The Extension of Roth's Theorem for Matrix Equations Over a Ring

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ABSTRACT

This paper extends Roth's similarity theorem as follows: Let $R$ be a ring with identity, $B(A) = \sum_{i=0}^{k} B_i A^i \in R_{mxq}$, if either $R$ is a division ring and $A \in R$, $A$ is algebraic, or $R$ is finitely generated as a module over its center, then the matrix equation $\Sigma_{i=0}^{k} A^i X B_i = C$ over $R$ has a solution if and only if

$$\begin{pmatrix} \lambda I - A & -C \\ O & B(\lambda) \end{pmatrix} \equiv \begin{pmatrix} \lambda I - A & O \\ O & B(\lambda) \end{pmatrix}.$$

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Let $R$ be a ring with identity, and $\text{Cen } R = \{ r \in R \mid rx = xr, \ x \in R \}$ be the center of $R$. Let $R_{m \times n} (R_{m \times n}[\lambda])$ denote the set of all $m \times n$ matrices over $R (R[\lambda])$, $A(\lambda) \equiv B(\lambda)$ denote that $A(\lambda) \in R_{m \times n} [\lambda]$ is equivalent to

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$B(\lambda) \in R_{m \times n}[\lambda]$ over $R[\lambda]$, and $A \sim B$ denote that $A \in R_{n \times n}$ is similar to $B \in R_{n \times n}$ over $R$. Let $M(\lambda) = \sum_{i=0}^{k} M_i \lambda^i \in R_{m \times n}[\lambda]$, $A \in R_{m \times m}$, and define

$$[M(\lambda)]_L(A) = \sum_{i=0}^{k} A^i M_i.$$  

On the solvability of a matrix equation, we have the well-known Roth's theorems [1] as follows:

**Similarity Theorem.** Let $F$ be a field, $A \in F_{n \times n}$, $B \in F_{m \times m}$, and $C \in F_{n \times m}$. Then the matrix equation

$$AX - XB = C$$  

has a solution $X \in F_{n \times m}$ if and only if

$$\begin{pmatrix} A & C \\ O & B \end{pmatrix} \sim \begin{pmatrix} A & O \\ O & B \end{pmatrix},$$  

or equivalently, if and only if

$$\begin{pmatrix} \lambda I - A & -C \\ O & \lambda I - B \end{pmatrix} \equiv \begin{pmatrix} \lambda I - A & O \\ O & \lambda I - B \end{pmatrix}.$$  

**Equivalence Theorem.** Let $F$ be a field, $A \in F_{m \times r}$, $B \in F_{s \times n}$, and $C \in R_{r \times n}$. Then the matrix equation

$$AX - YB = C$$  

has a solution $X \in F_{r \times n}, Y \in R_{m \times s}$ if and only if

$$\begin{pmatrix} A & C \\ O & B \end{pmatrix} \equiv \begin{pmatrix} A & O \\ O & B \end{pmatrix}.$$  

Let $R$ be a ring with identity. If the matrix equation (2) [or (5)] over $R$ has a solution over $R$ if and only if (3) [or (6)] holds, then we say $R$ has the
similarity [or equivalence] property. On the extension of Roth's theorem, R. Guralnick [2] showed that a semisimple Artinian ring has the equivalence property, and

**Lemma 1** [2]. An Artinian principal ideal ring has the equivalence property (cf. [3]).

W. Gustafson and J. Zelmanowitz [4] showed that:

**Lemma 2**. Let $R$ be a ring with identity. If $R$ is finitely generated as module over its center, then $R$ has the equivalence property.

Some noncommutative results for the extension of Roth's theorem to the matrix equation $AXB + CYD = E$ or $AXB - GXD = E$ can be found in the author's papers [5] and [6].

In this paper, we discuss the extension of Roth's theorem for the matrix equation

$$
\sum_{i=0}^{k} A^iX B_i = C, \quad (7)
$$

where $A \in R_{n \times n}$, $B_i \in R_{m \times q}$, $i = 0, 1, \ldots, k$, and $C \in R_{n \times q}$.

Clearly, the equation (7) is the generalization of the following equations:

$$
AX - XB = C, \quad (8)
$$

$$
X - AXB = C, \quad (9)
$$

$$
AXB = C. \quad (10)
$$

Let $D$ be a division ring, $A \in D_{n \times n}$. If there exists a polynomial $f(\lambda) \in (\text{Cen } D)[\lambda]$ such that $f(A) = 0$, then $A$ is said to be algebraic [7]. If $A \in D_{n \times n}$ is algebraic, then there exists a unique monic polynomial $q(\lambda) \in (\text{Cen } D)[\lambda]$ of minimum degree such that $q(A) = 0$, and $q(\lambda)$ is called the minimal central polynomial of $A$.

Clearly, if $A \in D_{n \times n}$ is centralizable, i.e. there exists an invertible matrix $P$ over $D$ such that $P^{-1}AP$ is a matrix over $\text{Cen } D$, then $A$ is algebraic. If $D$ is a finite dimensional central division algebra over a field, then any $n \times n$ matrix $D$ is algebraic. Thus, the algebraic matrix over a division ring is more useful.
LEMMA 3 [8]. Let \( R \) be a ring with identity. Given \( f, g \in R[\lambda] \), if \( g \) is monic, then there exist unique elements \( q, r \in R[\lambda] \) such that
\[
f = gq + r, \quad \text{deg } r < \text{deg } g.
\] (11)

THEOREM 1. Let \( R \) be a ring with identity, \( B(\lambda) = \sum_{i=0}^{k} B_i \lambda^i \in R_{m \times q}[\lambda] \). Then the matrix equation (7) over \( R \) has a solution \( X \in R_{n \times m} \) if and only if the matrix equation
\[
(\lambda I - A)X(\lambda) + Y(\lambda)B(\lambda) = C
\] (12)
over \( R[\lambda] \) has a solution \( (X(\lambda), Y(\lambda)) \in R_{n \times (q+m)}[\lambda] \).

Proof. If the matrix equation (7) over \( R \) has a solution \( X_0 \in R_{n \times m} \), let \( \bar{B}_0 = X_0 B_0 - C, \bar{B}_i = X_0 B_i, i = 1, \ldots, k, \) and
\[
Q(\lambda) = \sum_{i=1}^{k} \sum_{j=0}^{i-1} \lambda^{i-j-1} A^j \bar{B}_i,
\] (13)
then it is easy to see that
\[
X_0 B(\lambda) - C = (\lambda I - A)Q(\lambda), \quad \text{or}
\]
(14)
Thus, the matrix equation (12) has a solution \( (X(\lambda), Y(\lambda)) = (-Q(\lambda), X_0) \in R_{n \times (q+m)}[\lambda] \).

Conversely, if (12) has a solution \( (X_0(\lambda), Y_0(\lambda)) \in R_{n \times (q+m)}[\lambda] \), then it is easy to see that
\[
[Y_0(\lambda) B(\lambda)]_L(A) = [C - (\lambda I - A)X_0(\lambda)]_L(A) = C.
\] (15)
Let \( Y_0(\lambda) = \sum_{j=0}^{r} D_j \lambda^j \in R_{n \times m}[\lambda], X_1 = \sum_{j=0}^{r} A^j D_j \in R_{n \times m} \). By (15), we have
\[
\sum_{i=0}^{k} A^i X_1 B_i = \sum_{i=0}^{r} \sum_{j=0}^{k} A^{i+j} D_j B_i
\]
\[
= \left( \sum_{j=0}^{r} \sum_{i=0}^{k} \lambda^{i+j} D_j B_i \right)_L(A)
\]
\[
= [Y_0(\lambda) B(\lambda)]_L(A)
\]
\[
= C.
\] (16)
Thus, matrix equation (7) over \( R \) has a solution \( X = X_1 \).
Now, we extend Roth's similarity theorem as follows:

**Theorem 2.** Let \( R \) be a ring with identity, and \( B(\lambda) = \sum_{i=0}^{k} B_i \lambda^i \in R_{m \times q}[\lambda] \). If either \( R \) is a division ring and \( A \subseteq R_{n \times n} \) is algebraic, or \( R \) is finitely generated as module over its center, then the matrix equation (7) over \( R \) has a solution \( X \in R_{n \times m} \) if and only if

\[
\begin{pmatrix}
\lambda I - A & -C \\
O & B(\lambda)
\end{pmatrix}
\equiv
\begin{pmatrix}
\lambda I - A & O \\
O & B(\lambda)
\end{pmatrix}. 
\]  

(17)

**Proof.** If matrix equation (7) has a solution \( X \in R_{n \times m} \), then by Theorem 1, there exists \((X_i(\lambda), Y_i(\lambda)) \in R_{n \times (q + m)}[\lambda] \) such that

\[
(\lambda I - A)X_i(\lambda) + Y_i(\lambda)B(\lambda) = C. 
\]  

(18)

Let

\[
P(\lambda) = \begin{pmatrix} I_n & Y_i(\lambda) \\ O & I_m \end{pmatrix} \quad \text{and} \quad Q(\lambda) = \begin{pmatrix} I_n & X_i(\lambda) \\ O & I_m \end{pmatrix}. 
\]  

(19)

Then we have

\[
P(\lambda) \begin{pmatrix}
\lambda I - A & -C \\
O & B(\lambda)
\end{pmatrix} Q(\lambda) = \begin{pmatrix}
\lambda I - A & O \\
O & B(\lambda)
\end{pmatrix}. 
\]  

(20)

Conversely, if the condition (17) holds, we prove that matrix equation (7) has a solution as follows:

**Case 1.** Suppose \( R \) is a division ring and \( A \in R_{n \times n} \) is algebraic. Let \( F = \text{Cen } R, \ q(\lambda) \in F[\lambda] \) be the minimum central polynomial of \( A \), and

\[
I_q = (q(\lambda)) = \{ q(\lambda)f(\lambda) | f(\lambda) \in R[\lambda] \}. 
\]  

(21)

Let \( \overline{R} = R[\lambda]/I_q \) be a quotient ring, by Lemma 3, \( \overline{R} \) can be written as \( R = \{ f + I_q | f \in R[\lambda] \text{ and } \deg f < \deg q(\lambda) \} \). Thus, it is easy to see that \( \overline{R} \) is an Artinian principal ideal ring. Without loss of generality, we assume that \( \lambda I - A, \ B(\lambda), \) and \( C \) are matrices over \( \overline{R} \). By (17), it is easy to see that

\[
\begin{pmatrix}
\lambda I - A & -C \\
O & B(\lambda)
\end{pmatrix}
\equiv
\begin{pmatrix}
\lambda I - A & O \\
O & B(\lambda)
\end{pmatrix} \quad \text{over } \overline{R}. 
\]
Thus, by Lemma 1, the matrix equation

\[(\lambda I - A)X(\lambda) + Y(\lambda)B(\lambda) = C\]  

(22)

over \(\overline{R}\) has a solution \((X_0(\lambda), Y_0(\lambda)) \in \overline{R}_{n \times (q + m)}\). Since \(q(A) = 0\), it makes sense to define \([I_q]_{N \times M}(A) = 0\) for all natural numbers \(N, M\). Let \(Y_0(\lambda) = Y_1(\lambda) + (I_q)_{n \times m}\), where \(Y_1(\lambda) = \sum_{j=0}^{r} D_j \lambda^j \in R_{n \times m}[\lambda]\); then we have

\[
[Y_1(\lambda)B(\lambda)]_L(A) = [Y_0(\lambda)B(\lambda)]_L(A) = C - (\lambda I - A)X_0(\lambda)]_L(A) = C.

(23)

Let \(X_1 = [Y_1(\lambda)]_L(A) = \sum_{j=0}^{r} A^jD_j \in R_{n \times m}\); then by (23) we have

\[
\sum_{i=0}^{k} A^iX_1B_i = \sum_{j=0}^{r} \sum_{i=0}^{k} \lambda^{i+j}D_j B_i(A) = [Y_1(\lambda)B(\lambda)]_L(A) = C.

(24)

This is, the matrix equation (7) has a solution \(X = X_1\).

Case 2. Suppose that \(R\) is finitely generated as module over \(\text{Cen } R\). Since \(\text{Cen}(R[\lambda]) = (\text{Cen } R)[\lambda]\), thus \(R[\lambda]\) is also a ring which is finitely generated as module over its center. By Lemma 2, we know that matrix equation (12) over \(R[\lambda]\) has a solution \((X_1(\lambda), Y_1(\lambda)) \in R_{n \times (q + m)}[\lambda]\). Thus, by Theorem 1, it is clear that the matrix equation (7) over \(R\) has a solution \(X \in R_{n \times m}\).

By Theorem 2, clearly we have

**Corollary 1.** Let \(A \in R_{m \times m}, B \in R_{n \times n}\), and \(C \in R_{m \times n}\). If either \(R\) is a division ring and \(A\) is algebraic, or \(R\) is a ring which is finitely generated as a module over \(\text{Cen } R\), then the matrix equation \(AX - XB = C\) over \(R\) has a solution \(X \in R_{m \times n}\) if and only if

\[
\begin{pmatrix}
\lambda I_m - A & -C \\
O & \lambda I_n - B
\end{pmatrix} \equiv \begin{pmatrix}
\lambda I_m - A & O \\
O & \lambda I_n - B
\end{pmatrix}.
\]

(25)
Corollary 2. Let \( A \in R_{m \times m} \), \( B \in R_{n \times n} \), and \( C \in R_{m \times n} \). If either \( R \) is a division ring and \( A \) is algebraic, or \( R \) is a ring which is finitely generated as a module over \( \text{Cen} R \), then matrix equation \( X - AXB = C \) has a solution \( X \in R_{m \times n} \) if and only if

\[
\begin{pmatrix}
\lambda I_m - A & -C \\
0 & I_n - \lambda B
\end{pmatrix} \equiv \begin{pmatrix}
\lambda I_m - A & O \\
O & I_n - \lambda B
\end{pmatrix}.
\]

Clearly, Corollary 2 is the extension of Theorem 2 in [9].

Corollary 3. Let \( A \in R_{m \times m} \), \( B \in R_{n \times q} \), and \( C \in R_{m \times q} \). If \( R \) is a division ring and \( A \) is algebraic, or \( R \) is a ring which is finitely generated as a module over \( \text{Cen} R \), then the matrix equation \( AXB = C \) over \( R \) has a solution \( X \in R_{m \times n} \) if and only if

\[
\begin{pmatrix}
\lambda I - A & -C \\
O & \lambda B
\end{pmatrix} \equiv \begin{pmatrix}
\lambda I - A & O \\
O & \lambda B
\end{pmatrix}.
\]

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