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Approximation theorems for group valued functions

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Abstract

Stone–Weierstrass-type theorems for groups of group-valued functions with a discrete range or a discrete domain are obtained. We study criteria for a subgroup of the group of continuous functions $C(X, G)$ (X compact, G a topological group) to be uniformly dense. These criteria are based on the existence of so-called condensing functions, where a continuous function $\phi: G \to G$ is said to be condensing (respectively, finitely condensing) if it does not operate on any proper, point separating, closed subgroup of $C(K, G)$, with K compact, (respectively, with K finite) that contains the constant functions.

The set $D_F(G)$ of finitely condensing functions in $C(G, G)$, is characterized, for any Abelian topological group G, as the set of those functions that are both non-affine and do not have nontrivial generalized periods (i.e. that do not factorize through nontrivial quotients of G). This provides approximation theorems for functions with discrete domain and arbitrary (topological group) range.

We also show that when G is discrete, every finitely condensing functions is condensing. The set of $D(G)$ of condensing functions is thus characterized for discrete Abelian G. This provides approximation theorems for functions with an arbitrary (compact) domain and a discrete range. Answering an old question of Sternfeld, the description of $D(\mathbb{Z})$ that follows is particularly simple: given $\phi: \mathbb{Z} \to \mathbb{Z}$, $\phi \in D(\mathbb{Z})$ if and only if for every $k \in \mathbb{N}$ with $k \geq 2$, there are $n_1, n_2 \in \mathbb{Z}$ such that $n_1 - n_2$ is a multiple of k, while $\phi(n_1) - \phi(n_2)$ is not.

Keywords: Stone–Weierstrass theorem; Group-valued functions; Condensing functions; Non-affine functions; Abelian topological groups

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1. Introduction

In his paper [2] Sternfeld introduced the concept of constructive topological group as an indicator of when (some weak form of) the Stone–Weierstrass theorem may hold in a group of group-valued functions.

Definition 1.1 (*Definition 3 of [2]*)**.** A topological group G is *constructive* if for every compact space K and every subgroup $\mathcal{H}$ of $C(K, G)$ which contains the constant functions and separates points, either $\mathcal{H}$ is dense or there is $\phi \in C(G, G)$ that does not operate on $\mathcal{H}$ (that is, with $\phi \circ h \notin \mathcal{H}$ for some $h \in \mathcal{H}$).

The paper [2] contains some examples of constructive groups such as the group $\mathbb{Z}$ of integers, or the additive group of a real Banach space and some examples of nonconstructive groups as the two-element group $\mathbb{Z}_2$ or the circle group $\mathbb{T}$. The authors of the present paper later showed, [1], that a locally compact group with more than 2 elements is constructive if and only if it is either totally disconnected or homeomorphic to $\mathbb{R}^n$ for some positive integer n.

Once a group G is known to be constructive it is natural to seek whether stronger forms of the Stone–Weierstrass hold for G, as happens with the classical case $G = \mathbb{R}$: notice that a linear subspace L of $C(K)$ is a subalgebra (respectively, a sublattice) if the function $f(x) = x^2$ (respectively, the function $f(x) = |x|$) operates on L. In general, if some non-affine function h in $C(\mathbb{R})$ operates on a linear subspace L of $C(K)$ which contains the constants, then the uniform closure of L is an algebra. Thus, if L separates the points of K, then it is dense in $C(K)$ [3, Theorem 1.1]. It is in this direction that the set of condensing functions is introduced.

Definition 1.2 (*Definition 1(ii) of [2]*)**.** A function $\phi \in C(G, G)$ is *condensing* (respectively, *finitely condensing*) if the following holds: for every compact (respectively, finite) space K and every subgroup $\mathcal{H}$ of $C(K, G)$ which separates the points of K and contains the constant functions, if ϕ operates on $\mathcal{H}$, then $\mathcal{H}$ is uniformly dense in $C(K, G)$.

The set of condensing (respectively, of finitely condensing) functions in $C(G, G)$ is denoted by $D(G)$ (respectively, by $D_F(G)$).

Each condensing function provides a simple criterion for the denseness of a subgroup. Our objective in this paper is to characterize $D(G)$. We address in particular the following question.

Question 1.3 (*Section 4 of [2]*)**.** For a constructive group G, find $D(G)$. In particular, find $D(\mathbb{Z})$.

For Abelian G, an obvious obstacle for a function $\phi \in C(G, G)$ to be condensing is to be affine. Recall that a function $\phi: G \to H$, with G and H Abelian groups, is affine if the associated function $\phi_a: G \to H$ given by $\phi_a(g) = \phi(g) - a$ with $a = \phi(0)$ is a homomorphism. Observe that ϕ is affine if and only if $\phi(a + b - c) = \phi(a) + \phi(b) - \phi(c)$, for every $a, b, c \in G$. Thus every affine function operates on the subgroup $\mathcal{H}$ of $C(\{1, 2, 3, 4\}, G)$ given by $\mathcal{H} = \{f: f(3) - f(4) = f(1) - f(2)\}$.

Another easily identified obstruction is the existence of a closed subgroup $H \subset G$ such that ϕ factorizes through G/H (think of the subgroup $\mathcal{H} \subset C(\{1, 2\}, G)$ defined by the condition $f \in \mathcal{H}$ if and only if $f(1) - f(2) \in H$, observe that this argument is also valid for non-Abelian groups). To mark this case we say, following [2], that a subgroup H of G is a *generalized period* of ϕ, if $\phi(xH)$ is contained in $\phi(x)H$ for every $x \in G$.

Note that H will be a generalized period of ϕ, if ϕ factors through G/H.

Definition 1.4. We say that a function $\phi \in C(G, G)$ *mixes cosets* if it does not have any nontrivial closed generalized periods. The set of functions in $C(G, G)$ that mixes cosets will be denoted by $\Lambda_m(G)$.

The main result of Section 3 (proved as Theorem 3.9) shows that being affine and admitting generalized periods are the only obstacles for a function to be finitely condensing. This turns to be a characterization of the set of condensing functions for discrete G (and hence an answer to Question 1.3 in this case) since, as shown in Section 2, $D_F(G) = D(G)$, if G is discrete. We have as a consequence that $\phi \in C(G, G)$ is in $D_F(G)$ if and only if it is not affine and mixes cosets.

Since affine maps between free Abelian groups of finite rank always have generalized periods, condensing functions on $\mathbb{Z}$ are easy to identify. In fact, we show in Section 3 that the set $D(\mathbb{Z})$ can be described as $\Lambda_m(\mathbb{Z})$ where

$$\Lambda_m(\mathbb{Z}) = \{\phi\colon \mathbb{Z} \to \mathbb{Z}\colon \text{ for every } k \in \mathbb{N}, k \geq 2, \text{ there are } n_1, n_2 \in \mathbb{Z} \\ \text{such that } n_1 - n_2 = 0 (\text{mod } k), \text{ but } \phi(n_1) - \phi(n_2) \neq 0 (\text{mod } k)\}.$$

Some of the conditions we impose are not natural for noncommutative groups. A function for instance may be condensing and yet admit a generalized period (see Example 4.2). This can be remedied by requiring generalized periods to be *normal* subgroups, no condensing function may then admit a normal generalized period. In Section 3 we find some other characterizations of $D_F(G)$ that are easily adaptable to the noncommutative case, it is not clear however whether these conditions can be related to easy algebraic properties such as being affine, as it happens in the commutative case.

2. Reduction to $C(F, G)$ with finite F

It is plain that $D(G) \subseteq D_F(G)$. The aim of this section is to establish, for a discrete group G, the equality $D_F(G) = D(G)$. Notice that, by [2, Proposition 1 (iv)], $D_F(\mathbb{Z}_2) = \emptyset$. Therefore, all groups are assumed to be different from $\mathbb{Z}_2$.

Let G be a group. Given a compact space K, a partition $\{V_1, V_2, \ldots, V_n\}$ is said to be *subordinated* to a function $h \in C(K, G)$ if $h(K) = \{z_1, z_2, \ldots, z_n\}$ and, for each $i = 1, 2, \ldots, n$, we have $V_i = h^{-1}(z_i)$. Our first result provides a useful tool for analyzing the relationship between condensing and finitely condensing functions.

Theorem 2.1. *Let G be a group and consider, for a compact space K, a subgroup $\mathcal{H}$ of $C(K, G)$ which separates the points of K and contains the constant functions. If a function $\phi \in D_F(G)$ operates on $\mathcal{H}$, then, given a partition $\{V_1, V_2, \ldots, V_n\}$ of K subordinated to a function $h \in \mathcal{H}$ and a finite subset $\{z_1, z_2, \ldots, z_n\} \subset G$, the function g defined as*

$$g|_{V_i} = z_i, \quad i = 1, 2, \ldots, n$$

belongs to $\mathcal{H}$.

Proof. Let $\mathcal{F}_h$ denote the subgroup of $\mathcal{H}$ defined as

$$\mathcal{F}_h = \left\{g \in \mathcal{H} : g|_{V_i} \text{ is constant, } 1 \leq i \leq n\right\}.$$

Let us take now the finite space $F_n = \{1, 2, \ldots, n\}$ and let φ be the function from $\mathcal{F}_h$ into $C(F_n, G)$ defined by the requirement that $\varphi(g)(i) = g(V_i)$ for each $i \in F_n$.

It is routine to verify that φ is a group monomorphism. In particular, $\varphi(\mathcal{F}_h)$ is a subgroup of $C(F_n, G)$. Moreover, as a matter of definition, $\varphi(\mathcal{F}_h)$ contains the constant functions and, since $h \in \mathcal{F}_h$, it separates the points of F_n. We next prove that ϕ operates on $\varphi(\mathcal{F}_h)$. For, consider $\varphi(g) \in \varphi(\mathcal{F}_h)$ where $g \in \mathcal{F}_h$, then the function $\phi \circ g$ belongs to $\mathcal{H}$ (ϕ operates on $\mathcal{H}$) which implies that $\phi \circ g \in \mathcal{F}_h$. Our claim now follows from the fact that $\phi \circ \varphi(g) = \varphi(\phi \circ g)$.

Since ϕ is finitely condensing, we have just proved that $\varphi(\mathcal{F}_h) = C(F_n, G)$. If we consider the function $t: F_n \to G$ defined as $t(i) = z_i$ for each $i = 1, 2, \ldots, n$, then $\varphi^{-1}(t)$ satisfies the desired properties. This completes the proof. □

The significance of the previous theorem for our purposes is captured by the next two lemmas. For the remainder of this section, we shall impose the blanket assumption of discreteness on all given groups. This fact implies that, for any finite space K, $C(K, G)$ is discrete and has no proper dense subsets. Also for the remainder of this section K will always denote a compact space and $\mathcal{H}$ a subgroup of $C(K, G)$ which contains the constant functions and separates the points of K.

Lemma 2.2. *Assume that there is $\phi \in D_F(G)$ which operates on $\mathcal{H}$. Let $\{V_i\}_{i=1}^n$ be a finite family of pairwise disjoint clopen subsets of K. If $x \notin \bigcup_{i=1}^n V_i$ and, for each $i = 1, 2, \ldots, n$, there is a function $h_i \in \mathcal{H}$ such that*

$$h_i(x) = r_i, \qquad h_i|_{V_i} \equiv s_i$$

with $r_i \neq s_i$, then, for each $g \in G \setminus \{e\}$, we can find a function $h_g \in \mathcal{H}$ such that $h_g(x) = g$, $h_g|_{\bigcup_{i=1}^n V_i} \equiv e$ and $h_g(K) = \{e, g\}$.

Proof. Let $g \in G \setminus \{e\}$. We proceed by induction on the number of the clopen sets in the family $\{V_i\}_{i=1}^n$. For $n = 1$, consider the partition $\mathcal{C}$ of K subordinated to h_1. Since $r_1 \neq s_1$, the pairwise disjoint clopen subsets $h_1^{-1}(r_1)$ and $h_1^{-1}(s_1)$ belong to $\mathcal{C}$ with $V_1 \subseteq h_1^{-1}(s_1)$. Thus, Theorem 2.1 tells us that there is a function m_1 in $\mathcal{H}$ satisfying $m_1|_{h^{-1}(r_1)} = g$ and $m_1|_{K \setminus h_1^{-1}(r_1)} \equiv e$. For $n > 1$, choose g_1 and g_2 in G with $g_1 \cdot g_2 \notin \{e, g_1, g_2\}$ (observe that G has more than two elements). By induction hypothesis, there are $m_1, m_2 \in \mathcal{H}$ such that

(i) $m_i(x) = g_i$ for $i = 1, 2$,
(ii) $m_1|_{\bigcup_{i=1}^{n-1} V_i} \equiv e$,
(iii) $m_2|_{V_n} \equiv e$,
(iv) $m_1(K) = \{e, g_1\}$ and $m_2(K) = \{e, g_2\}$.

Then $(m_1 \cdot m_2)(x) = g_1 \cdot g_2$ and $(m_1 \cdot m_2)(y) \in \{0, g_1, g_2\}$ when $y \in \bigcup_{i=1}^n V_i$. Since $g_1 \cdot g_2 \notin \{e, g_1, g_2\}$, we can apply Theorem 2.1 in order to obtain a function $h_g \in \mathcal{H}$ with $h_g(x) = g$, $h_g|_{\bigcup_{i=1}^n V_i} \equiv e$ and $h_g(K) = \{e, g\}$. □

Lemma 2.3. *Assume that there is $\phi \in D_F(G)$ which operates on $\mathcal{H}$. Let V be a clopen subset of K. If there exist functions $h_1, h_2, \ldots, h_n$ in $\mathcal{H}$ and points $x_1, x_2, \ldots, x_n$ in K such that*

$$h_i(x_i) = r_i \neq e, \qquad h_i|_{K \setminus V} \equiv e,$$

for each $i = 1, 2, \ldots, n$, then, for each $g \in G \setminus \{e\}$, there is a function $h_g \in \mathcal{H}$ satisfying $h_g|_{\bigcup_{i=1}^n h_i^{-1}(r_i)} \equiv g$ and $h_g|_{K \setminus (\bigcup_{i=1}^n h_i^{-1}(r_i))} \equiv e$. In particular, $h_g|_{K \setminus V} \equiv e$.

Proof. The proof proceeds along the same lines as the proof of Lemma 2.2: it follows from induction on the number of elements of the set $\{x_1, x_2, \ldots, x_n\}$. The step $n = 1$ is a consequence of Lemma 2.2. For $n > 1$, choose g_1 and g_2 in G with $g_1 \cdot g_2 \notin \{e, g_1, g_2\}$. By induction hypothesis, there are $m_1, m_2 \in \mathcal{H}$ such that

(i) $m_1|_{\bigcup_{i=1}^{n-1} h_i^{-1}(r_i)} \equiv g_1$.
(ii) $m_1|_{K \setminus \bigcup_{i=1}^{n-1} h_i^{-1}(r_i)} \equiv e$.
(iii) $m_2|_{h_n^{-1}(r_n)} \equiv g_2$.
(iv) $m_2|_{K \setminus h_n^{-1}(r_n)} \equiv e$.

It is easy to check that $(m_1 \cdot m_2)(x_i) \in \{g_1 \cdot g_2, g_1, g_2\}$ $(i = 1, 2, \ldots, n)$ and that $g|_{K \setminus (\bigcup_{i=1}^{n} h_i^{-1}(r_i))} \equiv e$. The result is now a consequence of Theorem 2.1. □

Lemmas 2.2 and 2.3 imply that $\mathcal{H}$ separates pairwise disjoint clopen sets.

Theorem 2.4. *Assume that there is $\phi \in D_F(G)$ which operates on $\mathcal{H}$. Let $g \in G \setminus \{e\}$. If V is a clopen subset of K, then the function*

$$h(x) = \begin{cases} g, & x \in V \\ e, & x \in K \setminus V \end{cases}$$

belongs to $\mathcal{H}$.

Proof. Fix $x \in V$. Since $\mathcal{H}$ separates points of K, a simple application of Theorem 2.1 implies that for each $y \in K \setminus V$, there is a function $h_y \in \mathcal{H}$ such that $h_y(x) = g$ and $h_y(y) = e$. By compactness, we can find a finite set $\{y_1, y_2, \ldots, y_n\} \subset K \setminus V$ such that $\left\{h_{y_i}^{-1}(e) : i = 1, 2, \ldots, n\right\}$ is a cover of $K \setminus V$. Since the functions $\left\{h_{y_i}\right\}_{i=1}^{n}$ and the family $\left\{h_{y_i}^{-1}(e) : i = 1, 2, \ldots, n\right\}$ satisfy the conditions of Lemma 2.2, there is a function $h_x \in \mathcal{H}$ such that $h_x(x) = g$ and $h_x|_{K \setminus V} \equiv e$.

Consider now the cover $\left\{h_x^{-1}(g) : x \in V\right\}$ of V. By compactness, there is a finite subcover $\left\{h_{x_1}^{-1}(g), h_{x_2}^{-1}(g), \ldots, h_{x_n}^{-1}(g)\right\}$. The set $\{x_1, x_2, \ldots, x_n\}$ and the clopen set V satisfy the conditions of Lemma 2.3, hence there is $h \in \mathcal{H}$ such that $h|_{\bigcup_{i=1}^{n} h_{x_i}^{-1}(g)} \equiv g$ and $h|_{K \setminus V} \equiv e$. The result now follows from the fact that $V = \bigcup_{i=1}^{n} h_{x_i}^{-1}(g)$. □

Now we are prepared to show the main result of this section.

Theorem 2.5. *If there is $\phi \in D_F(G)$ which operates on $\mathcal{H}$, then $\mathcal{H} = C(K, G)$.*

Proof. Let $g \in C(K, G)$ be given. Since G is discrete, there are $z_1, \ldots, z_n \in G$ with $g(K) = \{z_1, \ldots, z_n\}$. We now apply Theorem 2.4 to the clopen sets $V_j = g^{-1}(\{z_j\})$, $j = 1, \ldots, n$ and get a collection of functions $h_j \in \mathcal{H}$ with $h_j(V_j) \equiv z_j$ and $h_j(K \setminus V_j) \equiv e$. Then $g = \prod_{j=1}^{n} h_j$ and, hence, $g \in \mathcal{H}$. □

Corollary 2.6. *If G is discrete, then $D_F(G) = D(G)$.*

Given a topological (not necessarily discrete) group G, let G_d denote the underlying group equipped with the discrete topology. Assume that $\phi \in D_F(G_d)$ and that K is a finite space. If ϕ operates on a subgroup $\mathcal{H}$ of $C(K, G)$ which separates points and contains the constant functions, we can apply Corollary 2.6 in order to obtain $\mathcal{H} = C(K, G_d) = G^K$. Thus, $\phi \in D_F(G)$. We have just proved: *if G is not constructive and $D_F(G_d) \neq \emptyset$, then $D(G) \neq D_F(G)$*. Since a nonconstructive group G necessarily has, by definition $D(G) = \emptyset$, the following question naturally arises:

Question 2.7. Is there a group G which is not constructive and such that $D_F(G) \neq \emptyset$?

Another relevant question in this direction is the following

Question 2.8. Does there exist a constructive group such that $D(G) = \emptyset$?

3. Subgroups of $C(F, G)$ with finite F

Having proved that $D_F(G) = D(G)$ for discrete groups, we restrict our attention in this section to groups of continuous mappings on finite sets.

Recall that a subgroup H of G is a generalized period of ϕ, if for each $x \in G$, $\phi(xH)$ is contained in $\phi(x)H$. Note that H will be a generalized period of ϕ if and only if whenever $x^{-1}y \in H$, $\phi(x)^{-1}\phi(y) \in H$. Recall as well that we denote by $\Lambda_m(G)$ the set of functions in $C(G, G)$ that do not admit any nontrivial closed generalized periods, that is, the collection of functions that mix closed cosets.

To avoid undue repetition in the statements of our results, we adopt the convention that F always stands for a finite set with the discrete topology. To relate properties of a subgroup $\mathcal{H}$ of $C(F, G)$ to properties of G, we will use in the next two lemmas a collection of functions defined as follows.

Definition 3.1. Let $\mathcal{H}$ be a subgroup of $C(F, G)$. For every $A \subset F$ and every $j \in A$ we define the subgroup

$$G_j^A(\mathcal{H}) = \{x \in G\colon \text{ there is } f \in \mathcal{H} \text{ with } f(k) = e \text{ for all } k \in A \setminus \{j\} \text{ and } f(j) = x\}.$$

Lemma 3.2. *Let G be a topological group and let $\mathcal{H}$ be subgroup of $C(F, G)$. Let $A \subset F$ and $j \in A$ be given. If $\phi \in C(G, G)$ operates on $\mathcal{H}$, then $G_j^A(\mathcal{H})$ is a normal subgroup of G and a generalized period for ϕ.*

Proof. We first show that $G_j^A(\mathcal{H})$ is a generalized period. Let $x, y \in G$ be such that $x^{-1}y \in G_j^A(\mathcal{H})$, this means that there is a function $f \in \mathcal{H}$ with $f(j) = x^{-1}y$ and $f(k) = e$ for $k \in A \setminus \{j\}$. Then $f_2 = \overline{\phi(x)^{-1}} \cdot (\phi \circ (\overline{x} \cdot f)) \in \mathcal{H}$. It is clear that $f_2(k) = e$, for $k \in A \setminus \{j\}$ and that $f_2(j) = \phi(x)^{-1}\phi(y)$, therefore $\phi(x)^{-1}\phi(y) \in G_j^A(\mathcal{H})$ and $G_j^A(\mathcal{H})$ is a generalized period of ϕ.

To see that $G_j^A(\mathcal{H})$ is a normal subgroup of G we take $x \in G_j^A(\mathcal{H})$ and $g \in G$. Let $f \in \mathcal{H}$ be such that $f(j) = x$ and $f(k) = e$ if $k \in A \setminus \{j\}$, then $\bar{g}^{-1} f \bar{g} \in \mathcal{H}$ which shows that $g^{-1}xg \in G_j^A(\mathcal{H})$. □

Let $\mathcal{H}$ be a subgroup of $C(F, G)$. In the lemma and theorem that follow we will extend functions defined in the complement $F \setminus \{j\}$ of a point j to the whole F. The following notations will be useful to that effect:

(1) If $j \neq k \in F$ and $f\colon F \setminus \{j\} \to G$, we denote by $f_{j,k}\colon F \to G$ the function defined by the conditions:

$$f_{j,k} \restriction_{F \setminus \{j\}} = f \quad \text{and} \quad f_{j,k}(j) = f(k).$$

(2) If $x \in G$ and $B \subset F$ we denote by $f_B^x\colon F \to G$, the function defined by:

$$f_B^x(j) = x \quad \text{if } j \in B \text{ and } f_B(j) = e \text{ if } j \notin B.$$

(3) Given $j, k \in F$, $j \neq k$ we also define

$$\mathcal{H}_{j,k} = \left\{ f : F \setminus \{j\} \to G : f_{j,k} \in \mathcal{H} \right\}.$$

Lemma 3.3. *Let $\mathcal{H}$ be a subgroup of $C(F, G)$ that separates points of F. Let in addition $B \subset F$ be such that $f_B^x \in \mathcal{H}$ for every $x \in G$. If $j \in B$ and $k \in F \setminus B$, then either $\mathcal{H}_{j,k}$ separates points of $F \setminus \{j\}$ or there are $j_1, j_2 \in F$ such that $f(j_2) = f(j_1) \cdot f(j)^{-1} \cdot f(k)$, for every $f \in \mathcal{H}$.*

Proof. Take $j_1, j_2 \in F \setminus \{j\}$, $j_1 \neq j_2$. Since $\mathcal{H}$ separates points there is $f_1 \in \mathcal{H}$ with $f_1(j_1) \neq f_1(j_2)$.

Let first both $j_1, j_2 \in B$. If $f = f_1 \cdot f_B^{f_1(j)^{-1} \cdot f_1(k)}$, then $f \in \mathcal{H}$. Since $f(j) = f(k)$, $f \restriction_{F \setminus \{j\}} \in \mathcal{H}_{j,k}$. Observing that, for $i = 1, 2$,

$$f(j_i) = f_1(j_i) \cdot f_1(j)^{-1} \cdot f_1(k),$$

we see that f separates j_1 and j_2.

The same function works if $j_1, j_2 \notin B$, for in this case $f(j_i) = f_1(j_i)$, $i = 1, 2$.

Suppose finally that $j_1 \in B$ and $j_2 \notin B$ and assume that there is $f_2 \in \mathcal{H}$ with $f_2(j_2) \neq f_2(j_1) \cdot f_2(j)^{-1} \cdot f_2(k)$. Consider this time $f = f_2 \cdot f_B^{f_2(j)^{-1} \cdot f_2(k)}$ We see as above that $f \restriction_{F \setminus \{j\}} \in \mathcal{H}_{j,k}$. Now $f(j_1) = f_2(j_1) f_2(j)^{-1} \cdot f_2(k)$ and $f(j_2) = f_2(j_2)$. Our choice of f_2 then shows that f separates j_1 and j_2. □

The next lemma is our main technical tool.

Lemma 3.4. *Let G be a topological group, and $\mathcal{H} \subset C(F, G)$ be a closed subgroup that separates points and contains the constant functions.*

Assume that $\phi \in C(G, G)$ is non-affine. If ϕ operates on $\mathcal{H}$, then either:

(1) *ϕ has a nontrivial closed normal generalized period, or*
(2) *there is a dense subgroup D of G and a collection of homomorphisms $\rho_k : D \to G$, $1 \leq j \leq n$ with $n \leq |F|$ such that:*
   (a) *D is a generalized period of ϕ.*
   (b) *The homomorphisms ρ_k are injective for $k = 1, \ldots, n$.*
   (c) *The subgroup $\Delta = \{g \in D : \rho_k(g) = g$ for all $1 \leq k \leq n\}$ is closed in G and $\Delta \neq G$.*
   (d) *If $y^{-1}x \in D$, then*

$$\rho_k\left(\phi(x)\phi(y)^{-1}\right)\phi(y) = \phi\left(y\rho_k(y^{-1}x)\right), \quad \textit{for all } k = 1, \ldots, n. \tag{3.1}$$

Proof. Assume that (1) does not hold. Since by Lemma 3.2 $G_j^A(\mathcal{H})$ is a normal generalized period of ϕ for every $A \subset F$ and $j \in A$ and (1) fails to be valid, we have $\phi \in \Lambda_m(G)$ and the subgroups $G_j^A(\mathcal{H})$ are either trivial or dense.

We proceed to prove the lemma by induction on $N = |F|$. If $F = \{1, 2\}$, we choose $f \in \mathcal{H}$ with $f(1) \neq f(2)$. Then, since the function $\overline{f(2)}^{-1} f$ belongs to $\mathcal{H}$, we find that $f(2)^{-1} f(1) \in G_1^F(\mathcal{H})$. It follows that $G_1^F(\mathcal{H})$ is dense in G. The same argument shows that $G_2^F(\mathcal{H})$ is dense in G and therefore that $\mathcal{H}$ is dense in (and hence equal to) $C(F, G)$.

Assume now that $|F| = N$ and that the lemma is proved for subgroups $\mathcal{H} \subset C(X, G)$ with $|X| \leq N - 1$.

As the inductive hypothesis can be applied to the closure of the restriction groups $\mathcal{H} \restriction_A = \{f \restriction_A : f \in \mathcal{H}\}$, $A \subset F$, we can assume that $\mathcal{H} \restriction_A$ is dense in $C(A, F)$ for every $A \subset F$. In the same vein we have the following easily proved.

Claim. *If $\mathcal{H}_{j,k}$ separates the points of $F \setminus \{j\}$, then $\mathcal{H}_{j,k} = C(F \setminus \{j\})$. In addition, we may assume that $\mathcal{H}_{j,k}$ does not separate points whenever G_j^F is dense for otherwise $\mathcal{H}$ must be dense.*

Let now $m \leq |F|$ be chosen so that $G_{j_0}^{A_m}(\mathcal{H})$ is dense in G for some $A_m \subseteq F$ with $|A_m| = m$ and $G_j^A(\mathcal{H}) = \{0\}$, for every $A \subset F$ with $|A| \geq m + 1$.

If $A_m = F$, then choose $k \neq j_0$. Then the subgroup $\mathcal{H}_{j_0,k}$ separates points and we can apply our induction hypothesis to $\mathcal{H}_{j_0,k}$ so both $\mathcal{H}_{j_0,k}$ and $G_{j_0}^F(\mathcal{H})$ are dense which implies that $\mathcal{H} = C(F, G)$. Assume now that $n = N - m > 0$ and enumerate $F \setminus A_m$ as $F \setminus A_m = \{j_1, \ldots, j_n\}$. We then define, for each $1 \leq k \leq n$, a map $\rho_k : G_{j_0}^{A_m}(\mathcal{H}) \to G$, in such a way that for all $x \in G_{j_0}^{A_m}(\mathcal{H})$ there is a function $f \in \mathcal{H}$ with $f(j_0) = x$, $f(j) = e$ if $j_0 \neq j \in A_m$ and $f(j_k) = \rho_k(x)$.

The map ρ_k is well-defined: we would otherwise have two functions $f_1, f_2 \in \mathcal{H}$ with $f_i(j_0) = x$, $f_i(j) = e$ if $j_0 \neq j \in A_m$, $i = 1, 2$ and $f_1(j_k) \neq f_2(j_k)$. Then $f_2(j_k)^{-1} f_1(j_k) \in G_{j_k}^{A_m \cup \{j_k\}}(\mathcal{H})$. This goes against our choice of m since $|A_m \cup \{j_k\}| = m + 1$.

The proof of injectivity of ρ follows exactly the same pattern. Since $\mathcal{H}$ is a subgroup the maps ρ_k must be homomorphisms.

While the ρ_k's may be discontinuous, $\Delta = \{g \in D : \rho_k(g) = g \text{ for all } 1 \leq k \leq n\}$ is closed in G. It suffices to observe that

$$\Delta = \{x \in G : f^x_{\{j_0, j_1, \ldots, j_n\}} \in \mathcal{H}\},$$

where $f^x_{\{j_0, j_1, \ldots, j_n\}}$ is the map introduced just before Lemma 3.3. Since $\mathcal{H}$ is a closed subgroup it is clear that Δ must be closed.

Suppose now that $\Delta = G$. In that case the map $f^x_{\{j_0, j_1, \ldots, j_n\}}$ is in $\mathcal{H}$ for every $x \in G$ (i.e., the functions ρ_k are all equal to the identity mapping). Picking $k \in A_m \setminus \{j_0\}$ we may apply Lemma 3.3 (here $\{j_0, j_1, \ldots, j_n\}$ plays the role of B and $F \setminus \{j_0, j_1, \ldots, j_n\} = A_m \setminus \{j_0\}$) to find that either $\mathcal{H}_{j_0,k}$ separates points of $F \setminus \{j_0\}$ or there are $j_1, j_2 \in F$, $j_0 \neq j_1$, $j_0 \neq j_2$ such that $f(j_2) = f(j_1) f(j_0)^{-1} f(k)$ for all $f \in \mathcal{H}$.

Suppose now that $\mathcal{H}_{j_0,k}$ separates points. By the claim, $\mathcal{H}_{j_0,k} = C(F \setminus \{j_0\}, G)$ and $G_{j_0}^F(\mathcal{H}) = G$. It would follows that $\mathcal{H}$ does not separate points leading to a contradiction. We can therefore suppose that $\mathcal{H}_{j_0,k}$ does not separate points and thus that, by Lemma 3.3, there are $j_1, j_2 \in F$, $j_0 \neq j_1$, $j_0 \neq j_2$ such that $f(j_2) = f(j_1) f(j_0)^{-1} f(k)$ for all $f \in \mathcal{H}$. Then, recall that ϕ operates on $\mathcal{H}$, $\phi(f(j_1) f(j_0)^{-1} f(k)) = \phi(f(j_1))\phi(f(j_0))^{-1}\phi(f(k))$ for every $f(j_1)$, $f(j_0)$ and $f(k)$ with $f \in \mathcal{H}$. Since, by our inductive hypothesis, the group of restrictions $\mathcal{H}\restriction_{\{j_1, j_0, k\}}$ must be dense in $C(\{j_1, j_0, k\}, G)$ we conclude that $\phi(xy^{-1}z) = \phi(x)\phi(y)^{-1}\phi(z)$ for all $x, y, z \in G$ which implies that ϕ is affine. This goes against our hypothesis on ϕ and we have that $\Delta \neq G$.

Only assertion (d) remains now to be checked.

Let to that end $x, y \in G$ with $y^{-1}x \in G_{j_0}^{A_m}(\mathcal{H})$. Define the function $f : F \to G$ by $f(j_k) = y\rho_k(y^{-1}x)$, $1 \leq k \leq n$, $f(j_0) = x$ and $f(j) = y$ for $j_0 \neq j \in A_m$. Then $f \in \mathcal{H}$. Since ϕ operates on $\mathcal{H}$ and $\mathcal{H}$ contains the constants, the function $h : F \to G$ with $h(j_k) = \phi(y\rho_k(y^{-1}x))\phi(y)^{-1}$, $1 \leq k \leq n$, $h(j_0) = \phi(x)\phi(y)^{-1}$ and $h(j) = e$ for $j_0 \neq j \in A_m$ is also in $\mathcal{H}$. It follows from the definitions of the homomorphisms ρ_k that, for every $1 \leq k \leq n$,

$$\rho_k\left(\phi(x)\phi(y)^{-1}\right) = \phi(y\rho_k(y^{-1}x))\phi(y)^{-1}. \quad \square$$

Since our applications in this section of the previous lemma will concern Abelian groups we switch here to additive notation. For the rest of the present section, the identity element will consequently be denoted as 0. We remark that, in the setting of Abelian groups, the Eq. (3.1) becomes

$$\rho_k\big(\phi(x)-\phi(y)\big)+\phi(y)=\phi\big(\rho_k(x-y)+y\big),\quad \text{for all } k=1,\ldots,n \tag{3.2}$$

whenever $x-y\in D$.

Remark 3.5. Notice that, under the conditions in Lemma 3.4, Eq. (3.2) with $x=g$ and $y=0$ tells us that $\rho_k\phi(g)=\phi\rho_k(g)$ for every $g\in D$ ($1\le k\le n$). We shall freely use of this fact in the sequel.

Lemma 3.6. *Let G be an Abelian topological group, $\phi\colon G\to G$ a continuous function and $\rho_k\colon D\to G$, $1\le k\le n$ be a homomorphism defined on a subgroup D of G such that Eq. (3.2) holds. The subgroup $\Delta=\{g\in D\colon \rho_k(g)=g \text{ for all } 1\le k\le n\}$ is then a generalized period of ϕ.*

Proof. Let $x,y\in G$ with $x-y\in\Delta$. Then, applying Eq. (3.2), $\rho_k(\phi(x)-\phi(y))=\phi\big(\rho_k(x-y)+y\big)-\phi(y)$. Taking into account that $x-y\in\Delta$, $\rho_k(x-y)=x-y$ so that $\phi(x)-\phi(y)\in\Delta$. □

Lemma 3.7. *Let G be an Abelian topological group and $\phi\colon G\to G$ a continuous function with $\phi(0)=0$. If ϕ does not have any nontrivial closed generalized period but $\phi\notin D_F(G)$, then $\phi(g)=-\phi(-g)$ for every $g\in G$.*

Proof. Since $\phi\notin D_F(G)$, there is a finite set F such that ϕ operates on a proper closed subgroup of $\mathcal{H}\subset C(F,G)$ that separates points and contains the constant functions.

As ϕ does not admit closed nontrivial generalized periods, there are a dense subgroup D of G and a collection of homomorphisms $\rho_k\colon D\to G$, $1\le k\le n$, such that conditions (a)–(d) of Lemma 3.4 hold.

Let $g\in D$ and choose k, $1\le k\le n$. Applying Eq. (3.2) to $x=\rho_k(g)-g$ and $y=\rho_k(g)$, we have that

$$\rho_k\Big(\phi\big(\rho_k(g)-g\big)\Big)-\rho_k\Big(\phi\big(\rho_k(g)\big)\Big)+\phi\big(\rho_k(g)\big)=0.$$

After re-arranging terms we obtain

$$\rho_k\Big(\phi\big(\rho_k(g)-g\big)\Big)=\rho_k\Big(\rho_k\big(\phi(g)\big)-\phi(g)\Big).$$

Since ρ_k is injective, it follows that

$$\phi\big(\rho_k(g)-g\big)=\rho_k(\phi(g))-\phi(g). \tag{3.3}$$

Applying on the other hand the same equation to $x=0$ and $y=-g$, we have that

$$\phi\big(\rho_k(g)-g\big)=-\rho_k\big(\phi(-g)\big)+\phi(-g). \tag{3.4}$$

This identity can be used in Eq. (3.3) to yield

$$-\phi(g)+\rho_k\big(\phi(g)\big)=-\rho_k(\phi(-g))+\phi(-g)$$

or, what is the same,

$$\rho_k\big(\phi(g)+\phi(-g)\big)=\phi(g)+\phi(-g),$$

and this for every $k=1,\ldots,n$.

Thus $\phi(g)+\phi(-g)\in\Delta$ with Δ as defined in Lemma 3.4(c). Since the subgroup Δ is a proper closed subgroup and a generalized period of ϕ (Lemma 3.6) we have that $\Delta=\{0\}$, therefore $\phi(g)=-\phi(-g)$, for every $g\in D$. Since D is dense and ϕ is continuous the lemma follows. □

Theorem 3.8. *Let G be an Abelian topological group and $\phi\colon G\to G$ a continuous function. If ϕ does not have nontrivial closed generalized periods and is not affine, then $\phi\in D_F(G)$.*

Proof. Suppose $\phi\notin D_F(G)$. We shall assume that $\phi(0)=0$ and then prove that either ϕ has a generalized period or ϕ is a homomorphism. Once the theorem is proved for these functions the general result follows. If $\phi(0)=a\neq 0$, we can work with the function $\phi_a(g)=\phi(g)-a$. Since ϕ and ϕ_a have the same generalized periods we find that either ϕ has a generalized period or ϕ_a is a homomorphism, meaning that ϕ is affine.

We proceed as in Lemma 3.7. Let ϕ be a function which operates on a proper closed subgroup of $\mathcal{H}\subset C(F,G)$, that separates points and contains the constant functions.

There will be a dense subgroup D of G and a collection of homomorphisms $\rho_k\colon D\to G$, $1\leq k\leq n$ such that the properties (a)–(d) of Lemma 3.4 hold.

By Lemma 3.7 we have that $\phi(g)=-\phi(-g)$ for every $g\in G$.

Let $g_1,g_2\in D$ be arbitrarily chosen and take k, $1\leq k\leq n$. Putting $x=\rho_k(g_1-g_2)+g_2$ and $y=\rho_k(g_1-g_2)$ Eq. (3.2) becomes

$$\rho_k\Big(\phi\big(\rho_k(g_1-g_2)+g_2\big)-\rho_k\big(\phi(g_1-g_2)\big)\Big)+\rho_k\big(\phi(g_1-g_2)\big)=\rho_k\big(\phi(g_1)\big).$$

Since ρ_k is injective we obtain that

$$\phi\big(\rho_k(g_1-g_2)+g_2\big)-\rho_k\big(\phi(g_1-g_2)\big)+\phi(g_1-g_2)=\phi(g_1). \tag{3.5}$$

Using again Eq. (3.2) for the term $\phi\big(\rho_k(g_1-g_2)+g_2\big)$, Eq. (3.5) becomes

$$\rho_k\big(\phi(g_1)-\phi(g_2)\big)+\phi(g_2)-\rho_k\big(\phi(g_1-g_2)\big)+\phi(g_1-g_2)=\phi(g_1). \tag{3.6}$$

Regrouping the terms of (3.6) we obtain

$$\rho_k\big(-\phi(g_1-g_2)+\phi(g_1)-\phi(g_2)\big)=-\phi(g_1-g_2)+\phi(g_1)-\phi(g_2).$$

Since this is valid for every $k=1,\ldots,n$ and every $g_1,g_2\in D$, $\phi(g_1-g_2)-\phi(g_1)+\phi(g_2)\in\Delta$ with Δ as defined in Lemma 3.4(c). Since Δ is a proper closed subgroup and a generalized period of ϕ (Lemma 3.6) we have that $\Delta=\{0\}$, therefore $\phi(g_1-g_2)=\phi(g_1)-\phi(g_2)$, for every $g_1,g_2\in D$. Since D is dense and ϕ is continuous the theorem follows. □

Theorem 3.8 gives the harder direction of our desired characterization of $D_F(G)$.

Theorem 3.9. *Let G be an Abelian topological group. A continuous function $\phi\colon G\to G$ is not in $D_F(G)$ if and only if it is either affine or admits a nontrivial closed generalized period.*

Proof. If ϕ is not in $D_F(G)$, ϕ must either be affine or admit a closed generalized period by Theorem 3.8

As indicated in the introduction continuous affine functions are not in $D_F(G)$: if $\mathcal{H}\subset C(\{1,2,3,4\},G)$ is defined as $\mathcal{H}=\{f\in C(\{1,2,3,4\},G)\colon f(3)-f(4)=f(1)-f(2)\}$ then $\mathcal{H}$ is closed, nontrivial and ϕ operates on $\mathcal{H}$. In the same vein, if ϕ has a nontrivial closed generalized period $H \subset G$, then ϕ operates on the closed subgroup $\mathcal{H} = \{f \in C(\{1, 2\}, G): f(1)^{-1} f(2) \in H\}$. Therefore functions with closed nontrivial generalized periods are not in $D_F(G)$. □

For discrete G, we can use Corollary 2.6 to obtain a characterization of $D(G)$.

Corollary 3.10. *Let G be a discrete Abelian topological group. A function $\phi \in C(G, G)$ is in $D(G)$ if and only if it is not affine and mixes cosets.*

In certain cases the description of $D(G)$ is simpler, for instance when G is a free Abelian group of finite rank.

Lemma 3.11. *Let G be an Abelian topological group and let $\psi: G \to G$ be an affine map. If ρ is the homomorphism $\rho(g) = \psi(g) - \psi(0)$, then the subgroups $\rho(G)$ and $(Id - \rho)(G)$ are generalized periods of ψ.*

Proof. If $x, y \in G$ are such that $x - y = (Id - \rho)(g)$, $\psi(x) - \psi(y) = \rho(x) - \rho(y) = \rho(x - y) = (Id - \rho)(\rho(g)) \in (Id - \rho)(G)$. The latter subgroup is hence a generalized period of ψ. The case of $\rho(G)$ follows in exactly the same way. □

Lemma 3.12. *Let G denote a free Abelian group of finite rank with free generators $\{e_1, \ldots, e_n\}$. If $\rho: G \to G$ is an endomorphism of G, then either ρ or $(Id - \rho)$ is not surjective.*

Proof. Let M_f be the matrix whose columns are the coordinates of $\rho(e_i)$, $1 \leq i \leq n$. Then ρ is surjective if and only if $\left|\det(M_f)\right| = 1$. This equality cannot be satisfied by both M_f and $M_{Id-f} = I - M_f$. □

Theorem 3.13. *If G is a (discrete) free Abelian group of finite rank, then a function $\phi: G \to G$ belongs to $D(G)$ if and only if it does not have any nontrivial generalized periods.*

Proof. By Corollary 3.10 we only have to show that affine maps always admit nontrivial generalized periods.

Let $\psi: G \to G$ be affine and let $\rho(g) = \psi(g) - \psi(0)$ be the associated homomorphism. By Lemma 3.12, either ρ or $Id - \rho$ is not surjective. Since both $\rho(G)$ and $(Id - \rho)(G)$ are generalized periods of ψ (Lemma 3.11) we conclude that $\psi \notin \Lambda_m(G)$. □

We are now ready to answer Sternfeld's question on $D(\mathbb{Z})$.

Corollary 3.14. *The set $D(\mathbb{Z})$ can be described as follows:*

$$\begin{aligned} D(\mathbb{Z}) = \Lambda_m(\mathbb{Z}) = \{\phi: \mathbb{Z} \to \mathbb{Z}:\ & \textit{for every } k \in \mathbb{N}, k \geq 2 \textit{ there are } n_1, n_2 \in \mathbb{Z} \\ & \textit{such that } n_1 - n_2 = 0 (\mathrm{mod}\ k),\ \textit{but } \phi(n_1) - \phi(n_2) \neq 0 (\mathrm{mod}\ k)\}. \end{aligned}$$

Constructing functions in $D(\mathbb{Z})$ is easy:

Corollary 3.15. *If $\phi(0) = 0$ and for every $k \in \mathbb{N}$, $\phi(k) \neq 0 (\mathrm{mod}\ k)$, then $\phi \in D(\mathbb{Z})$.*

Corollary 3.16. *$D(\mathbb{Z})$ is dense in $\mathbb{Z}^{\mathbb{Z}}$ for the product topology of $\mathbb{Z}^{\mathbb{Z}}$.*

4. Odds and ends around $D(G)$

We discuss in this section the sharpness of the conditions we obtain in the previous sections.

Example 4.1. There are functions $\phi_i: \mathbb{Z}_2 \times \mathbb{Z}_2 \to Z_2 \times \mathbb{Z}_2$, $i = 1, 2$, such that:

(1) ϕ_1 does not have nontrivial generalized periods, yet is not condensing (it is affine).
(2) ϕ_2 is not affine but, yet is not condensing (it has nontrivial generalized periods).

These functions show that being non-affine or mixing cosets separately are not sufficient conditions to be in $D(G)$ (and thus that in general none of the conditions of Theorem 3.9 can be removed).

Proof. Let e_1, e_2 denote the generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Define ϕ_1 as the endomorphism of $\mathbb{Z}_2 \times \mathbb{Z}_2$ given by $\phi_1(e_1) = e_2, \phi_1(e_2) = e_1 + e_2$. The only proper subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are $\langle e_1 \rangle$, $\langle e_2 \rangle$ and $\langle e_1 + e_2 \rangle$, and it is readily checked that none of them is a generalized period of ϕ_1. Being a homomorphism, $\phi \notin D(G)$ by Corollary 3.10.

To find functions with generalized periods that are not affine is even easier, define for instance ϕ_2 as

$$\phi_2(0) = 0, \qquad \phi_2(e_1) = e_1, \qquad \phi_2(e_2) = e_1 \quad \text{and} \quad \phi_2(e_1 + e_2) = e_1.$$

Then the subgroup $\langle e_1 \rangle$ is a generalized period of ϕ_2 while ϕ_2 is obviously non-affine. □

A more intriguing question is to determine which functions are in $D(G)$ for noncommutative G. In this case, having a (non-normal) generalized period is not an obstacle any more.

Example 4.2. Let $G = \Sigma_3$ denote the group of permutations on the set $\{1, 2, 3\}$. There is a function $\phi \in D(G)$ that admits a generalized period.

Proof. Let $\phi: G \to G$ be the function defined as follows:

$$\phi(e) = e \qquad \phi((1\,2)) = (1\,2)$$
$$\phi((1\,3)) = (1\,2\,3) \qquad \phi((1\,2\,3)) = (13)$$
$$\phi((23)) = (1\,2\,3) \qquad \phi((1\,3\,2)) = (13).$$

Let $\mathcal{H}$ be a proper subgroup of $C(F, \Sigma_3)$, with F finite, that separates points and contains the constant functions. Suppose that ϕ operates on $\mathcal{H}$.

The only nontrivial normal subgroup of G is the subgroup $H = \{e, (123), (132)\}$. Since $\phi(e) = e$ and $\phi(123) \notin H$, we see that H is not a generalized period of ϕ. By Lemma 3.4, there are injective homomorphisms $\rho: G \to G$ different from the identity such that

$$\rho\Big(\phi(y)^{-1}\phi(x)\Big)\phi(y) = \phi\Big(y\rho(y^{-1}x)\Big).$$

Applying this equality to $x = (123)$ and $y = e$ one obtains

$$\rho\big((13)\big) = \phi\Big(\rho((123))\Big)$$

and therefore $\rho\big((13)\big) = (13)$ because $\rho(123)$ must have order 3.

Repeating the same process with $x = (13)$ and $y = e$, we have that $\rho(123) = \phi(\rho(13))$ and thus that $\rho((123)) = (123)$, for the only element of order three in the range of ϕ is (123). Since $\rho((13)) = (13)$ and $\rho((123)) = (123)$ and the permutations (13) and (123) generate G, we deduce that ρ is the identity. This goes against Lemma 3.4, showing that ϕ cannot operate on $\mathcal{H}$. Thus, $\phi \in D(G)$.

It is obvious that $K = \langle(1\,2)\rangle$ is a generalized period for ϕ. □

The generalized period of the above example is non-normal. Notice that a function with a normal generalized period cannot be in $D(G)$. In the light of this fact and Theorem 3.9 raise the following question:

Question 4.3. Let G be a non-Abelian topological (even discrete) group. Prove or disprove the following statement: a continuous function $\phi: G \to G$ is not in $D_F(G)$ if and only if it is either affine or admits a nontrivial, non-normal, closed generalized period.

We end by noting that the subgroup lattice of $C(F, G)$ for non-Abelian groups G can be very poor. This suggests that the whole approximation scheme should probably be considered from a different point of view in this case.

Example 4.4. Let G denote the alternating group A_5 of degree five. If F is finite, and $\mathcal{H}$ is a subgroup of $C(F, G)$ that separates points and contains the constant functions, then $\mathcal{H} = C(F, G)$.

Proof. The argument runs in parallel with that of Lemma 3.4.

Since the subgroups $G_j^A(\mathcal{H})$ $(A \subseteq F)$ are normal (Lemma 3.2) and G is a simple group [4, Theorem 2.11], we have that either $G_j^A(\mathcal{H}) = \{e\}$ or $G_j^A(\mathcal{H}) = G$, for every $F \subseteq G$.

We will argue by induction on $n = |F|$.

When $n = 2$, $G_j^F(\mathcal{H}) \neq \{e\}$ (see the proof of Lemma 3.4), for all $j \in F$. Hence, $G_j^F(\mathcal{H}) = G$ for every $j \in F$. This means that $\mathcal{H} = C(F, G)$.

Assume now that for every $K \subset F$, with $K \neq F$, $C(K, G)$ has no proper point-separating subgroups that contain the constant functions.

The same argument of Lemma 3.4 shows that $G_j^F(\mathcal{H}) = \{e\}$ for all $j \in F$.

We now fix $j_0 \in F$. Our inductive hypothesis implies that $\{h \restriction_{F\setminus\{j_0\}}: h \in \mathcal{H}\} = C(F \setminus \{j_0\}, G)$.

Fix another point $j_1 \in F$, $j_1 \neq j_0$ and for each $(a, x, y) \in G \times G \times G$ let $h_{a,x,y}: F \to G$ be defined as $h(j_0) = a$, $h(j_1) = y$ and $h(j) = x$ if $j \in F$, $j \neq j_0$, $j \neq j_1$. We can then define a map $\tilde{\rho}: G \times G \to G$ by assigning to each $(x, y) \in G \times G$ the only element $\tilde{\rho}(x, y)$ such that $h_{\tilde{\rho}(x,y),x,y} \in \mathcal{H}$. Observe that this element $\tilde{\rho}(x, y)$ is well-defined because $G_{j_0}^F(\mathcal{H}) = \{e\}$.

It is easy to see that $\tilde{\rho}$ is a homomorphism with $\tilde{\rho}(x, x) = x$ for every $x \in G$. This implies that there is a homomorphism $\rho: G \to G$ (namely $x \mapsto \tilde{\rho}(x, e)$) such that $\tilde{\rho}(x, y) = \rho(xy^{-1}) \cdot y$.

Since every endomorphism of G is an inner automorphism defined by an element $\sigma \in \Sigma_5$ [4, Theorem 2.17], we can find $\sigma \in \Sigma_5$ such that $\tilde{\rho}(\tau_1, \tau_2) = \rho(\tau_1\tau_2^{-1})\tau_2 = \sigma^{-1}\tau_1\tau_2^{-1}\sigma\tau_2$ for every $(\tau_1, \tau_2) \in G \times G$.

Let now $\tau \in G$ be chosen arbitrarily. Since $\tilde{\rho}$ is a homomorphism:

$$\begin{aligned} \tau^{-1}\sigma^{-1}\tau^{-1}\sigma\tau &= \tilde{\rho}\big((\sigma\tau^{-1}\sigma^{-1}, \tau)\big) \\ &= \tilde{\rho}\big((e, \tau)\big) \cdot \tilde{\rho}\big((\sigma\tau^{-1}\sigma^{-1}, e)\big) \\ &= \sigma^{-1}\tau^{-1}\sigma. \end{aligned} \tag{4.1}$$

Suppose now that σ is different from the identity. There are then $i_1, i_2 \in \{1, 2, 3, 4, 5\}$ with $i_1 \neq i_2$ such that $\sigma(i_1) = i_2$. Choose indices $i_3, i_4 \in \{1, 2, 3, 4, 5\}$ with $i_3, i_4 \notin \{i_1, i_2\}$ and $\sigma(i_3) = i_4$.

Let finally $\tau \in G$ be such that $\tau(i_1) = i_1, \tau(i_2) = i_3$ and $\tau(i_4) = i_2$ (we can choose τ to be $\tau = (i_2\, i_3\, i_4)$ if $i_3 \neq i_4$ and $\tau = (i_2\, i_3)(j\, k)$ with $j, k \in \{1, 2, 3, 4, 5\} \setminus \{i_1, i_2, i_3\}$ in case $i_3 = i_4$). Then

$$(\tau^{-1}\sigma^{-1}\tau^{-1}\sigma\tau)(i_1) = i_2$$

while

$$(\sigma^{-1}\tau^{-1}\sigma)(i_1) = i_3,$$

and equality (4.1) cannot hold. We conclude thus that $\sigma = e$ and, as a consequence, that ρ is the identity automorphism. It follows that $\tilde{\rho}(x, y) = x$ for every $x, y \in G$. But then the function $h_{x,x,y} \in \mathcal{H}$. Taking $y \in G$, $y \neq x$ and multiplying by the constant function x^{-1}, we find then that $h_{e,e,yx^{-1}} \in \mathcal{H}$, showing that $G^F_{j_1}(\mathcal{H}) \neq \{e\}$. This contradiction shows that, indeed, $\mathcal{H} = C(F, G)$. □

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