0$ and $[[[F_\Lambda, e_{-\beta}], e_{-\gamma}], F_\Lambda] \neq 0$, which proves the lemma.

**Lemma 4.1.6.** (a) Only one numerical mark of $\Lambda$ is different from 0, and that is equal to 1; in particular, $G_\alpha$ is simple.

(b) $\alpha$ is the highest root of one of the parts of the Dynkin diagram of $G_\alpha$ into which it is divided by the numerical mark of $\Lambda$.

**Proof.** (a) Since $(\Lambda, \alpha) = 0$, clearly there is a simple root $\beta$ for which $\alpha + \beta$ is a root, but $\alpha - \beta$ is not. If there is a simple root $\gamma \neq \beta$ for which $(\Lambda, \gamma) \neq 0$, then $\alpha - \gamma$ is not a root and by applying Lemma 4.1.5(c) we arrive at a contradiction.

Thus, the only nonzero numerical mark of $\Lambda$ corresponds to the simple root $\beta$. It is equal to 1 by Lemma 4.1.5(b).
(b) Suppose the contrary. Then there is a simple root $\beta$ for which $\alpha + \beta$ is a root and $(\alpha, \beta) = 0$. Multiplying both sides of (4.1.1) by $e_{-\beta}$ we have 

$$[[F_\alpha, e_{-\beta}], E_\beta] = e_{-\alpha - \beta},$$

from which it follows that $[F_\alpha, e_{-\beta}] \neq 0$; therefore $(\alpha, \beta) \neq 0$, which is a contradiction.

We denote by $s$ the number of the circle in the Dynkin diagram of $G_0$ against which the only nonzero numerical mark of $\Lambda$ is placed.

**Lemma 4.1.7.** Either the $s$th circle of the Dynkin diagram of $G_0$ is at an end, or it is joined to an end circle with the number $t$, and then $\alpha = \alpha_t$ is a simple root.

**Proof.** Suppose that the $s$th circle is joined both to the $(s - 1)$th and $(s + 1)$th. Applying Lemma 4.1.5(c) to $\beta = \alpha_s$ and $\gamma = \alpha_{s-1} + \alpha_s + \alpha_{s+1}$ (where $\alpha_i$ is a simple root corresponding to the $i$th circle), we see that $\alpha$ is a simple root. Lemma 4.1.7 now follows from Lemma 4.1.6(b).

**Conclusion of the proof of Proposition 4.1.1.** Unfortunately, I have not succeeded in avoiding case distinctions.

Let $\beta = \alpha_s$ be the unique simple root for which $(\alpha, \alpha_s) \neq 0$.

By Lemma 4.1.5(b):

$$-2(\alpha, \beta)(\beta, \beta) = 1. \quad (4.1.6)$$

In accordance with Lemma 4.1.7, we consider two cases separately.

**Case I.** The $s$th circle of the Dynkin diagram is at an end. If the $G_0$-module $G_{-1}$ is isomorphic to one of the linear Lie algebras $sl_n$ with $n > 2$ or $so_n$ with $n > 4$, $n \neq 6$, then we see evidently, by Lemma 4.1.6(b), that the local Lie superalgebra $G_{-1} \oplus G_0 \oplus G_1$ is isomorphic to the local part of $S(n)$ and $H(n)$, respectively.

We claim that all other cases are impossible. Let $t$ be the number of the circle in the Dynkin diagram that is determined by the following properties: $t \neq s$, $t$ is an end circle and belongs to the longest of the possible "tails" of the diagram. We denote by $\gamma$ the largest root for which in the decomposition into simple roots the coefficient of $\alpha_t$ is zero. It is not hard to check (using Table I) that in all cases satisfying (4.1.6), except the adjoint representation of $G_2$, neither $\alpha - \gamma$ nor $\beta - \gamma$ is a root and that $(\alpha, \gamma) \neq 0$. Therefore, by Lemma 4.1.5(c), this case cannot occur. For $G_2$ we set $\beta' = \alpha + \beta$. Then $\alpha + \beta'$ is a root; however, $(\alpha - \beta', \alpha + \beta') \neq 0$, as is easy to see, and this contradicts Lemma 4.1.5(a).

**Case II.** The $s$th circle of the Dynkin diagram is not at an end, but is joined by an edge to an end circle with the number $t$, and $\alpha = \alpha_t$. If the $G_0$-module $G_{-1}$ is isomorphic to $A^2sl_n$, $n > 3$, then we see clearly that the local Lie superalgebra $G_{-1} \oplus G_0 \oplus G_1$ is isomorphic to $P(n - 1)$.

We claim that all other cases are impossible. Let $\theta$ be the highest root of $G_0$. 
If \((\theta, \alpha_i) = (\theta, \alpha_i) = 0\), then by setting \(\gamma = \theta\) we arrive at a contradiction to Lemma 4.1.5(c). In the remaining cases satisfying (4.1.6), except the representation of \(C_n\) with highest weight \(\theta - \alpha_i\), we denote by \(\gamma\) the largest root for which in the decomposition into simple roots the coefficient of \(\alpha_i\) is zero, and we again use Lemma 4.1.5(c). In the case of \(C_n\) we set \(\beta' = \theta - 2\alpha_i\) and arrive at a contradiction to Lemma 4.1.5(a).

Thus, the local part of \(G\) is isomorphic to the local part of one of \(S(n), H(n)\), or \(P(n)\). The isomorphism of the \(\mathbb{Z}\)-graded Lie superalgebras themselves now follows from Propositions 1.2.3(c), 3.3.1(e), and 3.3.6(d).

This completes the proof of Proposition 4.1.1.

4.1.2. Classification of \(\mathbb{Z}\)-graded Lie superalgebras of depth 1.

We now describe two constructions of transitive \(\mathbb{Z}\)-graded Lie superalgebras.

Every \(\mathbb{Z}\)-graded Lie superalgebra \(G = \bigoplus G_i\) can be extended by means of an even derivation \(x\) defined by

\[
[z, x] = k x \quad \text{for} \quad x \in G_k.
\]

So we obtain a \(\mathbb{Z}\)-graded Lie superalgebra, which we denote by \(G^x = \bigoplus G^x_i\), where \(G^x_i = G_i^x\) for \(i \neq 0\) and \(G_0^x = G_0 \oplus \langle x \rangle\). If \(G_0\) is transitive and the center of \(G_0\) is trivial, then clearly \(G^x\) is also transitive.

The other construction goes as follows. Let \(H\) be a Lie algebra without center. On it we construct a Lie superalgebra \(H^x = G_{-1} \oplus G_0 \oplus G_1\) with a consistent \(\mathbb{Z}\)-grading, by setting \(G_{-1} = \xi H, G_0 = H, G_1 = \langle d/d\xi \rangle\), where the commutators are defined as follows: \([d/d\xi, \xi h] = h, [\xi h_1, h_2] = \xi[h_1, h_2], [d/d\xi, h] = 0\). Evidently, \(H^x\) is transitive.

Now we are in a position to state the main theorem of this section.

**Theorem 4.** A transitive irreducible Lie superalgebra \(G = \bigoplus G_i\) with a consistent \(\mathbb{Z}\)-grading and \(G_1 \neq 0\) is isomorphic as \(\mathbb{Z}\)-graded superalgebra to one of the following list:

I. \(A(m, n), C(n), P(n)\);

II. \(W(n), S(n), H(n), \hat{H}(n)\);

III. \(H^x\), where \(H\) is a simple Lie algebra;

IV. \(G^x\), where \(G\) is of type I, II, or III and the center of \(G_0\) is trivial.

**Proof.** Since the representation of \(G_0\) on \(G_{-1}\) is faithful and irreducible, \(G_0 = G_0^x \oplus C\), where \(G_0^x\) is semisimple, \(C\) is the center of \(G_0\), \(\dim C \leq 1\), and if \(\dim C = 1\), then \(C = \langle x \rangle\), with \([z, g] = kg\) for \(g \in G_k\) (see Proposition 1.2.12). Therefore, the representation of \(G_0\) on \(G_1\) is completely reducible; let

\[
G_1 = \bigoplus_{i} G_1^{(i)} \quad (4.1.7)
\]
be the decomposition of \( G_1 \) into \( G_0 \)-irreducible components. We denote by \( G^{(s)} \) the \( \mathbb{Z} \)-graded subalgebra of \( G \):

\[
G^{(s)} = G_{-1} \bigoplus [G_{-1}, G_1^{(s)}] \bigoplus G_1^{(s)} \bigoplus (G_1^{(s)})^2 \bigoplus \cdots.
\]

If we consider in \( G \) the \( \mathbb{Z} \)-graded subalgebra \( G_{-1} \bigoplus G_0 \bigoplus G_1^{(s)} \bigoplus (G_1^{(s)})^2 \bigoplus \cdots \), we can infer from Proposition 1.2.9 that \([G_1, G_1^{(s)}] \subseteq G'_0\). Therefore, \( G^{(s)} \) is a \( \mathbb{Z} \)-graded Lie superalgebra satisfying all the conditions of Theorem 4.

There are two possibilities.

1. The representations of \([G_1, G_1^{(s)}]\) on \( G_{-1} \) and \( G_1^{(s)} \) are contragredient. According to Proposition 1.2.10(a), \( G^{(s)} = G_{-1} \bigoplus [G_1, G_1^{(s)}] \bigoplus G_1^{(s)} \) is classical. Propositions 2.3.9 and 2.4.4 now show that \( G^{(s)} \) is isomorphic as \( \mathbb{Z} \)-graded algebra to one of \( A(m, n) \) or \( C(n) \).

2. The representations of \([G_1, G_1^{(s)}]\) on \( G_{-1} \) and \( G_1^{(s)} \) are not contragredient. According to Proposition 1.2.10(b), \([G_1, G_1^{(s)}] = G'_0\) is then simple. If the representation of \([G_1, G_1^{(s)}]\) on \( G_1^{(s)} \) is not faithful, that is, \( \dim G_1^{(s)} = 1 \), then, as is easy to see, \( G^{(s)} = G_{-1} \bigoplus [G_1, G_1^{(s)}] \bigoplus G_1^{(s)} \) is isomorphic as \( \mathbb{Z} \)-graded algebra to \((G'_0)^6\). If the representation of \( G'_0 \) on \( G_1^{(s)} \) is faithful, then according to Proposition 4.1.1 \( G^{(s)} \) is bitransitive, by Proposition 1.2.13, \( G^{(s)} \) is isomorphic as \( \mathbb{Z} \)-graded Lie superalgebra to one of \( S(n) \), \( H(n) \), or \( P(n) \).

Thus, when \( G_1 = G_1^{(s)} \) is an irreducible \( G_0 \)-module, then Lemma 4.1.8 below and Propositions 3.3.1 and 3.3.6 show that \( G \) is one of \( A(m, n) \), \( C(n) \), \( S(n) \), \( H(n) \), \( \hat{H}(n) \), \( P(n) \), \( H^6 \), or of type IV. (In Lemma 4.1.8 the case \( A(1, 1) \) is excluded; however, \( A(1, 1) \cong H(4) \).

We claim that in (4.1.7) all the \( G_0 \)-modules are pairwise inequivalent. Let \( F_A \) be the highest weight vector of the \( G_0 \)-module \( G_{-1} \), and \( E_{M_1^s} \) the lowest weight vector of \( G_1^{(s)} \). Suppose that in (4.1.7) there are two isomorphic \( G_0 \)-modules, say \( G_1^{(1)} \) and \( G_1^{(2)} \). If they are contragredient to \( G_{-1} \), then by Proposition 2.1.6, \( G^{(1)} \) and \( G^{(2)} \) are isomorphic. Therefore, the vectors \([F_A, E_{M_1^s}]\) and \([F_A, E_{M_2^s}]\) are proportional; consequently, \([F_A, E_{M_1^s} - cE_{M_2^s}] = 0\) for some \( c \in k \). But then, clearly, \([G_{-1}, E_{M_1^s} - cE_{M_2^s}] = 0\), which contradicts the fact that \( G \) is transitive. If \( G^{(1)} \) and \( G^{(2)} \) are not contragredient to \( G_{-1} \), then \([F_A, E_{M_1^s}]\) are root vectors of \( G'_0 \) corresponding to one and the same root and are, therefore, proportional. Again, this contradicts the transitivity of \( G \).

When we now compare the possibilities for \( G^{(s)} \) obtained above, we see that the \( G_0 \)-module \( G_1 \) can be reducible only if the \( G_0 \)-module \( G_{-1} \) is isomorphic to \( gl_n \) or the \( G_0^\prime \)-module \( G_{-1} \) to \( A^2sl_4 \simeq so_6 \). In the first case, Propositions 3.3.2 and 3.1.1(e) show that \( G \) is isomorphic to \( W(n) \). In the second case it follows from Proposition 3.3.6(j) that \( G \) is isomorphic to one of \( H(6) \), \( \hat{H}(6) \), \( H(6)^5 \), or \( \hat{H}(6)^5 \).

The proof of Theorem 4 is now complete.
LEMMA 4.1.8. Let $G = \bigoplus_{i \geq -1} G_i$ be a transitive $\mathbb{Z}$-graded Lie superalgebra and $G' = G_{-1} \oplus G_0 \oplus G_1$ isomorphic to one of

(a) $H^t$;

(b) $A(m, n)$ for $(m, n) \neq (1, 1)$ or $C(n)$;

(c) $P(n)$.

Then $G = G'$.

Proof. In all three cases we have to show that $G_2 = 0$.

(a) We recall that $H^t = \xi H \oplus H \oplus \langle d/d\xi \rangle$. Suppose that there is a $t \in G_2$, $t \neq 0$. Then $[t, \xi a] = \alpha(a)(d/d\xi)$, where $\alpha$ is a nonzero linear function on $H$ (because of transitivity). Therefore, $0 = [t, [\xi a, \xi a]] = \alpha(a)$ for every $a \in H$, which is impossible.

(b) Suppose the contrary. Then there is a nonzero weight vector $t_\lambda$ of the representation of $G_0$ on $G_2$. By the transitivity of $G$, there is a root vector $e_{-\beta} \in G_{-1}$ for which

$$[t_\lambda, e_{-\beta}] = e_{\lambda - \beta} \in G_1. \quad (4.1.8)$$

By Proposition 2.5.5(e) there is a root vector $e_{-\lambda + \beta} \in G_{-1}$ for which $[e_{-\lambda + \beta}, e_{-\beta}] = h_{\lambda - \beta}$. Multiplying both sides of (4.1.8) by $e_{-\lambda + \beta}$, we have

$$[[e_{-\lambda + \beta}, t_\lambda], e_{-\beta}] = h_{\lambda - \beta}. \quad (4.1.9)$$

Hence it follows that $[e_{-\lambda + \beta}, t_\lambda] = e_{\beta}$, so that (4.1.9) shows that $h_{\lambda - \beta}$ is proportional to $h_\beta$; and then $\lambda = c\beta$ (also by Proposition 2.5.5(e)). Moreover, as we have seen, $\lambda - \beta = (c - 1)\beta$ is a nonzero root of $G'$. However, as is clear from Section 2.5.4, multiples of $\beta$ can only be the roots $0$ and $-\beta$, that is, $c = 0$ or $2$. In the first case we see that $\dim G_\beta \geq 2$, which contradicts Proposition 2.5.5(b). Thus, any weight of the representation of $G_0$ on $G_2$ is equal to twice a root of $G_0$. But clearly this is impossible.

(c) We recall that the $G_0$-module $G_{-1}$ is isomorphic to $A^2sl_n$ or $S^2sl_n^*$, and the $G_0$-module $G_1$ to $S^2sl_n^*$ or $A^2sl_n^*$. Let $A_{\pm 1}$ be the highest and $M_{\pm 1}$ the lowest weights of $G_{\pm 1}$. By transitivity, $G_2$ is a $G_0$-submodule of $G_1 \otimes G_{-1}^*$. (A mapping $\psi: G_2 \to G_1 \otimes G_{-1}^* = \text{Hom}(G_{-1}, G_1)$ can be constructed in the obvious fashion: $\psi(g)(a) = [g, a], \ g \in G_2, \ a \in G_{-1}$. Thus, $G_2$ is a $G_0$-submodule of $A^2sl_{n+1}^* \otimes S^2sl_{n+1}^*$. This module splits into two irreducible components with the lowest weights $M^{(1)} = -A_{-1} + M_1$ and $M^{(2)} = -A_{-1} + M_1 + \alpha_1 + \alpha_2$, where $\alpha_1$ and $\alpha_2$ are the first two simple roots of the Lie algebra $A_n$.}

4.1.3. On extensions of some Lie superalgebras.
We now assume that one of the weight vectors $E_{M(i)}$ is contained in $G_2$. By transitivity we then have, respectively,

$$[F_{A-1}, E_{M(0)}] = E_{M_1},$$  
(4.1.10)

$$[F_{A-1}, E_{M(0)}] = [E_{M_1}, e_{\alpha_1+\alpha_2}].$$  
(4.1.11)

Now (4.1.10) also gives

$$[[F_{A-1}, e_{-\alpha_1}], E_{M(0)}] = [E_{M_1}, e_{\alpha_2}].$$  
(4.1.12)

Next, $[F_{A-1}, E_{M_1}] = e_{-\alpha_1}$ (or $e_{\alpha_1}$). Therefore, from (4.1.10):

$$0 = [F_{A-1}^2, E_{M(0)}] = 2[F_{A-1}, E_{M_1}] = e_{\alpha_1}.$$  
This is a contradiction of $E_{M(0)} \in G_2$. Similarly, we find from (4.1.11) (or (4.1.12)) that

$$0 = [F_{A-1}, e_{-\alpha_1}], [E_{M_1}, e_{\alpha_2}] = e_{\alpha_2};$$  

hence $E_{M(2)} \notin G_2$. This proves the lemma.

4.2. The Classification of the Simple Lie Superalgebras

4.2.1. The main theorem. The following theorem is the central result of the paper.

**Theorem 5.** A simple finite-dimensional Lie superalgebra over an algebraically closed field $k$ of characteristic 0 is isomorphic either to one of the simple Lie algebras or to one of the Lie superalgebras $A(m, n)$, $B(m, n)$, $C(n)$, $D(m, n)$, $D(2, 1; \alpha)$, $F(4)$, $G(3)$, $P(n)$, $Q(n)$, $W(n)$, $S(n)$, $\tilde{S}(n)$, or $H(n)$.

**Proof.** Let $L = L_0 \oplus L_1$ be a simple finite-dimensional Lie superalgebra over $k$. If the representation of $L_0$ on $L_1$ is irreducible, then $L$ is classical. Therefore, by Theorem 2, $L$ is isomorphic to one of $B(m, n)$, $D(m, n)$, $F(4)$, $G(3)$, $P(n)$, $Q(n)$, $W(n)$, $S(n)$, $\tilde{S}(n)$, or $H(n)$.

Suppose now that the representation of $L_0$ on $L_1$ is reducible. Then, by Proposition 1.3.2, $L$ has a filtration: $L = L_{-1} \supset L_0 \supset L_1 \supset \cdots$ for which the associated $\mathbb{Z}$-graded Lie superalgebra $\text{Gr} L = \bigoplus \text{Gr}_{L_i} L$ satisfies all the conditions of Theorem 4. Therefore, $\text{Gr} L$ can only be isomorphic to one of the Lie superalgebras of type I–IV listed there.

From the proof of Proposition 2.2.2, it is clear that if $\text{Gr} L \simeq H^2$, then $L$ is not simple; hence type III does not occur.
Proposition 1.3.1 shows that if the center of $Gr_0 L$ is nontrivial, then $L \simeq Gr L$. Hence, type IV does not occur either, because clearly no superalgebra of this type is simple; also $L \simeq W(n)$ if $Gr L \simeq W(n)$.

If $Gr L \simeq A(m, n)$, $C(n)$, or $P(n)$, then evidently, the representation of $L_0$ on $L_I$ is for $L$ the same as for $Gr L$. Therefore, Proposition 2.1.4 shows that $L \simeq Gr L$, so that $L$ is one of $A(m, n)$, $C(n)$, or $P(n)$.

If $Gr L \simeq H(n)$ or $\hat{H}(n)$, then $L \simeq Gr L$, by Proposition 3.3.7. But $\hat{H}(n)$ is not simple, so that this case is impossible, and $L$ is one of the $H(n)$.

Finally, if $Gr L \simeq S(n)$, then by Proposition 3.3.5, $L$ is isomorphic to an $S(n)$ or $S(n)$.

This completes the proof of the theorem.

4.2.2. Isomorphisms. It is not hard to list all the isomorphisms between simple Lie superalgebras. They are: $A(m, n) \simeq A(n, m)$; $A(1, 0) \simeq C(2) \simeq W(2)$; $A(1, 1) \simeq H(4)$; $P(2) \simeq S(3)$.

In the remaining cases, except for $D(2, 1; \alpha)$, $S(n)$, and $S(n)$, simple Lie superalgebras are pairwise nonisomorphic, because for them the $L_0$-modules $L_I$ are nonisomorphic. $S(n)$ and $S(n)$ are also nonisomorphic, according to Proposition 3.3.4. Conditions for isomorphisms of superalgebras in the family $D(2, 1; \alpha)$ were derived in Proposition 2.5.4(b).

The following is a list of the dimensions of all the simple Lie superalgebras.

<table>
<thead>
<tr>
<th>$L$</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A(m, n)$</td>
<td>$(m + n + 2)^2 - 1 - \delta_{m,n}$</td>
</tr>
<tr>
<td>$B(m, n)$</td>
<td>$2(m + n)^2 + m + 3n$</td>
</tr>
<tr>
<td>$C(n)$</td>
<td>$2n^2 + n - 2$</td>
</tr>
<tr>
<td>$D(m, n)$</td>
<td>$2(m + n)^2 - m + n$</td>
</tr>
<tr>
<td>$D(2, 1; \alpha)$</td>
<td>17</td>
</tr>
<tr>
<td>$F(4), G(3)$</td>
<td>40, 31</td>
</tr>
<tr>
<td>$P(n)$</td>
<td>$(n + 1)^2 - 1$</td>
</tr>
<tr>
<td>$Q(n)$</td>
<td>$2(n + 1)^2 - 2$</td>
</tr>
<tr>
<td>$W(n)$</td>
<td>$n \cdot 2^n$</td>
</tr>
<tr>
<td>$S(n)$</td>
<td>$(n - 1)2^n + 1$</td>
</tr>
<tr>
<td>$H(n)$</td>
<td>$2^n - 2$</td>
</tr>
</tbody>
</table>

4.2.3. Classification of finite-dimensional primitive Lie superalgebras. Let $L$ be a Lie superalgebra and $L_0$ be a distinguished subalgebra. The pair $(L, L_0)$ is called even primitive if $L_0$ is a maximal proper subalgebra, it contains no ideals of $L$, and $L_0 \subset L$. The same arguments as in the proof of Theorem 5 give a classification of the primitive even pairs (clearly, they are all finite-dimensional automatically, because $Gr L$ is embedded in $W(n)$, where $n = \dim L/L_0$). To state and prove the result, we need only make the following remarks.

As it is easy to see, if $L$ is a Lie superalgebra with a filtration for which $Gr L \simeq H^t$, then either $L \simeq H^t$ or $L \simeq \operatorname{der} Q(n) = Q(n) \oplus D$ is a semidirect sum of Lie superalgebras, $D$ being, up to a constant factor, the only odd outer derivation of $Q(n)$ (see Proposition 5.1.2(c)).

To an irreducible faithful representation of a Lie algebra $H$ in a space $V$
there corresponds the primitive Lie superalgebra $H^\vee = V \oplus H$, where $[V, V] = 0, [h, v] = h(v)$ for $h \in H, v \in V$, and $H_0^\vee = H, H_1^\vee = V$.

**Theorem 5'**. Let $(L, L_0)$ be a primitive even pair. Then $L$ is isomorphic to a Lie superalgebra in the following list:

I. $A(m, n), C(n), P(n)$;

II. $B(m, n), D(m, n), F(4), G(3), Q(n), \text{der} Q(n), D(2, 1; \alpha)$;

III. $W(n), S(n), S(n), H(n), \tilde{H}(n)$;

IV. $H^\vee$, where $H$ is a simple Lie algebra;

V. $G^\sharp$, where $G$ is one of the $\mathbb{Z}$-graded Lie superalgebras $A(n, n), P(n), S(n), H(n), \tilde{H}(n), H^\sharp$.

VI. $H^\vee$, where $H$ is a Lie algebra and $V$ a faithful irreducible $H$-module.

Each of these Lie superalgebras, except $P(n)$ and $P^\sharp(n)$, admits a unique structure of a primitive even pair. There are two such structures for $P(n)$ and $P^\sharp(n)$.

Note that Theorem 5' also gives a classification of the primitive transitive supergroups of transformations of a supermanifold whose stabilizer contains a maximal reduced subgroup.

### 5. Continuation of the Theory

#### 5.1. Description of Semisimple Lie Superalgebras in Terms of Simple Ones

**5.1.1. Definition.** Let $A = A_5 \oplus A_1$ be a superalgebra, $\text{der} A$ the Lie superalgebra of its derivations, and $L$ a subset of $\text{der} A$. Then $A$ is said to be $L$-simple if $A$ contains no nontrivial ideals that are invariant under all the derivations in $L$. If $A$ is der $A$-simple and $A^2 \neq 0$, then $A$ is called differentiably simple.

We define operators $l_s$ and $r_s$, $s \in A$, on $A$ by the formulas

$$l_s(a) = sa, \quad r_s(a) = (-1)^{(\deg a)(\deg s)}as, \quad a \in A.$$  

It is easy to verify that if $D \in \text{der} A$, then

$$[D, l_s] = l_{D(s)} \quad \text{and} \quad [D, r_s] = r_{D(s)}.$$  

We denote by $T(A)$ the associative subalgebra of $l(A)$ (all the endomorphisms of $A$) generated by all the $l_s$ and $r_s$, $s \in A$.  

Finally, the centroid of $A$ is the associative superalgebra $\Gamma(A) = \{ g \in L(A) \mid [g, h] = 0, h \in T(A) \}$. $A$ is called central if $\Gamma(A) = k$.

5.1.2. Differentiably simple superalgebras. A verbatim repetition of the arguments in [21], with the relevant definition replaced by those above, leads to the following result.

**Proposition 5.1.1.** Let $G$ be a finite-dimensional differentiably simple (not necessarily Lie) superalgebra. Then $G \simeq S \otimes A(n)$, where $S$ is a simple and $A(n)$ is the Grassmann superalgebra.

5.1.3. Description of semisimple Lie superalgebras. We recall that a superalgebra $A$ is said to be semisimple if $A^2 \neq 0$ and $A$ contains no nontrivial solvable ideals. In [21] the description of differentiably simple algebras is used to derive a description of semisimple Lie algebras over any field. The same arguments are suitable for Lie superalgebras.

**Theorem 6.** Let $S_1, \ldots, S_r$ be finite-dimensional Lie superalgebras, $n_1, \ldots, n_r$ be nonnegative integers, and $S = \bigoplus_{i=1}^r S_i \otimes A(n_i)$. Then

$$S = \text{inder } S = \bigoplus_{i=1}^r (\text{inder } S_i) \otimes A(n_i) \subseteq \text{der } S = \bigoplus_{i=1}^r ((\text{der } S_i) \otimes A(n_i) + 1 \otimes A(n_i)).$$

Let $L$ be a subalgebra of $\text{der } S$ containing $S$; we denote by $L_i$ the set of components of elements of $L$ in $1 \otimes \Lambda(n_i)$. Then:

(a) $L$ is semisimple if and only if $A(n_i)$ is $L_i$-simple for all $i$.

(b) All finite-dimensional semisimple Lie superalgebras arise in the manner indicated.

(c) $\text{der } L$ is the normalizer of $L$ in $\text{der } S$, provided that $L$ is semisimple.

5.1.4. Description of $\text{der } G$ for the simple Lie superalgebras $G$.

**Proposition 5.1.2.** Let $G$ be a simple Lie superalgebra, $\text{der } G$ be the Lie superalgebra of its derivations, and $\text{inder } G (\simeq G)$ be the ideal of $\text{der } G$ consisting of the inner derivations.

(a) If $G$ is one of the classical Lie superalgebras $A(m, n)$ with $m \neq n$, $B(m, n)$, $C(n)$, $D(m, n)$, $F(4)$, $G(3)$, or one of the Lie superalgebras of Cartan type $W(n)$ or $S(n)$, then $\text{der } G = \text{inder } G$.

(b) If $G = \bigoplus G_i$ is one of the $\mathbb{Z}$-graded Lie superalgebras $A(n, n)$ with
If \( n > 1 \), \( S(n) \), or \( P(n) \), then \( \text{der } G = \text{inder } G \oplus \langle z \rangle \) is a semidirect sum, where \( z \) is an even derivation of \( G \) such that \( \{z, g\} = kg \) for \( g \in G_k \).

(c) If \( G \cong \mathcal{Q}(n) = \mathcal{Q}(n)_0 \oplus \mathcal{Q}(n)_1 \), then \( \text{der } G = \text{inder } G \oplus \langle D \rangle \) is a semidirect sum, where \( D \) is the (within proportionality unique) endomorphism of \( \mathcal{Q}(n) \) for which \( D(\mathcal{Q}(n)_0) = 0 \), \( D(\mathcal{Q}(n)_1) = \mathcal{Q}(n)_0 \), and \( D: \mathcal{Q}(n)_1 \rightarrow \mathcal{Q}(n)_0 \) is an isomorphism of \( \mathcal{Q}(n)_0 \)-modules.

(d) If \( G \cong H(n) \), \( n \geq 5 \), then \( \text{der } G = \text{inder } G \oplus T \), where

\[
T = \left\langle \sum \xi_i \frac{\partial}{\partial \xi_i}, \sum \frac{\partial}{\partial \xi_i} (\xi_1 \cdots \xi_n) \frac{\partial}{\partial \xi_i} \right\rangle \subset \mathcal{W}(n)
\]
is a two-dimensional solvable Lie superalgebra.

(e) If \( G \cong A(1, 1) = G_{-1} \oplus G_0 \oplus G_1 \), then \( \text{der } G = \text{inder } G \oplus P \) is a semidirect sum, where \( P = \langle D_{-1}, z, D_{+1} \rangle \) is a three-dimensional simple Lie algebra \( [z, g] = kg \) for \( k \in G_k \), \( D_{\pm} \) are the (up to a constant factor unique) endomorphisms of \( G \) for which \( D_{\pm}(G_0) = 0 \), \( D_{\pm}(G_{1,1}) = 0 \), \( D_{\pm}(G_{1,1}) = G_{1,1} \), and \( D_{\pm}: G_{1,1} \rightarrow G_{1,1} \) are isomorphisms of \( G_0 \)-modules.

Proof. \( G \cong \text{inder } G \subset \text{der } G \) is an ideal of \( \text{der } G \). Let \( G_0 \) be the reductive part of \( G_0 \). Since the \( G_0 \)-module \( \text{der } G \) is completely reducible, we have that \( \text{der } G = \text{inder } G \oplus T \) is a direct sum of \( G_0 \)-modules and \( T \) is a \( \mathbb{Z}_2 \)-graded subspace; in particular, \( [G_0, T] \subset T \). On the other hand, \( [G_0, T] \subset \text{inder } G \), because \( \text{inder } G \) is an ideal of \( \text{der } G \). Therefore, \( [G_0, T] = 0 \) and if \( D \in T \), then \( \text{ad } D \) is an endomorphism of the \( G_0 \)-module \( \text{der } G \). Using this fact it is now easy to compute \( T \) in all cases.

5.2. Irreducible Finite-Dimensional Representations of Solvable and Simple Lie Superalgebras

5.2.1. Induced modules. Let \( G \) be a Lie superalgebra, \( U(G) \) its universal enveloping superalgebra (see Section 1.1.3), \( H \) a subalgebra of \( G \), and \( V \) an \( H \)-module. \( V \) can be extended to a \( U(H) \)-module. We consider the \( \mathbb{Z}_2 \)-graded space \( U(G) \otimes U(H) \ V \) (this is the factor space of \( U(G) \otimes V \) by the linear span of the elements of the form \( gh \otimes v - g \otimes h(v) \), \( g \in U(G) \), \( h \in U(H) \)). This space can be endowed with the structure of a \( G \)-module as follows: \( g(u \otimes v) = gu \otimes v \), \( g \in G \), \( u \in U(G) \), \( v \in V \). The so constructed \( G \)-module is said to be induced from the \( H \)-module \( V \) and is denoted by \( \text{Ind}_H^G \ V \).

We list some of the simplest properties of induced modules, which follow from the Poincaré–Birkhoff–Witt theorem (see Section 1.1.3).

**Proposition 5.2.1.** (a) Let \( G \) be a Lie superalgebra, \( H \) a subalgebra, \( V \) a simple \( G \)-module, and \( W \) an \( H \)-submodule of \( V \) considered as an \( H \)-module. Then \( V \) is a factor module of the \( G \)-module \( \text{Ind}_H^G \ W \).
(b) If \( H_2 \subset H_1 \subset G \) are subalgebras of \( G \) and \( W \) an \( H_2 \)-module, then 
\[
\text{Ind}_{H_2}^G (\text{Ind}_{H_1}^{H_2} W) \simeq \text{Ind}_{H_2}^G W.
\]

(c) Let \( H \subset G \) be a subalgebra of \( G \) containing \( G_0 \), and \( g_1, \ldots, g_t \), odd elements of \( G \) whose projections onto \( G/H \) form a basis. Let \( W \) be an \( H \)-module. Then 
\[
\text{Ind}_{H}^G W = \bigoplus_{1 \leq i_1 < \ldots < i_t \leq t} g_{i_1} \ldots g_{i_t} W
\]
is a direct sum of subspaces; in particular,
\[
dim \text{Ind}_{H}^G W = 2^t \dim W.
\]

The next result follows from Proposition 5.2.1(c) and Ado's theorem for Lie algebras.

**ADO'S THEOREM.** Every finite-dimensional Lie superalgebra has a finite-dimensional faithful representation.

5.2.2. Representations of solvable Lie superalgebras. Let \( G = G_0 \oplus G_1 \) be a Lie superalgebra. A linear form \( l \in G^* \) is said to be distinguished if \( l([G_0, G_0]) = l(G_1) = 0 \). We denote by \( \mathcal{L} \) the space of distinguished linear forms, by \( \mathcal{L}_0 \) the subspace consisting of those \( l \) for which \( l([G, G]) = l(G_1) = 0 \), and by \( \mathcal{L}_1 \) the subgroup of \( \mathcal{L}_0 \) generated by the linear forms given by the one-dimensional factors of the adjoint representation of \( G \).

Let \( \rho \) be a representation of \( G \) in a space \( V \), \( \mathcal{M} \) a subgroup of \( \mathcal{L}_0 \), and \( \lambda \in \mathcal{M} \). We define a representation \( \tilde{\rho} \) of \( G \) in \( V \) by the formula \( \tilde{\rho}(g)v = \rho(g)v + \lambda(g)v \) (i.e., \( \tilde{\rho} \) is a tensor product of \( \rho \) and a one-dimensional representation). The \( G \)-modules \( \rho \) and \( \tilde{\rho} \) are said to be \( \mathcal{M} \)-equivalent.

**LEMMA 5.2.2.** Let \( G \) be a Lie superalgebra, \( H \) be a subalgebra of codimension 1 containing \( G_0 \), and \( g \) be an odd element for which \( G = H \oplus \langle g \rangle \) is a direct sum of subspaces.

(a) If \( W \) is an irreducible \( H \)-module, then all the irreducible factors of the \( H \)-module \( \text{Ind}_{H}^G W = W \oplus gW \) are \( \mathcal{L}_0 \)-equivalent to \( W \) (\( \mathcal{L}_0 \subset H^* \)).

(b) If \( V \) is an irreducible \( G \)-module and \( W \) an irreducible \( H \)-submodule with \( W \neq V \), then \( V \simeq \text{Ind}_{H}^G W \).

**Proof.** (a) For \( h \in H \) we have \([h, g] = \lambda(h)g + h'\), where \( h' \in H, \lambda \in \mathcal{L}_0 \). Therefore, \( h(gv) = g(hv + \lambda(h)v) + h'v \); hence, the \( H \)-modules \( W \) and \( \text{Ind}_{H}^G W/W \) are \( \mathcal{L}_0 \)-equivalent.

(b) follows from (a) and Proposition 5.2.1.

Let \( l \in \mathcal{L} \) be a distinguished linear form, considered modulo \( \mathcal{L}_0 \); we set \( G_l = \{ g \in G \mid l([g, g]) = 0 \text{ for } g \in G \} \). Clearly, \( G_l \) is a subalgebra of \( G \) containing \( G_0 \), and \( l([G_l, G_l]) = 0 \). A subalgebra \( P \subset G \) is said to be subordinate to \( l \) if \( l([P, P]) = 0 \) and \( G_l \subset P \). Clearly, this concept is well defined.

We single out an important class of solvable Lie superalgebras—the completely solvable ones—for which all irreducible factors of the adjoint representa-
tion are one-dimensional. By Engel’s theorem, a nilpotent Lie superalgebra is completely solvable, and $\mathcal{L}_1 = 0$.

Finally, we denote by $\{H, l\}$ the one-dimensional $H$-module given by a linear form $l \in \mathcal{L}_0$ according to formula $h(\psi) = l(h)\psi$.

Now we are in a position to state a theorem that describes the finite-dimensional irreducible representations of solvable Lie superalgebras.

**Theorem 7.** Let $G = G_0 \oplus G_1$ be a solvable Lie superalgebra.

(a) If $V$ is an irreducible finite-dimensional $G$-module, then all the irreducible factors of $V$ considered as a $G_0$-module are one-dimensional, and their corresponding linear forms, extended by zero to $G_1$, lie in a single coset $l_V \in \mathcal{L}/\mathcal{L}_0$.

(b) Let $l \in \mathcal{L}/\mathcal{L}_0$, $P$ be a maximal subalgebra subordinate to $l$, and $\{P, l\}$ be the one-dimensional $P$-submodule given by the linear form $l \in l$. Then the $G$-module $V = \text{Ind}_P^G(P, l)$ is finite-dimensional and irreducible, and $l = l_V$. Two such $G$-modules $V_1$ and $V_2$ are $\mathcal{L}_0$-equivalent if and only if $l_1 = l_2$.

(c) Every finite-dimensional irreducible $G$-module $V$ is isomorphic to one of the modules $\text{Ind}_P^G(G, l)$, where $l \in l_V$ and $P$ is a maximal subalgebra subordinate to $l$.

(d) If $G$ is completely solvable, then $\mathcal{L}_0$ can be replaced everywhere by $\mathcal{L}_1$. In particular, if $G$ is nilpotent, we obtain a bijective correspondence $V \mapsto l_V$ between the set of classes of isomorphic finite-dimensional irreducible $G$-modules and $\mathcal{L}$.

**Proof.** (a) is proved by induction on $\dim G$. Let $V$ be an irreducible $G$-module, $H$ be a subalgebra of codimension 1 containing $[G, G]$, $G = H \oplus \langle g \rangle$, where $g \in G_1$, $s \in \mathbb{Z}_2$, and $W$ be an irreducible submodule of an $H$-module $V$. Clearly, then $V = \bigoplus_{s=0}^{s} \langle g \rangle W$ and since $H$ is an ideal of $G$, the proof of Lemma 5.2.2(a) shows that all the irreducible factors of $V$ are isomorphic to $W$. Therefore, if $g \in G_1$, then $V$ and $W$, as $G_0$-modules, have the same stock of irreducible factors, and (a) is true by induction. But if $g \in G_0$, then $G_0 = H_0 \oplus \langle g \rangle$, and only as $H_0$-modules do $V$ and $W$ have the same stock of irreducible factors. But then it follows that if $l_1, l_2 \in G^*$ are linear forms that are zero on $G_1$ and give irreducible factors of the $G_0$-module $V$; then by the inductive hypothesis, $l_1 - l_2 |_{[H_0, H_0]} = 0$, in particular, $l_1 - l_2 |_{[G_0, G_0]} = 0$. Since, of course, $l_1 - l_2 |_{[G_0, G_0]} = 0$ we see that $l_1 - l_2 \in \mathcal{L}_0$, as required.

(c) is proved by induction on $\dim G_1$. Let $V$ be a finite-dimensional irreducible $G$-module and $P$ be a maximal subalgebra subordinate to $l_V$. If $\dim G_1 = 0$, then (c) is true by Lie’s theorem. We may, therefore, assume that $\dim G_1 > 0$. We analyze first the case $G = P$, that is, $l_V = 0$. By induction on $\dim G_1$ we show that $\dim V = 1$. Let $G'$ be a subalgebra of $G$, of codimension 1, containing $G_0$, and $G - G' \oplus \langle g \rangle$. By the inductive hypothesis, the
G'-module V contains a one-dimensional submodule \( \langle v \rangle \). Suppose that dim \( V \) \( > 1 \); then, evidently, \( V = \langle v \rangle \oplus \langle gv \rangle \). Now \( g(gv) = \frac{1}{2}[g, g]v = \frac{1}{2}l([g, g])v = 0 \). Since V is irreducible, there is an element \( h \in G_1 \) for which \( h(gv) = v \). Replacing \( h \) by \( h + cg \) for a suitable \( c \in k \), we may assume that \( h(v) = 0 \). But then \( [h, g]v = v \) and \( l([h, g]) = 1 \), which contradicts our assumption.

Suppose now that \( G \neq P \). Then \( G \) has a subalgebra \( H \) of codimension 1 containing \( P \), and \( G = H \oplus \langle g \rangle \), \( g \in G_1 \). Let \( W \) be an irreducible submodule of the \( H \)-module \( V \). By the inductive hypothesis, \( W = \text{Ind}_P^H(P, l) \). If \( H = P \), then, as we have shown, \( W = \langle v \rangle \) is a one-dimensional \( H \)-module. In that case \( V \neq W \) because \( G \neq P \), and so \( V = \text{Ind}_P^G(P, l) \), by Lemma 5.2.2(b) and Proposition 5.2.1(b).

Suppose now that \( H \neq P \). Then \( l([h, h]) \neq 0 \) for some \( h \in H_1 \), and the quadratic equation in \( \alpha \),

\[
l([g + \alpha h, g + \alpha h]) = l([g, g]) + 2\alpha l([h, g]) + \alpha^2 l([h, h]) = 0,
\]

has a root \( \alpha_0 \). Replacing \( g \) by \( g + \alpha_0 h \), we may assume that \( l([g, g]) = 0 \).

What we have to show (according to Proposition 5.2.1(b)) is that the \( G \)-module \( \text{Ind}_P^G W = W + gW \) is irreducible. Suppose the contrary, i.e., that it contains a nonzero irreducible \( G \)-submodule \( W' \). By Lemma 5.2.2(a), the \( G \)-modules \( W \) and \( W' \) are \( \mathcal{L}_0 \)-equivalent. Hence, in particular, there is a one-dimensional \( P \)-submodule \( \{P, l_1\} = \langle v_1 + gv_2 \rangle \in W \), where \( l - l_1 \in \mathcal{L}_0 \). Now \( h(v_1 + gv_2) = l_1(h)v_1 + [h, g]v_2 = h(v_1) + [h, g]v_2 + ghv_2 \) for \( h \in G_0 \). Since \( [h, g] = c(h)g + h', h' \in H \), we see that \( h(v_2) = (l(h) - c(h))v_2 \). In particular, \( \langle v_2 \rangle \) is a one-dimensional submodule of the \( G_0 \)-module \( W \); hence by (a), \( h(v_2) = l_2(h)v_2 \) for \( h \in G_0 \), where \( l_2, l_0 \in \mathcal{L}_0 \). Therefore, in particular, \( l_0([g, g]) = 0 \). If \( v_1 \neq 0 \), then \( g(v_1 + gv_2) = gv_1 + \frac{1}{2}l([g, g])v_2 = gv_1 \neq 0 \); but if \( v_1 = 0 \), then \( gv_2 \neq 0 \). Thus, \( W' \cap gW \neq 0 \). It therefore follows from Lemma 5.2.2(a) that \( W' = gW \). But then \( h(gv) = l([h, g])v + ghv \) for \( h \in H_1 \).

Since \( gW \) is a \( G \)-module, we infer that \( l([h, g]) = 0 \) for \( h \in H_1 \), and since, furthermore, \( l([g, g]) = 0 \), it follows that \( l([G_1, g]) = 0 \), that is, \( g \in G_1 \subseteq P \). This contradicts the choice of \( g \).

(b) evidently follows from (a) and (c), and (d) follows from the fact that if \( G \) is a completely solvable Lie superalgebra, then all the irreducible factors of the \( G \)-module \( U(G) \) are one-dimensional and are given by linear forms in \( \mathcal{L}_1 \).

This completes the proof of the theorem.

The following propositions are consequences of Theorem 7.

**Proposition 5.2.3.** For an irreducible finite-dimensional representation of a solvable Lie superalgebra \( G = G_0 \oplus G_1 \) in a space \( V = V_0 \oplus V_1 \) we have:

Either \( \dim V_0 = \dim V_1 \) and \( \dim V = 2^s \), where \( s \leq \dim G_1 \), or \( \dim V = 1 \).
Proposition 5.2.4. All the irreducible finite-dimensional representations of a solvable Lie superalgebra $G = G_0 \oplus G_1$ are one-dimensional if and only if $[G_1, G_1] \subseteq [G_0, G_0]$.

Example 1. It follows from Theorem 7 that the families of representations $\rho_\alpha$ and $\rho'_\alpha$ of the Heisenberg superalgebras $N$ and $N'$, which were constructed in Section 1.1.6, contain all their finite-dimensional non-one-dimensional irreducible representations, and each precisely once.

Example 2. Let $G = l(1, 1)$ be the completely solvable Lie superalgebra with the basis $z = (1 0), h = (0 1), e = (0 1), f = (1 0)$. The set of representations of dimension > 1 is parameterized by the numbers $\alpha = l(h), \beta = l(z) \neq 0$:

$z \mapsto \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}, \quad h \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha - 1 \end{pmatrix}, \quad e \mapsto \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$

For $\beta = 0$ we obtain all the one-dimensional representations $h \mapsto \alpha, z, e, f \mapsto 0$.

The proof of Theorem 7 also works for infinite-dimensional representations of completely solvable Lie superalgebras. Two representations are called weakly equivalent if they have the same kernel in $U(G)$. We set $\mathcal{L} = \{l \in G^* | l(G_1) = 0\}$. Let $\mathcal{G}_0$ be the Zariski closure of $\text{Aut} G_0$ in $GL(G_0)$.

It is known (Dixmier) that there is a bijection between the set of $\mathcal{G}_0$-orbits in $G_0^*$ and the set of classes of weakly equivalent representations of $G_0$.

Theorem 7'. (a) If $G$ is a completely solvable Lie superalgebra and $V$ is an irreducible $G$-module, then there is an uncondensed Jordan–Holder series relative to $G_0$ and all its irreducible factors correspond to a single $\mathcal{G}_0$-orbit $\Omega_V$ in $\mathcal{L}/\mathcal{L}_1$.

(b) Let $l \in \mathcal{L}/\mathcal{L}_1$, $P$ be a maximal subalgebra subordinate to $l$, and $\dim P_0 = \frac{1}{2}(\dim G_0 + \dim(G_1)_0)$. Then the $G$-module $V_l = \text{Ind}_{\mathcal{P}}^G(\{P, l\})$ is irreducible. The correspondence $l \mapsto V_l$ induces a bijective correspondence between the set of classes of $\mathcal{L}_1$-weakly equivalent irreducible representations of $G$ and the set of $\mathcal{G}_0$-orbits in $\mathcal{L}/\mathcal{L}_1$. Here $\Omega_{V_l} = \mathcal{G}_0 \cdot l$.

5.2.3. Representations of simple Lie superalgebras.

Proposition 5.2.5. Let $G = \bigoplus_{i \geq -d} G_i$ be a $\mathbf{Z}$-graded Lie superalgebra of depth $d$, $N^\pm = \bigoplus_{i > 0} G_{-i}, B^\pm = N^\pm \ominus B_0$, where $B_0$ is a Borel subalgebra of $(G_0)_0$. Suppose that

$[B^+, B^-] \subseteq [B^+, B^-], \quad [B_0, (G^\pm)_0] = (G^\pm)_0, \quad \text{and} \quad G_{-1} \text{ generates } N^\pm.$

(a) Let $V$ be a finite-dimensional irreducible $G$-module; we set $V_0 = \{v \in V | N^-(v) = 0\}$. Then $V_0$ is an irreducible submodule of the $G_0$-module $V$. 
Two finite-dimensional $G$-modules $V$ and $V'$ are isomorphic if and only if the corresponding $G_0$-modules $V_0$ and $V_0'$ are isomorphic.

If the depth $d = 1$, then for any finite-dimensional irreducible $G_0$-module $V_0'$ there is a finite-dimensional irreducible $G$-module $V$ for which the $G_0$-module $V_0$ is isomorphic to $V_0'$.

Proof. Since $B^+$ is a solvable Lie superalgebra, it follows by Proposition 5.2.4 that any irreducible factor of the $B^+$-module $V$ is one-dimensional. Using the properties of $N^+$, we hence find that the $N^+$-module $V$ is nilpotent. In particular, $V_0 = 0$. The same is true for $N^{-}$.

Now it only remains to go through the arguments in [7] almost verbatim. First, we show that $W = V_0 \cap G_{-} V$ is equal to 0. Let $U'(G_{-})$ be the subalgebra of the enveloping superalgebra $U(G)$ generated by $G_{-}$ and $U(G_{-}) = U'(G_{-}) \oplus \langle 1 \rangle$. Since $U'(G_{-})$ is nilpotent, $U(G_{-}) W \subseteq U'(G_{-}) V \neq V$. $U(G_{-}) W$ is a $G$-submodule of $V$, and since $V$ is a simple $G$-module, $U(G_{-}) W = 0$; in particular, $W = 0$.

We set $V_i = G_{-} V_{i-1}$ for $i > 0$. Clearly, $V$ is the sum of the subspaces $V_i$. We show by induction that this sum is direct. Suppose the contrary, i.e., that $v \in V_{m+1} \cap (\bigoplus_{i=0}^{m} V_i)$, $v \neq 0$. Then $G_1 v \neq 0$ because $W = 0$. But $G_1 v \subseteq V_m \cap (\bigoplus_{i=0}^{m-1} V_i)$, which is impossible. Thus, $V = \bigoplus_{i>0} V_i$. From this it follows, obviously, that $V_0$ is an irreducible $G_0$-module.

Let $V$ be a finite-dimensional irreducible $G$-module; then $V \simeq \text{Ind}^{G_0}_{G_0} V_0/I$, where $V_0$ is an irreducible $G_0$-submodule and $I$ is a maximal submodule of the $G$-module $\text{Ind}^{G_0}_{G_0} V_0$. Since $V = \bigoplus_{i>0} V_i$, this $I$ is uniquely determined as the sum of all graded submodules of the $G$-module $\text{Ind}^{G_0}_{G_0} V_0$, and this proves (b).

We define an action of $G_0 \oplus N^+$ on $V'$, setting $N^+ V_0' = 0$. Since $d - 1$, the induced $G$-module $\text{Ind}^{G_0}_{G_0} V_0$ is finite-dimensional (see Proposition 5.2.1(c)). The required $G$-module is a factor module of this $G$-module.

We apply Proposition 5.2.5 to the following $\mathbb{Z}$-graded Lie superalgebras $G = \bigoplus G_i$: (a) $\mathfrak{p}(n)$, $\mathfrak{w}(n)$, $\mathfrak{s}(n)$, $\mathfrak{h}(n)$ with the “standard” $\mathbb{Z}$-grading; (b) $Q(n)$ with the $\mathbb{Z}$-grading in Section 2.5.7 with $k_1 = \cdots = k_n = 1$; (c) the contragredient Lie superalgebras with “standard” $\mathbb{Z}$-grading (from Section 2.5.7 with $k_1 = \cdots = k_r = 1$).

In cases (b) and (c) we set $H = (G_0)_0 = \langle h_1, \ldots, h_r \rangle$, $N^+ = \bigoplus_{i>0} G_i$, $B = H \oplus N^+$. Let $A \in H^*$, $a_i = A(h_i) \in k$, $\langle v_A \rangle$ be a one-dimensional $B$-module for which $N^+(v_A) = 0$, $h_i(v_A) = a_i v_A$. We set $V_A = \text{Ind}^{G}_G \langle v_A \rangle / I_A$, where $I_A$ is the (unique) maximal submodule of the $G$-module $V_A$. $A$ is called the highest weight of the $G$-module $V_A$. It follows from Proposition 5.2.5(b) that the $G$-modules $V_{A_1}$ and $V_{A_2}$ are isomorphic if and only if $A_1 - A_2$. Numbers $a_i = A(h_i)$ are called the numerical marks of $A$.

We let $\mathbb{Z}_+$ denote the set of nonnegative integers.
THEOREM 8. (a) Let $G = \bigoplus G_i$ be one of the following $\mathbb{Z}$-graded Lie superalgebras of depth $1$: $P(n)$, $W(n)$, $S(n)$, $H(n)$. Then the correspondence in Proposition 5.2.5 between finite-dimensional irreducible $G$-modules and finite-dimensional irreducible $G_0$-modules is bijective.

(b) For $Q(n)$ the set of numerical marks of the highest weight of the finite-dimensional module $V_A$ is characterized by the following conditions: $a_i \in \mathbb{Z}_+$ and if $a_i = 0$, then $a_1 + 2a_2 + \cdots + (i-1)a_{i-1} = a_n + 2a_{n-1} + \cdots + (n-i)a_{i+1}$. 

(c) For the contragredient Lie superalgebras in Table VI (the $s$th row of the Cartan matrix is normalized so that $a_{ss} = 1$ for $a_{ss} = 0$), the set $\{a_i\}$ of numerical marks of the highest weight of the finite-dimensional module $V_A$ is characterized by the following properties:

(1) $a_i \in \mathbb{Z}_+$ for $i \neq s$;

(2) $k \in \mathbb{Z}_+$, where $k$ is given by the following table:

<table>
<thead>
<tr>
<th>$G$</th>
<th>$k$</th>
<th>$h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B(0, n)$</td>
<td>$1/2a_n$</td>
<td>$0$</td>
</tr>
<tr>
<td>$B(m, n)$, $m &gt; 0$</td>
<td>$a_n - a_{n+1} - \cdots - a_{m+n-1} - 1/2a_{m+n}$</td>
<td>$m$</td>
</tr>
<tr>
<td>$D(m, n)$</td>
<td>$a_n - a_{n+1} - \cdots - a_{m+n-2} - 1/2(a_{m+n-1} + a_{m+n})$</td>
<td>$m$</td>
</tr>
<tr>
<td>$D(2, 1; \alpha)$</td>
<td>$(1 + \alpha)(2a_1 - a_2 - \alpha a_3)$</td>
<td>$2$</td>
</tr>
<tr>
<td>$F(4)$</td>
<td>$1/3(2a_1 - 3a_2 - 4a_3 - 2a_4)$</td>
<td>$4$</td>
</tr>
<tr>
<td>$G(3)$</td>
<td>$1/2(a_1 - 2a_2 - 3a_3)$</td>
<td>$3$</td>
</tr>
</tbody>
</table>

(3) for $k < b$ (in the table) there are the supplementary conditions:

- $B(m, n)$: $a_n - a_{n+1} - \cdots = a_{m+n}$.

- $D(m, n)$: $k = m - 1$.

- $D(2, 1; \alpha)$: $a_i = 0$ if $k = 0$; $(a_3 + 1)\alpha = \pm (a_2 + 1)$ if $k = 1$.

- $F(4)$: $a_i = 0$ if $k = 0$; $k \neq 1$; $a_2 = a_4 = 0$ if $k = 2$; $a_2 = 2a_4 + 1$ if $k = 3$.

- $G(3)$: $a_i = 0$ if $k = 0$; $k \neq 1$; $a_2 = 0$ if $k = 2$.

Proof. (a) follows from Proposition 5.2.5.

To prove (b) and (c), we let $\Pi_0$ denote the system of simple roots of the (reductive) Lie algebra $G_0$ determined by the induced $\mathbb{Z}$-grading. In order for the $G$-module $V_A$ to be finite-dimensional it is necessary and sufficient that

$$e_{\alpha}^{c+1}v_A = 0 \quad \text{for} \quad \alpha \in \Pi_0, \quad \text{where} \quad c = 2(\alpha, \alpha)/(\alpha, \alpha) \quad (5.2.1)$$

(it is well known that these relations generate the annihilator of $v_A$).

(b) If $G = Q(n)$, then $\Pi_0 = \{\alpha_1, \ldots, \alpha_s\}$, where $e_{\alpha_i} = e_i$; hence $a_i \in \mathbb{Z}_+$. It is easy to see that the supplementary conditions only arise when $a_i = 0$: 

This condition gives the equality $\bar{s}_i f_i \sigma = 0$, which is equivalent to the second equality in (b).

(c) If $G = A(m, n)$ or $C(n)$, then $\Pi_0 = \{\alpha_i, i \neq s\}$, and condition (5.2.1) is equivalent to condition (1). For the remaining contragredient Lie superalgebras, $\Pi_0 = \{\alpha_i, i \neq s, \beta\}$, where $\beta$ is the maximal root among the roots of the form $\sum_{\alpha > s} k \alpha_i$. By the same token, condition (5.2.1) shows that conditions (1) and (2) are necessary. It is also clear that (1) is sufficient for (5.2.1) when $\alpha = \alpha_i$. However, condition (2) turns out not to be sufficient for (5.2.1) when $\alpha = \beta$. When $\alpha = \beta$, using direct computations from (5.2.1), we can show that condition (3) is necessary. It is also not hard to verify that (1) and (2) are sufficient for (5.2.1) when $G = B(0, n)$.

It remains to show that conditions (2) and (3) are sufficient for (5.2.1) when $\alpha = \beta$. To do this, it suffices to find a set of highest weights $\Lambda$ of finite-dimensional modules $V_\Lambda$ with the property $2(\Lambda, \beta)/(\beta, \beta) \leq k$ which generates the plane defined by the equations in (3). It is clear that for $B(m, n)$ and $D(m, n)$ without loss of generality we may assume that $n = 1$; then for $B(m, 1)$ and $D(m, 1)$ the desired set is the exterior powers of the standard representation. For $D(2, 1; a)$, $F(4)$, and $G(3)$, we must take the exterior powers of the adjoint representation; then we need only verify that (3) is sufficient for (5.2.1) when $\alpha = \beta, c = 2$.

The theorem is proved.

Let $G$ be a simple finite-dimensional contragredient Lie superalgebra, and let $(\ , \ )$ be an invariant nondegenerate bilinear form on $G$. We let $\rho$ denote the difference between the half-sums of the positive even roots and the positive odd roots. It is not hard to show that $\rho(\alpha_i) = (\alpha_i, \alpha_i)/2$. We define the Casimir operator in the center of the enveloping superalgebra by the formula: $\Gamma = \sum (-1)^{\deg u_i} u_i u_i^t$, where $\{u_i\}$ is a dual bases of $G$ relative to the form $(\ , \ )$.

Let $V$ be a finite-dimensional irreducible $G$-module with highest weight $\Lambda$. The action of $\Gamma$ on $V$ can be written in the form: $\Gamma(v) = (\lambda, \lambda + 2\rho) v + \sum_{\alpha > 0} e_{-\alpha} e_\alpha v_\lambda$. In particular, $\Gamma(v_\Lambda) = (\Lambda, \Lambda + 2\rho)v_\Lambda$, and, by Schur's lemma, $\Gamma$ is a scalar operator; hence, $\Gamma(v) = (\Lambda, \Lambda + 2\rho)v, v \in V$. We define the supertrace form in the usual way: $(a, b)_\nu = \text{str}(ab)$. Since invariant forms on $G$ are proportional, we have $(a, b)_\nu = l_\nu(a, b)$, where $l_\nu \in k$ is the index of the representation $V$. We have: $\text{str}(\Gamma) = \sum (-1)^{\deg u_i} \text{str}(u_i u_i^t) = l_\nu (\dim G_3 - \dim G_1)$. On the other hand, $\text{str}(\Gamma) = (\dim V_0 - \dim V_1)(\Lambda, \Lambda + 2\rho)$. Thus, $l_\nu(\dim G_3 - \dim G_1) = (\dim V_0 - \dim V_1)(\Lambda, \Lambda + 2\rho)$, from which we obtain

**Proposition 5.2.6.** The supertrace form of a finite-dimensional irreducible representation of a simple contragredient Lie superalgebra with $\dim G_3 \neq \dim G_1$ in a space $V$ with highest weight $\Lambda$ is nondegenerate if and only if $\dim V_0 - \dim V_1(\Lambda, \Lambda + 2\rho) \neq 0$. 

Example (compare [22]). We consider the standard representation \( \text{osp}(1,2) \) of the spin superalgebra \( \mathbf{B}(0,1) \). Then \( V^k = \Lambda^k \text{osp}(1,2), \ k = 0, 1, \ldots \) are all the irreducible finite-dimensional representations of \( \mathbf{B}(0,1) \). The highest weight of \( V^k \) is \( 2k \), \( \dim V^k = 2k + 1 \), \( V^0_0 \) and \( V^1_1 \) are \( \mathbf{B}(0,1)_0 \)-irreducible, \( \dim V^0_0 = k \), \( \dim V^1_1 = k + 1 \). The supertrace form is always nondegenerate (Proposition 5.2.6).

5.3. Simple Lie Superalgebras Over Nonclosed Fields

In this section the ground field \( k \) is arbitrary, of characteristic 0.

5.3.1. Reduction of the classification of simple Lie superalgebras over \( k \) to finding of the forms. Let \( k \) be the algebraic closure of \( k \). We recall that a \( k \)-algebra \( G \) is said to be a form of a \( k \)-algebra \( G \) if \( G \otimes k \cong G \). If \( G \) is a form of \( G \) and \( V \) a \( G \)-module, where \( V \) is a vector space over \( k \), then \( V \) is called a form of the \( G \)-module \( V \otimes k \). We recall that if \( G \) is a semisimple Lie algebra over \( k \) and \( V \) is an irreducible \( G \)-module over \( k \), then the \( G \otimes k \)-module \( V \otimes k \) splits into the direct sum of irreducible submodules, which are equivalent up to a “twist” under an outer automorphism of \( G \otimes k \). We also remark that there is at most one irreducible form of a \( G \)-module \( V \) for a given form \( G \) of a Lie algebra \( G \).

The next result is proved just as for Lie algebras [10].

Proposition 5.3.1. A simple finite-dimensional Lie superalgebra over \( k \) is isomorphic either to \( G \otimes k' \), where \( k' \) is a finite extension of \( k \) and \( G \) is one of the \( k \)-algebras \( \mathbf{A}(m, n), \mathbf{B}(m, n), \mathbf{C}(n), \ldots, \mathbf{S}(n), \mathbf{S}(n), \mathbf{H}(n), \) or it is a form of one of these \( k \)-algebras.

If \( G = G_0 \oplus G_1 \) is a Lie superalgebra over \( k \), then for an element \( \alpha \in k^* \) mod \( k^{*2} \) we can construct another form \( G' \) for \( G \otimes k \), by setting \( [a, b]' = \alpha[a, b] \) for \( a, b \in G_1 \) and \([a, b]' = [a, b] \) otherwise. This form we call equivalent to the original one.

5.3.2. Forms of the classical Lie superalgebras.

Proposition 5.3.2. (a) If a Lie superalgebra \( G = G_0 \oplus G_1 \) over \( k \) is a form of \( \overline{G} = \overline{G}_0 \oplus \overline{G}_1 \), then \( G_0 \) is a form of \( \overline{G}_0 \), and the \( G_0 \)-module \( G_1 \) is a form of the \( G_0 \)-module \( \overline{G}_1 \).

(b) Suppose that \( \overline{G} = \overline{G}_0 \oplus \overline{G}_1 \) is a classical Lie superalgebra over \( k \); suppose also that \( G_0 \) is a form of \( \overline{G}_0 \) and a \( G_0 \)-module \( V \) is a form of the \( \overline{G}_0 \)-module \( \overline{G}_1 \). Then there is one and only one up to equivalence \( G \) isomorphic to \( V \).

(c) The Lie superalgebras \( \mathbf{B}(m, n), \mathbf{D}(m, n), \mathbf{D}(2, 1; \alpha), \mathbf{F}(4), \mathbf{G}(3), \mathbf{P}(n), \)
and Q(n), defined over k, have at most one up to equivalence form \( G = G_0 \oplus G_1 \) with a given subalgebra \( G_0 \).

(d) The Lie superalgebras \( A(m, n) \) and \( C(n) \) over \( k \) have at most one form \( G = G_0 \oplus G_1 \) with a given subalgebra \( G_0 \) for which the \( G_0 \)-module \( G_1 \) is irreducible (or reducible, respectively), up to equivalence.

**Proof.** (a) is obvious. (b) follows from the fact mentioned earlier (see Section 2.1.6) that the \( G_0 \)-module \( S^2 G_1 \) contains \( \text{ad} G_0 \) as a direct summand with multiplicity 1. The condition for the map \( S^2 G_1 \to \text{ad} G_0 \) to define a Lie superalgebra (see Section 1.1.2, (1.1.3)) is preserved under a change of field.

(c) and (d) follow from remarks made above in Section 5.3.1.

### 5.3.3. Forms of Lie superalgebras of Cartan type.

In Sections 3.1 and 3.3 it was shown that for all Lie superalgebras of Cartan type, that is, \( W(n) \) with \( n \geq 3 \), \( S(n) \) with \( n \geq 4 \), and \( H(n) \) with \( n \geq 5 \), the filtration is invariant under automorphisms, and the reductive part of the automorphism group is isomorphic to \( GL_n \) in the first two cases, \( SL_n \), and \( SO_n \), respectively. Hence by using the same arguments as in [18], we obtain the following result.

**Proposition 5.3.3.** Let \( G \) be a Lie superalgebra over \( k \) and a form of one of the Lie superalgebras of Cartan type \( W(n) \), \( S(n) \), \( S(n) \), or \( H(n) \), over \( k \). Then \( G \) is isomorphic to one of the following Lie superalgebras over \( k \), respectively \( W(n) \), \( S(n) \), or \( H(\sum_1^d e_i) \), where \( \alpha_i \in k^* \).

### 5.3.4. Classification of simple finite-dimensional real Lie superalgebras.

Let us begin by constructing some series of examples of Lie superalgebras over \( \mathbb{R} \).

We fix the standard embeddings of the fields of real and of complex numbers in the quaternion field: \( \mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \). Throughout what follows, the bar denotes the standard conjugation in \( \mathbb{C} \) and \( \mathbb{H} \).

(a) **The special linear Lie superalgebras** \( sl(m, n; k) \), \( k = \mathbb{R} \) or \( \mathbb{C} \) or \( \mathbb{H} \).

We consider the space \( l(m, n; k) \) of all square matrices of order \( m + n \) over \( k \). In it we single out the subspaces

\[
sl(m, n; k) = \left\{ a \in l(m, n; k) \mid \text{str}(a) = 0 \right\} \quad \text{for} \quad k = \mathbb{R} \text{ or } \mathbb{C} \\
sl(m, n; k) = \left\{ a \in l(m, n; k) \mid \text{Re str}(a) = 0 \right\}.
\]

where \( \alpha \) is an \( (m \times m) \)-, \( \delta \) an \( (n \times n) \)-, \( \beta \) an \( (m \times n) \)-, and \( \gamma \) an \( (n \times m) \)-matrix. We define the bracket in the usual way: \([a, b] = ab - (-1)^{\text{tr}} ba\) if \( a \in l(m, n; k) \), \( b \in l(m, n; k) \). This makes \( l(m, n; k) \) into a real Lie superalgebra. The special linear superalgebra \( sl(m, n; k) \) is the real subalgebra of \( l(m, n; k) \) distinguished by the conditions:
For \( m = n \) these superalgebras contain a one-dimensional center, which has to be factored out.

(b) **Unitary and orthogonal-symplectic Lie superalgebras.** Again let \( k = \mathbb{R} \) or \( \mathbb{C} \) or \( \mathbb{H} \). We consider the \( \mathbb{Z}_2 \)-graded space \( k^{m+n} = k^m \oplus k^n \); let \( f = f(x, y) \), be a function on \( k^{m+n} \) with values in \( k \), which is linear relative to the first variable, superhermitian, i.e., \( f(x, y) = (-1)^{(\deg x)(\deg y)}f(y, x) \), nondegenerate and consistent, i.e., \( f(x, y) = 0 \) if \( x \in k^m, y \in k^n \). We put

\[
\begin{align*}
su(m, n; f)_s &= \{ a \in sl(m, n; C) \mid f(a(x), y) = -(-1)^{s\deg x}f(x, a(y)) \}, \\
osp(m, n; f)_s &= \{ a \in sl(m, n; R)_s \mid f(a(x), y) = -(-1)^{s\deg x}f(x, a(y)) \}, \\
hosp(m, n; f)_s &= \{ a \in sl(m, n; H)_s \mid f(a(x), y) = -(-1)^{s\deg x}f(x, a(y)) \}, \quad s \in \mathbb{Z}_2
\end{align*}
\]

The Lie superalgebras \( su(m, n; f), osp(m, n; f) \) and \( hosp(m, n; f) \) are called unitary, orthogonal-symplectic, and quaternion orthogonal-symplectic, respectively.

(c) **The Lie superalgebras** \( \mathbf{UQ}(n, p) \) and \( \mathbf{HQ}(n) \). Let

\[
\begin{align*}
\mathbf{UQ}(n; p) &= \left\{ \left( \begin{array}{c|c} a & b \\ \hline ib & a \end{array} \right) \right\} \subset sl(n, n; C), \\
\mathbf{HQ}(n) &= \left\{ \left( \begin{array}{c|c} a & b \\ \hline b & a \end{array} \right) \right\} \subset l(m, n; H), \quad \Re \tr b = 0.
\end{align*}
\]

We put \( \mathbf{UQ}(n, p) = \mathbf{UQ}(n, p)/\langle 1_{2n} \rangle \), \( \mathbf{HQ}(n) = \mathbf{HQ}(n)/\{1_{2n}, \lambda 1_{2n}, \lambda \in \mathbb{R} \} \).

(d) **The Lie superalgebras** \( \mathbf{D}(2, 1; \alpha; p) \). For each of the representations of \( so(4, 4 - p; \mathbb{R}) \otimes sl(2; \mathbb{R}) \), \( p = 0, 1, 2 \), there is a family of real Lie superalgebras \( \mathbf{D}(2, 1; \alpha; p), \alpha \in \mathbb{R} \setminus \{0, -1\} \), that are forms of \( \mathbf{D}(2, 1; \alpha) \).

(e) **The Lie superalgebras** \( \mathbf{F}(4; p) \). Each of the Lie algebras \( so(p, 7 - p) \), \( p = 0, 1, 2, 3 \), has a spinor representation \( spin_{p,7-p} \), which is a real form of the \( \mathbf{B}_3 \)-module \( spin_7 \). For each of the four linear Lie algebras \( spin_{7-p} \) there is, by Proposition 5.3.2(b) one and only one real Lie superalgebra \( \mathbf{F}(4; p) \), \( p = 0, 1, 2, 3 \), which is a form of the complex Lie superalgebra \( \mathbf{F}(4) \).

(f) **The Lie superalgebras** \( \mathbf{G}(3; p) \). Each of the real forms \( \mathbf{G}_{2, p}, p = 0, 1, \) of the complex Lie algebra \( \mathbf{G}_2 \) has a 7-dimensional representation \( G_{2,p} \). For each of the two linear Lie algebras \( G_{2,p} \otimes sl_2 \) there is, by Proposition 5.3.2(b), one and only one real Lie superalgebra \( \mathbf{G}(3; p) \), \( p = 0, 1 \), which is a form of the complex Lie superalgebra \( \mathbf{G}(3) \).

(g) **The Lie superalgebras** \( \mathbf{H}(n; p; R) \):

\[ \mathbf{H}(n; p; R) = \left\{ D \in \mathbf{W}(n; R) \mid D \left( \sum_{i=1}^{p} (d\xi_i)^2 - \sum_{i=p+1}^{n} (d\xi_i)^2 \right) = 0 \right\} \]

(h) **The Lie superalgebras** \( \mathbf{P}(n; R), \mathbf{Q}(n; R), \mathbf{W}(n; R), \mathbf{S}(n; R), \) and \( \mathbf{S}(n; R) \). These are \( \mathbf{P}(n), \mathbf{Q}(n), \ldots \), defined for \( k = \mathbb{R} \).
Real Lie superalgebras obtained from one another by the construction in Section 5.3.1 (for $\alpha = -1$) are called dual. From the classification of simple real Lie algebras and from Propositions 5.3.1–5.3.3 we derive the following theorem.

**Theorem 9.** A simple finite-dimensional real Lie superalgebra that is not a Lie algebra is isomorphic either to one of the complex Lie superalgebras $A(m, n)$, $B(m, n)$, $C(n)$, $D(m, n)$, $D(2, 1; \alpha)$, $F(4)$, $G(3)$, $P(n)$, $Q(n)$, $W(n)$, $S(n)$, $\tilde{S}(n)$, or $H(n)$, regarded as real superalgebras, or, up to transition to its dual, to one of the forms of these Lie superalgebras listed below:

- **A** $sl(m, n; \mathbb{R})$, $su(m, n; p, q)$, $m, n \geq 1$, $m + n \geq 2$; $sl(m, n, \mathbb{H})$, $m, n \geq 1$; $H(4; p; \mathbb{R})$.
- **B** $osp(m, n; p; \mathbb{R})$, $m$ odd, $m \geq 1$, $n \geq 2$.
- **C** $osp(2m, n; p; \mathbb{R})$, $hosp(1, n; p)$, $n \geq 2$.
- **D** $osp(m, n; p; \mathbb{R})$, $m$ even, $m \geq 4$, $n \geq 2$; $hosp(m, n; p)$, $m \geq 2$, $D(2, 1; \alpha; p)$.
- **F** $F(4, p)$, $p = 0, 1, 2, 3$.
- **G** $G(3, p)$, $p = 0, 1$.
- **P** $P(n, \mathbb{R})$, $n \geq 3$.
- **Q** $Q(n, \mathbb{R})$; $UQ(n; p)$, $n \geq 3$; $HQ(n)$, $n \geq 2$.
- **W** $W(n, \mathbb{R})$, $n \geq 3$.
- **S** $S(n, \mathbb{R})$, $n \geq 4$.
- **\tilde{S}** $\tilde{S}(n, \mathbb{R})$, $n \geq 4$.
- **H** $H(n; p; \mathbb{R})$, $n \geq 5$.

5.4. **On the Classification of Infinite-Dimensional Primitive Lie Superalgebras**

In Section 4.2.3, we have given a classification of the primitive Lie superalgebras $(L, L_0)$ for which $L_0 \supseteq L_0$; they are all finite-dimensional.

In this section we state without proof some partial results on the classification of infinite-dimensional primitive Lie superalgebras. We recall that Lie superalgebra $L$ with a distinguished subalgebra $L_0$ is called primitive if $L_0$ is a maximal subalgebra and it does not contain nontrivial ideals of $L$.

5.4.1. **Two algebras of differential forms.** Let $\Omega_0(m)$ be the superalgebra of differential forms with coefficients from the polynomial algebra $k[x_1, \ldots, x_m]$; in other words, $\Omega_0(m)$ is the associative superalgebra over $k[x_1, \ldots, x_m]$ (with trivial $\mathbb{Z}_2$-grading) with the generators $dx_1, \ldots, dx_m$ with the defining relations

$$dx_i \wedge dx_j = -dx_j \wedge dx_i, \quad \deg dx_i = -1, \quad i, j = 1, \ldots, m.$$
On $\Omega_0(m)$ the differential $d$ is defined in the usual way, as a derivation of degree $\bar{1}$ for which $d(x_i) = dx_i$ and $d^2(x_i) = 0$, $i = 1, \ldots, m$, with the standard properties.

We now define the following superalgebras:

$$\Omega(m, n) = \Omega_0(m) \otimes \Omega(n) \quad \text{and} \quad \Theta(m, n) = \Omega_0(m) \otimes \Theta(n).$$

The differentials $d$ and $\theta$ are extended from $\Omega(n)$ and $\Theta(n)$ to $\Omega(m, n)$ and $\Theta(m, n)$ in the natural manner, namely, $d = d \otimes 1 + 1 \otimes d$, $\theta = d \otimes 1 + 1 \otimes \theta$. It is not hard to establish their properties, which are similar to those in Chapter 3.

We set $A(m, n) = k[x_1, \ldots, x_n] \otimes A(n)$. The relations $\deg x_i = \deg x_j = 1$ determine on $A(m, n)$ a $\mathbb{Z}$-grading (which is not consistent with the $\mathbb{Z}_2$-grading).

Every derivation $D$ of degree $s$ of $A(m, n)$ extends uniquely to a derivation of $\Theta(m, n)$ and $\Omega(m, n)$, subject to the conditions $[D, \theta]f = [D, d]f = 0$, $f \in \Lambda(m, n)$.

5.42. Six series of infinite-dimensional Lie superalgebras. We introduce the following differential forms:

$$v = dx_1 \wedge \cdots \wedge dx_m \wedge \theta \xi_1 \wedge \cdots \wedge \theta \xi_n \in \Theta(m, n),$$

$$h = 2 \sum_{i=1}^{k} dx_i \wedge dx_{k+i} + \sum_{i=1}^{n} (d \xi_i)^2 \in \Omega(m, n), \quad m = 2k,$$

$$k = dx_{2k+1} + \sum_{i=1}^{k} (x_i dx_{k+i} - x_{k+i} dx_i) + \sum_{i=1}^{k} \xi_i d \xi_i \in \Omega(m, n), \quad m = 2k + 1.$$

We now define six series of infinite-dimensional Lie superalgebras ($m > 0$):

I. $W(m, n) = \operatorname{der} A(m, n)$.

The other five series consist of Lie algebras inside $W(m, n)$, which are characterized by the following action on the differential forms

II. $S(m, n) = \{ D \in W(m, n) \mid Dv = 0 \}$,

II'. $CS(m, n) = \{ D \in W(m, n) \mid Dv = \lambda v, \lambda \in k \}$,

III. $H(m, n) = \{ D \in W(m, n) \mid Dh = 0 \}$,

III'. $CH(m, n) = \{ D \in W(m, n) \mid Dh = \lambda h, \lambda \in k \}$,

IV. $K(m, n) = \{ D \in W(m, n) \mid Dk = uk, u \in A(m, n) \}$.

Note that for $n = 0$ we obtain the six standard series of infinite-dimensional Lie algebras of Cartan type.

The $\mathbb{Z}$-grading of $A(m, n)$ induces an (inconsistent) $\mathbb{Z}$-grading in $W(m, n)$. We write it down in more detail. Every element $D \in W(m, n)$ can be expressed as a linear differential operator

$$D = \sum_{i=1}^{m} P_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} Q_j \frac{\partial}{\partial \xi_j}, \quad P_i, Q_j \in A(m, n). \quad (5.4.1)$$
The relations deg $x_i = \deg \xi_i = 1$, $\deg \partial/\partial x_i = \deg \partial/\partial \xi_i = -1$ also determine on $\mathbb{W}(m, n)$ a $\mathbb{Z}$-grading $\mathbb{W}(m, n) = \bigoplus_{i>1} \mathbb{W}(m, n)_i$, which corresponds in the canonical way to a filtration; the appropriate distinguished subalgebra is $\bigoplus_{i>0} \mathbb{W}(m, n)_i$. The filtration and the distinguished subalgebra induce on every subalgebra $L$ a filtration and a distinguished subalgebra $L_0 = L \cap \bigoplus_{i>0} \mathbb{W}(m, n)_i$.

The Lie superalgebra $S(m, n)$ consists of the operators of the form (5.4.1) satisfying the condition

$$\text{div } D = \sum_{i=1}^n \frac{\partial P_i}{\partial x_i} + (-1)^{\deg_a D_{i+1}} \sum_{j=1}^n \frac{\partial Q_j}{\partial t_j} = 0.$$ 

Hence we see that $S(m, n)$ is the linear span of the elements of the form

$$\frac{\partial a}{\partial \xi_i} \frac{\partial}{\partial \xi_i} + \frac{\partial a}{\partial x_i} \frac{\partial}{\partial x_i}, \quad \frac{\partial a}{\partial \xi_i} \frac{\partial}{\partial x_j} - \frac{\partial a}{\partial x_j} \frac{\partial}{\partial \xi_i},$$

$$\frac{\partial a}{\partial \xi_i} \frac{\partial}{\partial \xi_j} + (-1)^{\deg a} \frac{\partial a}{\partial x_j} \frac{\partial}{\partial \xi_i}, \quad a \in \Lambda(m, n).$$

The Lie superalgebra $H(m, n)$ consists of the operators of the form

$$D_a = \sum_{i=1}^n \frac{\partial a}{\partial \xi_i} \frac{\partial}{\partial \xi_i} + \sum_{i=1}^k \left( \frac{\partial a}{\partial x_i} \frac{\partial}{\partial x_{k+i}} - \frac{\partial a}{\partial x_{k+i}} \frac{\partial}{\partial x_i} \right), \quad a \in \Lambda(m, n).$$

Here $[D_a, D_b] = D_{\{a, b\}}$, where

$$\{a, b\} = (-1)^{\deg a} \sum_{i=1}^n \frac{\partial a}{\partial \xi_i} \frac{\partial}{\partial \xi_i} + \sum_{i=1}^k \left( \frac{\partial a}{\partial x_i} \frac{\partial}{\partial x_{k+i}} - \frac{\partial a}{\partial x_{k+i}} \frac{\partial}{\partial x_i} \right).$$

Next,

$$CS(m, n) = S(m, n) \oplus \left\langle \sum_{i=1}^n \xi_i \frac{\partial}{\partial \xi_i} + \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \right\rangle,$$

$$CH(m, n) = H(m, n) \oplus \left\langle \sum_{i=1}^n \xi_i \frac{\partial}{\partial \xi_i} + 2 \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \right\rangle.$$

Finally, the Lie superalgebra $K(m, n)$ consists of elements of the form

$$D_a' = \sum_{i=1}^n \left( \frac{\partial a}{\partial x_m} x_i - \frac{\partial a}{\partial \xi_i} \right) \frac{\partial}{\partial \xi_i} + \sum_{i=1}^k \left( \frac{\partial a}{\partial x_m} x_i + \frac{\partial a}{\partial x_{k+i}} \right) \frac{\partial}{\partial x_i}$$

$$+ \left( \frac{\partial a}{\partial x_m} x_{k+i} - \frac{\partial a}{\partial x_{k+i}} \right) \frac{\partial}{\partial x_{k+i}}$$

$$+ \left( 2(-1)^{\deg a} \sum_{i=1}^n \frac{\partial a}{\partial \xi_i} \xi_i - \sum_{i=1}^{m-1} \frac{\partial a}{\partial x_i} x_i \right) \frac{\partial}{\partial x_m}, \quad a \in \Lambda(m, n).$$
The $\mathbb{Z}$-grading of $\mathcal{W}(m, n)$ also induces a $\mathbb{Z}$-grading of the form $G = \bigoplus_{i \geq -d} G_i$ on the Lie superalgebras of series II, II', III, and III'. This is not so in the case of the last series $K(m, n)$. However, if we set $\deg x_i = \deg \xi_j = 1$, $\deg \partial/\partial x_i = \deg \partial/\partial \xi_j = -1$ for $1 \leq i \leq m - 1, 1 \leq j \leq n$ and $\deg x_m = 2$, $\deg \partial/\partial x_m = -2$, then the resulting $\mathbb{Z}$-grading of $\mathcal{W}(m, n)$ also induces a $\mathbb{Z}$-grading of the form $G = \bigoplus_{i \geq -d} G_i$ on $K(m, n)$.

Note that the $G_0$-modules $G_{-1}$ for these six series are isomorphic to the following Lie superalgebras:

- $\mathfrak{l}(m, n)$ for $\mathcal{W}(m, n)$ and $CS(m, n)$, $\mathfrak{sl}(m, n)$ for $\mathfrak{S}(m, n)$, $\mathfrak{osp}(n, m)$ for $\mathfrak{H}(m, n)$,
- $\mathfrak{cosp}(n, m)$ for $\mathfrak{CH}(n, m)$ and $\mathfrak{cosp}(n, m - 1)$ for $K(m, n)$.

The superalgebras of all six series are transitive and irreducible, and those of series I, II, III, and IV are even simple.

5.43. On the classification of primitive Lie superalgebras. Let $L$ be an infinite-dimensional primitive Lie superalgebra and $L_0$ be the distinguished subalgebra. Let $L_{-1}$ be some minimal ($\mathbb{Z}_s$-graded) subspace of $L$ that contains $L_0$ and is different from $L_0$ and $\text{ad} L_0$-invariant. We construct a filtration in $L$ of the form

$$L = L_{-d} \supset L_{-d+1} \supset \cdots \supset L_{-1} \supset L_0 \supset L_1 \supset \cdots,$$

by setting (see [6]):

$$L_{-(s+1)} = [L_{-1}, L_s] + L_{-s}, \quad L_s = \{a \in L_{s-1} \mid [a, L_{-1}] \subset L_{s-1}\}, \quad s > 0.$$ 

The corresponding associate $\mathbb{Z}$-graded superalgebra $\text{Gr} L = \bigoplus_{i \geq -d} G_i$ has the following properties:

1°. $\text{Gr} L$ is transitive and irreducible,

2°. $G_{-s} = G_{s-1}$ for $s > 0$.

3°. $G_1 \neq 0$.

We may also assume that

4°. $\bigoplus_{i < 0} G_i$ does not contain nonzero ideals of $\text{Gr} L$ (because we can factor out such an ideal if it exists).

If the $\mathbb{Z}$-grading is consistent, then

5°. $[G_0, G_0]$ is a contragredient Lie superalgebra.

Apparently, 5° holds in general, but I have not been able to prove this.

Now we can state (without proof) the main result of this section.

**Theorem 10.** Let $G = \bigoplus_{i \geq -d} G_i$ be an infinite-dimensional $\mathbb{Z}$-graded Lie superalgebra having properties 1° 5°. Then $G$ is isomorphic as $\mathbb{Z}$-graded superalgebra to one of $\mathcal{W}(m, n)$, $\mathfrak{S}(m, n)$, $CS(m, n)$, $\mathfrak{H}(m, n)$, $\mathfrak{CH}(m, n)$, or $K(m, n)$ with $m > 0$. 
The proof uses the same methods as in Chapter 4 and relies on Theorem 3.

A primitive Lie superalgebra $L$ with distinguished subalgebra $L_0$ is called complete if it is complete in the topology defined by the subspaces of the transitive filtration of the pair $(L, L_0)$ (see Section 1.3.1). The superalgebra $A(m, n) = k[[x_1, \ldots, x_m]] \otimes A(n)$ is complete in the topology defined by its natural filtration.

We denote by $\mathfrak{W}(m, n)$ the Lie superalgebra of all continuous derivations of $A(m, n)$. Then $\mathfrak{W}(m, n)$ is a complete primitive Lie superalgebra with the natural distinguished subalgebra. Complete and primitive are also $S(m, n)$, $K(m, n)$, which are characterized by the same action on the differential forms $v, h, k$, as for $S(m, n)$, $K(m, n)$.

A well-known result of Cartan asserts that $\mathfrak{W}(m, 0)$, $K(m, 0)$ are the only infinite-dimensional complete primitive Lie algebras.

**Conjecture 1.** An infinite-dimensional complete primitive Lie superalgebra is isomorphic to one of $\mathfrak{W}(m, n)$, $S(m, n)$, $CS(m, n)$, $A(m, n)$, $CH(m, n)$, or $K(m, n)$ for $m > 0$.

**5.4.4. Remarks.** (a) In Chapter 4 we have, in fact, proved that if $G = \bigoplus_{i \geq - \infty} G_i$ is an infinite-dimensional Lie superalgebra with a consistent $\mathbb{Z}$-grading having properties 1°–4°, then $G$ is isomorphic to $K(1, n)$.

(b) There are general embedding theorems, which generalize standard theorems for Lie algebras and Propositions 3.1.2 and 3.1.3.

If $G = \bigoplus_{i \geq - \infty} G_i$ is a transitive $\mathbb{Z}$-graded Lie superalgebra, and $m = \dim(G_{-1})_0$, $n = \dim(G_{-1})_1$, then there is an embedding $G \rightarrow \mathfrak{W}(m, n)$ preserving the $\mathbb{Z}$-grading.

Let $L = L_{-1} \supset L_0 \supset L_1 \supset \cdots$ be a transitive Lie superalgebra with a filtration, $m = \dim(L/L_0)_0$, $n = \dim(L/L_0)_1$. Then there is an embedding $\alpha : L \rightarrow \mathfrak{W}(m, n)$ preserving the filtration. If $\beta$ is another such embedding and $(\alpha - \beta)L \subseteq L_0$, then there exists one and only one (continuous) automorphism $\varphi$ of $\mathfrak{W}(m, n)$ for which $\alpha = \varphi \circ \beta$; $\varphi$ can be induced by an automorphism of $A(m, n)$.

(c) By the same method as Proposition 3.3.8, it can be proved that every nondegenerate closed differential form from $\mathfrak{W}(m, n)$ of degree 2 is reduced to the form $h$ by an appropriate automorphism of $A(m, n)$.

**5.5. Some Unsolved Problems**

**5.5.1. Classification of infinite-dimensional primitive Lie superalgebras.** About this topic, see Conjecture 1 and Theorem 10.

**5.5.2. Formulas for the characters and dimensions of irreducible representations.** The most urgent task is to prove a formula for the characters in the case of contragredient Lie superalgebras. For contragredient Lie algebras (including
these of infinite dimension) this is done in [14]. However, the proof in [14] only works for $B(0, n)$.

5.5.3. Cohomology. For the definition of the cohomology group $H^k(G, V)$ of a Lie superalgebra $V$ with coefficients in a $G$-module $V$, see [17]. As usual, it is shown that if $V$ is a finite-dimensional irreducible $G$-module and $\Gamma$ is the Casimir operator (the existence of an invariant bilinear form is assumed), then $H^k(G, V) = 0$ for $\Gamma(V) \neq 0$. In the case of contragredient Lie superalgebras, the latter condition is equivalent to $(\Lambda, \Lambda + 2\rho) \neq 0$, where $\Lambda$ is the highest weight (see Section 5.2.3), and it is not violated in any nontrivial representation only for $B(0, n)$.

Now some questions arise at once: the cohomology of the simple finite-dimensional Lie superalgebras with trivial coefficients, and the cohomology of the infinite-dimensional complete primitive Lie superalgebras.

Closely connected with the problem of the triviality of $H^1(G, V)$ is the full reducibility of representations and the theorems of Levi and Mal'tsev. A counterexample to Levi's theorem is $s\mathfrak{u}(n, n)$, and one to full reducibility is the adjoint representation of $A(n, n)$. As we have already mentioned, full reducibility always holds for $B(0, n)$. It is not hard to show that if $G$ is a classical Lie superalgebra, then $H^1(G, V) = 0$ for all irreducible representations, with the exception of a finite set $S$. It would be interesting to find this $S$ and also to classify all indecomposable representations of the classical Lie superalgebras.

5.5.4. Infinite-dimensional representations. Undoubtedly, Kirillov's orbits method extends to Lie superalgebras. (In particular, Theorem 7' on infinite-dimensional representations of solvable Lie algebras points to this.) We mention that Kirillov's differential form $\omega(x, y) = l([x, y])$ on an orbit of the co-adjoint representation of a Lie superalgebra is a form in $dx$ and $d\xi$ (see Section 5.4). On infinite-dimensional representations of the simple Lie superalgebras almost nothing is known. First in line is, of course, the dispin algebra $B(0, 1)$.

5.5.5. Generalized Lie superalgebras. We consider the ring $M = \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$ (s times). An $M$-graded algebra is called a generalized superalgebra. If $\alpha = (\alpha_1, \ldots, \alpha_s) \in M$, we set $(-1)^\alpha = (-1)^{\alpha_1} \cdots (-1)^{\alpha_s}$. Now all the definitions and assertions of Section 1.1 carry over to generalized superalgebras, in particular, the definitions of a Lie superalgebra, of the supertrace, and the Killing form. Just as in Section 2.1, we can define series of generalized Lie superalgebras $sl(n_1, \ldots, n_s)$, $osp(n_1, \ldots, n_s)$, $Q(n)$, and as in Chapter 3, the series $W$, $S$, $\tilde{S}$, $H$. The same problems arise here as for Lie superalgebras, first and foremost, the problem of classifying the simple generalized Lie superalgebras.

Additional remarks. To Section 5.2. In my recent article, "Characters of Typical Representations of Classical Lie Superalgebras" (Commun. Algebra 5, No. 8, 889–897(1977)), the formulas for the character and supercharacter of finite-dimensional irreducible representations in "general position" (so-called typical
representations) are obtained. For example, all the representations of $B(0, n)$ are typical and all the representations of $A(1, 0)$ are typical except for $S^{\otimes \infty}(2, 1)$ and its dual.

To Section 5.3. D. Z. Djokovic and G. Hochshild proved, in their article "Semi-simplicity of Z-graded Lie algebras, II" (Illinois J. Math. 20 (1976), 134–143), that every finite-dimensional representation of a Lie superalgebra $G$ is fully reducible if and only if $G$ is a direct sum of a semisimple Lie algebra and several copies of $B(0, n)$.

From my article, mentioned in Section 5.2, it follows that $H^1(G, V) = 0$ for a typical representation $V$. By the way, $H^1(A(1, 0), V) = 0$ only for one irreducible representation-standard representation.

To Section 5.4. B. Kostant in his recent paper, "Graded manifolds, graded Lie theory and prequantization," gave a more correct definition of a supermanifold than the one in [5]. This definition allowed him to develop the theory of homogeneous supermanifolds, and "orbits method" for supergroups.

REFERENCES