On existence of periodic solutions of Rayleigh equation of retarded type

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Abstract

Existence of periodic solutions for a kind of non-autonomous Rayleigh equations of retarded type

\[ x''(t) + f(t, x'(t - \sigma)) + g(t, x(t - \tau(t))) = p(t) \]

is studied, and some new results are obtained. Our work generalizes and improves the known results in the literature. © 2006 Elsevier B.V. All rights reserved.

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1. Introduction

In recent years, the existence of periodic solutions for a kind of Rayleigh equations was studied by some researchers (see [1–11]). In [4,3], continuation theorems are introduced and applied to the existence of solutions of differential equations. In particular, a specific example is given in [4, p. 99] (see also [3, p. 175]) on how periodic solutions can be obtained by means of these theorems for the differential equation

\[ x''(t) + f(x'(t)) + h(t, x(t)) = 0 \]  \quad (1.1)

where \( f \in C(\mathbb{R}), f(0) = 0, \) and \( h \in C(\mathbb{R}^2, \mathbb{R}) \) is \( 2\pi \)-periodic in \( t, \) and \( h(t, x)x < 0 \) for \( |x| \geq r, t \in [0, 2\pi]. \) In the course of derivations, it is realized that once appropriate a priori bounds for the \( 2\pi \)-periodic solutions of the auxiliary equations

\[ x''(t) + \lambda f(x'(t)) + \lambda h(t, x(t)) = 0 \]

are known for each \( \lambda \in (0, 1), \) then standard procedures will allow these theorems to imply existence of periodic solutions to Eq. (1.1).

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Employing these approach, the existence of $2\pi$-periodic solutions for a kind of Rayleigh equations with a deviating argument

$$x''(t) + f(x'(t)) + g(x(t - \tau(t))) = p(t),$$

(1.3)

where $f$, $g$, $p$ and $\tau$ are real continuous functions defined on $\mathbb{R}$, and $\tau$ and $p$ are periodic with period $2\pi$, and $f(0) = 0$ and $\int_0^{2\pi} p(s) \, ds = 0$, was studied in [7–9].

In [10], Wang discussed the following non-autonomous Rayleigh equation of retarded type

$$x''(t) + f(t, x'(t-\sigma)) + g(t, x(t-\tau)) = p(t),$$

(1.4)

where $\sigma \geq 0$, $\tau \geq 0$, $f, g \in C(\mathbb{R}^2, \mathbb{R})$ and $f(t, x), g(t, x)$ are $2\pi$-periodic in $t$, $p \in C(\mathbb{R}, \mathbb{R})$ is periodic with period $2\pi$, and got the following results under the assumptions of $f(t, 0) = 0$ for $t \in \mathbb{R}$ and $\int_0^{2\pi} p(s) \, ds = 0$.

**Theorem A.** Assume that there exist constants $K > 0$, $M > 0$ and $d > 0$ such that

(A1) $|f(t, x)| \leq K$, for $(t, x) \in \mathbb{R}^2$;

(A2) $xg(t, x) > 0$ and $|g(t, x)| > K$ for $t \in \mathbb{R}$, $|x| \geq d$;

(A3) $g(t, x) \geq -M$ for $t \in \mathbb{R}, x \leq -d$.

(A4) $\sup_{(t,x)\in\mathbb{R}\times[-d,d]} |g(t, x)| < +\infty$.

Then (1.4) has at least a periodic solution with period $2\pi$.

**Theorem B.** In Theorem A, if (A3) is replaced by

(A5) $g(t, x) \leq M$ for $t \in \mathbb{R}, x \geq d$.

Then the conclusion still holds.

**Remark 1.1.** In Theorems A and B, condition (A4) is always satisfied because $g(t, x)$ is continuous and periodic in $t$, and so we can omit it.

In this paper, we will discuss the existence of $2\pi$-periodic solutions of non-autonomous Rayleigh equation of retarded type

$$x''(t) + f(t, x'(t-\sigma)) + g(t, x(t-\tau)) = p(t),$$

(1.5)

where $\sigma \geq 0$, $f, g \in C(\mathbb{R}^2, \mathbb{R})$ and $f(t, x), g(t, x)$ are $2\pi$-periodic in $t$, $f(t, 0) = 0$ for $t \in \mathbb{R}$, $\tau, p \in C(\mathbb{R}, \mathbb{R})$ are periodic with period $2\pi$, and $\int_0^{2\pi} p(t) \, dt = 0$. Using the theory in [4,3] and a improved prior estimate, we obtain better sufficient conditions for the existence of periodic solution of Eq. (1.5). These results generalize and improve those in [10] even if in the case of $\tau(t) \equiv \tau($constant$)$, and all of conditions are more weaker than those in [7–9] even if in $f(t, x) \equiv f(x), g(t, x) \equiv g(x)$.

For the sake of convenience, we denote by $C_{2\pi}$ the space of continuous $2\pi$-periodic functions, endowed the $\|x\|_0 = \max_{t\in[0,2\pi]}|x(t)|$.

2. Main results

**Lemma 2.1.** Let $x(t)$ be continuous derivable $T$-periodic function ($T > 0$). Then for any $t_0 \in (-\infty, \infty)$

$$\max_{t \in [t_0, t_0 + T]} |x(t)| \leq |x(t_0)| + \frac{1}{2} \int_0^T |x'(s)| \, ds.$$  

(2.1)
Proof. Choose $t^* \in [t_*, t_* + T]$ such that $|x(t^*)| = \max_{t \in [t_*, t_* + T]} |x(t)|$. Then

$$|x(t^*)| = \left| x(t_*) + \int_{t_*}^{t^*} x'(s) \, ds \right| \leq |x(t_*)| + \int_{t_*}^{t^*} |x'(s)| \, ds$$

and

$$|x(t^*)| = |x(t^* - T)| = \left| x(t_*) - \int_{t^* - T}^{t_*} x'(s) \, ds \right| \leq |x(t_*)| + \int_{t^* - T}^{t_*} |x'(s)| \, ds.$$

Combing the above two inequalities, we have

$$|x(t^*)| \leq |x(t_*)| + \frac{1}{2} \int_{t^* - T}^{t_*} |x'(s)| \, ds = |x(t_*)| + \frac{1}{2} \int_{0}^{T} |x'(s)| \, ds.$$

The proof is complete. □

Theorem 2.1. Assume that there exist constants $r_1, r_2 \geq 0$, $d > 0$, $K > 0$ and $M > 0$ such that

(H1) $|f(t, x)| \leq r_1 |x| + K$, $\forall (t, x) \in \mathbb{R}^2$;
(H2) $x g(t, x) > 0$ and $|g(t, x)| > r_1 |x| + K$ for $t \in \mathbb{R}, |x| > d$;
(H3) $g(t, x) \geq r_2 x - M$ for $t \in \mathbb{R}, x \leq -d$.

If

$$2\pi(r_1 + (\pi + 1)r_2) < 1, \quad (2.2)$$

then Eq. (1.5) has at least a $2\pi$-periodic solution.

Proof. Consider the auxiliary equation

$$x''(t) + \lambda f(t, x'(t - \sigma)) + \lambda g(t, x(t - \tau(t))) = \lambda p(t), \quad \lambda \in (0, 1). \quad (2.3)$$

From the results (degree theory) in [4,3] (see also proof of [10]), it is sufficient to show that there are positive constants $M_0$ and $M_1$, independent of $\lambda$, such that if $x(t)$ is a $2\pi$-periodic solution of Eq. (2.3), then $\|x\|_0 < M_0$ and $\|x'\|_0 < M_1$. Now, let $x = x(t)$ be any $2\pi$-periodic solution of Eq. (2.3). Integrating both sides of (2.3) on $[0, 2\pi]$, we have

$$\int_{0}^{2\pi} (f(s, x'(s - \sigma)) + g(s, x(s - \tau(s)))) \, ds = 0. \quad (2.4)$$

It follows that there exists a $t_1 \in [0, 2\pi]$ such that

$$f(t_1, x'(t_1 - \sigma)) + g(t_1, x(t_1 - \tau(t_1))) = 0. \quad (2.5)$$

We assert that there exists a $t^* \in [0, 2\pi]$ such that

$$|x(t^*)| \leq \|x'\|_0 + d. \quad (2.6)$$

Case 1: $r_1 = 0$. From (2.5) and (H1), we have

$$|g(t_1, x(t_1 - \tau(t_1)))| \leq K,$$

which, together with (H2), implies that

$$|x(t_1 - \tau(t_1))| \leq d. \quad (2.7)$$
Thus, we have

\[ |x(t_1 - \tau(t_1))| \leq \|x'\|_0. \]  

(2.8)

Combining (2.7) and (2.8), we see that

\[ |x(t_1 - \tau(t_1))| \leq \|x'\|_0 + d. \]

\[ \text{Note that } x(t) \text{ is periodic, there exists a } t^* \in [0, 2\pi] \text{ such that (2.6) holds. By Lemma 2.1, we have} \]

\[ \|x\|_0 \leq |x(t^*)| + \frac{1}{2} \int_0^{2\pi} |x'(s)| \, ds \]

\[ \leq d + (\pi + 1)\|x'\|_0. \]  

(2.9)

(2.10)

Let \( E_1 = \{ t : t \in [0, 2\pi], x(t - \tau(t)) > d \}, E_2 = \{ t : t \in [0, 2\pi], x(t - \tau(t)) < -d \}, E_3 = \{ t : t \in [0, 2\pi], |x(t - \tau(t))| \leq d \}. \) From (2.4), we obtain

\[ \int_{E_1} |g(s, x(s - \tau(s)))| \, ds \leq \int_0^{2\pi} |f(s, x'(s - \sigma))| \, ds + \left( \int_{E_2} + \int_{E_3} \right) |g(s, x(s - \tau(s)))| \, ds. \]

Thus,

\[ \|x'\|_0 \leq \frac{1}{2} \int_0^{2\pi} |x''(s)| \, ds \]

\[ \leq \frac{1}{2} \left[ \int_0^{2\pi} |f(s, x'(s - \sigma))| \, ds + \int_0^{2\pi} |g(s, x(s - \tau(s)))| \, ds + \int_0^{2\pi} |p(s)| \, ds \right] \]

\[ \leq \frac{1}{2} \left[ \int_0^{2\pi} |f(s, x'(s - \sigma))| \, ds + \left( \int_{E_2} + \int_{E_3} \right) |g(s, x(s - \tau(s)))| \, ds + 2\pi\|p\|_0 \right] \]

\[ \leq \int_0^{2\pi} |f(s, x'(s - \sigma))| \, ds + \left( \int_{E_2} + \int_{E_3} \right) |g(s, x(s - \tau(s)))| \, ds + \pi\|p\|_0 \]

\[ \leq 2\pi(r_1\|x'\|_0 + r_2\|x\|_0) + \pi(2(K + M + g_d) + \|p\|_0) \]

\[ \leq 2\pi(r_1 + (\pi + 1)r_2)\|x'\|_0 + \pi(2(K + M + g_d + r_2d) + \|p\|_0), \]

where \( g_d = \max_{t \in [0, 2\pi], |x| \leq d}|g(t, x)|. \) Therefore,

\[ \|x'\|_0 \leq \frac{\pi(2(K + M + g_d + r_2d) + \|p\|_0)}{1 - 2\pi(r_1 + r_2(\pi + 1))} \triangleq M_1. \]

It follows from (2.10) that

\[ \|x\|_0 \leq d + M_1(\pi + 1) \triangleq M_0. \]

This completes the proof. \( \Box \)
Theorem 2.2. Assume that there exist constants \( r_1, r_2 \geq 0, d > 0, K > 0 \) and \( M > 0 \) such that

\[
\begin{align*}
\text{(H1)} & \quad |f(t, x)| \leq r_1 |x| + K, \forall (t, x) \in \mathbb{R}^2; \\
\text{(H2)} & \quad x g(t, x) > 0 \text{ and } |g(t, x)| > r_1 |x| + K \text{ for } t \in \mathbb{R}, |x| > d; \\
\text{(H4)} & \quad g(t, x) \leq r_2 x + M \text{ for } t \in \mathbb{R}, x \geq d.
\end{align*}
\]

If (2.2) holds, then Eq. (1.5) has at least a \( 2\pi \)-periodic solution.

**Remark 2.1.** When \( \tau(t) \equiv \tau(\text{constant}) \) and \( r_1 = r_2 = 0 \), Theorems 2.1 and 2.2 are Theorems A and B, respectively, and so our results generalize and improve the corresponding results in [10].

**Remark 2.2.** Note that the condition \( 2\pi (r_1 + (\pi + 1) r_2) < 1 \) is weaker than \( 4\pi (r_1 + (2\pi + 1) r_2) < 1 \) and \( 8\pi (r_1 + \pi r_1 r_2) < 1 \). Therefore, even if in the case that \( f(t, x) \equiv f(x) \) and \( g(t, x) \equiv g(x) \), Theorems 2.1 and 2.2 still improve Theorems 1 and 2 in [10], and Theorems 1 and 2 in [8] as well.

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**References**


