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Analyticity of a class of degenerate evolution equations on the canonical simplex of $\mathbb{R}^d$ arising from Fleming–Viot processes

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ABSTRACT

We study the analyticity of the semigroups generated by a class of degenerate second order differential operators in the space $C(S_d)$, where S_d is the canonical simplex of $\mathbb{R}^d$. The semigroups arise from the theory of Fleming–Viot processes in population genetics.

1. Introduction

In this paper we are dealing with the class of degenerate second order elliptic differential operators

$$\mathcal{A}_d u(x) = \frac{1}{2}\sum_{i,j=1}^{d} x_i(\delta_{ij} - x_j)\partial^2_{x_i x_j} u(x), \quad x \in S_d, \tag{1.1}$$

and $m\mathcal{A}_d$, where $S_d = \{x \in [0,1]^d \mid \sum_{i=1}^d x_i \leqslant 1\}$ is the canonical simplex of $\mathbb{R}^d$ and m is a strictly positive function in the space $C(S_d)$ of all continuous functions on S_d. The operator (1.1) arises in the theory of Fleming–Viot processes as a generator of a Markov C_0-semigroup defined on $C(S_d)$. Fleming–Viot processes are measure-valued processes that can be viewed as diffusion approximations of empirical processes associated with some classes of discrete time Markov chains in population genetics. We refer to [15,16,19] for more details on the topic. In particular, the operator (1.1) is the generator corresponding to the diffusion model in population genetics in which neither mutation, migration, nor selection affects. This is the simplest case of a Wright–Fisher model. Actually, the generators corresponding to more general diffusion models in population genetics are of the following type

$$Au(x) = \frac{1}{2}\sum_{i,j=1}^{d} x_i(\delta_{ij} - x_j)\partial^2_{x_i x_j} u(x) + \sum_{i=1}^{d} b_i(x)\partial_{x_i} u(x), \quad x \in S_d, \tag{1.2}$$

where the coefficients b_i belong to the space $C(S_d)$ and depend on factors as mutation, selection and migration. So, the operators (1.2) are of degenerate elliptic type with the elliptic part as in (1.1).

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The operators (1.2) arising from Fleming–Viot processes have been largely studied using an analytic approach by several authors in different settings, see [1–3,6,8–10,14,23,26–29] and the references quoted therein. The interest is that the equations describing the diffusion processes are of degenerate type and hence, the classical techniques for the study of (parabolic) elliptic operators on smooth domains cannot be applied. In particular, the difficulty in studying these operators is twofold: the operators (1.2) degenerate on the boundary ∂S_d of S_d in a very natural way and the domain S_d is not smooth as its boundary presents sides and corners.

As it is shown in the Feller theory for the one-dimensional case, the behavior of the diffusion process on the boundary constitutes one of its main characteristics. So, the appropriate setting for studying the equations describing the diffusion process is the space of continuous functions on the simplex S_d.

In the one-dimensional case, the study of such type of degenerate (parabolic) elliptic problems on $C([0,1])$ started in the fifties with the papers by Feller [17,18]. The subsequent work of Clément and Timmermans [11] clarified which conditions on the coefficients of the operator mA, with A defined according to (1.2) and $0 < m \in C([0,1])$, guarantee the generation of a C_0-semigroup in $C([0,1])$. The problem of the regularity of the generated semigroup in $C([0,1])$ has been considered by several authors, [4,6,5,23]. In particular, Metafune [23] established the analyticity of the semigroup under suitable conditions on the coefficients of the operator mA. Thus, he obtained the analyticity of the semigroup generated by $x(1-x)D^2$ on $C([0,1])$, which was a problem left open for a long time. We refer to [7] for a survey on this topic.

In the d-dimensional case, the problem of generation of a C_0-semigroup in $C(S_d)$ has been studied by different authors. In more generality, the problem was solved by Ethier [14]. Actually, Ethier [14] (see also [15, p. 375]) proved the existence of a C_0-semigroup of positive contractions on $C(S_d)$ under mild conditions on the drift terms b_i. In the following, we state such a result in the case of our interest.

Theorem 1.1. *(See Ethier [14].) The closure $(\mathcal{A}_d, D(\mathcal{A}_d))$ of $(\mathcal{A}_d, C^2(S_d))$ generates a positive and contractive C_0-semigroup $(T(t))_{t\geqslant 0}$ on $C(S_d)$. Moreover, the space $C^n(S_d)$ is a core for the infinitesimal generator of $(T(t))_{t\geqslant 0}$ for every $n \geqslant 2$.*

On the other hand, Shimakura [27] (see [28, Ch. VIII, p. 221]) gave concrete representation formulas for the semigroups of diffusion processes associated to a class of Wright–Fisher models including the simplest case. In particular, Shimakura [28, Ch. VIII, p. 221] showed that the eigenvalues of $\mathcal{A}_d$ are given by

$$\lambda_n = -\frac{n(n-1)}{2}, \quad n \in \mathbb{N}, \tag{1.3}$$

and that the corresponding process is replicated on every face of S_d in the following way. Denote by $\mathcal{A}_{d,F}$ the restriction of $\mathcal{A}_d$ to a face F of S_d and by $F(V)$ the face of S_d having V as a set of vertices. If V contains $p+1$ vertices of S_d with $p < d$, then $F(V)$ can be identified with the simplex S_p and the differential operator $\mathcal{A}_{d,F}$ with the differential operator $\mathcal{A}_p$ on S_p, i.e.,

$$\mathcal{A}_{d,F(V)} u = \mathcal{A}_p(u|_{F(V)}), \quad u \in D(\mathcal{A}_d). \tag{1.4}$$

Moreover, in [28, Ch. VIII, p. 221] it was proved that the restriction of the semigroup $(T(t))_{t\geqslant 0}$ to every face $F(V)$ with $p+1$ vertices and $p < d$ satisfies

$$\big(T(t)f\big)\big|_{F(V)} = T_{F(V)}(t)(f|_{F(V)}), \quad f \in C(S_d), \tag{1.5}$$

where $(T_{F(V)}(t))_{t\geqslant 0}$ denotes the semigroup on $C(F(V))$ generated by $\mathcal{A}_{d,F(V)}$. As the process is preserved under restriction to faces, Campiti and Rasa [8] pointed out that the domain $D(\mathcal{A}_d)$ can be described recursively as follows

$$D(\mathcal{A}_d) = \Big\{ u \in C(S_d) \;\Big|\; u \in \bigcap_{q\geqslant 1} W^{2,q}_{\mathrm{loc}}(\mathring{S}_d),\ \mathcal{A}_d u \in C(S_d) \text{ and for every proper face } F \subseteq S_d\colon u|_F \in D(\mathcal{A}_{d,F}) \text{ and } \mathcal{A}_{d,F}(u|_F) = (\mathcal{A}_d u)|_F \Big\}. \tag{1.6}$$

If $V_d = \{v_0, \ldots, v_d\}$ denotes the set of vertices of S_d, then (1.6) implies that $\mathcal{A}_d u(v_i) = 0$ for every $u \in D(\mathcal{A}_d)$ and $i = 0, \ldots, d$.

There are few results about the regularity of the generated semigroup in $C(S_d)$, [1,3]. In the papers [1,3] it was established the differentiability and the compactness of the generated semigroups related to some classes of operators of type mA in $C(S_d)$, including the generators of diffusion processes associated to a class of Wright–Fisher models, but not the generator (1.1) corresponding to the simplest case. The main aim of this paper is to prove the analyticity of the semigroup generated by the closure of $(\mathcal{A}_d, C^2(S_d))$ on $C(S_d)$ and hence, extending the result of Metafune to several variables. The proof of the result is given by induction on the integer d. Actually, we provide a method which allows us to reduce the proof to the one-dimensional case and which gives information on this particular class of operators.

The paper is organized as follows. In Section 2 we consider the problem of the analyticity of the semigroup generated by the closure of $(\mathcal{A}_2, C^2(S_2))$ on $C(S_2)$, deepening and solving the 2-dimensional case. The end of this is to clarify in details the necessary techniques to give the inductive step. In Section 3 we prove the analyticity of the semigroup generated by the closure of $(\mathcal{A}_d, C^2(S_d))$ on $C(S_d)$ by induction. Finally, by using the method of approximate resolvents, we show the analyticity of the semigroup generated by the closure of $(m\mathcal{A}_d, C^2(S_d))$ on $C(S_d)$.

1.1. Notation

The function spaces considered in this paper consist of complex-valued functions.

Let $K \subseteq \mathbb{R}^d$ be a compact set. Denote by $C^n(K)$ the space of all n-times continuously differentiable functions u on K such that $\lim_{x\to x_0} D^\alpha u(x)$ exists and is finite for all $|\alpha| \leqslant n$ and $x_0 \in \partial K$. In particular, $C(K)$ denotes the space of all continuous functions u on K. The norm on $C(K)$ is the supremum norm and is denoted by $\| \ \|_K$. The norm $\| \ \|_{n,K}$ on $C^n(K)$ is defined by $\|u\|_{n,K} = \sum_{|\alpha|\leqslant n} \|D^\alpha u\|_K$.

For easy reading, in some cases we will adopt the notation $\|\varphi(x)u\|_K$ to still denote $\sup_{x\in K} |\varphi(x)\ u(x)|$.

A *bounded analytic semigroup of angle* θ with $0 < \theta \leqslant \pi/2$ is an analytic semigroup defined in the sector $\Sigma_\theta = \{z \in \mathbb{C} \mid |\arg z| < \theta\}$.

For other undefined notation and results on the theory of semigroups we refer to [13,22,25].

In the present paper we will use some results about injective tensor products. We refer to [20,21,30,24] for definitions and basic results in this topic and for related applications.

2. The 2-dimensional case

2.1. Auxiliary results

We first consider the one-dimensional second order differential operator

$$Au(x) = m(x)xu''(x), \quad x \in [0,b], \tag{2.1}$$

and suppose that $b > 0$ and m is a strictly positive function in $C([0,b])$. The operator A with domain $D(A)$, defined by

$$D(A) = \Big\{u \in C\big([0,1]\big) \cap C^2\big(]0,b]\big) \ \Big| \ \lim_{x\to 0^+} Au = 0,\ u'(b) = 0\Big\}, \tag{2.2}$$

generates a bounded analytic C_0-semigroup $(T(t))_{t\geqslant 0}$ of angle $\pi/2$ on $C([0,b])$ which is contractive, [23,6,7,11].

Proposition 2.1. *Let $b > 0$ and let m be a strictly positive function in $C([0,b])$. Then the operator A with domain $D(A)$ defined according to* (2.2) *satisfies the following properties*:

(1) *There exist $\varepsilon_b > 0$, $C_b > 0$ and $D_b > 0$ such that, for every $0 < \varepsilon < \varepsilon_b$ and $u \in C([0,b]) \cap C^2(]0,b])$ with $Au \in C([0,b])$, we have*

$$\big\|\sqrt{x}u'\big\|_{[0,b]} \leqslant \frac{C_b}{\varepsilon}\|u\|_{[0,b]} + D_b\varepsilon\|Au\|_{[0,b]}.$$

(2) *There exist $K_b > 0$ and $t_b > 0$ such that, for every $0 < t < t_b$, we have*

$$\big\|\sqrt{x}\big(T(t)u\big)'\big\|_{[0,b]} \leqslant \frac{K_b}{\sqrt{t}}\|u\|_{[0,b]}, \quad u \in C[0,b],$$

and such that, for every $t \geqslant t_b$, we have

$$\big\|\sqrt{x}\big(T(t)u\big)'\big\|_{[0,b]} \leqslant K_b\|u\|_{[0,b]}, \quad u \in C[0,b].$$

(3) *For each $0 < \theta < \pi$ there exists a constant $C_b > 0$ such that, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| > 1$, we have*

$$\big\|\sqrt{x}\big(R(\lambda, A)u\big)'\big\|_{[0,b]} \leqslant \frac{C_b}{\sqrt{|\lambda|}}\|u\|_{[0,b]}, \quad u \in C\big([0,b]\big).$$

Proof. Denote by $m_0 = \min_{x\in[0,b]} m(x)$. Then $m_0 > 0$.

(1) Let $u \in C([0,b]) \cap C^2(]0,b])$ with $Au \in C([0,b])$. Then we have, for every $z, h \in]0,b/2]$, that

$$u(z+h) = u(z) + hu'(z) + \int_0^h (h-s)u''(z+s)\,ds. \tag{2.3}$$

Let $0<\varepsilon<\sqrt{\frac{b}{2}}$ and $h=\varepsilon\sqrt{z}$. Then $h<\frac{b}{2}$ and hence, from (2.3) it follows

$$\sqrt{z}u'(z)=\frac{1}{\varepsilon}\big(u(z+\varepsilon\sqrt{z})-u(z)\big)-\frac{1}{\varepsilon}\int_0^{\varepsilon\sqrt{z}}\frac{\varepsilon\sqrt{z}-s}{z+s}u''(z+s)(z+s)\,ds,$$

where

$$\int_0^{\varepsilon\sqrt{z}}\frac{\varepsilon\sqrt{z}-s}{z+s}\,ds\leqslant\frac{1}{z}\int_0^{\varepsilon\sqrt{z}}(\varepsilon\sqrt{z}-s)\,ds=\frac{\varepsilon^2}{2}.$$

Therefore

$$\big\|\sqrt{z}u'\big\|_{[0,b/2]}\leqslant\frac{2}{\varepsilon}\|u\|_{[0,b]}+\frac{\varepsilon}{2}\big\|zu''\big\|_{[0,b]}\leqslant\frac{2}{\varepsilon}\|u\|_{[0,b]}+\frac{\varepsilon}{2m_0}\|Au\|_{[0,b]}.$$

On the other hand, if $z\in[b/2,b]$ and $\varepsilon\in]0,b/4]$ (and hence, $z-\varepsilon\in[b/4,b[$), there exists $\xi\in[b/4,b]$ such that

$$u(z-\varepsilon)=u(z)-\varepsilon u'(z)+\frac{\varepsilon^2}{2}u''(\xi)$$

and hence,

$$u'(z)=\frac{1}{\varepsilon}\big(u(z)-u(z-\varepsilon)\big)+\frac{\varepsilon}{2}u''(\xi).$$

It follows that

$$\begin{aligned}\big|\sqrt{z}u'(z)\big|&\leqslant\frac{2\sqrt{b}}{\varepsilon}\|u\|_{[0,b]}+\frac{\sqrt{b}}{2}\varepsilon\frac{4}{b}\big|\xi u''(\xi)\big|\leqslant\frac{2\sqrt{b}}{\varepsilon}\|u\|_{[0,b]}+\frac{2\varepsilon}{\sqrt{b}}\big\|zu''\big\|_{[b/4,b]}\\&\leqslant\frac{2\sqrt{b}}{\varepsilon}\|u\|_{[0,b]}+\frac{2\varepsilon}{\sqrt{b}m_0}\|Au\|_{[b/4,b]}.\end{aligned}$$

So,

$$\big\|\sqrt{z}u'\big\|_{[b/2,b]}\leqslant\frac{2\sqrt{b}}{\varepsilon}\|u\|_{[0,b]}+\frac{2\varepsilon}{\sqrt{b}m_0}\|Au\|_{C[0,b]}.$$

We then obtain, for every $0<\varepsilon<\varepsilon_b:=\min\{\sqrt{\frac{b}{2}},\frac{b}{4},1\}$, that

$$\big\|\sqrt{z}u'\big\|_{[0,b]}\leqslant\frac{2+2\sqrt{b}}{\varepsilon}\|u\|_{[0,b]}+\frac{\varepsilon}{m_0}\left(\frac{1}{2}+\frac{2}{\sqrt{b}}\right)\|Au\|_{[0,b]}.$$

(2) Let $u\in C([0,b])$. Since $(T(t))_{t\geqslant 0}$ is a bounded analytic C_0-semigroup in $C([0,b])$, we have $T(t)u\in D(A)$ and there exists $M>0$ such that $t\|AT(t)\|\leqslant M$ for every $t>0$. Applying the property (1) above, we then obtain

$$\big\|\sqrt{x}\big(T(t)u\big)'\big\|_{[0,b]}\leqslant\frac{C_b}{\varepsilon}\big\|T(t)u\big\|_{[0,b]}+D_b\varepsilon\big\|AT(t)u\big\|_{[0,b]}\leqslant\frac{C_b}{\varepsilon}\|u\|_{[0,b]}+D_b\varepsilon\frac{M}{t}\|u\|_{[0,b]}.$$

Set $t_b:=\varepsilon_b^2$. Then there exists $K_b=\max\{C_b+MD_b,\sqrt{t_b}(C_b+MD_b)\}>0$ such that we obtain, for every $0<t<t_b$ and taking $\varepsilon=\sqrt{t}$, that

$$\big\|\sqrt{x}\big(T(t)u\big)'\big\|_{[0,b]}\leqslant\frac{K_b}{\sqrt{t}}\|u\|_{[0,b]},$$

and such that, for every $t\geqslant t_b$,

$$\big\|\sqrt{x}\big(T(t)u\big)'\big\|_{[0,b]}\leqslant K_b\|u\|_{[0,b]}.$$

(3) By property (2) above the operator $\sqrt{x}DT(t)$ is bounded on $C([0,b])$ with norm less or equal to $K_b/\sqrt{t}$ if $0<t<t_b$ and to K_b if $t\geqslant t_b$. It follows, for every $\eta>1$, $u\in C([0,b])$ and $x\in]0,b]$, that

$$\sqrt{x}D\left(\int_0^{+\infty}e^{-\eta t}T(t)u\,dt\right)=\int_0^{\infty}e^{-\eta t}\sqrt{x}\big(T(t)u\big)'\,dt$$

and hence,

$$\left\|\sqrt{x}\big(R(\eta, A)u\big)'\right\|_{[0,b]} \leqslant K_b \|u\|_{[0,b]} \left(\int_0^{t_b} t^{-1/2} e^{-\eta t}\, dt + \int_{t_b}^{+\infty} e^{-\eta t}\, dt \right) = K_b \left(\frac{C_1}{\sqrt{\eta}} + \frac{C_2}{\eta} \right) \|u\|_{[0,b]} \leqslant \frac{C_b}{\sqrt{\eta}} \|u\|_{[0,b]}.$$

Consequently, if $v \in D(A)$ and $\eta > 1$, then

$$\left\|\sqrt{x} v'\right\|_{[0,b]} \leqslant \frac{C_b}{\sqrt{\eta}} \|\eta v - Av\|_{[0,b]} \leqslant C_b \left(\sqrt{\eta} \|v\|_{[0,b]} + \frac{1}{\sqrt{\eta}} \|Av\|_{[0,b]} \right).$$

Let $0 < \theta < \pi$ be a fixed angle. If $v = R(\mu, A)u$ for some $\mu \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\mu| > 1$ and $u \in C([0, b])$, then by the sectoriality of A it follows

$$\begin{aligned} \left\|\sqrt{x}\big(R(\mu, A)u\big)'\right\|_{[0,b]} &\leqslant C_b \left(\sqrt{\eta} \left\|R(\mu, A)u\right\|_{[0,b]} + \frac{1}{\sqrt{\eta}} \left\|AR(\mu, A)u\right\|_{[0,b]} \right) \\ &= C_b \left(\sqrt{\eta} \left\|R(\mu, A)u\right\|_{[0,b]} + \frac{1}{\sqrt{\eta}} \left\|\mu R(\mu, A)u - u\right\|_{[0,b]} \right) \\ &\leqslant C_b M \left(\frac{\sqrt{\eta}}{|\mu|} \|u\|_{[0,b]} + \frac{1}{\sqrt{\eta}} \|u\|_{[0,b]} \right), \end{aligned}$$

where the constant M depends only on θ. By taking $\eta = |\mu|$, we get the assertion. □

Remark 2.2. The inclusion $(D(A), \|\ \|_A) \hookrightarrow C([0, b])$ is compact and hence, $(A, D(A))$ has compact resolvent (here, $\|\ \|_A$ denotes the graph norm). Indeed, if $u \in D(A)$, then via Proposition 2.1(1) we obtain, for every $0 < x, y \leqslant b$, that

$$\left|u(x) - u(y)\right| = \left| \int_x^y u'(t)\, dt \right| = \left| \int_x^y \sqrt{t} \frac{1}{\sqrt{t}} u'(t)\, dt \right| \leqslant \left| \int_x^y \frac{1}{\sqrt{t}}\, dt \right| \left\|\sqrt{s} u'\right\|_{[0,b]} \leqslant C|\sqrt{x} - \sqrt{y}| \|u\|_A \tag{2.4}$$

for some constant $C > 0$. Next, let $(u_n)_n \subseteq D(A)$ with $\sup_{n \in \mathbb{N}} \|u_n\|_A = K < \infty$. Then (2.4) implies that the sequence $(u_n)_n$ is equicontinuous in $C([0, b])$. Since $(u_n)_n$ is also equibounded in $C([0, b])$, we can apply Ascoli–Arzelà theorem to conclude that $(u_n)_n$ contains a subsequence $(u_{n'})_{n'}$ converging to some u in $C([0, b])$. This proves the claim.

Since $(A, D(A))$ generates an analytic C_0-semigroup $(T(t))_{t \geqslant 0}$ on $C([0, b])$ (and hence, a norm-continuous C_0-semigroup) and has compact resolvent, $(T(t))_{t \geqslant 0}$ is also compact.

We now consider the one-dimensional second order differential operator

$$\mathcal{A}_1 u(x) = m(x) x(1 - x) u''(x), \quad x \in [0, 1], \tag{2.5}$$

with domain $D(\mathcal{A}_1)$ defined by

$$D(\mathcal{A}_1) = \left\{ u \in C^2\big(]0, 1[\big) \cap C\big([0, 1]\big) \;\Big|\; \lim_{x \to 0,1} \mathcal{A}_1 u(x) = 0 \right\}. \tag{2.6}$$

The operator $(\mathcal{A}_1, D(\mathcal{A}_1))$ generates a bounded analytic C_0-semigroup $(T(t))_{t \geqslant 0}$ of angle $\pi/2$ on $C([0, 1])$ which is positive and contractive, [6,23]. Using Proposition 2.1, we can show that the operator $(\mathcal{A}_1, D(\mathcal{A}_1))$ also satisfies the following properties.

Corollary 2.3. *Let m be a strictly positive function in $C([0, 1])$. Let $(\mathcal{A}_1, D(\mathcal{A}_1))$ be the differential operator on $[0, 1]$ defined according to (2.5). Then the differential operator $(\mathcal{A}_1, D(\mathcal{A}_1))$ satisfies the following properties*:

(1) *There exist $\bar{\varepsilon} > 0$, $C > 0$ and $D > 0$ such that, for every $0 < \varepsilon < \bar{\varepsilon}$ and $u \in C([0, 1]) \cap C^2(]0, 1[)$ with $\mathcal{A}_1 u \in C([0, 1])$, we have*

$$\left\|\sqrt{x(1 - x)} u'\right\|_{[0,1]} \leqslant \frac{C}{\varepsilon} \|u\|_{[0,1]} + D\varepsilon \|\mathcal{A}_1 u\|_{[0,1]}.$$

(2) *There exist $K > 0$ and $\bar{t} > 0$ such that, for every $0 < t < \bar{t}$, we have*

$$\left\|\sqrt{x(1 - x)}\big(T(t)u\big)'\right\|_{[0,1]} \leqslant \frac{K}{\sqrt{t}} \|u\|_{[0,1]}, \quad u \in C\big([0, 1]\big),$$

and such that, for every $t \geqslant \bar{t}$, we have

$$\left\|\sqrt{x(1 - x)}\big(T(t)u\big)'\right\|_{[0,1]} \leqslant K \|u\|_{[0,1]}, \quad u \in C\big([0, 1]\big).$$

(3) *For each* $0<\theta<\pi$ *there exists a constant* $C>0$ *such that, for every* $\lambda\in\{z\in\mathbb{C}\mid |\arg z|<\theta\}$ *with* $|\lambda|>1$, *we have*

$$\bigl\|\sqrt{x(1-x)}\bigl(R(\lambda,\mathcal{A}_1)u\bigr)'\bigr\|_{[0,1]}\leqslant\frac{C}{\sqrt{|\lambda|}}\|u\|_{[0,1]},\quad u\in C\bigl([0,1]\bigr).$$

Proof. (1) Let $b=\frac{1}{2}$. Since $m(x)(1-x)$ is a strictly positive function in $C([0,b])$, the differential operator $\mathcal{A}_1|_{[0,b]}$ is of the same type of (2.1) and hence, we can apply Proposition 2.1(1) to conclude that there exist $\varepsilon_1>0$, $C_1>0$ and $D_1>0$ such that, for every $0<\varepsilon<\varepsilon_1$ and $u\in C([0,b])\cap C^2(]0,b])$ with $\mathcal{A}_1u\in C([0,b])$, we have

$$\bigl\|\sqrt{x}u'\bigr\|_{[0,b]}\leqslant\frac{C_1}{\varepsilon}\|u\|_{[0,b]}+D_1\varepsilon\|\mathcal{A}_1u\|_{[0,b]}. \tag{2.7}$$

Next, let A be the differential operator on $[0,b]$ defined by $Av(x)=m(1-x)x(1-x)v''(x)$ for $x\in[0,b]$, and let $\Phi:C([b,1])\to C([0,b])$ be the surjective isometry defined by $\Phi(u)(x):=u(1-x)$ for $u\in C([b,1])$. Then the differential operator A is of the same type of (2.1). In particular, we have

$$(A\circ\Phi)(u)(x)=m(1-x)x(1-x)u''(1-x),\quad x\in[0,b],\ u\in C\bigl([0,b]\bigr)\cap C^2\bigl(]0,b]\bigr),$$

and hence,

$$\bigl(\Phi^{-1}\circ A\circ\Phi\bigr)(u)(x)=m(x)(1-x)xu''(x),\quad x\in[b,1],\ u\in C\bigl([b,1]\bigr)\cap C^2\bigl([b,1[\bigr).$$

Thus, we can apply again Proposition 2.1(1) to conclude that there exist $\varepsilon_2>0$, $C_2>0$ and $D_2>0$ such that, for every $0<\varepsilon<\varepsilon_2$ and $u\in C([b,1])\cap C^2([b,1[)$ with $\mathcal{A}_1u\in C([b,1])$, we have $v=\Phi(u)\in C([0,b])\cap C^2(]0,b])$ with $Av\in C([0,b])$ and

$$\bigl\|\sqrt{1-x}u'\bigr\|_{[b,1]}=\bigl\|\sqrt{x}v'\bigr\|_{[0,b]}\leqslant\frac{C_2}{\varepsilon}\|v\|_{[0,b]}+D_2\varepsilon\|Av\|_{[0,b]}=\frac{C_2}{\varepsilon}\|u\|_{[b,1]}+D_2\varepsilon\|\mathcal{A}_1u\|_{[b,1]}. \tag{2.8}$$

Combining (2.7) and (2.8) and setting $\bar\varepsilon=\min\{\varepsilon_1,\varepsilon_2\}$, we obtain, for every $0<\varepsilon<\bar\varepsilon$ and $u\in C([0,1])\cap C^2(]0,1[)$ with $\mathcal{A}_1u\in C([0,1])$, that

$$\begin{aligned}\bigl\|\sqrt{x(1-x)}u'\bigr\|_{[0,1]}&\leqslant\bigl\|\sqrt{x}u'\bigr\|_{[0,b]}+\bigl\|\sqrt{1-x}u'\bigr\|_{[b,1]}\\&\leqslant\frac{C_1}{\varepsilon}\|u\|_{[0,b]}+D_1\varepsilon\|\mathcal{A}_1u\|_{[0,b]}+\frac{C_2}{\varepsilon}\|u\|_{[b,1]}+D_2\varepsilon\|\mathcal{A}_1u\|_{[b,1]}\\&\leqslant\frac{C_1+C_2}{\varepsilon}\|u\|_{[0,1]}+(D_1+D_2)\varepsilon\|\mathcal{A}_1u\|_{[0,1]}.\end{aligned}$$

Then, the proof of property (1) is complete.

Properties (2) and (3) follow as in the proof of Proposition 2.1. □

2.2. Consequences for a class of two-dimensional elliptic differential operators

Using the previous results and some basic properties of injective tensor products in the setting of Banach spaces, [20,21,30,24], in this subsection we are able to provide resolvent estimates for the two-dimensional second order differential operators of the following type

$$A_2u(x,y)=m_1(x)x(1-x)\partial_x^2u(x,y)+m_2(y)y\partial_y^2u(x,y),\quad (x,y)\in[0,1]\times[0,b], \tag{2.9}$$

with $b>0$, m_1 and m_2 strictly positive functions in $C([0,1])$ and in $C([0,b])$, respectively. To this end, we proceed as follows.

We consider the one-dimensional differential operators

$$B_1u(x)=m_1(x)x(1-x)u''(x),\quad x\in[0,1],\quad\text{and}\quad B_2v(y)=m_2(y)yv''(y),\quad y\in[0,b],$$

with domains $D(B_1)$ and $D(B_2)$, where $D(B_1)$ is defined according to (2.6) and $D(B_2)$ is defined according to (2.2), respectively. The operators $(B_1,D(B_1))$ and $(B_2,D(B_2))$ generate bounded analytic C_0-semigroups of angle $\pi/2$ on $C([0,1])$ and on $C([0,b])$ respectively, which are both contractive. Denote such semigroups respectively by $(S_1(t))_{t\geqslant0}$ and $(S_2(t))_{t\geqslant0}$. Then the injective tensor product $(T(t))_{t\geqslant0}=(S_1(t)\mathbin{\widehat{\otimes}_\varepsilon}S_2(t))_{t\geqslant0}$ is also a bounded analytic C_0-semigroup of angle $\pi/2$ on $C([0,1]\times[0,b])=C([0,1])\mathbin{\widehat{\otimes}_\varepsilon}C([0,b])$, which is contractive, [24]. Moreover, the infinitesimal generator of $(T(t))_{t\geqslant0}$ is the closure of the operator

$$\bigl((B_1\otimes I_y)+(I_y\otimes B_2),D(B_1)\otimes D(B_2)\bigr),$$

where I_x and I_y denote the identity map on $C([0,1])$ and on $C([0,b])$ with respect to the variables x and y respectively, and admits the space $D(B_1)\otimes D(B_2)$ as a core. Observing that

$$A_2u = (B_1 \otimes I_y)u + (I_y \otimes B_2)u, \quad u \in D(B_1) \otimes D(B_2),$$

we can denote such a closure by $(A_2, D(A_2))$. Since $D(B_1) \otimes D(B_2)$ is a core for $(A_2, D(A_2))$, we have $C^2([0,1] \times [0,b]) \subseteq D(A_2) \subseteq C([0,1] \times [0,b]) \cap C^2(]0,1[\times]0,b])$, [30, Ch. 44].

Since the semigroups $(S_1(t))_{t\geqslant 0}$ and $(S_2(t))_{t\geqslant 0}$ are also compact, see [6,23] and Remark 2.2, their injective tensor product $(T(t))_{t\geqslant 0}$ shares too the compactness property, [21, §44, p. 285]. Hence, its generator $(A_2, D(A_2))$ has compact resolvent or equivalently, the canonical injection $(D(A_2), \| \ \|_{A_2}) \hookrightarrow C([0,1] \times [0,b])$ is compact, where $\| \ \|_{A_2}$ denotes the graph norm.

Next, setting $T_{2,b} = [0,1] \times [0,b]$ and using Proposition 2.1 and Corollary 2.3, we obtain:

Proposition 2.4. *Let $b > 0$ and let m_1 and m_2 be two strictly positive functions in $C([0,1])$ and in $C([0,b])$ respectively. Then the operator $(A_2, D(A_2))$ defined according to (2.9) satisfies the following properties:*

(1) *There exist $H > 0$ and $\underline{t} > 0$ such that, for every $0 < t < \underline{t}$ and $u \in C(T_{2,b})$, we have*

$$\left\| \sqrt{x(1-x)}\partial_x\big(T(t)u\big)\right\|_{T_{2,b}} \leqslant \frac{H}{\sqrt{t}}\|u\|_{T_{2,b}}, \qquad \left\| \sqrt{y}\partial_y\big(T(t)u\big)\right\|_{T_{2,b}} \leqslant \frac{H}{\sqrt{t}}\|u\|_{T_{2,b}},$$

and such that, for every $t \geqslant \underline{t}$ and $u \in C(T_{2,b})$, we have

$$\left\| \sqrt{x(1-x)}\partial_x\big(T(t)u\big)\right\|_{T_{2,b}} \leqslant H\|u\|_{T_{2,b}}, \qquad \left\| \sqrt{y}\partial_y\big(T(t)u\big)\right\|_{T_{2,b}} \leqslant H\|u\|_{T_{2,b}}.$$

(2) *For each $0 < \theta < \pi$ there exists a constant $C > 0$ such that, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| > 1$ and for every $u \in C(T_{2,b})$, we have*

$$\left\| \sqrt{x(1-x)}\partial_x\big(R(\lambda, A_2)u\big)\right\|_{T_{2,b}} \leqslant \frac{C}{\sqrt{|\lambda|}}\|u\|_{T_{2,b}}, \qquad \left\| \sqrt{y}\partial_y\big(R(\lambda, A_2)u\big)\right\|_{T_{2,b}} \leqslant \frac{C}{\sqrt{|\lambda|}}\|u\|_{T_{2,b}}.$$

Proof. (1) By Proposition 2.1(2) and Corollary 2.3 the operators $\sqrt{x(1-x)}\partial_x S_1(t)$ and $\sqrt{y}\partial_y S_2(t)$ are bounded on $C([0,1])$ and on $C([0,b])$ respectively, with norm less or equal to $\max\{K, K_b\}/\sqrt{t}$ if $0 < t < \underline{t} := \min\{\bar{t}, t_b\}$ and to $\max\{K, K_b\}/\sqrt{\underline{t}}$ if $t \geqslant \underline{t}$. Then the operators $(\sqrt{x(1-x)}\partial_x S_1(t))\widehat{\otimes}_\varepsilon S_2(t)$ and $S_1(t)\widehat{\otimes}_\varepsilon(\sqrt{y}\partial_y S_2(t))$ are also bounded on $C(T_{2,b})$ with norm less or equal to $\max\{K, K_b\}/\sqrt{t}$ if $0 < t \leqslant \underline{t}$ or to $\max\{K, K_b\}/\sqrt{\underline{t}}$ if $t \geqslant \underline{t}$, [20]. So, the thesis follows, after having observed that, for every $u \in C(T_{2,b})$, we have

$$\sqrt{x(1-x)}\partial_x\big(T(t)u\big) = \big(\big(\sqrt{x(1-x)}\partial_x S_1(t)\big)\widehat{\otimes}_\varepsilon S_2(t)\big)(u), \qquad \sqrt{y}\partial_y\big(T(t)u\big) = \big(S_1(t)\widehat{\otimes}_\varepsilon\big(\sqrt{y}\partial_y S_2(t)\big)\big)(u).$$

Property (2) follows analogously to the one-dimensional case, i.e., it suffices to repeat the argument already used in the proof of Proposition 2.1(3). □

We now consider the more general case

$$m(y)A_2u(x,y) = m(y)\big[m_1(x)x(1-x)\partial_x^2 u(x,y) + m_2(y)y\partial_y^2 u(x,y)\big], \quad (x,y) \in T_{2,b},$$

and observe that:

Proposition 2.5. *Let m be a strictly positive function in $C([0,b])$. Then the operator $(m(y)A_2, D(A_2))$ generates a contractive C_0-semigroup on $C(T_{2,b})$ and has compact resolvent. In particular, $D(B_1) \otimes D(B_2)$ is a core for $(m(y)A_2, D(A_2))$.*

Proof. Since $(A_2, D(A_2))$ generates a contractive C_0-semigroup on $C(T_{2,b})$ and m is a strictly positive function in $C([0,b])$, we can apply a result of Dorroh [12, Theorem] to conclude that $(m(y)A_2, D(A_2))$ also generates a contractive C_0-semigroup on $C(T_{2,b})$. Hence, the fact that $(m(y)A_2, D(A_2))$ has compact resolvent follows easily, after having observed that the norms $\| \ \|_{A_2}$ and $\| \ \|_{mA_2}$ are equivalent. □

Thanks to Propositions 2.4 and 2.5 we can use the method of approximate resolvents to prove the following result.

Proposition 2.6. *Let m be a strictly positive function in $C([0,b])$. Then the operator $(m(y)A_2, D(A_2))$ generates an analytic C_0-semigroup of angle $\pi/2$ on $C(T_{2,b})$. The semigroup is compact.*

Proof. For the sake of simplicity, we suppose $b = 1$ and set $m_0 =: \min_{y\in[0,1]} m(y)$. Moreover, we denote by Q the square $T_{2,1}$.

For each $n \in \mathbb{N}$ let $I_n^i := [\frac{i-1}{n}, \frac{i+1}{n}]$, $i = 1,\dots,n-1$. Then we choose $\phi_n^i \in C^\infty(\mathbb{R})$ for all $i = 1,\dots,n-1$, such that $\operatorname{supp}(\phi_n^i) \subseteq I_n^i$ and $\sum_{i=1}^{n-1}(\phi_n^i)^2 = 1$. We observe that, if $v_i \in C(Q)$, for $i = 1,\dots,n-1$, and $y \in [0,1]$, then there exists

$j \in \{1, \dots, n-1\}$ such that $y \in I_n^j$ and hence

$$\sum_{i=1}^{n-1} \phi_n^i(y) v_i(x,y) = \phi_n^{j-1}(y) v_{j-1}(x,y) + \phi_n^j(y) v_j(x,y) + \phi_n^{j+1}(y) v_{j+1}(x,y).$$

Therefore, we have

$$\left\| \sum_{i=1}^{n-1} \phi_n^i v_i \right\|_Q \leqslant 3 \sup_{i=1,\dots,n-1} \left\| \phi_n^i v_i \right\|_Q. \tag{2.10}$$

Since the operator $(A_2, D(A_2))$ generates a bounded analytic C_0-semigroup of angle $\pi/2$ on $C(Q)$, for each $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, $n \in \mathbb{N}$ and $i = 1, \dots, n-1$, we can define

$$R_{in}(\lambda) = \left(\lambda - m\left(\frac{i-1}{n}\right) A_2 \right)^{-1},$$

and hence, for a fixed angle $0 < \theta < \pi$, there exists $M > 0$ such that, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$, $n \in \mathbb{N}$ and $i = 1, \dots, n-1$, we have

$$\left\| R_{in}(\lambda) \right\| = \left[m\left(\frac{i-1}{n}\right) \right]^{-1} \left\| R\left(\frac{\lambda}{m(\frac{i-1}{n})}, A_2 \right) \right\| \leqslant \frac{M}{|\lambda|}. \tag{2.11}$$

If we set $\mu = \lambda[m(\frac{i-1}{n})]^{-1}$, then we also have

$$\begin{aligned} A_2 R_{in}(\lambda) &= \left[m\left(\frac{i-1}{n}\right) \right]^{-1} A_2 R(\mu, A_2) \\ &= \left[m\left(\frac{i-1}{n}\right) \right]^{-1} \big((A_2 - \mu) R(\mu, A_2) + \mu R(\mu, A_2) \big) \\ &= \left[m\left(\frac{i-1}{n}\right) \right]^{-1} \big(-I + \lambda R_{in}(\lambda) \big) \end{aligned}$$

and hence,

$$\left\| A_2 R_{in}(\lambda) \right\| \leqslant \left[m\left(\frac{i-1}{n}\right) \right]^{-1} (1+M) \leqslant \frac{1+M}{m_0}. \tag{2.12}$$

We now consider the approximate resolvents of the operator mA_2 defined by

$$S_n(\lambda) u = \sum_{i=1}^{n-1} \phi_n^i \cdot R_{in}(\lambda) \big(\phi_n^i u \big), \quad u \in C(Q).$$

Combining (2.11) with (2.10), we obtain, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ and $n \in \mathbb{N}$, that

$$\left\| S_n(\lambda) \right\| \leqslant \frac{3M}{|\lambda|}. \tag{2.13}$$

Since we have, for every $\phi, \eta \in D(A_2)$, that

$$A_2(\phi\eta) = \eta A_2(\phi) + \phi A_2(\eta) + 2\big[m_1(x) x(1-x) \partial_x \phi \partial_x \eta + m_2(y) y \partial_y \phi \partial_y \eta \big],$$

the operators $S_n(\lambda)$ satisfy, for every $u \in C(Q)$,

$$\begin{aligned} (\lambda - mA_2) S_n(\lambda) u &= (\lambda - mA_2) \sum_{i=1}^{n-1} \phi_n^i \cdot R_{in}(\lambda) \big(\phi_n^i u \big) \\ &= \sum_{i=1}^{n-1} \phi_n^i \cdot (\lambda - mA_2) R_{in}(\lambda) \big(\phi_n^i u \big) - \sum_{i=1}^{n-1} mA_2 \big(\phi_n^i \big) \cdot R_{in}(\lambda) \big(\phi_n^i u \big) \\ &\quad - 2m \sum_{i=1}^{n-1} \big[x(1-x) m_1(x) \partial_x \phi_n^i \partial_x \big(R_{in}(\lambda) \big(\phi_n^i u \big) \big) + y m_2(y) \partial_y \phi_n^i \partial_y \big(R_{in}(\lambda) \big(\phi_n^i u \big) \big) \big] \end{aligned}$$

$$= u + \sum_{i=1}^{n-1} \phi_n^i \cdot \left(m\left(\frac{i-1}{n}\right) - m\right) A_2\big(R_{in}(\lambda)\big(\phi_n^i u\big)\big)$$
$$- \sum_{i=1}^{n-1} m A_2\big(\phi_n^i\big) \cdot R_{in}(\lambda)\big(\phi_n^i u\big) - 2m \sum_{i=1}^{n-1} y m_2(y) \partial_y \phi_n^i \partial_y \big(R_{in}(\lambda)\big(\phi_n^i u\big)\big)$$
$$=: \big(I + C_1(\lambda) + C_2(\lambda) + C_3(\lambda)\big)u.$$

We now fix $\bar n \in \mathbb{N}$ such that $\sup_{I^i_{\bar n}} |m(y) - m(\frac{i-1}{\bar n})| \leqslant \varepsilon =: \frac{m_0}{6(1+M)}$ for $i = 1, \dots, \bar n - 1$. Then, from (2.10)–(2.12) and Proposition 2.4(2) it follows, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| > 1$ and $u \in C(Q)$, that

$$\big\|C_1(\lambda)u\big\|_Q \leqslant 3\varepsilon \sup_{i=1,\dots,\bar n-1} \big\|\phi_{\bar n}^i A_2\big(R_{i\bar n}(\lambda)\big(\phi_{\bar n}^i u\big)\big)\big\|_Q \leqslant 3\varepsilon \frac{1+M}{m_0} \|u\|_Q < \frac{1}{2}\|u\|_Q,$$

$$\big\|C_2(\lambda)u\big\|_Q = \left\| \sum_{i=1}^{\bar n-1} m A_2\big(\phi_{\bar n}^i\big) \cdot R_{i\bar n}(\lambda)\big(\phi_{\bar n}^i u\big)\right\|_Q \leqslant \sum_{i=1}^{\bar n-1} \max_{y\in[0,1]} m(y) \big\|A_2\big(\phi_{\bar n}^i\big)\big\|_Q \big\|R_{i\bar n}(\lambda)\big(\phi_{\bar n}^i u\big)\big\|_Q$$
$$\leqslant C \max_{y\in[0,1]} m(y) \sup_{i=1,\dots,\bar n-1} \big\|R_{i\bar n}(\lambda)\big(\phi_{\bar n}^i u\big)\big\|_Q \leqslant \frac{K}{|\lambda|}\|u\|_Q,$$

$$\big\|C_3(\lambda)u\big\|_Q \leqslant 6 \max_{y\in[0,1]} m(y) \sup_{i=1,\dots,\bar n-1} \big\|y m_2(y) \partial_y \phi_{\bar n}^i \partial_y\big(R_{i\bar n}(\lambda)(\phi_{\bar n^i} u)\big)\big\|_Q \leqslant H \sup_{i=1,\dots,\bar n-1} \big\|\sqrt{y}\partial_y\big(R_{i\bar n}(\lambda)(\phi_{\bar n^i} u)\big)\big\|_Q$$
$$= H \sup_{i=1,\dots,\bar n-1} \left[m\left(\frac{i-1}{\bar n}\right)\right]^{-1} \left\|\sqrt{y}\partial_y\left(R\left(\lambda m\left(\frac{i-1}{\bar n}\right)^{-1}, A_2\right)\big(\phi_{\bar n}^i u\big)\right)\right\|_Q \leqslant \frac{K'}{|\lambda|}\|u\|_Q,$$

for some positive constants K, K' independent of λ and u. Now, if $|\lambda| \geqslant R$ for some $R > 0$ large enough, then we get $\|C_1(\lambda) + C_2(\lambda) + C_3(\lambda)\| < 1$ and hence, the operator $B = (\lambda - mA_2)S_{\bar n}(\lambda)$ is invertible in $\mathcal{L}(C(Q))$. So, there exists $R(\lambda, mA_2) = S_{\bar n}(\lambda)B^{-1}$ in $\mathcal{L}(C(Q))$ and by (2.13)

$$\big\|R(\lambda, mA_2)\big\| = \big\|S(\lambda)B^{-1}\big\| \leqslant \frac{M'}{|\lambda|} \tag{2.14}$$

for some $M' > 0$ independent of λ, provided $\lambda - m(y)A_2$ is injective and, in particular, for $\lambda > 0$ as $m(y)A_2$ is dissipative by Proposition 2.5.

Observing that if $\lambda \in \rho(m(y)A_2)$ and $|\mu - \lambda| \leqslant \|R(\lambda, mA_2)\|^{-1}$ then $\mu \in \rho(m(y)A_2)$, it is not difficult to conclude via (2.14) and an argument of connectness that

$$\rho\big(m(y)A_2\big) \supseteq \big\{z \in \mathbb{C} \bigm| |\arg z| < \theta,\ |z| > R\big\}.$$

This fact together with (2.14) imply that mA_2 generates an analytic semigroup of angle $\pi/2$.

Since the semigroup is analytic, hence norm-continuous, and the differential operator $(mA_2, D(A_2))$ has compact resolvent, the semigroup is also compact. □

Moreover, an analogous result of Proposition 2.4 holds in the case of the differential operator $(mA_2, D(A_2))$. Indeed, we have:

Proposition 2.7. *Let $b > 0$ and let m be a strictly positive function in $C([0, b])$. Then the operator $(mA_2, D(A_2))$ satisfies the following properties*:

(1) *There exist $K_b > 0$ and $t_b > 0$ such that, for every $0 < t < t_b$ and $u \in C(T_{2,b})$, we have*

$$\big\|\sqrt{x(1-x)}\partial_x\big(T(t)u\big)\big\|_{T_{2,b}} \leqslant \frac{K_b}{\sqrt{t}}\|u\|_{T_{2,b}}, \qquad \big\|\sqrt{y}\partial_y\big(T(t)u\big)\big\|_{T_{2,b}} \leqslant \frac{K_b}{\sqrt{t}}\|u\|_{T_{2,b}}$$

and such that, for every $t \geqslant t_b$ and $u \in C(T_{2,b})$, we have

$$\big\|\sqrt{x(1-x)}\partial_x\big(T(t)u\big)\big\|_{T_{2,b}} \leqslant K_b\|u\|_{T_{2,b}}, \qquad \big\|\sqrt{y}\partial_y\big(T(t)u\big)\big\|_{T_{2,b}} \leqslant K_b\|u\|_{T_{2,b}}.$$

(2) *For each $0 < \theta < \pi$ there exist two constants $C_b > 0$ and $c_b > 0$ such that, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta|\}$ with $|\lambda| > c_b$ and $u \in C(T_{2,b})$, we have*

$$\big\|\sqrt{x(1-x)}\partial_x\big(R(\lambda, mA_2)u\big)\big\|_{T_{2,b}} \leqslant \frac{C_b}{\sqrt{|\lambda|}}\|u\|_{T_{2,b}}, \qquad \big\|\sqrt{y}\partial_y\big(R(\lambda, mA_2)u\big)\big\|_{T_{2,b}} \leqslant \frac{C_b}{\sqrt{|\lambda|}}\|u\|_{T_{2,b}}.$$

Proof. We first prove property (2). Fixed an angle $0<\theta<\pi$, let $\lambda\in\{z\in\mathbb{C}\mid |\arg z|<\theta|\}$ with $|\lambda|>1$ and $u\in D(mA_2)=D(A_2)$. Then there exists $v\in C(T_{2,b})$ such that $R(\lambda,A_2)v=u$ and hence, by Proposition 2.4(2) we have

$$\big\|\sqrt{x(1-x)}\partial_x u\big\|_{T_{2,b}}=\big\|\sqrt{x(1-x)}\partial_x\big(R(\lambda,A_2)v\big)\big\|_{T_{2,b}}\leqslant\frac{C}{\sqrt{|\lambda|}}\|v\|_{T_{2,b}}=\frac{C}{\sqrt{|\lambda|}}\|A_2u-\lambda u\|_{T_{2,b}}. \tag{2.15}$$

If $|\lambda|$ is large enough, by Proposition 2.6 there exists also $w\in C(T_{2,b})$ such that $R(\lambda,mA_2)w=u$ with $\|\lambda R(\lambda,mA_2)\|\leqslant M$ for some $M>0$. We then obtain

$$A_2u-\lambda u=\frac{1}{m}(mA_2u-\lambda u)+\left(\frac{1}{m}-1\right)\lambda u,$$

and hence,

$$\begin{aligned}\|A_2u-\lambda u\|_{T_{2,b}}&\leqslant\frac{1}{m_0}\|mA_2u-\lambda u\|_{T_{2,b}}+\left(\frac{1}{m_0}+1\right)|\lambda|\|u\|_{T_{2,b}}\\&=\frac{1}{m_0}\|mA_2u-\lambda u\|_{T_{2,b}}+\left(\frac{1}{m_0}+1\right)|\lambda|\big\|R(\lambda,mA_2)w\big\|_{T_{2,b}}\\&\leqslant\frac{1}{m_0}\|mA_2u-\lambda u\|_{T_{2,b}}+M\left(\frac{1}{m_0}+1\right)\|w\|_{T_{2,b}}\\&\leqslant M'\left(\frac{1}{m_0}+1\right)\|w\|_{T_{2,b}},\end{aligned} \tag{2.16}$$

with $M'=2M(\frac{1}{m_0}+1)$ (assuming that $M\geqslant 1$). Combining (2.15) with (2.16), we get

$$\big\|\sqrt{x(1-x)}\partial_x u\big\|_{T_{2,b}}\leqslant\frac{K}{\sqrt{|\lambda|}}\|mA_2u-\lambda u\|_{T_{2,b}}.$$

The other inequality follows proceeding in analogous way.

(1) Fix $0<\theta<\pi$. By property (2) above there exist C_b, $c_b>0$ such that, for every $\lambda\in\{z\in\mathbb{C}\mid|\arg z|<\theta\}$ with $|\lambda|>c_b$ and $u\in C(T_{2,b})$,

$$\big\|\sqrt{x(1-x)}\partial_x\big(R(\lambda,mA_2)u\big)\big\|_{T_{2,b}}\leqslant\frac{C_b}{\sqrt{|\lambda|}}\|u\|_{T_{2,b}}.$$

So, we obtain, for every $\lambda\in\{z\in\mathbb{C}\mid|\arg z|<\theta\}$ with $|\lambda|>c_b$ and $v\in D(A_2)$, that

$$\big\|\sqrt{x(1-x)}\partial_x v\big\|_{T_{2,b}}\leqslant\frac{C_b}{\sqrt{|\lambda|}}\|\lambda v-mA_2v\|_{T_{2,b}}\leqslant\frac{C_b}{\sqrt{|\lambda|}}|\lambda|\|v\|_{T_{2,b}}+\frac{C_b}{\sqrt{|\lambda|}}\|mA_2v\|_{T_{2,b}}. \tag{2.17}$$

Since $(mA_2,D(A_2))$ generates an analytic C_0-semigroup $(T(t))_{t\geqslant 0}$ of angle $\pi/2$ on $C(T_{2,b})$, for every $u\in C(T_{2,b})$ we have $T(t)u\in D(A_2)$ and there exist $M>0$, $w>0$ such that $t\|mA_2T(t)\|\leqslant Me^{wt}$ for $t>0$. Applying (2.17) with $v=T(t)u$, we then obtain that

$$\big\|\sqrt{x(1-x)}\partial_x\big(T(t)u\big)\big\|_{T_{2,b}}\leqslant C_b\sqrt{|\lambda|}\|u\|_{T_{2,b}}+\frac{C_b}{\sqrt{|\lambda|}}M\frac{e^{wt}}{t}\|u\|_{T_{2,b}}. \tag{2.18}$$

Set $t_b:=\frac{1}{c_b}>0$. Then there exists $K_b=\max\{C_b(1+Me^{wt_b}),\frac{C_b}{\sqrt{t_b}}(1+M)\}$ such that we get, for every $0<t<t_b$ and taking $\lambda=t^{-1}$, that

$$\big\|\sqrt{x(1-x)}\partial_x\big(T(t)u\big)\big\|_{T_{2,b}}\leqslant\frac{C_b}{\sqrt{t}}\big(1+Me^{wt_b}\big)\|u\|_{T_{2,b}}\leqslant\frac{K_b}{\sqrt{t}}\|u\|_{T_{2,b}},$$

and such that, for every $t\geqslant t_b$,

$$\big\|\sqrt{x(1-x)}\partial_x\big(T(t)u\big)\big\|_{T_{2,b}}\leqslant\frac{C_b}{\sqrt{t_b}}(1+M)e^{wt}\|u\|_{T_{2,b}}\leqslant K_be^{wt}\|u\|_{T_{2,b}}.$$

The other inequality follows proceeding in analogous way. □

We establish now the following notation: for every $0<\delta<1$ set

$$T_{2,1-\delta}=[0,1]\times[0,1-\delta],\qquad T_{1-\delta,2}=[0,1-\delta]\times[0,1]. \tag{2.19}$$

Example 2.8. The above results apply to the following second order differential operators

$$\begin{aligned} A_{2,1}u(x,y) &= \frac{1}{1-y}\frac{x(1-x)}{2}\partial_x^2 u + \frac{y(1-y)}{2}\partial_y^2 u \\ &= \frac{1}{1-y}\left(\frac{x(1-x)}{2}\partial_x^2 u + \frac{y(1-y)^2}{2}\partial_y^2 u\right), \quad (x,y) \in T_{2,1-\delta}, \end{aligned} \tag{2.20}$$

with domain $D(B_1) \otimes D(B_2)$, and

$$\begin{aligned} A_{2,2}u(x,y) &= \frac{x(1-x)}{2}\partial_x^2 u + \frac{1}{1-x}\frac{y(1-y)}{2}\partial_y^2 u \\ &= \frac{1}{1-x}\left(\frac{x(1-x)^2}{2}\partial_x^2 u + \frac{y(1-y)}{2}\partial_y^2 u\right), \quad (x,y) \in T_{1-\delta,2}, \end{aligned} \tag{2.21}$$

with domain $D(B_2) \otimes D(B_1)$, where $D(B_1)$ and $D(B_2)$ are defined by

$$D(B_1) := \Big\{u \in C\big([0,1]\big) \cap C^2\big(]0,1[\big) \;\Big|\; \lim_{x \to 0^+,1^-} x(1-x)u''(x) = 0\Big\},$$

$$D(B_2) := \Big\{u \in C\big([0,1-\delta]\big) \cap C^2\big(]0,1-\delta]\big) \;\Big|\; \lim_{y \to 0^+} yu''(y) = 0,\ u'(1-\delta) = 0\Big\}.$$

Indeed, as $\delta \leqslant 1-y \leqslant 1$ for every $y \in [0,1-\delta]$, by the previous considerations we can conclude that the closure $(A_{2,1}, D(A_{2,1}))$ of $(A_{2,1}, D(B_1) \otimes D(B_2))$ generates an analytic C_0-semigroup of angle $\pi/2$ on $C(T_{2,1-\delta})$, which is contractive and compact, and shares properties (1) and (2) in Proposition 2.7. Analogously, as $\delta \leqslant 1-x \leqslant 1$ for every $x \in [0,1-\delta]$, the closure $(A_{2,2}, D(A_{2,2}))$ of $(A_{2,2}, D(B_2) \otimes D(B_1))$ generates an analytic C_0-semigroup of angle $\pi/2$ on $C(T_{1-\delta,2})$, which is contractive and compact, and shares properties (1) and (2) in Proposition 2.7 with respect to y and x.

2.3. Analyticity of a class of degenerate evolution equations on the canonical simplex of $\mathbb{R}^2$

Let S_2 be the simplex of $\mathbb{R}^2$ defined by

$$S_2 = \big\{(x,y) \in \mathbb{R}^2 \;\big|\; x, y \geqslant 0,\ x+y \leqslant 1\big\}.$$

We are here concerned with the second order degenerate elliptic differential operator

$$\mathcal{A}_2 u(x,y) = \frac{1}{2}x(1-x)\partial_x^2 u(x,y) + \frac{1}{2}y(1-y)\partial_y^2 u(x,y) - xy\partial_{xy}^2 u(x,y), \quad (x,y) \in S_2. \tag{2.22}$$

The aim of this subsection is to show the analyticity of the semigroup $(T(t))_{t \geqslant 0}$ generated by the closure $(\mathcal{A}_2, D(\mathcal{A}_2))$ of $(\mathcal{A}_2, C^2(S_2))$ on $C(S_2)$ (see Theorem 1.1). In order to prove this, we use suitable changes of coordinates as follows.

Fix $0 < \delta < \frac{1}{2}$. Then, we set

$$\Omega_1 := \big\{(x,y) \in S_2 \;\big|\; 0 \leqslant y \leqslant 1-\delta\big\}, \qquad \Omega_2 := \big\{(x,y) \in S_2 \;\big|\; 0 \leqslant x \leqslant 1-\delta\big\}. \tag{2.23}$$

Then $S_2 = \bigcup_{i=1}^2 \Omega_i$. Next, we introduce the maps

$$\begin{aligned} &\varphi_1 : T_{2,1-\delta} \to \Omega_1, \quad (r,s) \to \varphi_1(r,s) = \big(r(1-s), s\big), \\ &\varphi_2 : T_{1-\delta,2} \to \Omega_2, \quad (r,s) \to \varphi_2(r,s) = \big(r, s(1-r)\big), \end{aligned} \tag{2.24}$$

where $T_{2,1-\delta}$ and $T_{1-\delta,2}$ are defined as in (2.19).

Lemma 2.9. *The map φ_i is bijective and a C^∞-diffeomorphism for $i = 1, 2$.*

Lemma 2.10. *Let $\Phi_1 : C(\Omega_1) \to C(T_{2,1-\delta})$ and $\Phi_2 : C(\Omega_2) \to C(T_{1-\delta,2})$ be the operators defined by*

$$\Phi_i(u) = u \circ \varphi_i, \quad u \in C(\Omega_i),\ i = 1, 2.$$

Then Φ_i is a surjective isometry for $i = 1, 2$. In particular, $\Phi_1(C^n(\Omega_1)) = C^n(T_{2,1-\delta})$ and $\Phi_2(C^n(\Omega_2)) = C^n(T_{1-\delta,2})$ for every $n \in \mathbb{N}$.

For each $i \in \{1, 2\}$ we define

$$\mathcal{A}_{2,i} := \Phi_i^{-1} \circ A_{2,i} \circ \Phi_i, \qquad D(\mathcal{A}_{2,i}) = \Phi_i^{-1}\big(D(A_{2,i})\big), \tag{2.25}$$

with $(A_{2,i}, D(A_{2,i}))$ the second order differential operators defined in Example 2.8. Then the operator $(\mathcal{A}_{2,i}, D(\mathcal{A}_{2,i}))$ generates an analytic C_0-semigroup of angle $\pi/2$ on $C(\Omega_i)$ (which is also contractive and compact) for every $i \in \{1,2\}$. We observe that, for $i=1$ and $v = \Phi_1(u)$ for some $u \in \Phi_1^{-1}(D(A_{2,1}))$, we have

$$\partial_r v = (1-s)\partial_x u, \qquad \partial_r^2 v = (1-s)^2 \partial_x^2 u,$$
$$\partial_s v = -r\partial_x u + \partial_y u, \qquad \partial_s^2 v = r^2\partial_x^2 u - 2r\partial_{xy}^2 u + \partial_y^2 u.$$

So, we obtain

$$\begin{aligned} A_{2,1}\big(\Phi_1(u)\big)(r,s) = A_{2,1}v(r,s) &= \frac{r(1-r)(1-s)}{2}\partial_x^2 u + \frac{s(1-s)}{2}\big(r^2\partial_x^2 u - 2r\partial_{xy}^2 u + \partial_y^2 u\big) \\ &= \frac{r(1-s)[1-r(1-s)]}{2}\partial_x^2 u - rs(1-s)\partial_{xy}^2 u + \frac{s(1-s)}{2}\partial_y^2 u, \quad (r,s) \in T_{2,1-\delta}, \end{aligned}$$

and hence,

$$\mathcal{A}_{2,1}(x,y) = \frac{x(1-x)}{2}\partial_x^2 u(x,y) + \frac{y(1-y)}{2}\partial_y^2 u(x,y) - xy\partial_{xy} u(x,y), \quad (x,y) \in \Omega_1,$$

i.e., $\mathcal{A}_2|_{\Omega_1} = \mathcal{A}_{2,1}$.

On the other hand, for $i=2$ and $v = \Phi_2(u)$ for some $u \in \Phi_2^{-1}(D(A_{2,2}))$, we have

$$\partial_r v = \partial_x u - s\partial_y u, \qquad \partial_r^2 v = \partial_x^2 u - 2s\partial_{xy}^2 u + s^2\partial_y^2 u,$$
$$\partial_s v = (1-r)\partial_y u, \qquad \partial_s^2 v = (1-r)^2\partial_y^2 u.$$

So, we obtain

$$\begin{aligned} A_{2,2}\big(\Phi_2(u)\big)(r,s) = A_{2,2}v(r,s) &= \frac{r(1-r)}{2}\big(\partial_x^2 u - 2s\partial_{xy}^2 u + s^2\partial_y^2 u\big) + \frac{s(1-s)(1-r)}{2}\partial_y^2 u \\ &= \frac{r(1-r)}{2}\partial_x^2 u - rs(1-r)\partial_{xy}^2 u + \frac{s(1-r)[1-s(1-r)]}{2}\partial_y^2 u, \quad (r,s) \in T_{1-\delta,2}, \end{aligned}$$

and hence,

$$\mathcal{A}_{2,2}(x,y) = \frac{x(1-x)}{2}\partial_x^2 u(x,y) + \frac{y(1-y)}{2}\partial_y^2 u(x,y) - xy\partial_{xy} u(x,y), \quad (x,y) \in \Omega_2,$$

i.e., $\mathcal{A}_2|_{\Omega_2} = \mathcal{A}_{2,2}$.

We may now prove the main theorem of this section.

Theorem 2.11. *The closure $(\mathcal{A}_2, D(\mathcal{A}_2))$ of $(\mathcal{A}_2, C^2(S_2))$ generates an analytic C_0-semigroup $(T(t))_{t\geqslant 0}$ of angle $\pi/2$ on $C(S_2)$. The semigroup is compact.*

Proof. Fix $0 < \delta < \frac{1}{2}$. Let $\{\psi_i\}_{i=1,2} \subseteq C_c^\infty(\mathbb{R}^2)$ be such that $\sum_{i=1}^2 (\psi_i)^2 = 1$ on S_2 and

$$\operatorname{supp}(\psi_1) \subseteq \big\{(x,y) \in \mathbb{R}^2 \;\big|\; y < 1-\delta\big\}, \qquad \operatorname{supp}(\psi_2) \subseteq \big\{(x,y) \in \mathbb{R}^2 \;\big|\; x < 1-\delta\big\}.$$

For the sake of simplicity, we still denote by ψ_i the restriction of ψ_i to Ω_i, for $i=1,2$.

By Proposition 2.6 the operators $\mathcal{A}_{2,i}$ generate analytic C_0-semigroups of angle $\pi/2$. So, if $0 < \theta < \pi$ is a fixed angle, we can find two positive constants C and R such that, for $|\lambda| \geqslant R$ with $|\arg\lambda| < \theta$, the resolvents $R(\lambda, \mathcal{A}_{2,i})$ exist and satisfy

$$\big\|R(\lambda, \mathcal{A}_{2,i})\big\| \leqslant \frac{C}{|\lambda|}, \quad i = 1,2. \tag{2.26}$$

So, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| \geqslant R$, we can define the operator $S(\lambda): C(S_2) \to C(S_2)$ via

$$S(\lambda)u = \sum_{i=1}^{2} \psi_i R(\lambda, \mathcal{A}_{2,i})(\psi_i u), \quad u \in C(S_2), \tag{2.27}$$

and hence,

$$\big\|S(\lambda)\big\| \leqslant \frac{3C}{|\lambda|}.$$

We observe that the previous considerations on the differential operators $\mathcal{A}_{2,i}$ ensure, for every $i = 1, 2$ and $u \in C(S_2)$, that

$$\mathcal{A}_2\big(\psi_i R(\lambda, \mathcal{A}_{2,i})(\psi_i u)\big) = \mathcal{A}_{2,i}\big(\psi_i R(\lambda, \mathcal{A}_{2,i})(\psi_i u)\big), \tag{2.28}$$

and, for every $f, g \in D(\mathcal{A}_{2,i})$, that

$$\mathcal{A}_{2,i}(fg) = (\mathcal{A}_{2,i} f)g + f(\mathcal{A}_{2,i} g) + \big[x(1-x-y)\partial_x f \partial_x g + (1-x-y)y\partial_y f \partial_y g + xy(\partial_x f - \partial_y f)(\partial_x g - \partial_y g)\big]. \tag{2.29}$$

By (2.28) and (2.29) we obtain, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| \geqslant R$ and $u \in C(S_2)$, that

$$\begin{aligned}
(\lambda - \mathcal{A}_2)S(\lambda)u &= \lambda S(\lambda)(u) - \sum_{i=1}^{2} \mathcal{A}_2\big(\psi_i R(\lambda, \mathcal{A}_{2,i})(\psi_i u)\big) \\
&= \lambda S(\lambda)(u) - \sum_{i=1}^{2} \mathcal{A}_{2,i}\big(\psi_i\big(R(\lambda, \mathcal{A}_{2,i})(\psi_i u)\big)\big) \\
&= \sum_{i=1}^{2} \psi_i(\lambda - \mathcal{A}_{2,i})R(\lambda, \mathcal{A}_{2,i})(\psi_i u) - \sum_{i=1}^{2} \mathcal{A}_{2,i}(\psi_i) R(\lambda, \mathcal{A}_{2,i})(\psi_i u) \\
&\quad - \sum_{i=1}^{2}\big[x(1-x-y)\partial_x\big(R(\lambda, \mathcal{A}_{2,i})(\psi_i u)\big)\partial_x \psi_i + y(1-x-y)\partial_y\big(R(\lambda, \mathcal{A}_{2,i})(\psi_i u)\big)\partial_y \psi_i \\
&\quad + xy\big(\partial_x\big(R(\lambda, \mathcal{A}_{2,i})(\psi_i u)\big) - \partial_y\big(R(\lambda, \mathcal{A}_{2,i})(\psi_i u)\big)\big)(\partial_x \psi_i - \partial_y \psi_i)\big] \\
&=: \big(I + B(\lambda) + C(\lambda)\big)(u).
\end{aligned}$$

Applying (2.26) we obtain, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| \geqslant R$ and $u \in C(S_2)$, that

$$\big\|B(\lambda)u\big\|_{S_2} \leqslant \frac{M}{|\lambda|}\|u\|_{S_2},$$

for some $M > 0$. In order to estimate $C(\lambda)$ we proceed as follows.

Let $f \in D(\mathcal{A}_{2,1})$ and $v = \Phi_1(f)$. Then $v \in D(A_{2,1})$ and the following holds

$$\partial_r v = (1-s)\partial_x f, \qquad \partial_s v = -r\partial_x f + \partial_y f, \tag{2.30}$$

and hence,

$$\partial_x f = \frac{\partial_r v}{1-s}, \qquad \partial_y f = \partial_s v + \frac{r}{1-s}\partial_r v, \qquad \partial_x f - \partial_y f = \frac{1-r}{1-s}\partial_r v - \partial_s v. \tag{2.31}$$

So, by Lemma 2.10 and Proposition 2.7(2) (combined with Example 2.8) we obtain, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| > R_1 := \max\{R, c_b\}$, that

$$\begin{aligned}
&\big\|x(1-x-y)\partial_x f \partial_x \psi_1 + y(1-x-y)\partial_y f \partial_y \psi_1 + xy(\partial_x f - \partial_y f)(\partial_x \psi_1 - \partial_y \psi_1)\big\|_{S_2} \\
&\quad \leqslant \big\|x(1-x-y)\partial_x f \partial_x \psi_1\big\|_{\Omega_1} + \big\|y(1-x-y)\partial_y f \partial_y \psi_1\big\|_{\Omega_1} + \big\|xy(\partial_x f - \partial_y f)(\partial_x \psi_1 - \partial_y \psi_1)\big\|_{\Omega_1} \\
&\quad \leqslant C_1\left(\left\|r(1-s)(1-r)(1-s)\frac{v_r}{1-s}\right\|_{T_{2,1-\delta}} + \left\|(1-r)(1-s)s\left(v_s + \frac{r}{1-s}v_r\right)\right\|_{T_{2,1-\delta}}\right. \\
&\qquad \left. + \left\|r(1-s)s\left(v_r\frac{1-r}{1-s} - v_s\right)\right\|_{T_{2,1-\delta}}\right) \\
&\quad \leqslant \frac{C_1'}{\sqrt{|\lambda|}}\|\lambda v - A_{2,1}v\|_{T_{2,1-\delta}} = \frac{C_1'}{\sqrt{|\lambda|}}\|\lambda f - \mathcal{A}_{2,1} f\|_{\Omega_1}.
\end{aligned}$$

If $f = R(\lambda, \mathcal{A}_{2,1})(\psi_1 u)$ for some $u \in C(S_2)$, then it follows, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| > R_1$, that

$$\begin{aligned}
&\big\|x(1-x-y)\partial_x\big(R(\lambda, \mathcal{A}_{2,1})(\psi_1 u)\big)\partial_x \psi_1 + y(1-x-y)\partial_y\big(R(\lambda, \mathcal{A}_{2,1})(\psi_1 u)\big)\partial_y \psi_1 \\
&\qquad + xy\big(\partial_x\big(R(\lambda, \mathcal{A}_{2,1})(\psi_1 u)\big) - \partial_y\big(R(\lambda, \mathcal{A}_{2,1})(\psi_1 u)\big)\big)(\partial_x \psi_1 - \partial_y \psi_1)\big\|_{S_2} \\
&\quad \leqslant \frac{C_1'}{\sqrt{|\lambda|}}\|u\|_{\Omega_1} \leqslant \frac{C_1'}{\sqrt{|\lambda|}}\|u\|_{S_2},
\end{aligned} \tag{2.32}$$

with C_1' a positive constant independent of λ and f.

By the symmetry of the change of variables, one analogously shows that, there exists $C_2 > 0$ such that, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| > R_1$ and $u \in C(S_2)$, we have

$$\begin{aligned}&\big\|x(1-x-y)\partial_x\big(R(\lambda,\mathcal{A}_{2,2})(\psi_2 u)\big)\partial_x\psi_2 + (1-x-y)y\partial_y\big(R(\lambda,\mathcal{A}_{2,2})(\psi_2 u)\big)\partial_y\psi_2\\ &\quad + xy\big(\partial_x\big(R(\lambda,\mathcal{A}_{2,2})(\psi_2 u)\big) - \partial_y\big(R(\lambda,\mathcal{A}_{2,1})(\psi_2 u)\big)\big)(\partial_x\psi_2 - \partial_y\psi_2)\big\|_{S_2}\\ &\leqslant \frac{C_2}{\sqrt{|\lambda|}}\|u\|_{S_2}.\end{aligned} \tag{2.33}$$

Combining (2.33) and (2.32), we obtain that there exists $K > 0$ such that, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| > R_1$ and $u \in C(S_2)$,

$$\big\|C(\lambda)u\big\|_{S_2} \leqslant \frac{K}{\sqrt{|\lambda|}}\|u\|_{S_2}.$$

If $|\lambda| \gg 1$, then the operator $B = (\lambda - \mathcal{A}_2)S(\lambda)$ is invertible in $\mathcal{L}(C(S_2))$. So, there exists $R(\lambda,\mathcal{A}_2) = S(\lambda)B^{-1}$ and

$$\big\|R(\lambda,\mathcal{A}_2)\big\| = \big\|S(\lambda)B^{-1}\big\| \leqslant \frac{C'}{|\lambda|},$$

with C' a positive constant independent of λ, provided $\lambda - \mathcal{A}_2$ is injective and, in particular, for $\lambda > 0$ as $\mathcal{A}_2$ is dissipative. To conclude that the semigroup is analytic of angle $\pi/2$ it now suffices to repeat the argument already used in the proof of Proposition 2.6.

Since $R(\lambda,\mathcal{A}_2) = S(\lambda)B^{-1}$ for some $\lambda > 0$ and the operator $S(\lambda)$ is compact by Proposition 2.6, the differential operator $(\mathcal{A}_2, D(\mathcal{A}_2))$ has compact resolvent. Thus, the semigroup is also compact, being analytic and hence, norm continuous. □

Recalling that the eigenvalues of the operator $\mathcal{A}_2$ are given by $\lambda_n = -\frac{n(n-1)}{2}$, $n \geqslant 1$, [28, Ch. VIII, p. 221], and using Theorem 2.11 together with [23, Proposition 5.6] we obtain the following result.

Theorem 2.12. *The semigroup generated by $(\mathcal{A}_2, D(\mathcal{A}_2))$ is bounded analytic of angle $\pi/2$.*

Moreover, the differential operator $(\mathcal{A}_2, D(\mathcal{A}_2))$ satisfies:

Proposition 2.13. *The closure $(\mathcal{A}_2, D(\mathcal{A}_2))$ of the differential operator $(\mathcal{A}_2, C^2(S_2))$ defined in* (2.22) *satisfies the following properties*:

(1) *There exist $K_b > 0$ and $t_b > 0$ such that, for every $0 < t < t_b$ and $u \in C(S_2)$, we have*

$$\big\|\sqrt{x(1-x)}\partial_x\big(T(t)u\big)\big\|_{S_2} \leqslant \frac{K_b}{\sqrt{t}}\|u\|_{S_2}, \qquad \big\|\sqrt{y(1-y)}\partial_y\big(T(t)u\big)\big\|_{S_2} \leqslant \frac{K_b}{\sqrt{t}}\|u\|_{S_2}$$

and such that, for every $t \geqslant t_b$ and $u \in C(S_2)$, we have

$$\big\|\sqrt{x(1-x)}\partial_x\big(T(t)u\big)\big\|_{S_2} \leqslant K_b\|u\|_{S_2}, \qquad \big\|\sqrt{y(1-y)}\partial_y\big(T(t)u\big)\big\|_{S_2} \leqslant K_b\|u\|_{S_2}.$$

(2) *For each $0 < \theta < \pi$ there exists two constants $C > 0$ and $l > 1$ such that, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| > l$ and $u \in C(S_2)$, we have*

$$\big\|\sqrt{x(1-x)}\partial_x\big(R(\lambda,\mathcal{A}_2)u\big)\big\|_{S_2} \leqslant \frac{C}{\sqrt{|\lambda|}}\|u\|_{S_2}, \qquad \big\|\sqrt{y(1-y)}\partial_y\big(R(\lambda,\mathcal{A}_2)u\big)\big\|_{S_2} \leqslant \frac{C}{\sqrt{|\lambda|}}\|u\|_{S_2}.$$

Proof. We first prove property (2). According to the notation in the proof of Theorem 2.11, fixed an angle $0 < \theta < \pi$ there exists $l > 1$ such that, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| > l$, we have

$$R(\lambda,\mathcal{A}_2) = S(\lambda)B^{-1},$$

where the operators $S(\lambda)$ are defined according to (2.27) and $\|B^{-1}\| \leqslant 2$. So, we obtain, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| > l$ and $u \in C(S_2)$, that

$$\begin{aligned}\big\|\sqrt{x(1-x)}\partial_x\big(R(\lambda,\mathcal{A}_2)u\big)\big\|_{S_2} &\leqslant \sum_{i=1}^{2}\big\|\sqrt{x(1-x)}\partial_x\big(\psi_i R(\lambda,\mathcal{A}_{2,i})\big(\psi_i B^{-1}u\big)\big)\big\|_{\Omega_i}\\ &\leqslant \sum_{i=1}^{2}\big(\big\|\sqrt{x(1-x)}\partial_x(\psi_i)R(\lambda,\mathcal{A}_{2,i})\big(\psi_i B^{-1}u\big)\big\|_{\Omega_i}\end{aligned}$$

$$
\begin{aligned}
&\quad + \big\| \sqrt{x(1-x)}\psi_i \partial_x\big(R(\lambda,\mathcal{A}_{2,i})\big(\psi_i B^{-1}u\big)\big)\big\|_{\Omega_i}\big) \\
&\leqslant c\sum_{i=1}^{2}\bigg(\frac{C}{|\lambda|}\big\|\psi_i B^{-1}u\big\|_{\Omega_i} + \big\|\sqrt{x(1-x)}\psi_i\partial_x\big(R(\lambda,\mathcal{A}_{2,i})\big(\psi_i B^{-1}u\big)\big)\big\|_{\Omega_i}\bigg) \\
&\leqslant c\sum_{i=1}^{2}\bigg(\frac{C}{\sqrt{|\lambda|}}\big\|B^{-1}\big\|\|u\|_{S_2} + \big\|\sqrt{x(1-x)}\partial_x\big(R(\lambda,\mathcal{A}_{2,i})\big(\psi_i B^{-1}u\big)\big)\big\|_{\Omega_i}\bigg),
\end{aligned}
\tag{2.34}
$$

where $c := \sup_{i=1}^{2}\|\psi_i\|_{1,\Omega_i}$. To estimate the second addend on the right in (2.34) we proceed as follows.

For $i=1$ set $f = R(\lambda,\mathcal{A}_{2,1})(\psi_1 B^{-1}u)$ and $v = \Phi_1(f)$. Then by (2.30) and (2.31)

$$
\begin{aligned}
\big\|\sqrt{x(1-x)}\partial_x f\big\|_{\Omega_1} &= \bigg\|\frac{\sqrt{r(1-s)[1-r(1-s)]}}{1-s}\partial_r v\bigg\|_{T_{2,1-\delta}} \\
&\leqslant \frac{C_1''}{\sqrt{|\lambda|}}\|\lambda v - A_{2,1}v\|_{T_{2,1-\delta}} = \frac{C_1}{\sqrt{|\lambda|}}\|\lambda f - \mathcal{A}_{2,1}f\|_{\Omega_1} \\
&= \frac{C_1}{\sqrt{|\lambda|}}\big\|\psi_1 B^{-1}u\big\|_{\Omega_1} \leqslant \frac{C_1}{\sqrt{|\lambda|}}\big\|B^{-1}\big\|\|u\|_{S_2}.
\end{aligned}
\tag{2.35}
$$

Next, for $i=2$ set $f = R(\lambda,\mathcal{A}_{2,2})(\psi_2 B^{-1}u)$ and $v = \Phi_2(f)$. Then $\partial_x f = \partial_r v + \frac{s}{1-r}\partial_s v$ and hence,

$$
\begin{aligned}
\big\|\sqrt{x(1-x)}\partial_x f\big\|_{\Omega_2} &= \bigg\|\sqrt{r(1-r)}\bigg(\partial_r v + \frac{s}{1-r}\partial_s v\bigg)\bigg\|_{T_{1-\delta,2}} \\
&\leqslant \big\|\sqrt{r(1-r)}\partial_r v\big\|_{T_{1-\delta,2}} + \bigg\|\sqrt{r(1-r)}\frac{s}{1-r}\partial_s v\bigg\|_{T_{1-\delta,2}} \\
&\leqslant \frac{C_1''}{\sqrt{|\lambda|}}\|\lambda v - A_{2,2}v\|_{T_{2,1-\delta}} = \frac{C_2}{\sqrt{|\lambda|}}\|\lambda f - \mathcal{A}_{2,2}f\|_{\Omega_2} \\
&= \frac{C_2}{\sqrt{|\lambda|}}\big\|\psi_1 B^{-1}u\big\|_{\Omega_1} \leqslant \frac{C_2}{\sqrt{|\lambda|}}\big\|B^{-1}\big\|\|u\|_{S_2}.
\end{aligned}
\tag{2.36}
$$

Combining (2.34)–(2.36) we obtain, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| > l$ and $u \in C(S_2)$, that

$$
\big\|\sqrt{x(1-x)}\partial_x\big(R(\lambda,\mathcal{A}_2)u\big)\big\|_{S_2} \leqslant \frac{C}{\sqrt{|\lambda|}}\|u\|_{S_2}
$$

for some constant $C > 0$ independent of u and λ.

The proof of the other case is analogue.

Property (1) follows as in the proof of Proposition 2.7(1). □

3. The *d*-dimensional case

The aim of this section is to show that the semigroup $(T(t))_{t\geqslant 0}$ generated by the closure $(\mathcal{A}_d, D(\mathcal{A}_d))$ of the operator $(\mathcal{A}_d, C^2(S_d))$ on $C(S_d)$ (see Theorem 1.1) is also analytic for $d > 2$. We prove this using an argument by induction as follows.

3.1. Inductive hypotheses and consequences

We suppose that the following holds.

Hypotheses 3.1 *(Inductive hypothesis). Suppose that the closure $(\mathcal{A}_d, D(\mathcal{A}_d))$ of $(\mathcal{A}_d, C^2(S_d))$ satisfies the following properties*:

(1) *$(\mathcal{A}_d, D(\mathcal{A}_d))$ generates a bounded analytic C_0-semigroup $(T(t))_{t\geqslant 0}$ of angle $\pi/2$ on $C(S_d)$. The semigroup is compact.*
(2) *For each $0 < \theta < \pi$ there exist $C > 0$ and $l > 1$ such that, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| > l$, $i = 1, \dots, d$ and $u \in C(S_d)$, we have*

$$
\big\|\sqrt{x_i(1-x_i)}\partial_{x_i}\big(R(\lambda,\mathcal{A}_d)u\big)\big\|_{S_d} \leqslant \frac{C}{\sqrt{|\lambda|}}\|u\|_{S_d}.
$$

In order to prove the inductive step, we need to provide some auxiliary results as follows.

Fix $0<\delta<\frac{1}{2}$. Then, we define the sets $T_{d+1,1-\delta} := S_d \times [0,1-\delta]$, $T_{1-\delta,d+1} := [0,1-\delta]\times S_d$, and consider the second order differential operators

$$A_{d+1,1} = \frac{1}{2(1-x_{d+1})}\sum_{i,j=1}^{d} x_i(\delta_{ij}-x_j)\partial^2_{x_ix_j}u + \frac{1}{2}x_{d+1}(1-x_{d+1})\partial^2_{x_{d+1}}u, \quad x\in T_{d+1,1-\delta}, \tag{3.1}$$

$$A_{d+1,2} = \frac{1}{2}x_1(1-x_1)\partial^2_{x_1}u + \frac{1}{2(1-x_1)}\sum_{i,j=2}^{d+1} x_i(\delta_{ij}-x_j)\partial^2_{x_ix_j}u, \quad x\in T_{1-\delta,d+1}. \tag{3.2}$$

Recalling that $C(T_{d+1,1-\delta}) = C(S_d)\,\widehat{\otimes}_\varepsilon\, C([0,1-\delta])$ and $C(T_{1-\delta,d+1}) = C([0,1-\delta])\,\widehat{\otimes}_\varepsilon\, C(S_d)$ and the discussion prior to Proposition 2.4, we can prove via Hypotheses 3.1, Proposition 2.1 and analogously to the proofs of Propositions 2.4, 2.6 and 2.7 the following facts.

Setting $D(B) := \{u\in C([0,1-\delta])\cap C^2(]0,1-\delta]) \mid \lim_{y\to 0^+} yu''(y)=0,\ u'(1-\delta)=0\}$, we have:

Proposition 3.2. *Suppose that Hypotheses* 3.1 *hold. Then the following properties are satisfied*:

(1) *The closure $(A_{d+1,1}, D(A_{d+1,1}))$ of $(A_{d+1,1}, D(\mathcal{A}_d)\otimes D(B))$ generates an analytic C_0-semigroup of angle $\pi/2$ on $C(T_{d+1,1-\delta})$ which is contractive and compact.*

(2) *The closure $(A_{d+1,2}, D(A_{d+1,2}))$ of $(A_{d+1,2}, D(B)\otimes D(\mathcal{A}_d))$ generates an analytic C_0-semigroup of angle $\pi/2$ on $C(T_{1-\delta,d+1})$ which is contractive and compact.*

Proposition 3.3. *Suppose that Hypotheses* 3.1 *hold. Then the following properties are satisfied:*

(1) *For each $0<\theta<\pi$ there exist $C_1>0$ and $l_1>1$ such that, for every $\lambda\in\{z\in\mathbb{C} \mid |\arg z|<\theta\}$ with $|\lambda|>l_1$ and $u\in C(T_{d+1,1-\delta})$, we have*

$$\left\|\sqrt{x_i(1-x_i)}\,\partial_{x_i}\big(R(\lambda, A_{d+1,1})u\big)\right\|_{T_{d+1,1-\delta}} \leqslant \frac{C_1}{\sqrt{|\lambda|}}\|u\|_{T_{d+1,1-\delta}}, \quad i=1,\dots,d,$$

and

$$\left\|\sqrt{x_{d+1}}\,\partial_{x_{d+1}}\big(R(\lambda, A_{d+1,1})u\big)\right\|_{T_{d+1,1-\delta}} \leqslant \frac{C_1}{\sqrt{|\lambda|}}\|u\|_{T_{d+1,1-\delta}}.$$

(2) *For each $0<\theta<\pi$ there exist $C_2>0$ and $l_2>1$ such that, for every $\lambda\in\{z\in\mathbb{C} \mid |\arg z|<\theta\}$ with $|\lambda|>l_2$ and $u\in C(T_{1-\delta,d+1})$, we have*

$$\left\|\sqrt{x_1}\,\partial_{x_1}\big(R(\lambda, A_{d+1,2})u\big)\right\|_{T_{1-\delta,d+1}} \leqslant \frac{C_2}{\sqrt{|\lambda|}}\|u\|_{T_{1-\delta,d+1}},$$

and

$$\left\|\sqrt{x_i(1-x_i)}\,\partial_{x_i}\big(R(\lambda, A_{d+1,2})u\big)\right\|_{T_{1-\delta,d+1}} \leqslant \frac{C_2}{\sqrt{|\lambda|}}\|u\|_{T_{1-\delta,d+1}}, \quad i=2,\dots,d+1.$$

3.2. Analyticity of a class of degenerate evolution equations on the canonical simplex of $\mathbb{R}^{d+1}$

For the inductive step, we also need to perform the following changes of coordinates.

Fix $0<\delta<\frac{1}{2}$. Then, we set

$$\Omega_1 := \{x\in S_{d+1} \mid 0\leqslant x_{d+1}\leqslant 1-\delta\}, \qquad \Omega_2 := \{x\in S_{d+1} \mid 0\leqslant x_1\leqslant 1-\delta\}. \tag{3.3}$$

Then $S_{d+1} = \bigcup_{i=1}^{2}\Omega_i$. Next, we consider the maps $\varphi_1 : T_{d+1,1-\delta}\to\Omega_1$ and $\varphi_2 : T_{1-\delta,d+1}\to\Omega_2$ defined by

$$\begin{aligned}
\varphi_1(r) &:= \big(r_1(1-r_{d+1}), r_2(1-r_{d+1}),\dots, r_d(1-r_{d+1}), r_{d+1}\big),\\
\varphi_2(r) &:= \big(r_1, r_2(1-r_1),\dots, r_d(1-r_1), r_{d+1}(1-r_1)\big).
\end{aligned} \tag{3.4}$$

Lemma 3.4. *The map φ_i is bijective and a C^∞-diffeomorphism for $i=1,2$.*

Lemma 3.5. *Let $\Phi_1 : C(\Omega_1) \to C(T_{d+1,1-\delta})$ and $\Phi_2 : C(\Omega_2) \to C(T_{1-\delta,d+1})$ be the operators defined by*

$$\Phi_i(u) = u \circ \varphi_i, \quad u \in C(\Omega_i),\ i = 1, 2.$$

Then Φ_i is a surjective isometry for $i = 1, 2$. In particular, $\Phi_1(C^n(\Omega_1)) = C^n(T_{d+1,1-\delta})$ and $\Phi_2(C^n(\Omega_2)) = C^n(T_{1-\delta,d+1})$ for every $n \in \mathbb{N}$.

For each $i \in \{1, 2\}$ we define

$$\mathcal{A}_{d+1,i} := \Phi_i^{-1} \circ A_{d+1,i} \circ \Phi_i, \qquad D(\mathcal{A}_{d+1,i}) = \Phi_i^{-1}\big(D(A_{d+1,i})\big), \tag{3.5}$$

with $(A_{d+1,i}, D(A_{d+1,i}))$ the second order differential operators defined in (3.1) and (3.2). Then, by Proposition 3.2 and Lemma 3.5 the operator $(\mathcal{A}_{d+1,i}, D(\mathcal{A}_{d+1,i}))$ generates an analytic semigroup of angle $\pi/2$ on $C(\Omega_i)$ for every $i \in \{1, 2\}$. We observe that, for $i = 1$ and $v = \Phi_1(u)$ for some $u \in \Phi_1^{-1}(D(A_{d+1,1}))$, we have

$$\begin{aligned}
&\partial_{r_i} v = (1 - r_{d+1})\partial_{x_i} u, \qquad \partial^2_{r_i} v = (1 - r_{d+1})^2 \partial^2_{x_i} u, \quad i = 1, \ldots, d,\\
&\partial^2_{r_i r_j} v = (1 - r_{d+1})^2 \partial^2_{x_i x_j} u, \quad i, j = 1, \ldots, d,\\
&\partial_{r_{d+1}} v = -\sum_{i=1}^{d} r_i \partial_{x_i} u + \partial_{x_{d+1}} u, \qquad \partial^2_{r_{d+1}} v = \sum_{i,j=1}^{d} r_i r_j \partial^2_{x_i x_j} u - 2\sum_{i=1}^{d} r_i \partial^2_{x_i x_{d+1}} u + \partial^2_{x_{d+1}} u.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
A_{d+1,1}\big(\Phi_1(u)\big)(r) &= A_{d+1,1} v(r)\\
&= \frac{1}{2(1 - r_{d+1})} \sum_{i,j=1}^{d} r_i(\delta_{ij} - r_j)(1 - r_{d+1})^2 \partial^2_{x_i x_j} u\\
&\quad + \frac{1}{2} r_{d+1}(1 - r_{d+1}) \left(\sum_{i,j=1}^{d} r_i r_j \partial^2_{x_i x_j} u - 2\sum_{i=1}^{d} r_i \partial^2_{x_i x_{d+1}} u + \partial^2_{x_{d+1}} u \right)\\
&= \frac{1}{2} \sum_{i,j=1}^{d} r_i(1 - r_{d+1})\big[\delta_{ij} - r_j(1 - r_{d+1})\big] \partial^2_{x_i x_j} u\\
&\quad - \sum_{i=1}^{d} r_i(1 - r_{d+1}) r_{d+1} \partial^2_{x_i x_{d+1}} u + \frac{1}{2} r_{d+1}(1 - r_{d+1}) \partial^2_{x_{d+1}} u, \quad r \in T_{d+1,1-\delta},
\end{aligned}$$

and hence,

$$\begin{aligned}
\mathcal{A}_{d+1,1} u(x) &= \frac{1}{2} \sum_{i,j=1}^{d} x_i(\delta_{ij} - x_j)\partial^2_{x_i x_j} u(x) - \sum_{i=1}^{d} x_i x_{d+1} \partial^2_{x_i x_{d+1}} u(x) + \frac{1}{2} x_{d+1}(1 - x_{d+1}) \partial^2_{x_{d+1}} u(x)\\
&= \frac{1}{2} \sum_{i,j=1}^{d+1} x_i(\delta_{ij} - x_j)\partial^2_{x_i x_j} u(x), \quad x \in \Omega_1,
\end{aligned}$$

i.e., $\mathcal{A}_{d+1}|_{\Omega_1} = \mathcal{A}_{d+1,1}$.

On the other hand, for $i = 2$ and $v = \Phi_2(u)$ for some $u \in \Phi_2^{-1}(D(A_{d+1,2}))$, we have

$$\begin{aligned}
&\partial_{r_1} v = \partial_{x_1} u - \sum_{i=2}^{d+1} r_i \partial_{x_i} u, \qquad \partial^2_{r_1} v = \partial^2_{x_1} u - 2\sum_{i=2}^{d+1} r_i \partial^2_{x_1 x_i} u + \sum_{i,j=2}^{d+1} r_i r_j \partial^2_{x_i x_j} u,\\
&\partial_{r_i} v = (1 - r_1)\partial_{x_i} u, \qquad \partial^2_{r_i} v = (1 - r_1)^2 \partial^2_{x_i} u, \quad i = 2, \ldots, d + 1,\\
&\partial^2_{r_i r_j} v = (1 - r_1)^2 \partial^2_{x_i x_j} u, \quad i, j = 2, \ldots, d + 1.
\end{aligned}$$

So, we obtain

$$A_{d+1,2}\big(\Phi_2(u)\big)(r) = A_{d+1,2}v(r) = \frac{1}{2}r_1(1-r_1)\left(\partial^2_{x_1}u - 2\sum_{i=2}^{d+1} r_i\partial^2_{x_1x_i}u + \sum_{i,j=2}^{d+1} r_ir_j\partial^2_{x_ix_j}u\right)$$
$$+\frac{1}{1-r_1}\frac{1}{2}\sum_{i,j=2}^{d+1} r_i(\delta_{ij}-r_j)(1-r_1)^2\partial^2_{x_ix_j}u = \frac{1}{2}r_1(1-r_1)\partial^2_{x_1}u - \sum_{i=2}^{d+1} r_1r_i(1-r_1)\partial^2_{x_1x_i}u$$
$$+\frac{1}{2}\sum_{i,j=2}^{d+1} r_i(1-r_1)\big[\delta_{ij}-r_j(1-r_1)\big]\partial^2_{x_ix_j}u, \quad r\in T_{1-\delta,d+1},$$

and hence,

$$\mathcal{A}_{d+1,2}u(x) = \frac{1}{2}x_1(1-x_1)\partial^2_{x_1}u(x) - \sum_{i=2}^{d+1} x_1x_i\partial^2_{x_1x_i}u(x) + \frac{1}{2}\sum_{i,j=2}^{d+1} x_i(\delta_{ij}-x_j)\partial^2_{x_ix_j}u(x)$$
$$= \frac{1}{2}\sum_{i,j=1}^{d+1} x_i(\delta_{ij}-x_j)\partial^2_{x_ix_j}u(x), \quad x\in\Omega_2,$$

i.e., $\mathcal{A}_{d+1}|_{\Omega_2} = \mathcal{A}_{d+1,2}$.

We now may prove the main theorem of this section.

Theorem 3.6. *Suppose that Hypotheses* 3.1 *hold. Then the closure* $(\mathcal{A}_{d+1}, D(\mathcal{A}_{d+1}))$ *of* $(\mathcal{A}_{d+1}, C^2(S_{d+1}))$ *generates an analytic* C_0*-semigroup* $(T(t))_{t\geqslant 0}$ *of angle* $\pi/2$ *on* $C(S_{d+1})$*. The semigroup is compact.*

Proof. Fix $0<\delta<\frac{1}{2}$. Let $\{\psi_i\}_{i=1,2}\subseteq C_c^\infty(\mathbb{R}^{d+1})$ such that $\sum_{i=1}^{2}(\psi_i)^2 = 1$ on S_{d+1} and

$$\operatorname{supp}(\psi_1)\subseteq\big\{x\in\mathbb{R}^{d+1}\;\big|\;x_{d+1}<1-\delta\big\},\qquad \operatorname{supp}(\psi_2)\subseteq\big\{x\in\mathbb{R}^{d+1}\;\big|\;x_1<1-\delta\big\}.$$

For the sake of simplicity, we still denote by ψ_i the restriction of ψ_i to Ω_i, for $i=1,2$.

By Proposition 3.2 the operators $\mathcal{A}_{d+1,i}$ defined according to (3.5) generate analytic semigroups of angle $\pi/2$. So, if $0<\theta<\pi$ is a fixed angle, we can find two positive constant C and R such that, for $|\lambda|\geqslant R$ with $|\arg\lambda|<\theta$, the resolvents $R(\lambda,\mathcal{A}_{2,i})$ exist and satisfies

$$\big\|R(\lambda,\mathcal{A}_{2,i})\big\|\leqslant\frac{C}{|\lambda|},\quad i=1,2. \tag{3.6}$$

Then, for every $\lambda\in\{z\in\mathbb{C}\mid |\arg z|<\theta\}$ with $|\lambda|\geqslant R$, we can define the operator $S(\lambda): C(S_{d+1})\to C(S_{d+1})$ via

$$S(\lambda)u = \sum_{i=1}^{2}\psi_i R(\lambda,\mathcal{A}_{d+1,i})(\psi_i u),\quad u\in C(S_{d+1}), \tag{3.7}$$

and hence,

$$\big\|S(\lambda)\big\|\leqslant\frac{3C}{|\lambda|}.$$

We observe that the previous considerations on the differential operators $\mathcal{A}_{d+1,i}$ ensure, for every $i=1,2$ and $u\in C(S_{d+1})$, that

$$\mathcal{A}_{d+1}\big(\psi_i R(\lambda,\mathcal{A}_{d+1,i})(\psi_i u)\big) = \mathcal{A}_{d+1,i}\big(\psi_i R(\lambda,\mathcal{A}_{d+1,i})(\psi_i u)\big), \tag{3.8}$$

and, for every $f,g\in D(\mathcal{A}_{d+1,i})$, that

$$\mathcal{A}_{d+1,i}(fg) = g(\mathcal{A}_{d+1,i}f) + f(\mathcal{A}_{d+1,i}g) + \sum_{i,j=1}^{d+1} x_i(\delta_{ij}-x_j)\partial_{x_i}f\,\partial_{x_j}g. \tag{3.9}$$

By (3.8) and (3.9) we obtain, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| \geqslant R$ and $u \in C(S_{d+1})$, that

$$\begin{aligned}(\lambda - \mathcal{A}_{d+1})S(\lambda)u &= \lambda S(\lambda)(u) - \sum_{i=1}^{2} \mathcal{A}_{d+1}\big(\psi_i R(\lambda, \mathcal{A}_{d+1,i})(\psi_i u)\big) \\ &= \lambda S(\lambda)(u) - \sum_{i=1}^{2} \mathcal{A}_{d+1,i}\big(\psi_i\big(R(\lambda, \mathcal{A}_{d+1,i})(\psi_i u)\big)\big) \\ &= \sum_{i=1}^{2} \psi_i(\lambda - \mathcal{A}_{d+1,i})R(\lambda, \mathcal{A}_{d+1,i})(\psi_i u) - \sum_{i=1}^{2} \mathcal{A}_{d+1,i}(\psi_i)R(\lambda, \mathcal{A}_{d+1,i})(\psi_i u) \\ &\quad - \sum_{i=1}^{2} \sum_{j,h=1}^{d+1} x_j(\delta_{jh} - x_h)\partial_{x_j}\big(R(\lambda, \mathcal{A}_{d+1,i})(\psi_i u)\big)\partial_{x_h}\psi_i \\ &=: \big(I + B(\lambda) + C(\lambda)\big)(u).\end{aligned}$$

Applying (3.6) we obtain, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| \geqslant R$ and $u \in C(S_{d+1})$, that

$$\big\|B(\lambda)u\big\|_{S_{d+1}} \leqslant \frac{M}{|\lambda|}\|u\|_{S_{d+1}},$$

for some $M > 0$. In order to estimate $C(\lambda)$ we proceed as follows.

We first observe, for every $f, g \in D(\mathcal{A}_{d+1})$, that

$$\begin{aligned}\sum_{i,j=1}^{d+1} x_i(\delta_{ij} - x_j)\partial_{x_i} f \partial_{x_j} g &= \sum_{i=1}^{d+1} x_i(1 - x_i)\partial_{x_i} f \partial_{x_i} g - \sum_{i=1}^{d+1} \sum_{j=1, j\neq i}^{d+1} [x_i x_j \partial_{x_i} f \partial_{x_j} g - x_i x_j \partial_{x_i} f \partial_{x_i} g + x_i x_j \partial_{x_i} f \partial_{x_i} g] \\ &= \sum_{i=1}^{d+1} x_i(1 - x_i)\partial_{x_i} f \partial_{x_i} g - \sum_{i=1}^{d+1} \sum_{j=1, j\neq i}^{d+1} x_i x_j \partial_{x_i} f(\partial_{x_j} g - \partial_{x_i} g) - \sum_{i=1}^{d+1} x_i \partial_{x_i} f \partial_{x_i} g \sum_{j=1, j\neq i}^{d+1} x_j \\ &= \sum_{i=1}^{d+1} x_i\left(1 - \sum_{j=1}^{d+1} x_j\right)\partial_{x_i} f \partial_{x_i} g - \sum_{i=1}^{d+1} \sum_{j=1, j\neq i}^{d+1} x_i x_j \partial_{x_i} f(\partial_{x_j} g - \partial_{x_i} g).\end{aligned}$$

Next, let $f \in D(\mathcal{A}_{d+1,1})$ and $v = \Phi_1(f)$. Then $v \in D(A_{d+1,1})$ and the following holds

$$\partial_{r_i} v = (1 - r_{d+1})\partial_{x_i} f, \quad i = 1, \dots, d, \qquad \partial_{r_{d+1}} v = -\sum_{i=1}^{d} r_i \partial_{x_i} f + \partial_{x_{d+1}} f, \tag{3.10}$$

and hence,

$$\partial_{x_i} f = \frac{1}{1 - r_{d+1}}\partial_{r_i} v, \quad i = 1, \dots, d, \qquad \partial_{x_{d+1}} f = \partial_{r_{d+1}} v + \sum_{i=1}^{d} \frac{r_i}{1 - r_{d+1}}\partial_{r_i} v. \tag{3.11}$$

So, by Proposition 3.3 we obtain, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| > R_1 := \max\{R, l_1, l_2\}$, that

$$\begin{aligned}\left\|\sum_{i,j=1}^{d+1} x_i(\delta_{ij} - x_j)\partial_{x_i} f \partial_{x_j} \psi_1\right\|_{S_{d+1}} &= \left\|\sum_{i=1}^{d+1} x_i\left(1 - \sum_{j=1}^{d+1} x_j\right)\partial_{x_i} f \partial_{x_i} \psi_1 - \sum_{i=1}^{d+1} \sum_{j=1, j\neq i}^{d+1} x_i x_j \partial_{x_i} f(\partial_{x_j} \psi_1 - \partial_{x_i} \psi_1)\right\|_{S_{d+1}} \\ &\leqslant c\left(\sum_{i=1}^{d+1}\left\|x_i\left(1 - \sum_{j=1}^{d+1} x_j\right)\partial_{x_i} f\right\|_{\Omega_1} + \sum_{i=1}^{d+1} \sum_{j=1, j\neq i}^{d+1} \|x_i x_j \partial_{x_i} f\|_{\Omega_1}\right) \\ &= c\left(\sum_{i=1}^{d}\left\|r_i(1 - r_{d+1})^2\left(1 - \sum_{j=1}^{d} r_j\right)\frac{1}{1 - r_{d+1}}\partial_{r_i} v\right\|_{T_{d+1,1-\delta}}\right. \\ &\quad + \left\|r_{d+1}(1 - r_{d+1})\left(1 - \sum_{j=1}^{d} r_j\right)\left(\partial_{r_{d+1}} v + \sum_{i=1}^{d} \frac{r_i}{1 - r_{d+1}}\partial_{r_i} v\right)\right\|_{T_{d+1,1-\delta}}\end{aligned}$$

$$+\sum_{i=1}^{d}\sum_{j=1,j\neq i}^{d}\left\|r_ir_j(1-r_{d+1})^2\frac{1}{1-r_{d+1}}\partial_{r_i}v\right\|_{T_{d+1,1-\delta}}$$
$$+\sum_{i=1}^{d}\left\|r_i(1-r_{d+1})\frac{1}{1-r_{d+1}}\partial_{r_i}v\right\|_{T_{d+1,1-\delta}}$$
$$+\sum_{i=1}^{d}\left\|r_{d+1}r_i(1-r_{d+1})\left(\partial_{r_{d+1}}v+\sum_{i=1}^{d}\frac{r_i}{1-r_{d+1}}\partial_{r_i}v\right)\right\|_{T_{d+1,1-\delta}}\Bigg)$$
$$\leqslant\frac{C_1}{\sqrt{|\lambda|}}\|\lambda v-A_{d+1,1}v\|_{T_{d+1,1-\delta}}=\frac{C_1}{\sqrt{|\lambda|}}\|\lambda f-\mathcal{A}_{d+1,1}f\|_{\Omega_1}$$
$$\leqslant\frac{C_1}{\sqrt{|\lambda|}}\|\lambda f-\mathcal{A}_{d+1}f\|_{S_{d+1}}.$$

If $f=R(\lambda,\mathcal{A}_{d+1,1})(\psi_1u)$ for some $u\in C(S_{d+1})$, it follows, for every $\lambda\in\{z\in\mathbb{C}\mid|\arg z|<\theta\}$ with $|\lambda|>R_1$, that

$$\left\|\sum_{j,h=1}^{d+1}x_j(\delta_{jh}-x_h)\partial_{x_j}\big(R(\lambda,\mathcal{A}_{d+1,1})(\psi_1u)\big)\partial_{x_h}\psi_i\right\|_{S_{d+1}}\leqslant\frac{C_1}{\sqrt{|\lambda|}}\|u\|_{S_{d+1}}. \tag{3.12}$$

By the symmetry of change of variables, we obtain in analogous way that there exists $C_2>0$ such that, for every $\lambda\in\{z\in\mathbb{C}\mid|\arg z|<\theta\}$ with $|\lambda|>R_1$ and $u\in C(S_{d+1})$, we have

$$\left\|\sum_{j,h=1}^{d+1}x_j(\delta_{jh}-x_h)\partial_{x_j}\big(R(\lambda,\mathcal{A}_{d+1,2})(\psi_2u)\big)\partial_{x_h}\psi_i\right\|_{S_{d+1}}\leqslant\frac{C_2}{\sqrt{|\lambda|}}\|u\|_{S_{d+1}}. \tag{3.13}$$

Combining (3.12) and (3.13), we obtain that there exists $K>0$ such that, for every $\lambda\in\{z\in\mathbb{C}\mid|\arg z|<\theta\}$ with $|\lambda|>R_1$ and $u\in C(S_{d+1})$,

$$\|C(\lambda)u\|_{S_{d+1}}\leqslant\frac{K}{\sqrt{|\lambda|}}\|u\|_{S_{d+1}}.$$

If $|\lambda|\gg1$, then the operator $B=(\lambda-\mathcal{A}_{d+1})S(\lambda)$ is invertible in $\mathcal{L}(C(S_{d+1}))$. Hence, there exists $R(\lambda,\mathcal{A}_{d+1})=S(\lambda)B^{-1}$ and

$$\|R(\lambda,\mathcal{A}_{d+1})\|=\|S(\lambda)B^{-1}\|\leqslant\frac{C'}{|\lambda|}, \tag{3.14}$$

with C' a positive constant independent of λ, provided $\lambda-\mathcal{A}_{d+1}$ is injective and, in particular, for $\lambda>0$ as $\mathcal{A}_{d+1}$ is dissipative. To conclude that the semigroup is analytic of angle $\pi/2$ it now suffices to repeat the argument already used in the proof of Proposition 2.6.

Since $R(\lambda,\mathcal{A}_{d+1})=S(\lambda)B^{-1}$ for some $\lambda>0$ and the operator $S(\lambda)$ is compact by Proposition 3.2, the differential operator $(\mathcal{A}_{d+1},D(\mathcal{A}_{d+1}))$ has compact resolvent. Thus, the semigroup is also compact, being analytic and hence, norm continuous. □

Recalling that the eigenvalues of the operator $\mathcal{A}_{d+1}$ are given by $\lambda_n=-\frac{n(n-1)}{2}$, $n\geqslant1$, [28, Ch. VIII, p. 221], and using Theorem 3.6 together with [23, Proposition 5.6] we obtain the following result.

Theorem 3.7. *The semigroup generated by $(\mathcal{A}_{d+1},D(\mathcal{A}_{d+1}))$ is bounded analytic of angle $\pi/2$.*

Moreover, the differential operator $(\mathcal{A}_{d+1},D(\mathcal{A}_{d+1}))$ satisfies:

Proposition 3.8. *Suppose that Hypotheses* 3.1 *hold.*

Then the closure $(\mathcal{A}_{d+1},D(\mathcal{A}_{d+1}))$ of the differential operator $(\mathcal{A}_{d+1},C^2(S_{d+1}))$ defined in (1.1) *satisfies the following properties*:

(1) *There exist $K_b>0$ and $t_b>0$ such that, for every $0<t<t_b$, $i=1,\dots,d+1$ and $u\in C(S_{d+1})$, we have*

$$\left\|\sqrt{x_i(1-x_i)}\partial_{x_i}\big(T(t)u\big)\right\|_{S_{d+1}}\leqslant\frac{K_b}{\sqrt{t}}\|u\|_{S_{d+1}},$$

and such that, for every $t\geqslant t_b$, $i=1,\dots,d+1$ and $u\in C(S_{d+1})$, we have

$$\left\|\sqrt{x_i(1-x_i)}\partial_{x_i}\big(T(t)u\big)\right\|_{S_{d+1}}\leqslant K_b\|u\|_{S_{d+1}}.$$

(2) *For each* $0 < \theta < \pi$ *there exist two constants* $C > 0$ *and* $l > 1$ *such that, for every* $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ *with* $|\lambda| > l$, $i = 1, \dots, d+1$ *and* $u \in C(S_{d+1})$, *we have*

$$\left\| \sqrt{x_i(1-x_i)}\partial_{x_i}\bigl(R(\lambda, \mathcal{A}_{d+1})u\bigr) \right\|_{S_{d+1}} \leqslant \frac{C}{\sqrt{|\lambda|}} \|u\|_{S_{d+1}}.$$

Proof. We first prove property (2). According to the notation in the proof of Theorem 3.6, fixed an angle $0 < \theta < \pi$ there exists $l > 1$ such that, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| > l$, we have

$$R(\lambda, \mathcal{A}_{d+1}) = S(\lambda)B^{-1},$$

where the operators $S(\lambda)$ are defined in (3.7) and $\|B^{-1}\| \leqslant 2$. Fix $i \in \{1, \dots, d+1\}$. So, we obtain, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| > l$ and $u \in C(S_{d+1})$, that

$$\begin{aligned}
&\left\| \sqrt{x_i(1-x_i)}\partial_{x_i}\bigl(R(\lambda, \mathcal{A}_{d+1})u\bigr) \right\|_{S_{d+1}} \\
&\quad\leqslant \sum_{j=1}^{2} \left\| \sqrt{x_i(1-x_i)}\partial_{x_i}\bigl(\psi_i R(\lambda, \mathcal{A}_{d+1,j})\bigl(\psi_j B^{-1}u\bigr)\bigr) \right\|_{\Omega_j} \\
&\quad\leqslant \sum_{j=1}^{2} \Bigl(\left\| \sqrt{x_i(1-x_i)}\partial_{x_i}(\psi_j) R(\lambda, \mathcal{A}_{d+1,j})\bigl(\psi_j B^{-1}u\bigr) \right\|_{\Omega_j} + \left\| \sqrt{x_i(1-x_i)}\psi_j \partial_{x_i}\bigl(R(\lambda, \mathcal{A}_{d+1,j})\bigl(\psi_j B^{-1}u\bigr)\bigr) \right\|_{\Omega_j} \Bigr) \\
&\quad\leqslant c \sum_{j=1}^{2} \left(\frac{C}{|\lambda|} \left\| \psi_j B^{-1}u \right\|_{\Omega_j} + \left\| \sqrt{x_i(1-x_i)}\psi_j \partial_{x_i}\bigl(R(\lambda, \mathcal{A}_{d+1,j})\bigl(\psi_j B^{-1}u\bigr)\bigr) \right\|_{\Omega_j} \right) \\
&\quad\leqslant c \sum_{j=1}^{2} \left(\frac{C}{\sqrt{|\lambda|}} \left\| B^{-1} \right\| \|u\|_{S_{d+1}} + \left\| \sqrt{x_i(1-x_i)}\partial_{x}\bigl(R(\lambda, \mathcal{A}_{d+1,j})\bigl(\psi_j B^{-1}u\bigr)\bigr) \right\|_{\Omega_j} \right)
\end{aligned} \tag{3.15}$$

with $c := \sup_{j=1}^{2} \|\psi_j\|_{1,\Omega_i}$. To estimate the second addend on the right in (3.15) we proceed as follows.

For $j = 1$ set $f = R(\lambda, \mathcal{A}_{d+1,1})(\psi_1 B^{-1}u)$ and $v = \Phi_1(f)$. Then, by (3.10) and (3.11) we have, for $i = 1, \dots, d$, that

$$\begin{aligned}
\left\| \sqrt{x_i(1-x_i)}\partial_{x_i} f \right\|_{\Omega_1} &= \left\| \frac{\sqrt{r_i(1-r_{d+1})[1-r_i(1-r_{d+1})]}}{1-r_{d+1}} \partial_{r_i} v \right\|_{T_{d+1,1-\delta}} \\
&\leqslant \frac{C_1''}{\sqrt{|\lambda|}} \|\lambda v - A_{d+1,1} v\|_{T_{d+1,1-\delta}} = \frac{C_1}{\sqrt{|\lambda|}} \|\lambda f - \mathcal{A}_{d+1,1} f\|_{\Omega_1} \\
&= \frac{C_1}{\sqrt{|\lambda|}} \left\| \psi_1 B^{-1}u \right\|_{\Omega_1} \leqslant \frac{C_1}{\sqrt{|\lambda|}} \left\| B^{-1} \right\| \|u\|_{S_{d+1}},
\end{aligned} \tag{3.16}$$

and we have, for $i = d+1$, that

$$\left\| \sqrt{x_{d+1}(1-x_{d+1})}\partial_{x_{d+1}} f \right\|_{\Omega_1} \tag{3.17}$$

$$\begin{aligned}
&= \left\| \sqrt{r_{d+1}(1-r_{d+1})} \left(\partial_{r_{d+1}} v + \sum_{i=1}^{d} \frac{r_i}{1-r_{d+1}} \partial_{r_i} v \right) \right\|_{T_{d+1,1-\delta}} \\
&\leqslant \left\| \sqrt{r_{d+1}(1-r_{d+1})}\partial_{r_{d+1}} v \right\|_{T_{d+1,1,\delta}} + \sum_{i=1}^{d} \left\| \sqrt{r_{d+1}(1-r_{d+1})} \frac{r_i}{1-r_{d+1}} \partial_{r_i} v \right\|_{T_{d+1,1-\delta}} \\
&\leqslant \frac{C_1''}{\sqrt{|\lambda|}} \|\lambda v - A_{d+1,1} v\|_{T_{d+1,1-\delta}} = \frac{C_1'}{\sqrt{|\lambda|}} \|\lambda f - \mathcal{A}_{d+1,1} f\|_{\Omega_1} \\
&= \frac{C_1'}{\sqrt{|\lambda|}} \left\| \psi_1 B^{-1}u \right\|_{\Omega_1} \leqslant \frac{C_1'}{\sqrt{|\lambda|}} \left\| B^{-1} \right\| \|u\|_{S_{d+1}}.
\end{aligned} \tag{3.18}$$

By the symmetry of the change of variables, one analogously shows that there exists $C_2 > 0$ such that, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| > l$, $i = 1, \dots, d+1$ and $u \in C(S_{d+1})$, we have

$$\left\| \sqrt{x_i(1-x_i)}\partial_{x_i}\bigl(R(\lambda, \mathcal{A}_{d+1,2})\bigl(\psi_2 B^{-1}u\bigr)\bigr) \right\|_{\Omega_2} \leqslant \frac{C_2}{\sqrt{|\lambda|}} \left\| B^{-1} \right\| \|u\|_{S_{d+1}}. \tag{3.19}$$

Combining (3.15)–(3.17) and (3.19) we obtain, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ with $|\lambda| > l$, $i = 1, \dots, d+1$ and $u \in C(S_{d+1})$, that

$$\left\| \sqrt{x_i(1-x_i)}\partial_{x_i}\big(R(\lambda, \mathcal{A}_{d+1})u\big)\right\|_{S_{d+1}} \leqslant \frac{C}{\sqrt{|\lambda|}}\|u\|_{S_{d+1}}$$

for some constant $C > 0$ independent of u and λ.

Property (1) follows as in the proof of Proposition 2.7(1). □

3.3. The main results

Finally, we can state and prove the main results of this paper.

Theorem 3.9. *The closure $(\mathcal{A}_d, D(\mathcal{A}_d))$ of the operator $(\mathcal{A}_d, C^2(S_d))$ generates a bounded analytic C_0-semigroup of angle $\pi/2$ on $C(S_d)$ for every $d \geqslant 1$. The semigroup is compact.*

Proof. The proof is by induction on the integer $d \geqslant 1$. The case $d = 1$ is given in [6,23]. Suppose that the result holds for $d \geqslant 2$. Then we can apply Theorem 3.6 and conclude that the result holds for $d + 1$. Thus, the proof is complete. □

Therefore, a similar argument as in the proof of Proposition 2.6 together with Theorem 3.9 allow us to show that the following holds.

Theorem 3.10. *Let $d \geqslant 1$ and m be a strictly positive function in $C(S_d)$. Then the operator $(m\mathcal{A}_d, D(\mathcal{A}_d))$ generates an analytic C_0-semigroup of angle $\pi/2$ on $C(S_d)$. The semigroup is contractive and compact.*

Proof. By Theorem 1.1 and [12, Theorem] we can conclude that $(m\mathcal{A}_d, D(\mathcal{A}_d))$ generates a contractive C_0-semigroup on $C(S_d)$. We claim that the semigroup is analytic of angle $\pi/2$. To show this we can proceed as in the proof of Proposition 2.6 and hence, we indicate only the main changes.

For each $n \in \mathbb{N}$ let $I_n^j := [\frac{j-1}{n}, \frac{j+1}{n}]$, $j = 1, \dots, n-1$, and let $L = \{1, \dots, n-1\}^d$. Then, for every $j = (j_1, j_2, \dots, j_d) \in L$, we define the set

$$J_n^j = \big(I_n^{j_1} \times I_n^{j_2} \times \cdots \times I_n^{j_d}\big) \cap S_d.$$

Set $M_n = \{j \in L \mid J_n^j \neq \emptyset\}$ and fix $V_n^j \in J_n^j$ for all $j \in M_n$. Then, we choose $\phi_n^j \in C^\infty(\mathbb{R}^d)$ for all $j \in M_n$ such that $\operatorname{supp}(\phi_n^j) \subseteq J_n^j$ and $\sum_{j \in M_n} (\phi_n^j)^2 = 1$. We observe that, if $v_j \in C(Q)$, for $j \in M_n$, and $x \in S_d$, then there exists $\bar{j} \in M_n$ such that $x \in J_n^{\bar{j}}$ and hence,

$$\sum_{j \in M_n} \phi_n^j(x) v_j(x) = \sum_{j \in M_{n,0}} \phi_n^j(x) v_j(x),$$

where $M_{n,0} = \{j = (\bar{j}_1 + h_1, \bar{j}_2 + h_2, \dots, \bar{j}_d + h_d) \mid \forall i \in \{1, \dots, d\}\ k_i \in \{-1, 0, 1\}\}$ so that $M_{n,0}$ contains exactly 3^d elements. Therefore, we have

$$\left\| \sum_{j \in M_n} \phi_n^j v_j \right\|_{S_d} \leqslant 3^d \sup_{j \in M_n} \big\|\phi_n^j v_j\big\|_{S_d}. \tag{3.20}$$

Since $(\mathcal{A}_d, D(\mathcal{A}_d))$ generates a bounded analytic semigroup of angle $\pi/2$ on $C(S_d)$, for each $\lambda \in \mathbb{C} \setminus (-\infty, 0)$, $n \in \mathbb{N}$ and $j \in M_n$, we can define

$$R_{jn}(\lambda) = \big(\lambda - m\big(V_n^j\big)\mathcal{A}_d\big)^{-1},$$

and hence, for a fixed angle $0 < \theta < \pi$, there exists $K > 0$ such that, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$, $n \in \mathbb{N}$ and $j \in M_n$, we have

$$\big\|R_{jn}(\lambda)\big\| = \big[m\big(V_n^j\big)\big]^{-1} \left\| R\left(\frac{\lambda}{m(V_n^j)}, \mathcal{A}_d\right)\right\| \leqslant \frac{K}{|\lambda|}. \tag{3.21}$$

Moreover, if we set $\mu = \lambda[m(V_n^j)]^{-1}$, then we have

$$\mathcal{A}_d R_{jn}(\lambda) = \big[m\big(V_n^j\big)\big]^{-1}\big(-I + \lambda R_{jn}(\lambda)\big)$$

and hence,

$$\|\mathcal{A}_d R_{jn}(\lambda)\| \leqslant \left[m\left(V_n^j\right)\right]^{-1}(1+K) \leqslant \frac{1+K}{m_0} \tag{3.22}$$

with $m_0 = \min_{x\in S_d} m(x) > 0$. We now consider the approximate resolvents of the operator $m\mathcal{A}_d$ defined by

$$S_n(\lambda)u = \sum_{j\in M_n} \phi_n^j \cdot R_{jn}(\lambda)\left(\phi_n^j u\right), \quad u \in C(S_d).$$

Combining (3.21) with (3.20), we obtain, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta\}$ and $n \in \mathbb{N}$, that

$$\|S_n(\lambda)\| \leqslant \frac{3^d K}{|\lambda|}. \tag{3.23}$$

Since we have, for every $\phi, \eta \in D(\mathcal{A}_d)$ that

$$\mathcal{A}_d(\phi\eta) = \eta\mathcal{A}_d(\phi) + \phi\mathcal{A}_2(\eta) + \sum_{i,j=1}^{d} x_i(\delta_{ij} - x_j)\partial_{x_i}\phi\partial_{x_j}\eta,$$

the operators $S_n(\lambda)$ satisfy, for every $u \in C(S_d)$,

$$\begin{aligned}(\lambda - m\mathcal{A}_d)S_n(\lambda)u &= u + \sum_{j\in M_n} \phi_n^j\left(m\left(V_n^j\right) - m\right)\mathcal{A}_d\left(R_{jn}(\lambda)\left(\phi_n^j u\right)\right)\\ &\quad - \sum_{j\in M_n} m\mathcal{A}_d\left(\phi_n^j\right) \cdot R_{jn}(\lambda)\left(\phi_n^j u\right) - \sum_{j\in M_n}\sum_{i,h=1}^{d} x_i(\delta_{ih} - x_h)\partial_{x_i}\left(R_{jn}(\lambda)\left(\phi_n^j u\right)\right)\partial_{x_h}\phi_n^j\\ &=: \left(I + C_1(\lambda) + C_2(\lambda) + C_3(\lambda)\right)u.\end{aligned}$$

We fix $\bar{n} \in \mathbb{N}$ such that $\sup_{J_{\bar{n}}^j} |m(x) - m(V_{\bar{n}}^j)| \leqslant \varepsilon =: \frac{m_0}{2.3^d(1+K)}$ for $j \in M_{\bar{n}}$. Then, from (3.20)–(3.22) and Proposition 3.8(2), and argumenting as in proof of Proposition 2.6 we obtain, for every $\lambda \in \{z \in \mathbb{C} \mid |\arg z| < \theta|\}$ with $|\lambda| > l$ and $u \in C(S_d)$, that

$$\|C_1(\lambda)u\|_{S_d} < \frac{1}{2}\|u\|_{S_d}, \qquad \|C_2(\lambda)u\|_{S_d} \leqslant \frac{K'}{|\lambda|}\|u\|_{S_d}, \qquad \|C_3(\lambda)u\|_{S_d} \leqslant \frac{K''}{\sqrt{\lambda}}\|u\|_{S_d},$$

for some positive constants K', K'' independent of λ and u. Now, if $|\lambda|$ is large enough, then we get $\|C_1(\lambda) + C_2(\lambda) + C_3(\lambda)\| < 1$ and hence, the operator $B = (\lambda - mA_2)S_{\bar{n}}(\lambda)$ is invertible in $\mathcal{L}(C(S_d))$. So, there exists $R(\lambda, m\mathcal{A}_d) = S_{\bar{n}}(\lambda)B^{-1}$ in $\mathcal{L}(C(S_d))$ and, by (3.23) $\|R(\lambda, m\mathcal{A}_d)\| = \|S(\lambda)B^{-1}\| \leqslant \frac{M'}{|\lambda|}$ for some $M' > 0$ independent of λ, provided $\lambda - m\mathcal{A}_d$ is injective and, in particular, for $\lambda > 0$ as $m\mathcal{A}_d$ is dissipative. To conclude the proof it now suffices to repeat the argument already used in the proof of Proposition 2.6. □

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