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Conjugacy for normal or abnormal linear Bolza problems

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Abstract

For certain Bolza problems with linear dynamics, two sets extending the notion of conjugate points in the calculus of variations are introduced. Independently of nonsingularity assumptions, their emptiness, in one case without normality assumptions, is shown to be equivalent to a second order necessary condition. A comparison with other notions available in the literature is given.

Keywords: Optimal control; Conjugate points; Second order necessary conditions; Normality

1. Introduction

In this paper we shall be concerned with the problem (see the details in Section 2) of minimizing a functional of the form $I(x, u) = g(x(t_1)) + \int_{t_0}^{t_1} L(t, x(t), u(t))\,dt$ subject to:

(a) $(x, u) : [t_0, t_1] \to \mathbf{R}^n \times \mathbf{R}^m$ with x piecewise C^1, u piecewise continuous;
(b) $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ $(t \in [t_0, t_1])$;
(c) $x(t_0) = \xi_0$, $Cx(t_1) + b = 0$.

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For this problem, a second order necessary condition for optimality can be easily derived without normality assumptions, and our aim is to find an appropriate notion of conjugate points which characterizes that condition. In the classical theory of calculus of variations, this is usually achieved by means of nontrivial solutions of Jacobi's equation (Euler's equation for the second integrand) vanishing at two points which are called conjugate to each other. The condition of Jacobi states that the set of points conjugate to an endpoint is empty in the underlying open time interval, and it is equivalent to a second order necessary condition assuming the trajectory under consideration satisfies the strengthened condition of Legendre and, therefore, it is nonsingular.

For the problem posed above, let us denote by $\mathcal{H}$ the set of processes (x, u) satisfying the second order necessary condition. In this paper we first introduce a set of points $\mathcal{S}_1(x, u)$ with the property that $(x, u) \in \mathcal{H}$ if and only if $\mathcal{S}_1(x, u)$ is empty. It extends the classical theory not only to more general problems but also to singular trajectories. In fact, one can easily show that, in the calculus of variations context, this set contains the classical set of conjugate points in the open interval under nonsingularity assumptions. We compare it with a set $\mathcal{G}_1(x, u)$ of "generalized conjugate points" defined in [4] and show that $\mathcal{G}_1(x, u) \subset \mathcal{S}_1(x, u)$. For normal problems, we introduce a second set $\mathcal{S}_2(x, u)$ whose emptiness is again equivalent to the condition $(x, u) \in \mathcal{H}$, but it might be easier to check its nonemptiness than that of $\mathcal{S}_1(x, u)$. We compare it with a set $\mathcal{G}_2(x, u)$ of "generalized coupled points" introduced in [7] and show that $\mathcal{G}_1(x, u) \subset \mathcal{G}_2(x, u) \subset \mathcal{S}_2(x, u)$. All these sets of "extended conjugate points" are intervals in $\mathbf{R}$, and $\mathcal{S}_1$ and $\mathcal{S}_2$ correspond to a generalization of two sets first introduced in [1,6].

Now, for the "conjugate intervals" defined in [4,7], their emptiness has been established merely as a sufficient condition for the existence of negative second variations. Also (we hope most readers will agree) the main idea of characterizing a condition is, generally, to obtain a simpler way of verifying it. However, even in the calculus of variations, simple examples show that to solve the question of nonemptiness of these sets may be much more difficult than checking if that condition holds. For the two sets introduced in this paper, however, these two undesirable features are completely solved.

2. The problem and necessary conditions

Suppose we are given an interval $T := [t_0, t_1]$ in $\mathbf{R}$, open sets $O \subset \mathbf{R}^n$ and $V \subset \mathbf{R}^m$, $\xi_0 \in \mathbf{R}^n$, $b \in \mathbf{R}^k$ $(k \leqslant n)$, a constant matrix C of dimension $k \times n$, functions $g : O \to \mathbf{R}$ and $L : T \times O \times V \to \mathbf{R}$, and matrices of continuous functions A, B of dimensions $n \times n$ and $n \times m$, respectively. Denote by $X(T, O)$ the space of piecewise C^1 functions mapping T to O, by $\mathcal{U}(T, V)$ the space of piecewise continuous functions mapping T to V, set $Z := X(T, O) \times \mathcal{U}(T, V)$, and define

$$D := \{(x, u) \in Z \mid \dot{x}(t) = A(t)x(t) + B(t)u(t)\ (t \in T)\},$$
$$Z_e := \{(x, u) \in D \mid x(t_0) = \xi_0,\ Cx(t_1) + b = 0\}.$$

Let $I : Z \to \mathbf{R}$ be given by $I(x, u) := g(x(t_1)) + \int_{t_0}^{t_1} L(\tilde{x}(t))\, dt$, where $(\tilde{x}(t))$ is short for $(t, x(t), u(t))$. The problem we shall deal with, which we label (P), is that of minimizing I

over Z_e. We assume throughout the paper that L and g are of class C^2, and the matrix C is of full rank.

Elements of Z will be called *processes*, and a process (x, u) *solves* (P) if $(x, u) \in Z_e$ and $I(x, u) \leqslant I(y, v)$ for all $(y, v) \in Z_e$. Without loss of generality, the theory to follow will be applied to global solutions since the necessary conditions we state hold for any open sets $O \subset \mathbf{R}^n$ and $V \subset \mathbf{R}^m$. Thus, shrinking these sets if necessary, the same conditions remain valid for local minima.

To state well-known first and second order conditions for (P) (see, for example, [4,7]), we shall find convenient to introduce the following notation.

- For any $S \subset T$ and $r \in \mathbf{N}$, let $X(S, \mathbf{R}^r)$ (respectively $\mathcal{U}(S, \mathbf{R}^r)$) be the space of piecewise C^1 (respectively piecewise continuous) functions mapping S to $\mathbf{R}^r$. For simplicity, set $X := X(T, \mathbf{R}^n)$, $\mathcal{U} := \mathcal{U}(T, \mathbf{R}^m)$.
- Define the set of *admissible variations* as
$$Y := \big\{(y, v) \in X \times \mathcal{U} \mid \dot{y}(t) = A(t)y(t) + B(t)v(t)\ (t \in T),\ y(t_0) = 0,\\ Cy(t_1) = 0\big\}.$$
- The problem (P) will be called *normal* if there is no nonzero solution p on T of the system
$$\dot{p}(t) + A^*(t)p(t) = 0, \quad B^*(t)p(t) = 0, \quad t \in T, \quad -p(t_1) \in \mathcal{N},$$
where $\mathcal{N} := \{p \in \mathbf{R}^n \mid p = C^*\gamma \text{ for some } \gamma \in \mathbf{R}^k\}$ and '$*$' denotes transpose.
- For all $(x, u) \in Z$, let $M(x, u)$ be the set of all $(\lambda_0, p) \in \mathbf{R} \times X$ satisfying:
  (i) $\lambda_0 \geqslant 0$ and $\lambda_0 + |p| \neq 0$;
  (ii) $\dot{p}(t) + A^*(t)p(t) = \lambda_0 L_x^*(\tilde{x}(t))\ (t \in T)$;
  (iii) $B^*(t)p(t) = \lambda_0 L_u^*(\tilde{x}(t))\ (t \in T)$;
  (iv) $-[p(t_1) + \lambda_0 g'(x(t_1))^*] \in \mathcal{N}$.
- Consider the sets
$$\mathcal{E} := \big\{(x, u, p) \in Z \times X \mid (x, u) \in D \text{ and } (1, p) \in M(x, u)\big\},$$
$$\mathcal{H} := \big\{(x, u) \in Z \mid I''\big((x, u); (y, v)\big) \geqslant 0 \text{ for all } (y, v) \in Y\big\},$$
where
$$I''\big((x, u); (y, v)\big) = \big\langle y(t_1), \Lambda y(t_1)\big\rangle + \int_{t_0}^{t_1} 2\Omega\big(t, y(t), v(t)\big)\, dt, \quad (y, v) \in X \times \mathcal{U},$$
$\Lambda := g''\big(x(t_1)\big)$ and, for all $(t, y, v) \in T \times \mathbf{R}^n \times \mathbf{R}^m$,
$$2\Omega(t, y, v) := \big\langle y, L_{xx}\big(\tilde{x}(t)\big)y\big\rangle + 2\big\langle y, L_{xu}\big(\tilde{x}(t)\big)v\big\rangle + \big\langle v, L_{uu}\big(\tilde{x}(t)\big)v\big\rangle.$$

Theorem 2.1. *Suppose* (x, u) *solves* (P). *Then* $M(x, u) \neq \emptyset$. *If* (P) *is normal then there exists a unique* $p \in X$ *such that* $(x, u, p) \in \mathcal{E}$ *and, moreover,* $(x, u) \in \mathcal{H}$.

For the problem we are dealing with, the normality assumption is no longer required for the second order condition to hold. Let us give a simple proof of this fact.

Theorem 2.2. *If* (x,u) *solves* (P) *then* $(x,u)\in\mathcal{H}$.

Proof. Let $(y,v)\in Y$, $\epsilon\in\mathbf{R}$, and set $(z,w):=(x+\epsilon y,u+\epsilon v)$. Clearly $\dot z(t)=A(t)z(t)+B(t)w(t)$ $(t\in T)$, $z(t_0)=\xi_0$ and $Cz(t_1)+b=0$. Thus, since O and V are open, there exists $\delta>0$ such that, for all $|\epsilon|<\delta$, $(x+\epsilon y,u+\epsilon v)$ belongs to Z_e. The function $h:(-\delta,\delta)\to\mathbf{R}$ given by $h(\epsilon)=I(x+\epsilon y,u+\epsilon v)$ $(|\epsilon|<\delta)$ has therefore a minimum at $\epsilon=0$ and so

$$\begin{aligned}0 &= h'(0)=I'\big((x,u);(y,v)\big)\\ &:= g'\big(x(t_1)\big)y(t_1)+\int_{t_0}^{t_1}\big\{L_x\big(\tilde x(t)\big)y(t)+L_u\big(\tilde x(t)\big)v(t)\big\}\,dt\end{aligned}$$

and $0\leqslant h''(0)=I''((x,u);(y,v))$. In particular, $(x,u)\in\mathcal{H}$. □

3. Conjugate intervals

For all $s\in[t_0,t_1)$ let $T_s=[s,t_1]$, $X_s=X(T_s,\mathbf{R}^n)$, $\mathcal{U}_s=\mathcal{U}(T_s,\mathbf{R}^m)$, and

$$\begin{aligned}Y_s := \big\{(y,v)\in X_s\times\mathcal{U}_s \mid\ & \dot y(t)=A(t)y(t)+B(t)v(t)\ (t\in T_s),\ y(s)=0,\\ & Cy(t_1)=0\big\}.\end{aligned}$$

Whenever we are given $(x,u)\in Z$ and $(y,v)\in X_s\times\mathcal{U}_s$, we shall consider the functions $\sigma:T_s\to\mathbf{R}^n$ and $\rho:T_s\to\mathbf{R}^m$ defined by

$$\begin{aligned}\sigma[y,v](t)=\sigma(t)&:=L_{xx}\big(\tilde x(t)\big)y(t)+L_{xu}\big(\tilde x(t)\big)v(t),\\ \rho[y,v](t)=\rho(t)&:=L_{ux}\big(\tilde x(t)\big)y(t)+L_{uu}\big(\tilde x(t)\big)v(t).\end{aligned}$$

Also, given $(x,u)\in Z$ and (y,v) and $(z,w)\in X_s\times\mathcal{U}_s$, consider the bilinear form

$$\mathcal{F}_s\big((z,w),(y,v)\big)=\big\langle z(t_1),\Lambda y(t_1)\big\rangle+\int_s^{t_1}\big\{\big\langle z(t),\sigma(t)\big\rangle+\big\langle w(t),\rho(t)\big\rangle\big\}\,dt.$$

Definition 3.1. For any $(x,u)\in Z$ let $\mathcal{S}_1(x,u)$ be the set of points $s\in[t_0,t_1)$ for which there exists $(y,v)\in Y_s$ such that either (i) or (ii) holds:

(i) $\mathcal{F}_s((y,v),(y,v))<0$.
(ii) There exists $(z,w)\in Y$ such that

$$\mathcal{F}_s\big((z,w),(y,v)\big)^2>\mathcal{F}_s\big((y,v),(y,v)\big)I''\big((x,u);(z,w)\big).$$

Theorem 3.2. $(x,u)\in\mathcal{H}\Leftrightarrow\mathcal{S}_1(x,u)=\emptyset$.

Proof. $\Rightarrow$. Suppose there exists $s\in\mathcal{S}_1(x,u)$. Let $(y,v)\in Y_s$ be as in 3.1. Since $(x,u)\in\mathcal{H}$, 3.1(i) cannot hold. Thus $a:=\mathcal{F}_s((y,v),(y,v))\geqslant 0$ and there exists $(z,w)\in Y$ such that $\beta^2>ak$ where $\beta=\mathcal{F}_s((z,w),(y,v))$ and $k=I''((x,u);(z,w))$. Set $(\zeta(t),\eta(t)):=(0,0)$

if $t \in [t_0, s]$, $(\zeta(t), \eta(t)) := (y(t), v(t))$ if $t \in [s, t_1]$, and note that $(y_\alpha, v_\alpha) := (z + \alpha\zeta, w + \alpha\eta)$ belongs to Y for any $\alpha \in \mathbf{R}$. If $a = 0$, so that $\beta \neq 0$, set $\alpha := -(\beta + k/2\beta)$. Then $I''((x,u); (y_\alpha, v_\alpha)) = k + a\alpha^2 + 2\alpha\beta = -2\beta^2 < 0$. If $a > 0$, set $\alpha := -\beta/a$. Then $I''((x,u); (y_\alpha, v_\alpha)) = (ak - \beta^2)/a < 0$.

$\Leftarrow$. Suppose $(x,u) \notin \mathcal{H}$. Let $(y, v) \in Y$ be such that $I''((x,u); (y,v)) < 0$. Then 3.1(i) holds with $s = t_0$ and so $t_0 \in \mathcal{S}_1(x,u)$. □

In view of this result and Theorem 2.2, the nonemptiness of $\mathcal{S}_1(x,u)$ implies non-optimality of (x,u), independently of normality assumptions. For normal problems, however, we shall now introduce a set $\mathcal{S}_2(x,u)$ for which verifying membership may be easier than that of $\mathcal{S}_1(x,u)$.

The definition of this second set is strongly based on the so-called *accessory problem* to (P), which we label (AP). Given $(x,u) \in Z$, it corresponds to the problem of minimizing $I''((x,u); \cdot)/2$ over Y, that is,

$$\text{Minimize } K(y,v) := \frac{1}{2}\big\langle y(t_1), \Lambda y(t_1)\big\rangle + \int_{t_0}^{t_1} \Omega\big(t, y(t), v(t)\big)\,dt$$

subject to:

(a) $(y,v) \in X \times \mathcal{U}$;
(b) $\dot{y}(t) = A(t)y(t) + B(t)v(t)$ $(t \in T)$;
(c) $y(t_0) = 0$, $Cy(t_1) = 0$.

For this problem, let us proceed as with (P) by defining, for the normal case, the sets M and $\mathcal{E}$. Given $(x,u) \in Z$, let $\tilde{M}(y,v)$ be the set of all $q \in X$ satisfying:

(i) $\dot{q}(t) + A^*(t)q(t) = \Omega_y^*(\tilde{y}(t))$ $[= L_{xx}(\tilde{x}(t))y(t) + L_{xu}(\tilde{x}(t))v(t)]$ $(t \in T)$;
(ii) $B^*(t)q(t) = \Omega_u^*(\tilde{y}(t))$ $[= L_{ux}(\tilde{x}(t))y(t) + L_{uu}(\tilde{x}(t))v(t)]$ $(t \in T)$;
(iii) $-[q(t_1) + \Lambda y(t_1)] \in \mathcal{N}$,

and let $\tilde{\mathcal{E}}(x,u)$ be the set of all $(y,v,q) \in X \times \mathcal{U} \times X$ such that $\dot{y}(t) = A(t)y(t) + B(t)v(t)$ $(t \in T)$ and $q \in \tilde{M}(y,v)$. For any $s \in [t_0, t_1)$, let $\tilde{\mathcal{E}}_s(x,u)$ be as above with (y,v,q) defined on T_s. The following result gives first order conditions for a solution of the normal accessory problem. It does follow from Theorem 2.1 (see, for example, [5]), but under assumptions weaker than those imposed before since the integrand Ω can only be assured to be piecewise continuous.

Lemma 3.3. *Suppose* (P) *is normal and* $(x,u) \in Z$. *If* (y,v) *solves* (AP) *then there exists* $q \in X$ *such that* $(y,v,q) \in \tilde{\mathcal{E}}(x,u)$.

Definition 3.4. For any $(x,u) \in Z$ let $\mathcal{S}_2(x,u)$ be the set of points $s \in [t_0, t_1)$ for which there exists $(y,v) \in Y_s$ such that:

(i) $\mathcal{F}_s((y,v),(y,v)) \leqslant 0$.

(ii) If there exists $q \in X_s$ such that $(y, v, q) \in \tilde{\mathcal{E}}_s(x, u)$, then $s > t_0$ and either (a) or (b) holds:
  (a) $L_{uu}(\tilde{x}(s+))v(s) \neq 0$;
  (b) there exists $w \in \mathcal{U}([t_0, s], \mathbf{R}^m)$ with $\langle z(s), q(s)\rangle \neq 0$ where $z : [t_0, s] \to \mathbf{R}^n$ is the solution of $\dot{z}(t) = A(t)z(t) + B(t)w(t)$, $z(t_0) = 0$.

Theorem 3.5. *Suppose* (P) *is normal. Then* $(x, u) \in \mathcal{H} \Leftrightarrow \mathcal{S}_2(x, u) = \emptyset$.

Proof. $\Rightarrow$. Suppose there exists $s \in \mathcal{S}_2(x, u)$. Let $(y, v) \in Y_s(x, u)$ be as in 3.4 and define $(\zeta(t), \eta(t)) := (0, 0)$ if $t \in [t_0, s]$, $(\zeta(t), \eta(t)) := (y(t), v(t))$ if $t \in [s, t_1]$. Note that, by 3.4(i),

$$I''\big((x, u); (\zeta, \eta)\big) = \mathcal{F}_s\big((y, v), (y, v)\big) \leqslant 0.$$

Strict inequality contradicts the assumption $(x, u) \in \mathcal{H}$ and, therefore, (ζ, η) solves (AP). By Lemma 3.3, there exists $q_1 \in X$ such that $(\zeta, \eta, q_1) \in \tilde{\mathcal{E}}(x, u)$. Thus, for all $t \in T$,

$$\dot{q}_1(t) + A^*(t)q_1(t) = L_{xx}\big(\tilde{x}(t)\big)\zeta(t) + L_{xu}\big(\tilde{x}(t)\big)\eta(t),$$
$$B^*(t)q_1(t) = L_{ux}\big(\tilde{x}(t)\big)\zeta(t) + L_{uu}\big(\tilde{x}(t)\big)\eta(t),$$

and $-[q(t_1) + \Lambda y(t_1)] \in \mathcal{N}$. Let q be the restriction of q_1 to $[s, t_1]$. Thus $(y, v, q) \in \tilde{\mathcal{E}}_s(x, u)$ and, by 3.4(ii), $s > t_0$. Suppose that 3.4(ii)(a) holds. Then $B^*(s-)q_1(s-) = 0 \neq B^*(s+)q_1(s+)$, contradicting that $t \mapsto B^*(t)q_1(t)$ is continuous. Suppose that 3.4(ii)(b) holds. Let $\Phi : T \to \mathbf{R}^{n\times n}$ satisfy $\dot{\Phi}(t) = -\Phi(t)A(t)$ $(t \in T)$, $\Phi(t_1) = I_n$. Since $\dot{q}_1(t) + A^*(t)q_1(t) = 0$ $(t \in [t_0, s])$, we have $q_1(s) = \Phi^*(s)\Phi^{*-1}(t)q_1(t)$ $(t \in [t_0, s])$, and so

$$0 \neq \big\langle z(s), q(s)\big\rangle = \int_{t_0}^{s} \big\langle \Phi^{-1}(s)\Phi(t)B(t)w(t), q_1(s)\big\rangle\, dt = \int_{t_0}^{s} \big\langle w(t), B^*(t)q_1(t)\big\rangle\, dt.$$

But $\langle w(t), B^*(t)q_1(t)\rangle = 0$ for all $t \in [t_0, s]$ and we reach a contradiction.

$\Leftarrow$. Suppose $(x, u) \notin \mathcal{H}$. Let $(y, v) \in Y$ be such that $I''((x, u); (y, v)) < 0$. Clearly, condition 3.4(ii) does not apply since, otherwise, $I''((x, u); (y, v))$, which coincides with $\mathcal{F}_{t_0}((y, v), (y, v))$, would vanish. Thus $t_0 \in \mathcal{S}_2(x, u)$. $\square$

4. Comparison with other notions

For problem (P) let us introduce the set of "generalized conjugate points" defined in [4].

Definition 4.1. For any $(x, u) \in Z$ let $\mathcal{G}_1(x, u)$ be the set of points $s \in [t_0, t_1)$ for which there exist $(y, v) \in Y_s$ and $q \in X_s$ such that, if $\lambda(t) := B^*(t)q(t) - \rho(t)$ $(t \in T_s)$, then

(i) $\dot{q}(t) + A^*(t)q(t) = \sigma(t)$ $(t \in T_s)$;
(ii) $q(s) \neq 0$, $-[q(t_1) + \Lambda y(t_1)] \in \mathcal{N}$;
(iii) $\langle v(t), \lambda(t)\rangle \geqslant 0$ $(t \in T_s)$;

and either (a) or (b) holds:

(a) $\langle v(t), \lambda(t)\rangle > 0$ on a set of positive measure;
(b) there exists $(z, w) \in Y$ such that $\langle z(s), q(s)\rangle > 0$ and $\langle w(t), \lambda(t)\rangle \geqslant 0$ $(t \in T_s)$.

Theorem 4.2. *For any* $(x, u) \in Z$, $\mathcal{G}_1(x, u) \subset \mathcal{S}_1(x, u)$.

Proof. Let $s \in \mathcal{G}_1(x, u)$ and let $(y, v) \in Y_s$ and $q \in X_s$ be as in 4.1. Observe first that

$$\begin{aligned}
&\mathcal{F}_s\big((y, v), (y, v)\big) \\
&\quad = \big\langle y(t_1), \Lambda y(t_1)\big\rangle \int_s^{t_1} \big\{\big\langle y(t), \dot q(t) + A^*(t)q(t)\big\rangle + \big\langle v(t), B^*(t)q(t) - \lambda(t)\big\rangle\big\}\, dt \\
&\quad = \big\langle y(t_1), \Lambda y(t_1) + q(t_1)\big\rangle - \int_s^{t_1} \big\langle v(t), \lambda(t)\big\rangle\, dt \leqslant 0.
\end{aligned}$$

If 4.1(a) holds, so does 3.1(i). If 4.1(a) does not hold, then $\mathcal{F}_s((y, v), (y, v)) = 0$ and there exists $(z, w) \in Y$ such that $\langle z(s), q(s)\rangle > 0$ and $\langle w(t), \lambda(t)\rangle \geqslant 0$ $(t \in T_s)$. Therefore,

$$\mathcal{F}_s\big((z, w), (y, v)\big) \leqslant \big\langle z(t_1), \Lambda y(t_1) + q(t_1)\big\rangle - \big\langle z(s), q(s)\big\rangle - \int_s^{t_1} \big\langle w(t), \lambda(t)\big\rangle\, dt < 0$$

and so 3.1(ii) holds. □

Let us turn now, for problem (P), to the set of "generalized coupled points" defined in [7].

Definition 4.3. For any $(x, u) \in Z$ let $\mathcal{G}_2(x, u)$ be the set of points $s \in [t_0, t_1)$ for which there exist $(y, v) \in Y_s$ and $q \in X_s$ such that if $\lambda(t) := B^*(t)q(t) - \rho(t)$ $(t \in T_s)$ then

(i) $\dot q(t) + A^*(t)q(t) = \sigma(t)$ $(t \in T_s)$;
(ii) $-[q(t_1) + \Lambda y(t_1)] \in \mathcal{N}$;
(iii) $\langle v(t), \lambda(t)\rangle \geqslant 0$ $(t \in T_s)$;
(iv) if the inequality in (iii) is equality for all $t \in T_s$ then, for any $\alpha \in \mathbf{R}^k$ satisfying $B^*(t)\Phi^*(t)C^*\alpha = \lambda(t)$ $(t \in T_s)$, there exists $w \in \mathcal{U}([t_0, s], \mathbf{R}^m)$ such that $\langle z(s), \Phi^*(s)C^*\alpha - q(s)\rangle < 0$, where $\Phi : T \to \mathbf{R}^{n\times n}$ satisfies $\dot\Phi(t) = -\Phi(t)A(t)$ $(t \in T)$, $\Phi(t_1) = I_n$, and z is the solution of $\dot z(t) = A(t)z(t) + B(t)w(t)$, $z(t_0) = 0$.

Theorem 4.4. *For any* $(x, u) \in Z$, $\mathcal{G}_1(x, u) \subset \mathcal{G}_2(x, u) \subset \mathcal{S}_2(x, u)$.

Proof. Let $s \in \mathcal{G}_1(x, u)$ and let $(y, v) \in Y_s$ and $q \in X_s$ satisfy 4.1. If 4.1(a) holds then $s \in \mathcal{G}_2(x, u)$. If 4.1(a) does not hold then there exists $(z, w) \in Y$ satisfying 4.1(b). Let $M(t) := B^*(t)\Phi^*(t)C^*$ $(t \in T)$ and let $\alpha \in \mathbf{R}^k$ satisfy $M(t)\alpha = \lambda(t)$ $(t \in T_s)$. Since $\langle z(t_1), C^*\alpha\rangle = 0$, $\langle M(t)\alpha, w(t)\rangle \geqslant 0$ $(t \in T_s)$, and

$$z(t) = \int_{t_0}^{t} \Phi(t)^{-1}\Phi(\tau)B(\tau)w(\tau)\,d\tau, \quad t \in T,$$

we have

$$\big\langle z(s), \Phi^*(s)C^*\alpha\big\rangle = \int_{t_0}^{s} \big\langle M(t)\alpha, w(t)\big\rangle\,dt = -\int_{s}^{t_1} \big\langle M(t)\alpha, w(t)\big\rangle\,dt \leqslant 0$$

and so $\langle z(s), \Phi^*(s)C^*\alpha - q(s)\rangle < 0$. This proves the first contention.

Now, let $s \in \mathcal{G}_2(x,u)$ and let $(y,v) \in Y_s$ and $q \in X_s$ be as in 4.3. As in the proof of 4.2, we have that 3.4(i) holds. Suppose there exists $q_1 \in X_s$ such that $(y,v,q_1) \in \tilde{\mathcal{E}}_s(x,u)$. Therefore

$$0 = \mathcal{F}_s\big((y,v),(y,v)\big) = -\int_{s}^{t_1} \big\langle v(t), \lambda(t)\big\rangle\,dt \leqslant 0$$

and the inequality in 4.3(iii) is equality for all $t \in T_s$. Let $l, l_1 \in \mathbf{R}^k$ be such that $q^*(t_1) = -y^*(t_1)\Lambda - l^*C$ and $q_1^*(t_1) = -y^*(t_1)\Lambda - l_1^*C$, and define $r(t) := q(t) - q_1(t)$ $(t \in T_s)$ and $\alpha := l_1 - l$. Since $\dot{r}(t) + A^*(t)r(t) = 0$ we have $r(t) = \Phi^*(t)r(t_1) = \Phi^*(t)C^*\alpha$ $(t \in T_s)$. Hence $B^*(t)\Phi^*(t)C^*\alpha = \lambda(t) + \rho(t) - B^*(t)q_1(t) = \lambda(t)$ $(t \in T_s)$. By 4.3(iv), there exists $w \in \mathcal{U}([t_0,s],\mathbf{R}^m)$ such that

$$0 > \big\langle z(s), \Phi^*(s)C^*\alpha - q(s)\big\rangle = \big\langle z(s), r(s) - q(s)\big\rangle = -\big\langle z(s), q_1(s)\big\rangle,$$

where $z : [t_0,s] \to \mathbf{R}^n$ is the solution of $\dot{z}(t) = A(t)z(t) + B(t)w(t)$, $z(t_0) = 0$. Thus $s > t_0$, 3.4(ii)(b) holds, and $s \in \mathcal{S}_2(x,u)$. □

5. An example

In this section we provide an example of a simple problem which illustrates how the theory related to the sets $\mathcal{S}_1(x,u)$ and $\mathcal{S}_2(x,u)$ can be applied. Also, we briefly explain some of the difficulties that arise in trying to prove nonemptiness of $\mathcal{G}_1(x,u)$ and $\mathcal{G}_2(x,u)$.

Example 5.1. Let $\alpha \in \mathbf{R}$ and consider the problem (P_α) of minimizing

$$I(x,u) = \frac{1}{2}\left(x_2^2(\pi) + \int_0^{\pi} \big\{t^3u^2(t) + 2t^3x_1(t)u(t) - 3tx_1^2(t)\big\}\,dt\right) \quad \big(x = (x_1,x_2)\big)$$

subject to $\dot{x}_1(t) = \dot{x}_2(t) = x_1(t) + u(t)$ $(t \in [0,\pi])$, $x(0) = 0$ and $x_1(\pi) + \alpha x_2(\pi) = 0$.

In this case we have $T = [0,\pi]$, $n = 2$, $m = k = 1$, $\xi_0 = (0,0)$, $b = 0$, $O = \mathbf{R}^2$, $V = \mathbf{R}$, $C = (1,\alpha)$,

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$g(x_1,x_2) = x_2^2/2$, and $L(t,x_1,x_2,u) = [t^3u^2 + 2t^3x_1u - 3tx_1^2]/2$.

Observe first that (P_α) is normal if $(p_1, p_2) \equiv 0$ is the only solution of the system

$$\begin{pmatrix} \dot p_1(t) + p_1(t) + p_2(t) \\ \dot p_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$p_1(t) + p_2(t) = 0, \quad t \in T, \qquad p_2(\pi) = \alpha p_1(\pi).$$

Therefore, (P_α) is normal if and only if $\alpha \neq -1$.

Now, we have $L_x = (t^3u - 3tx_1, 0)$, $L_u = t^3u + t^3x_1$, $L_{ux} = L_{xu}^* = (t^3, 0)$, $L_{uu} = t^3$,

$$L_{xx} = \begin{pmatrix} -3t & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Lambda = g''(x_1, x_2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, for $(y_1, y_2, v) \in X_s \times \mathcal{U}_s$,

$$\sigma(t) = \begin{pmatrix} -3ty_1(t) + t^3v(t) \\ 0 \end{pmatrix}, \qquad \rho(t) = t^3y_1(t) + t^3v(t),$$

the set Y_s is given by those $(y, v) = (y_1, y_2, v) \in X_s \times \mathcal{U}_s$ satisfying

$$\begin{pmatrix} \dot y_1(t) \\ \dot y_2(t) \end{pmatrix} = \begin{pmatrix} y_1(t) + v(t) \\ y_1(t) + v(t) \end{pmatrix}, \quad t \in [0, \pi],$$
$$y(s) = 0, \qquad y_1(\pi) + \alpha y_2(\pi) = 0,$$

and the bilinear form $\mathcal{F}_s$ corresponds to

$$\mathcal{F}_s\big((z, w), (y, v)\big) = z_2(\pi)y_2(\pi) + \int_s^\pi \big\{t^3v(t)z_1(t) - 3ty_1(t)z_1(t) + t^3y_1(t)w(t) + t^3v(t)w(t)\big\}\, dt.$$

Let us prove that, for any $(x, u) \in Z$, the point $s = 0$ belongs to $\mathcal{S}_2(x, u)$. Suppose that, for some $s \in [0, \pi)$, $y = (y_1, y_2) \in X_s$ is of class C^2 with $y(s) = y(\pi) = 0$. Set $v \equiv \dot y_1 - y_1$ so that $(y, v) \in Y_s$. In this event, integrating by parts, we obtain

$$\begin{aligned} \mathcal{F}_s\big((y, v), (y, v)\big) &= \int_s^\pi \big\{t^3\big(\dot y_1(t) - y_1(t)\big)^2 + 2t^3y_1(t)\big(\dot y_1(t) - y_1(t)\big) - 3ty_1^2(t)\big\}\, dt \\ &= \int_s^\pi \big\{t^3\dot y_1^2(t) - \big(t^3 + 3t\big)y_1^2(t)\big\}\, dt = -\int_s^\pi t^3y_1(t)\big[\ddot y_1(t) + y_1(t)\big]\, dt. \end{aligned}$$

Therefore, if $(y(t), v(t)) := (\sin t, \sin t, \cos t - \sin t)$ $(t \in T)$, then $(y, v) \in Y_0$ and $\mathcal{F}_0((y, v), (y, v)) = 0$. Thus, the first condition in the definition of $\mathcal{S}_2(x, u)$ is satisfied. For the second, suppose that there exists $(q_1, q_2) \in X$ such that $(y, v, q) \in \tilde{\mathcal{E}}(x, u)$. By definition, this means that $q_2(\pi) = \alpha q_1(\pi)$ and, for all $t \in T$,

$$\begin{pmatrix} \dot q_1(t) + q_1(t) + q_2(t) \\ \dot q_2(t) \end{pmatrix} = \begin{pmatrix} t^3v(t) - 3ty_1(t) \\ 0 \end{pmatrix},$$
$$q_1(t) + q_2(t) = t^3y_1(t) - t^3v(t).$$

This implies that

$$\dot{q}_1(t) = 3t^2 \cos t - t^3 \sin t = -3t \sin t - t^3 \sin t, \quad t \in T,$$

which is not the case. Hence $0 \in \mathcal{S}_2(x,u)$. An application of Theorems 2.2 and 3.5 shows that, for any $\alpha \neq -1$, the problem (P_α) has no solution.

For the case $\alpha = -1$, let us prove that $s = 0$ belongs also to $\mathcal{S}_1(x,u)$. Indeed, with the function (y,v) used before, note that it suffices to show that there exists $(z,w) = (z_1, z_2, w) \in Y$ such that $\mathcal{F}_0((z,w),(y,v)) \neq 0$. By definition, (z,w) should satisfy

$$\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} z_1(t) + w(t) \\ z_1(t) + w(t) \end{pmatrix}, \quad t \in [0,\pi], \qquad z(0) = 0, \qquad z_1(\pi) = z_2(\pi)$$

and therefore

$$\begin{aligned} \mathcal{F}_0\big((z,w),(y,v)\big) &= \int_0^\pi \big\{t^3 v(t) z_1(t) + t^3 v(t) w(t) + t^3 y_1(t) w(t) - 3t y_1(t) z_1(t)\big\}\, dt \\ &= \int_0^\pi \big\{t^3 \dot{y}_1(t) \dot{z}_1(t) - \big(t^3 + 3t\big) y_1(t) z_1(t)\big\}\, dt. \end{aligned}$$

If, for example, we set $(z_1(t), z_2(t), w(t)) := (t, t, 1-t)$, $t \in T$, then

$$\mathcal{F}_0\big((z,w),(y,v)\big) = \int_0^\pi \big\{t^3 \cos t - \big(t^4 + 3t^2\big) \sin t\big\}\, dt = -\int_0^\pi \big(t^4 + 6t^2\big) \sin t\, dt \neq 0.$$

Hence $0 \in \mathcal{S}_1(x,u)$ and, by Theorems 2.2 and 3.2, (P_α) with $\alpha = -1$ has no solution.

Let us turn now to the sets $\mathcal{G}_1(x,u)$ and $\mathcal{G}_2(x,u)$. Note first that the problem (P_α) with $\alpha = -1$, being abnormal, lies beyond the scope of Zeidan [7] but not that of Loewen and Zheng [4]. Now, by definition, if a point $s \in [0,\pi)$ belongs to any of these sets, there exist $(y,v) \in Y_s$ and $q \in X_s$ such that $q_1(\pi) = -q_2(\pi) - y_2(\pi)$ and, for all $t \in T_s$,

$$\begin{pmatrix} \dot{q}_1(t) + q_1(t) + q_2(t) \\ \dot{q}_2(t) \end{pmatrix} = \begin{pmatrix} t^3 v(t) - 3t y_1(t) \\ 0 \end{pmatrix} \quad \text{and} \quad v(t)\lambda(t) \geqslant 0,$$

where $\lambda(t) = q_1(t) + q_2(t) - t^3 y_1(t) - t^3 v(t)$. As one readily verifies, the additional conditions defining membership of $\mathcal{G}_1(x,u)$ or $\mathcal{G}_2(x,u)$ rule out the choice $y_1 \equiv 0$. Also (we omit the details) the function used to prove nonemptiness of $\mathcal{S}_1(x,u)$ and $\mathcal{S}_2(x,u)$ contradicts the above relations, and the question of how such functions can be found is left unsolved.

6. Nonemptiness of $\mathcal{S}_2(x,u)$

In the introduction we mentioned that there are problems for which checking that (x,u) does not belong to $\mathcal{H}$ may be easier than verifying nonemptiness of $\mathcal{G}_1(x,u)$ or $\mathcal{G}_2(x,u)$. In fact, there are examples where one can trivially find an admissible variation $(y,v) \in Y$ with $I''((x,u);(y,v)) < 0$, so that $(x,u) \notin \mathcal{H}$, but there does not exist $q \in X$ such that (y,v,q)

satisfies the conditions defining membership of $\mathcal{G}_1(x,u)$ or $\mathcal{G}_2(x,u)$. This does not occur with the sets introduced in this paper since the use of the same admissible variation (y,v) shows that $s = t_0$ belongs to $\mathcal{S}_1(x,u)$ and $\mathcal{S}_2(x,u)$.

Similarly, if a point s belongs to $\mathcal{G}_1(x,u)$ or $\mathcal{G}_2(x,u)$, the functions appearing in the definitions of these two sets can be used to prove that s belongs also to $\mathcal{S}_1(x,u)$ or $\mathcal{S}_2(x,u)$. The converse, however, may not occur (see Example 5.1) since one may find $(y,v) \in Y_s$ satisfying the conditions defining membership of $\mathcal{S}_1(x,u)$ or $\mathcal{S}_2(x,u)$, but there does not exist $q \in X_s$ such that (y,v,q) satisfies those of $\mathcal{G}_1(x,u)$ or $\mathcal{G}_2(x,u)$.

To illustrate these facts, let us provide a simple calculus of variations problem. For a more complete study of a class of problems including this particular one, we refer to [2,3].

Example 6.1. Let $a > 0$ and consider the problem of minimizing $I(x,u) = \frac{1}{2}\int_0^a t\{u^2(t) - x^2(t)\}\,dt$ subject to $\dot{x}(t) = u(t)$ $(t \in [0,a])$ and $x(0) = x(a) = 0$.

In this case we have $T = [0,a]$, $n = m = k = 1$, $\xi_0 = b = 0$, $O = V = \mathbf{R}$, $A \equiv g \equiv 0$, $B \equiv 1$, $C = 1$, and $L(t,x,u) = t(u^2 - x^2)/2$.

Let $(x,u) \in Z$. Observe first that, for any $s \in [0,a)$,

$$Y_s = \big\{(y,v) \in X \times \mathcal{U} \mid \dot{y}(t) = v(t)\ (t \in T_s),\ y(s) = y(a) = 0\big\},$$

and, for any (y,v) and $(z,w) \in X_s \times \mathcal{U}_s$,

$$\mathcal{F}_s\big((z,w),(y,v)\big) = \int_s^a t\big\{w(t)v(t) - z(t)y(t)\big\}\,dt.$$

Let $(y_1(t), v_1(t)) := (t, 1)$ for $t \in [0, a/2]$ and $(y_1(t), v_1(t)) := (a-t, -1)$ for $t \in (a/2, a]$. Then, as one readily verifies,

$$I''\big((x,u);(y_1,v_1)\big) = \mathcal{F}_0\big((y_1,v_1),(y_1,v_1)\big) = a^2/2 - a^4/24$$

which is negative if $a > \sqrt{12}$. In other words, if $a > \sqrt{12}$, there exists $(y,v) \in Y = Y_0$ with a negative second variation, showing that $(x,u) \notin \mathcal{H}$. Also, the proof shows that $0 \in \mathcal{S}_1(x,u) \cap \mathcal{S}_2(x,u)$. Since this holds independently of (x,u), it follows that, if $a > \sqrt{12}$, then $\mathcal{H} = \emptyset$, and so the problem has no solution.

On the other hand, if a point $s \in [0,a)$ belongs to $\mathcal{G}_1(x,u)$ or $\mathcal{G}_2(x,u)$ then, necessarily, there exist $(y,v) \in Y_s$ with $y \not\equiv 0$, and $q \in X_s$, such that $\dot{q}(t) = -ty(t)$ and $v(t)(q(t) - tv(t)) \geqslant 0$ for all $t \in [s,a]$. Thus there exists $c \in \mathbf{R}$ such that

$$\dot{y}(t)\left(c - \int_s^t \tau y(\tau)\,d\tau - t\dot{y}(t)\right) \geqslant 0 \quad \text{for all } t \in [s,a].$$

For the function defined above it is required for $c \in \mathbf{R}$ to satisfy both

$$c - \int_0^{a/2} \tau y_1(\tau)\,d\tau \geqslant a/2 > 0 \quad \text{and} \quad c - \int_0^{a/2} \tau y_1(\tau)\,d\tau \leqslant -a/2 < 0$$

which is a contradiction.

A similar situation occurs with the function $(y_2(t), v_2(t)) := (\sin t, \cos t)$ if $t \in [0, \pi]$ and $(y_2(t), v_2(t)) := (0, 0)$ if $t \in (\pi, a]$. Indeed, for the sets $\mathcal{G}_1(x, u)$ and $\mathcal{G}_2(x, u)$, one requires the existence of $q \in X$ such that

$$\cos t \left(c - \int_0^t \tau \sin \tau \, d\tau - t \cos t \right) \geqslant 0 \quad \text{for all } t \in [0, \pi].$$

This implies, in particular, that $c - \sin t \geqslant 0$ for $t \in [0, \pi/2)$ and $c - \sin t \leqslant 0$ for any $t \in (\pi/2, \pi]$, which is impossible. On the other hand,

$$I''\big((x, u); (y_2, v_2)\big) = \mathcal{F}_0\big((y_2, v_2), (y_2, v_2)\big) = \int_0^\pi t\{\cos^2 t - \sin^2 t\} \, dt = 0$$

and so $(y_2, v_2) \in Y$ does not have a negative second variation. It does satisfy, however, the condition 3.4(i) in the definition of $\mathcal{S}_2(x, u)$. Moreover, there does not exist $q \in X$ such that $(y_2, v_2, q) \in \tilde{\mathcal{E}}$ since, if this were the case, we would have $q(t) = t \cos t$ and $\dot{q}(t) = -t \sin t$ ($t \in [0, \pi]$), which is not the case. This implies that the point $s = 0$ belongs to $\mathcal{S}_2(x, u)$ and therefore, if $a \geqslant \pi$, the problem has no solution. Let us finally mention that, by using (y_2, v_2) to show that $s = 0$ belongs also to $\mathcal{S}_1(x, u)$, we require to prove the existence of $(z, w) \in Y$ such that

$$\mathcal{F}_s\big((z, w), (y_2, v_2)\big) = \int_0^\pi t\big\{\dot{z}(t) \cos t - z(t) \sin t\big\} \, dt = -\int_0^\pi z(t) \cos t \, dt \neq 0,$$

and this inequality clearly occurs if, for example, $z(t) = \sin t \cos t$, $t \in [0, \pi]$.

This example motivates a simple device for proving nonemptiness of $\mathcal{S}_2(x, u)$. Let us consider, for simplicity of exposition, the basic fixed-endpoint problem in the calculus of variations.

Suppose that we are given an interval $T = [t_0, t_1]$ in $\mathbf{R}$, two points ξ_0, ξ_1 in $\mathbf{R}$, and a function L mapping $T \times \mathbf{R} \times \mathbf{R}$ to $\mathbf{R}$. The problem is to minimize $I(x, u) = \int_{t_0}^{t_1} L(\tilde{x}(t)) \, dt$ subject to $\dot{x}(t) = u(t)$ $(t \in T)$, $x(t_0) = \xi_0$ and $x(t_1) = \xi_1$.

With respect to problem (P), we have $n = m = k = 1$, $b = -\xi_1$, $O = V = \mathbf{R}$, $A \equiv g \equiv 0$, $B \equiv 1$ and $C = 1$.

Proposition 6.2. *Let $(x, u) \in Z$, $s \in [t_0, t_1)$, and define*

$$p(t) := L_{uu}\big(\tilde{x}(t)\big), \qquad r(t) := \left[\frac{d}{dt} L_{xu}\big(\tilde{x}(t)\big)\right] - L_{xx}\big(\tilde{x}(t)\big), \quad t \in T.$$

Suppose that $p \in C^2$, r is continuous, and there exists $y \in C^2(T_s, \mathbf{R})$ with $y(s) = y(t_1) = 0$ such that

(i) $p(t)\ddot{y}(t) + [r(t) - \ddot{p}(t)/2]y(t) = 0$ *for all $t \in T_s$;*

(ii) $\dot{p}(t)\dot{y}(t) + [\ddot{p}(t)/2]y(t) \neq 0$ *for some $t \in T_s$.*

Then $s \in S_2(x, u)$.

Proof. Let $v \equiv \dot{y}$ so that $(y, v) \in Y_s$, and define

$$\mathcal{K}(y) := \mathcal{F}_s\big((y, v), (y, v)\big) = \int_s^{t_1} \big\{\dot{y}(t)\rho(t) + y(t)\sigma(t)\big\}\, dt.$$

From the definition, and integrating $\int L_{xu}(\tilde{x}(t))y(t)\dot{y}(t)\, dt$ by parts, we have

$$\mathcal{K}(y) = \int_s^{t_1} \big\{L_{uu}\big(\tilde{x}(t)\big)\dot{y}^2(t) + 2L_{xu}\big(\tilde{x}(t)\big)y(t)\dot{y}(t) + L_{xx}\big(\tilde{x}(t)\big)y^2(t)\big\}\, dt$$
$$= \int_s^{t_1} \big\{p(t)\dot{y}^2(t) - r(t)y^2(t)\big\}\, dt.$$

Integrating again by parts, we have

$$\int_s^{t_1} p(t)\dot{y}^2(t)\, dt = -\int_s^{t_1} y(t)\big[p(t)\ddot{y}(t) + \dot{p}(t)\dot{y}(t)\big]\, dt$$

and therefore

$$\mathcal{K}(y) = -\int_s^{t_1} y(t)\big[p(t)\ddot{y}(t) + \dot{p}(t)\dot{y}(t) + r(t)y(t)\big]\, dt.$$

Integrating once more by parts, now $\int \dot{p}(t)\dot{y}(t)y(t)\, dt$, we obtain

$$\mathcal{K}(y) = -\int_s^{t_1} y(t)\big[p(t)\ddot{y}(t) + \beta(t)y(t)\big]\, dt,$$

where $\beta(t) = r(t) - \ddot{p}(t)/2$. By (i) we have $\mathcal{K}(y) = 0$, and condition 3.4(i) in the definition of $\mathcal{S}_2(x, u)$ is satisfied. Now, suppose there exists $q \in X_s$ such that $(y, v, q) \in \tilde{\mathcal{E}}_s(x, u)$. Then, for all $t \in T_s$,

$$0 = \dot{\rho}(t) - \sigma(t) = p(t)\ddot{y}(t) + \dot{p}(t)\dot{y}(t) + r(t)y(t) = \dot{p}(t)\dot{y}(t) + \big[r(t) - \beta(t)\big]y(t)$$

contradicting (ii). Thus $s \in \mathcal{S}_2(x, u)$. □

Note that, in Example 6.1, $p(t) = r(t) = t$, so that 6.2(i) and 6.2(ii) correspond, respectively, to

$$t\big(\ddot{y}(t) + y(t)\big) = 0 \quad \text{for all } t \in [s, \pi], \quad \text{and} \quad \dot{y}(t) \neq 0 \quad \text{for some } t \in [s, \pi].$$

Thus, setting $y(t) = \sin t$ $(t \in [0, \pi])$, it follows that $s = 0$ belongs to $\mathcal{S}_2(x, u)$. Let us end with two more examples of this nature. The first is a generalization of Example 6.1.

Example 6.3. Let n be a positive integer, and consider the problem of minimizing

$$I(x, u) = \frac{1}{2}\int_0^{\pi} \big\{t^n u^2(t) - \big(t^n + n(n-1)t^{n-2}/2\big)x^2(t)\big\}\, dt$$

subject to $\dot{x}(t) = u(t)$, $t \in [0, \pi]$, and $x(0) = x(\pi) = 0$.

In this case we have $p(t) = t^n$ and $r(t) = t^n + n(n-1)t^{n-2}/2$. Thus 6.2(i) and 6.2(ii) correspond to

(i) $t^n(\ddot{y}(t) + y(t)) = 0$ for all $t \in [s, \pi]$;
(ii) $nt^{n-1}\dot{y}(t) + [n(n-1)t^{n-2}/2]y(t) \neq 0$ for some $t \in [s, \pi]$.

As before, setting $y(t) = \sin t$ $(t \in [0, \pi])$, it follows that $s = 0$ belongs to $\mathcal{S}_2(x, u)$. On the other hand, to show that a point $s \in [0, \pi)$ belongs to either $\mathcal{G}_1(x, u)$ or $\mathcal{G}_2(x, u)$, we need to prove the existence of a constant $c \in \mathbf{R}$ and $y \in X_s$ with $y(s) = y(\pi) = 0$ and $y \not\equiv 0$, such that

$$\dot{y}(t)\left(c - \int_s^t \left\{\tau^n + n(n-1)\tau^{n-2}/2\right\} y(\tau)\, d\tau - t^n \dot{y}(t)\right) \geqslant 0 \quad \text{for all } t \in [s, \pi].$$

Clearly the solution of this question is much more complicated than that of proving nonemptiness of $\mathcal{S}_2(x, u)$.

Example 6.4. Consider the problem of minimizing

$$I(x, u) = \frac{1}{2}\int_0^\pi \left\{u^2(t)\sin^2 t - x^2(t)\cos^2 t\right\} dt$$

subject to $\dot{x}(t) = u(t)$, $t \in [0, \pi]$, and $x(0) = x(\pi) = 0$.

In this case we have $p(t) = \sin^2 t$ and $r(t) = \cos^2 t$ so that 6.2(i) and 6.2(ii) correspond, respectively, to

$$\sin^2 t\big(\ddot{y}(t) + y(t)\big) = 0 \quad \text{for all } t \in [s, \pi] \quad \text{and}$$
$$2\sin t \cos t\, \dot{y}(t) + \cos 2t\, y(t) \neq 0 \quad \text{for some } t \in [s, \pi].$$

Setting $y(t) = \sin t$ $(t \in [0, \pi])$, we conclude that $s = 0$ belongs to $\mathcal{S}_2(x, u)$. Now, for the nonemptiness of $\mathcal{G}_1(x, u)$ and $\mathcal{G}_2(x, u)$, we require to prove the existence of $s \in [0, \pi)$, a constant $c \in \mathbf{R}$, and $y \in X_s$ with $y(s) = y(\pi) = 0$ and $y \not\equiv 0$, such that

$$\dot{y}(t)\left(c - \int_s^t y(\tau)\cos^2\tau\, d\tau - \dot{y}(t)\sin^2 t\right) \geqslant 0 \quad \text{for all } t \in [s, \pi],$$

a question which we leave open again.

We refer to [3] for a wide range of problems for which verifying membership of the set $\mathcal{S}_2(x, u)$ may be trivial, while checking if $(x, u) \notin \mathcal{H}$ or if the conditions defining membership of $\mathcal{G}_1(x, u)$ or $\mathcal{G}_2(x, u)$ hold, may be extremely difficult.

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