Standard Bases of Perfect Homogeneous Polynomial Ideals of Height 2

GIULIO CAMPANELLA*

Istituto Matematico “G. Castelnuovo,”
Università di Roma “La Sapienza,” 00185 Rome, Italy

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INTRODUCTION

This paper concerns minimal systems of homogeneous generators (i.e., standard bases) of homogeneous ideals \( I \) in a commutative polynomial ring \( K \) with coefficients in a field \( k \). A basic numerical question about a standard basis \( B \) of \( I \) is to determine its cardinality or, more generally, the number of the elements in \( B \) of degree \( t \), for each \( t \geq 0 \); this number depends only on \( I \) (and not on the choice of \( B \)) and will be denoted by \( v_t(I) \).

In this paper we obtain a sharp estimate for the numbers \( v_t(P) \), when \( P \) is a perfect homogeneous ideal of height 2 in \( K \). Precisely, in (2.1) we provide upper and lower bounds for \( v_t(P) \) in terms of the Hilbert function \( H = H(K/P, -) \) of \( K/P \) and of an integer \( \beta \), which ranges in an interval defined by \( H \). Then in (3.3) we show the sharpness of these bounds: in fact we prove a more general result by exhibiting examples of ideals \( P \) (with given \( H \) and \( \beta \) having, for each degree \( t \), one of the possible values \( v_t(P) \) expected from (2.1). Finally, in (4.1) we modify the preceding examples and obtain, under certain hypotheses, radical ideals with the same properties.

The preceding results offer refined versions of well known theorems. In fact, from (2.1) we obtain (cf. (2.3)) a more complete form of Dubreil’s inequalities (cf. [2, 11]); moreover (3.3) and (4.1) improve, in the particular hypotheses under consideration, a classic result of Macaulay (cf. [10, 14]) and its reduced version (cf. [6, 11, 13]). Finally, particular cases of our estimates in (2.1) can be found in [7, 8, 5, 12] (for points “in generic position” in the projective plane), in [1] (for “Hilbert-function-complete-intersections”), and in [4] (for monomial ideals of height 2).

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1. Basics and Notation

In this paper $K$ or $K(n+2)$ always denotes the commutative algebra of polynomials in $n+2$ indeterminates ($n \geq 0$) and coefficients in a given ground field $k$; $K$ is $\mathbb{N}$-graded by degree in the natural way. If $I$ is a non-unit homogeneous ideal of $K$, $A = K/I$ is a finitely generated $k$-algebra with gradation $\{A_t, t \geq 0\}$ induced by $K$ (notice: $A_0 = k$, $A_t$ is a $k$-vector space, and $A$ is generated, as $k$-algebra, by elements of $A_1$). In the sequel, an ideal of $A$ will always be a non-unit homogeneous ideal and a form of $A$ a homogeneous element of $A$.

We quickly review some standard facts about the rings under consideration (for more details, cf. [11]).

(1.A) (Standard bases). Let $\mathcal{I} = \sum_{t \geq 0} I_t$ be an ideal of $A$. A standard basis $B$ of $\mathcal{I}$ is a minimal system of homogeneous generators of $\mathcal{I}$; equivalently, $B$ is a set of homogeneous generators of $\mathcal{I}$ such that for each $t \geq 1$ the set $B_t = B \cap A_t$ is $k$-linearly independent modulo $A_i \mathcal{I}_{t-1}$.

If $\nu(\mathcal{I})$ is the minimal cardinality of a set of generators of $\mathcal{I}$ and, for each $t \geq 0$, $\#(B_t) = \nu_t(\mathcal{I})$ (note that this number is independent of $B$), then $\nu(\mathcal{I}) = \sum_{t \geq 0} \nu_t(\mathcal{I})$. The smallest (resp. largest) degree in a standard basis of $\mathcal{I}$ will be denoted $\alpha(\mathcal{I})$ (resp. $\omega(\mathcal{I})$).

(1.B) (Hilbert function). Fix the following notation: $\lambda(-)$ denotes the dimension of $k$-vector spaces; $\Delta$ denotes the difference operator, which associates to a sequence $S = S(-)$ of integer numbers the sequence $\Delta S$ defined by

$$(\Delta S)(0) = S(0); \quad (\Delta S)(t) = S(t) - S(t-1), \quad \text{for each } t \geq 1.$$ 

The Hilbert function $H(A, -)$ of $A$ is the sequence $\{H(A, t) = \lambda(A_t), t \geq 0\}$. We assume known the basic properties of $H(A, -)$ (see [15]). In particular, if $m = \dim(A)$ is the Krull dimension of $A$, then $\Delta^m H(A, t) = 0$ for each $t \geq 0$. Hence we denote by $\sigma = \sigma(A)$ the integer such that $\Delta^m H(A, t) = 0$ for each $t \geq \sigma$ and $\Delta^m H(A, \sigma - 1) \neq 0$. Notice that, with the notation in (A), $\alpha(\mathcal{I}) = \min\{t: H(A/\mathcal{I}, t) \neq H(A, t)\}$: hence $\alpha(\mathcal{I})$ is defined by the Hilbert function of $A/\mathcal{I}$. Finally, if $\dim(A/\mathcal{I}) = 0$, $\omega(\mathcal{I}) \leq \sigma(A/\mathcal{I})$. For brevity we will write $\sigma(\mathcal{I})$ instead of $\sigma(A/\mathcal{I})$.

(1.C) (Height 2, perfect polynomial ideals). Let $I = \sum_{t \geq 0} I_t$ be an ideal of height 2 in $K$. We call $\beta = \beta(I)$ the largest integer such that the set $\bigcup_t \{I_t, t < \beta\}$ has a non-trivial common divisor. If $k$ is an infinite field, it follows from a standard "prime avoidance" argument (cf. [1, Sect. 1]) that there exist two relatively prime forms $F$ and $G$ in $I$ of degrees (resp.) $\alpha(I)$ and $\beta(I)$: we will say that $\{F, G\}$ is a standard regular sequence in $I$. 


Observe that a standard regular sequence is always contained in a standard basis $B$ of $I$ and $B_{B} \neq \emptyset$: hence $\alpha(I) \leq \beta(I) \leq \omega(I)$.

We recall that an ideal $P$ in $K$ is perfect if $K/P$ is a Cohen–Macaulay ring. Throughout this paper $P$ always denotes a perfect (homogeneous) ideal of height 2 in $K = K(n + 2)$ ($n \geq 0$) (notice that $\dim(K/P) = n$).

(1.D) (Reduction to the case $n = 0$). In three circumstances we will be able to deduce a result about ideals $P$ in $K$ from an analogous result about ideals of height 2 in $K(2)$. Precisely:

(a) In (2.1) we will obtain upper and lower bounds for the functions $\nu_{s}(P)$ in terms of $\Delta^{n}H(K/P, -)$ and $\beta(P)$. Since these quantities do not change by an extension of the ground field $k$, we may assume, with no loss of generality, that $k$ is an infinite field. As in [1, (3.4, 5)], let $S$ denote a set of $n$ linear forms in $K$ and $\overline{P}$ (resp. $\overline{B}$) the image of $P$ (resp. of a standard basis $B$ of $P$) with respect to the canonical map $K \to K/SK = \overline{K}$. Then, for a "sufficiently general" set $S$: $\overline{K} \cong K(2)$; $\dim(\overline{K}/\overline{P}) = 0$; $\Delta^{n}H(K/P, -) = H(\overline{K}/\overline{P}; -)$; $\overline{B}$ is a standard basis of $\overline{P}$; $\beta(P) = \beta(\overline{P})$. (In particular, observe: $\alpha(P) = \alpha(\overline{P}) \leq \beta(\overline{P}) = \beta(P) \leq \omega(P) = \omega(\overline{P}) \leq \sigma(\overline{P}) = \sigma(P)$ and $\nu_{s}(P) = \nu_{s}(\overline{P})$, for each $t \geq 0$.) Therefore, without loss of generality, we may assume $n = 0$.

(b) In (3.3) we will produce an ideal $P$ in $K$ with a given sequence $H = \Delta^{n}H(K/P, -)$, a given integer $\beta = \beta(P)$, and a given sequence $\{\nu_{s}\} = \{\nu_{s}(P)\}$ (see (3.1), (3.2)). It is enough to construct an ideal $P$ of height 2 in $K(2)$ with the same data. In fact, if $K$ is obtained by adding $n$ indeterminates to $K(2)$, then the ideal $PK$ is perfect, of height 2, with same standard bases as $P$ and $\Delta^{n}H(K/PK, -) = H(K(2)/PK, -)$, $\beta(PK) = \beta(P)$.

(c) In (4.1) we will show that if $k$ is an infinite field and $n \geq 1$, the same result as in (b) can be achieved with the further condition: $P$ radical. For this purpose it is sufficient to prove the case $n = 1$ (and then add indeterminates) and this case can be deduced from the case $n = 0$ using a construction due to R. Hartshorne (see [9, 4]).

Summarizing then: as far as our results are concerned, the crucial case is that of height 2 homogeneous ideals in the polynomial algebra in two variables over an infinite field.

2. ESTIMATES OF $\nu_{s}(P)$ AND DUBREIL'S INEQUALITIES

In this section we let $P$ be a perfect homogeneous ideal of height 2 in $K = K(n + 2)$ ($n \geq 0$) and $A = K/P$, $\alpha = \alpha(P)$, $\beta = \beta(P)$. For the crucial case in which $k$ is infinite and $n = 0$, $A$ is the quotient by a homogeneous ideal of $R = K/(F, G)$, where $\{F, G\}$ is a standard regular sequence in $P$. The Hilbert function of $R$ depends only on $\alpha$ and $\beta$ (cf. [1, (2.1)]): so we let
\[ H(R, -) = H(\alpha, \beta; -) \], where, for each \( a, b \in \mathbb{N} \) and \( 1 \leq a \leq b \), \( H(a, b; -) \) denotes the sequence of integers defined by

\[
\Delta H(a, b; t) = \begin{cases} 
1, & \text{if } 0 \leq t < a - 1; \\
-1, & \text{if } b \leq t \leq a + b - 1; \\
0, & \text{otherwise}.
\end{cases}
\]

(2.1) Theorem. Let \( \lambda_t = H(\alpha, \beta; t) - \Delta^n H(A, t) \), for each \( t \geq 0 \). Then:

(a) \( v_\lambda(P) = 0 \), if \( 0 \leq t < \alpha \) and \( t > \sigma \);

(b) \( v_\lambda(P) = 1 + \lambda_\alpha \), if \( \alpha < \beta \); \( v_\lambda(P) = 2 + \lambda_\beta \), if \( \alpha = \beta \);

(c) \( \max(0, \Delta^2 \lambda_t) \leq v_\lambda(P) \leq \Delta \lambda_t \), if \( \alpha < t \leq \sigma \) and \( t \neq \beta \);

(d) \( 1 + \max(0, \Delta^2 \lambda_t) \leq v_\lambda(P) \leq 1 + \Delta \lambda_t \), if \( \beta \neq \alpha \).

Proof. By (1.1)(a) we may assume that \( k \) is infinite and \( n = 0 \). Let \( \mathcal{I} = P/(F, G) \). For each \( t \geq 0 \), \( \lambda(\mathcal{I}_t) = H(R, t) - H(A, t) = \lambda_t \), and

\[
v_\lambda(\mathcal{I}) = v_\lambda(P), \quad \text{if } t \neq \alpha, \beta;
\]

\[
v_\lambda(P) - 1, \quad \text{if } t = \alpha \neq \beta \text{ or } t = \beta \neq \alpha;
\]

\[
v_\lambda(P) - 2, \quad \text{if } t = \alpha = \beta
\]

(observe that if \( B \) is a standard basis containing \( F, G \) and \( \mathcal{B} \) is the canonical image of \( B - \{F, G\} \) in \( R \), then \( \mathcal{B} \) is a standard basis of \( \mathcal{I} \). Therefore the assertions (a) and (b) are evident and to prove (c) and (d) it is enough to verify

\[
\lambda(\mathcal{I}) \leq \lambda(R, \mathcal{I}) \leq \lambda(\mathcal{I}) + \Delta \lambda(\mathcal{I}), \quad \text{with } 0 \leq t \leq \sigma - 1
\]

(whence \( \Delta^2 \lambda(\mathcal{I}) \leq v_\lambda(\mathcal{I}) \leq \Delta \lambda(\mathcal{I}) \), with \( 1 \leq t \leq \sigma \).

From [1, (2.3)], there exist \( x, y \) in \( R_1 \) such that \( xk + yk = R_1 \) and the multiplications by \( x \) and \( y \), \( \mathcal{I}_t \rightarrow x\mathcal{I}_t \), \( \mathcal{I}_t \rightarrow y\mathcal{I}_t \), are injective for each \( t : 0 \leq t \leq \sigma - 1 \). So \( \lambda(\mathcal{I}_t) = \lambda(x\mathcal{I}_t) \leq \lambda(R_1, \mathcal{I}_t) \). To prove the second inequality of (1), let \( U \), denote a complementary \( k \)-vector space to the subspace \( x\mathcal{I}_{t-1} - (\mathcal{B}_t) \) of \( \mathcal{I}_t \) (\( 1 \leq t \leq \sigma - 1 \)). Then \( R_1, \mathcal{I}_t = x\mathcal{I}_t + xy\mathcal{I}_{t-1} + y(\mathcal{B}_t) + yU_t = x\mathcal{I}_t + y(\mathcal{B}_t) + yU_t \). But \( \lambda(x\mathcal{I}_t) = \lambda(\mathcal{I}_t), \quad \lambda(x\mathcal{I}_{t-1}) = \lambda(\mathcal{I}_{t-1}), \quad \lambda(y(\mathcal{B}_t)) = \lambda((\mathcal{B}_t)), \quad \lambda(yU_t) = \lambda(U_t) \); hence \( \lambda(R_1, \mathcal{I}_t) \leq \lambda(\mathcal{I}_t) + \lambda((\mathcal{B}_t)) + \lambda(U_t) = \lambda(\mathcal{I}_t) + \Delta \lambda(\mathcal{I}_t) \).

As a simple application of (2.1), we will prove in (2.3) a refined version of the following Dubreil's inequalities (cf. [2, 1]):

\[ v(P) \leq \alpha(P) + \beta(P) - \sigma(P) + 1 \leq \alpha(P) + 1. \]
(2.2) **Remark–Definition.** Let $H = A^n H(A, -)$. Then (cf. [1, (3.9)]):

1. $\Delta H(t) = 1$, if $0 \leq t < \alpha - 1$.
2. $\Delta H(t) \leq 0$, if $t \geq \alpha$.
3. $\Delta H(t) < 0$, if $\beta \leq t \leq \sigma$.

Hence we introduce the following notation:

(a) $\beta_0 = \min \{ h \in \mathbb{N} : \Delta H(t) < 0, \text{ if } h \leq t \leq \sigma \}$.
(b) For each $t \geq 0$, $\lambda^0 = H(\alpha, \beta_0; t) - H(t)$.

Clearly $\alpha \leq \beta_0 \leq \beta$; moreover $\beta_0$ and $\lambda^0$ depend only on $H$; hence the Hilbert function imposes to $\beta$ the conditions $\beta_0 \leq \beta \leq \sigma$. Finally, from (2) and the definition of $\sigma$, we obtain:

4. $\Delta H(\sigma) < 0$ and $H(t) = 0$, if $t \geq \sigma$.

Note that (1), (2), and (4) are independent of $\beta$.

(2.3) **Theorem** (Dubreil's Inequalities). Let $N_0(P) = 2 + \sum_{0 \leq t \leq \sigma} \max(0, A^2 \lambda^0_t)$ and $N(P) = 2 + \sum_{0 \leq t \leq \sigma} \max(0, A^2 \lambda_t)$. Then:

$$N_0(P) \leq N(P) \leq v(P) \leq \alpha(P) + \beta(P) - \sigma(P) + 1 \leq \alpha(P) + 1.$$  

Moreover, $N(P) \leq N_0(P) + 1$.

**Proof.** The second and third inequalities are easily obtained from (2.1) (note that $\sum_t \{ \lambda, 0 \leq t \leq \sigma \} = \lambda_\sigma = \alpha + \beta - 1 - \sigma$); the fourth one is clear because $\beta \leq \sigma$. For the first inequality we may assume $\beta \neq \beta_0$. Then, from the fact that $\sigma \leq \alpha + \beta_0 - 1$,

$$A^2 \lambda_t = A^2 \lambda^0_t,$$

if $t \neq \beta, \beta_0$ and $t \leq \sigma$;

$$= 1 + A^2 \lambda^0_{\beta_0},$$

if $t = \beta_0$;

$$= -1 + A^2 \lambda^0_{\beta},$$

if $t = \beta$.

Moreover, $\max(0, A^2 \lambda_{\beta_0}) = 1 + \max(0, A^2 \lambda^0_{\beta_0})$ (observe that $A^2 \lambda^0_{\beta_0} \geq 0$) and

$$\max(0, A^2 \lambda_{\beta}) = -1 + \max(0, A^2 \lambda^0_{\beta}),$$

if $A^2 \lambda_{\beta} \geq 1$;

$$= \max(0, A^2 \lambda^0_{\beta}),$$

if $A^2 \lambda_{\beta} < 0$.

It follows that

$$N(P) = N_0(P),$$

if $A^2 \lambda_{\beta} \geq 1$;

$$= N_0(P) + 1,$$

if $A^2 \lambda_{\beta} < 0$,

and this concludes the proof.
Notice that the weaker bounds \( N_0(P) \leq v(P) \leq \alpha(P) + 1 \) depend only on the Hilbert function of \( A \), while the sharper bounds \( N(P) \leq v(P) \leq \alpha(P) + \beta(P) - \sigma(P) + 1 \) depend also on \( \beta(P) \).

### 3. Sharpness of the Estimates

In this section we assume that a sequence \( H \), an integer \( \beta \), and a sequence \( \{v_t, t \geq 0\} \) verify the conditions in (2.2) and (2.1). Then, for each \( n \geq 0 \), we will produce (cf. (3.3)) an ideal \( P \) in \( K = K(n + 2) \) such that \( \Delta^nH(K/P, -) = \Delta^nH \), \( \beta(P) = \beta \), and \( v_t(P) = v_t \), for each \( t \geq 0 \); clearly this result proves the sharpness of the estimates in (2.1).

To describe exactly our hypotheses we introduce some definitions; in particular, the term chosen in (3.1) is suggested by the terminology used in [14, 6].

**Definition.** Let \( \alpha, \sigma \) be integers such that \( 1 \leq \alpha \leq \sigma \). A sequence \( H = H(-) \) of natural integers is called an elementary \( O \)-sequence if the conditions (1), (2), and (4) in (2.2) are verified; sometimes we will write \((H, \alpha, \sigma)\) instead of \( H \). For a given elementary \( O \)-sequence \( H \), \( \beta_0 \) is the integer defined as in (2.2).

**Definition.** Let \((H, \alpha, \sigma)\) be an elementary \( O \)-sequence and \( \beta, v_\beta, \ldots, v_\sigma \in \mathbb{N} \) such that \( \alpha \leq \beta \); for each \( t \geq 0 \), let \( \lambda_t = H(\alpha, \beta; t) - H(t) \). We say that \( \{\beta; v_\beta, \ldots, v_\sigma\} \) is a set of allowable values for \( H \) if: \( \beta_0 \leq \beta \leq \sigma \) and \( \max(0, \Delta^2 \lambda_t) \leq v_t \leq \Delta \lambda_t \), for each \( t: \alpha \leq t \leq \sigma \).

**Theorem.** Let \( k \) be a field, \((H, \alpha, \sigma)\) an elementary \( O \)-sequence, and \( \{\beta; v_\beta, \ldots, v_\sigma\} \) a set of allowable values for \( H \). For each \( n \geq 0 \) there exists a perfect homogeneous ideal \( P \) of height 2 in \( K = K(n + 2) \) such that:

\[
\begin{align*}
(a) & \quad \Delta^nH(K/P, -) = H; \\
(b) & \quad \beta(P) = \beta; \\
(c) & \quad v_t(P) = \begin{cases} v_t & \text{if } 0 \leq t < \alpha \text{ and } t > \sigma; \\ v_t + 1, & \text{if } \alpha < t \leq \sigma, t \neq \beta; \\ v_t + 2, & \text{if } t = \alpha = \beta. 
\end{cases}
\end{align*}
\]

Without loss of generality (cf. (1.1), (b)) we may prove (3.3) by assuming \( n = 0 \), hence \( K = k[X, Y] \). If \( \alpha = \sigma \), the ideal \( P = K, K \) clearly verifies the conditions in (3.3): hence we will assume, when necessary, \( \alpha < \sigma \).

To prove (3.3) we shall produce a monomial ideal \( P \) (i.e., an ideal
generated by monomials), containing \( \{X^\alpha, Y^\beta\} \) as standard regular sequence. The ideal we construct will have an additional special feature. In order to explain that feature we introduce a graphical point of view.

The \( K \)-algebra \( R = K/(X^\alpha, Y^\beta) \) is generated, as \( K \)-vector space, by the canonical images in \( R \) of the monomials \( X^iY^j \), with \( 0 \leq i < \alpha \), \( 0 \leq j < \beta \); by abuse of notation we call these elements monomials in \( R \). We order the monomials by \( X^iY^j \leq X^uY^v \) if \( i \leq u \). It is useful to identify the monomial \( X^iY^j \) with the point \((i, j) \in \mathbb{R} \times \mathbb{R}\). The integer points on the line \( i + j = t \) (\( t \) an integer \( \geq 0 \)) with \( i \geq 0 \), \( j \geq 0 \) correspond to the monomials of degree \( t \). With this ordering, if two monomials have degree \( t \), the one closer to the “\( y \)-axis” is the lesser.

Let \( P \) be a monomial ideal in \( K(2) \), \( \{X^\alpha, Y^\beta\} \subset P \), \( I = P/(X^\alpha, Y^\beta) \). If \( B \) is the monomial standard basis of \( P \), the canonical image \( \mathcal{B} \) of \( B - \{X^\alpha, Y^\beta\} \), in \( R \), is the monomial standard basis of \( I \).

(3.4) Remark. Let \( P \) be a monomial ideal in \( K(2) \) with standard regular sequence \( \{X^\alpha, Y^\beta\} \) \( (\alpha \leq \beta) \). Let \( I = P/(X^\alpha, Y^\beta) \subset R = K(2)/(X^\alpha, Y^\beta) \). Let \( \mathcal{B} \) the monomial standard basis of \( I \). We say that \( \mathcal{B} \) is a special basis for \( I \) if

\[
\text{the monomials of } \mathcal{B}_t \text{ precede all the monomials of } R_1 I_{t-1}, \text{ for every } t.
\]

If \( I \) has such a basis we can say a great deal about the generators of \( I \). In order to see this we introduce the following notation:

\[
\text{for } i, j \in \mathbb{N}, \alpha \leq i \leq \sigma - 1, j \geq 1, \text{ let } x_{ij} \text{ denote the cardinality of the set } \{T \in \mathcal{B}_t \mid TX^iY^{-j} \in I, TX^iY^{-j} \notin I \text{ if } j \geq 2, 1 \leq l < j\}.
\]

Notice that multiplying a monomial of degree \( t \) by \( X^iY^{-j} \) moves it down the line \( u + v = i \) by \( j \) units. So, \( x_{ij} \) counts how many elements of \( \mathcal{B}_t \) are exactly \( j \) units from the nearest other element of \( I_t \) which is less than it.

If we have an ideal with a special basis, then:

(i) \( x_{ij} = 0 \), if \( j \geq \sigma - i + 2 \);
(ii) \( v_{t}(I) = \sum_{j} \{x_{ij} \mid 1 \leq j \leq \sigma - i + 1\} \), if \( \alpha \leq i \leq \sigma - 1 \);
(iii) \( \lambda(I_{t}) - v_{t}(I) = \sum_{\alpha \leq i \leq \sigma - 1, \alpha \leq j \leq t - 1, 1 \leq l \leq t - j, \text{ and } t - j + 1 \leq m \leq \sigma - j + 1} \sum_{m} x_{mj} \), with \( \alpha + 1 \leq t \leq \sigma \), \( \alpha \leq j \leq t - 1 \), \( 1 \leq l \leq t - j \), and \( t - j + 1 \leq m \leq \sigma - j + 1 \).

For example, let \( I \) be the ideal in \( R = K/(X^{16}, Y^{18}) \) with special basis \( \mathcal{B} = \{X^4Y^{16}, X^7Y^{12}, X^4Y^{18}, X^{10}Y^8, X^{13}Y^4, X^{14}Y^2\} \). Then \( \sigma(I) = 21 \), \( \{\lambda(I_{16}), \ldots, \lambda(I_{21})\} = \{1, 3, 6, 8, 11, 12\} \), \( x_{16,2} = x_{17,1} = x_{18,1} = x_{20,2} = 1 \), \( x_{18,3} = 2 \), and \( x_{ij} = 0 \) in the other cases. Figure 1 illustrates the obvious properties (i), (ii), and (iii).
Proof of (3.3). Let $\mathcal{I}$ be an ideal in $R$, such that:

(a) $\mathcal{I}$ is generated by monomials $X^rY^s$ with $r, s \geq 1$ and $r + s \geq \alpha$.

Then the pull back of $\mathcal{I}$ in $K$ is a monomial ideal $P$ with $\{X^r, Y^s\}$ as standard regular sequence. Moreover, for each $t \geq 0$, $\lambda(\mathcal{I}_t) = H(\alpha, \beta; t) - H(K/P, t)$ and $v_t(\mathcal{I})$ verifies the relation (\cdot) in the proof of (2.1).

Therefore, to obtain an ideal $P$ satisfying (3.3) it is enough to construct a monomial ideal $\mathcal{I}$ verifying (a) and such that:

(b) $\lambda(\mathcal{I}_t) = \lambda_t$, if $0 \leq t \leq \sigma$ (note that $\lambda(\mathcal{I}_\sigma) = \lambda_\sigma$ implies $\lambda(\mathcal{I}_t) = \lambda_t$, for each $t \geq \sigma$).

(c) $v_t(\mathcal{I}) = v_t$, if $\alpha \leq t \leq \sigma$.

Let $S$ be the system of linear equations obtained from the $2(\sigma - \alpha)$ relations (ii) and (iii) in (3.4) by replacing the integers $x_{ij}$ (resp. $\lambda(\mathcal{I}_t)$, $v_t(\mathcal{I})$) with indeterminates $X_{ij}$ (resp. with the integers $\lambda_t$, $v_t$). We will show:

(3.5) Lemma. $S$ has a solution with entries in $\mathbb{N}$.

Even when (3.5) is proved, there remains the problem of translating a solution to the system of equations $S$ into a choice of suitable monomials. In order to make that translation we shall make repeated use of the following lemma.
(3.6) **Lemma.** Let $k$ be a field, $(x_1,\ldots, x_t) \in \mathbb{N}^t$, and $n, u, h$ be integers such that $\sum_{i} \{ x_i, 1 \leq l \leq t \} = n \leq u < h$. There exists a set $\mathcal{M}$ of monomials of degree $h$ in $K$ such that:

1. $\#(\mathcal{M}) = \sum_{i} \{ x_i, 1 \leq l \leq t \}$;
2. If $x^a Y^h \in \mathcal{M}$, $u - n \leq a < u$;
3. If $\mathcal{M} \neq \emptyset$, then $M = \mathcal{X}^a Y^h - (u-n) \in \mathcal{M}$;
4. If $N = x^a Y^h - u$, then $\lambda((\mathcal{M}, N)/(N)_{h+j})$

\[ \sum_{i} \{ x_i, 1 \leq l \leq t \}, \quad \text{if } j = 0; \]
\[ \sum_{i} \{ x_i, 1 \leq l \leq j \} + (j+1) \sum_{m} \{ x_m, j+1 \leq m \leq t \}, \text{if } 1 \leq j < t-1, \quad t \geq 2; \]
\[ n, \quad \text{if } j \geq t - 1. \]

Before proving the lemma we indicate how it may be used to provide an appropriate set of monomials. To this end we introduce the following notation: $X_{\sigma} = v_{\sigma}; n_{i} = \sum_{j} \{ jx_{j}, 1 \leq j \leq \sigma - i + 1 \}$ (with $\sigma \leq i \leq \sigma$); $u_{a} = \alpha; u_{i} = \alpha - \sum_{j} \{ n_{j}, \alpha \leq l \leq i - 1 \}$ (with $\alpha + 1 \leq i \leq \sigma$).

From the last equation of $S$, $\sum_{i} \{ n_{i}, \alpha \leq i \leq \sigma \} = \lambda_{a} = \alpha + \beta - \sigma - 1 \leq \alpha - 1$; hence $n_{i} \leq u_{i} < i$ (with $\alpha \leq i \leq \sigma$) and (3.6) applied to $(x_{i1},\ldots, x_{i,\sigma - i + 1})$, $n_{i}, u_{i}, i$, provides, for each $i$ (with $\alpha \leq i \leq \sigma$) a set $\mathcal{M}_{i}$ of monomials of degree $i$. These monomials generate a monomial ideal $\mathcal{I}$ in $R$ (notice that, with notation as in (3.6), $N_{a} = X^a, N_{i} = YM_{i-1}$, and $M_{a} = X^a - (\beta - 1) Y^{\beta - 1}$).

Thus we must only verify the conditions (a), (b), and (c). The conditions (a) and (c) are clear; for (b) observe that $\mathcal{I}$ verifies the condition described in (3.4); hence $\mathcal{S}_{i} = \oplus_{\alpha \leq j \leq t} ((\mathcal{M}_{j}, N)/N_{j})$, and the conclusion follows from the equalities in (3.6)(4).

Thus, to finish the proof of (3.3) we must prove (3.5) and (3.6).

**Proof of (3.5).** The solutions of $S$ coincide with those of the following system:

\[ T \begin{cases} \sum_{j} \{ X_{j}, 1 \leq j \leq \sigma - i + 1 \} = \lambda_{i}, \quad (\alpha \leq i \leq \sigma - 1); \\
\sum_{j} \{ X_{j,t-j}, \alpha \leq j \leq t - 1 \} = \lambda_{i} - \lambda_{t} \lambda_{i}, \quad (\alpha + 1 \leq t \leq \sigma). \end{cases} \]

In fact, if we set $G_{t} = \lambda_{t} \lambda_{i} - v_{t} + \sum_{j} X_{j,t-j}; \quad H_{t} = \lambda_{t} - v_{t} -$
\[ \sum_j \{ \sum_i lX_{jk} + (t-j+1) \sum_m X_{jm} \} \] (with the same indices as in \( S, T \)) and \( H_t' = 0 \) if \( t \leq \alpha \), then it is enough to verify that
\[ H_t - 2H_{t-1} + H_{t-2} = G_t \] (with \( \alpha + 1 \leq t \leq \sigma \)).

Now, for each \( l \) such that \( \alpha + 1 \leq l \leq \sigma \), let \( T_l \) be the system obtained from \( T \) by substituting \( l \) (hence \( T_{\sigma} = T \)). We show, by induction on \( l \), that \( T_l \) has a solution \((x_{ij})\), with entries in \( \mathbb{N} \). If \( l = \alpha + 1 \), \((v_{x+1} - d^2 \lambda_{x+1}, v_x - v_{x+1} + \lambda_{x+1})\) is the required solution of \( T_{x+1} \). Assume that \( \alpha + 1 < l \leq \sigma \) and that \((y_{ij})\) is a solution, with entries in \( \mathbb{N} \), of \( T_{l-1} \). To simplify the notation, we set \( y_{l-1,1} = v_{l-1} \) and \( b = v_l - d^2 \lambda_l \).

Claim. \( b \leq \sum_h \{ y_{h,l-h}, \alpha \leq h \leq l-1 \} \).

In fact, by summing the equations of \( T_{l-1} \), we obtain \( \sum_t \{ v_t - d^2 \lambda_t, \alpha + 1 \leq t \leq l-1 \} = \sum_h \{ v_h - y_{h,l-h}, \alpha \leq h \leq l-2 \} \). Hence \( v_{l-1} - \sum_t \{ d^2 \lambda_t, \alpha + 1 \leq t \leq l-1 \} = v_a - \sum_h \{ y_{h,l-h}, \alpha \leq h \leq l-2 \} \) and then \( d^2 \lambda_{l-1} = \sum_h \{ y_{h,l-h}, \alpha \leq h \leq l-1 \} \}. Clearly \( b \leq d^2 \lambda_{l-1} \) and this proves the claim.

Let now \( x_{ij} = y_{ij} \), if \( \alpha \leq i \leq l-2 \) and \( 1 \leq j \leq l-i-1 \). Hence we must only define \( \tau_i = (x_{i-l-i}, x_{i-l-i+1}) \) (with \( \alpha \leq i \leq l-1 \)). If \( b \leq y_{x_{i-\alpha}} \) we set
\[ \tau_a = (b, y_{x_{i-\alpha}} - b); \]
\[ \tau_i = (0, y_{i,l-i}), \quad \text{if} \quad \alpha + 1 \leq i \leq l - 1. \]

If \( b > y_{x_{i-\alpha}} \), from the claim, there exists \( u \in \mathbb{N} \) such that \( \alpha + 1 \leq u \leq l - 1 \) and \( \sum_h \{ y_{h,l-h}, \alpha \leq h \leq u - 1 \} < b \leq \sum_h \{ y_{h,l-h}, \alpha \leq h \leq u \} \). Then we set
\[ \tau_i = (y_{i,l-i}, 0), \quad \text{if} \quad \alpha \leq i \leq u - 1; \]
\[ \tau_u = \left( b - \sum_h \{ y_{h,l-h}, \alpha \leq h \leq u - 1 \}, y_{u,l-u} - x_{u,l-u} \right); \]
\[ \tau_i = (0, y_{i,l-i}), \quad \text{if} \quad u + 1 \leq i \leq l - 1 \) and \( u < l - 1 \).

One easily verifies that \((x_{ij})\) is a solution of \( T_l \).

**Proof of (3.6).** We proceed by induction on \( n \). The cases \( n = 0, 1 \) are trivial (\( \mathcal{M} = \emptyset \), if \( n = 0 \); \( \mathcal{M} = \{ M \} \) if \( n = 1 \)). Let \( n > 1 \) and \( s \in \mathbb{N} \) such that \( x_s \neq 0 \), \( x_{s+1} = \cdots = x_t = 0 \) (\( 1 \leq s \leq t \)). We define \((y_1, ..., y_t) \in \mathbb{N}^t \) in this way: \( y_t = x_t \) if \( 1 \leq t \leq s \) and \( l = s \); \( y_s = x_s - 1 \). Obviously \( \sum \{ y_t, 1 \leq l \leq t \} = n - s < n \). So the induction hypothesis, applied to \((y_1, ..., y_t)\), \( n - s, u, h \), produces a set \( \mathcal{M}' \) of monomials of degree \( h \) verifying properties analogous to (1), (2), (3), and (4). One easily verifies that \( \mathcal{M} = \mathcal{M}' \cup \{ M \} \).

**(3.7) Remark.** We suggest a graphical method to construct a set \( \mathcal{M} \) as in (3.6) (with notation and ordering introduced before (3.4)). If we identify \( X^*Y^h \) (with \( 0 \leq t \leq h \)) with \((t, h-t)\), the "first coordinate" of the
monomials in $\mathcal{M}$ must range in the closed interval $[u - n; u - 1]$ of the line $i + j = h$ (we are assuming $n \geq 1$ to avoid trivial cases). Consider the following partition $\{E_1, \ldots, E_t\}$ of this interval:

$$\#(E_i) = ix_i; \quad E_i \text{ is empty or a closed interval};$$

$$E_i \text{ precedes } E_{i+1} (1 \leq i \leq t - 1).$$

Then we choose, for each $E_i \neq \emptyset$, $x_i$ integers $a_{ij}$ ($1 \leq j \leq x_i$) in this way: $a_{ii}$ is the left bound of $E_i$ and, if $x_i \geq 2$, $a_{ij} = a_{ii} + i(j + 1)$ (with $2 \leq j \leq x_i$). Clearly $\{a_{ij}\}$ is a set of first coordinates of $\mathcal{M}$.

We conclude this section with a numerical example of the preceding construction.

(3.8) Example. Consider the following elementary $O$-sequence $H$: $H(t) = t + 1$ if $0 \leq t \leq 15$ and $\{H(16), H(17), \ldots\} = \{15, 13, 9, 6, 2, 0, 0, \ldots\}$.

Clearly $\alpha = \beta = 16$ and $\sigma = 21$. Assume $\beta = 18$. Then:

\[
\begin{array}{ccccccc}
  t & 16 & 17 & 18 & 19 & 20 & 21 \\
  \lambda_t & 1 & 3 & 6 & 8 & 11 & 12 \\
  \delta \lambda_t & 1 & 2 & 3 & 2 & 3 & 1 \\
  \delta^2 \lambda_t & 1 & 1 & 1 & -1 & 1 & -2 \\
\end{array}
\]

Among the 108 sets of allowable values for $H$ with $\beta = 18$, we choose $\{18; 1, 1, 3, 0, 1, 0\}$. With the method used in the proof of (3.5) we obtain the following solution $(x_{ij})$ of $S$:

$$(0, 1, 0, 0, 0, 0; 1, 0, 0, 0, 0; 1, 0, 2, 0; 0, 0, 0; 0, 1)$$

(notice that these integers $x_{ij}$ are exactly those in the example of (3.4)).

Now, from (3.7) we get, for each $t$ ($16 \leq t \leq 21$), the following sets:

- $\mathcal{M}_{16} = \{X^{14}Y^2\}$, $\mathcal{M}_{17} = \{X^{13}Y^4\}$, $\mathcal{M}_{18} = \{X^{10}Y^8, X^7Y^{11}, X^6Y^{12}\}$, $\mathcal{M}_{19} = \emptyset$, $\mathcal{M}_{20} = \{X^4Y^{16}\}$, $\mathcal{M}_{21} = \emptyset$. The set $\mathcal{M} = \mathcal{M}_{16} \cup \cdots \cup \mathcal{M}_{21}$ is the special basis required (and coincides with $\mathcal{B}$ of the example in (3.4)). Hence

$$P = (Y^{18}, X^4Y^{16}, X^6Y^{12}, X^7Y^{11}, X^{10}Y^8, X^{13}Y^4, X^{14}Y^2, X^{15})$$

is the ideal verifying the conditions in (3.3) (with $n = 0$).

4. Further Remarks

In this section we discuss two results linked to our previous results (3.3) and (2.1).
Let \( k \) be an infinite field, \( (H, \alpha, \sigma) \) an elementary \( O \)-sequence, and \( \{ \beta; v_1, \ldots, v_\sigma \} \) a set of allowable values for \( H \). For each \( n \geq 1 \) there exists a radical perfect homogeneous ideal \( P \) of height 2 in \( K = K(n+2) \) verifying the conditions (a), (b), and (c) in (3.3).

The obvious geometrical translation of (4.1) is the following:

\[
\text{(4.2) Corollary. Let } k, (H, \alpha, \sigma), \{ \beta; v_1, \ldots, v_\sigma \} \text{ be as in (4.1) and } \delta = \sum_{t \geq 0} H(t). \text{ For each } n \geq 1 \text{ there exists a set } V \text{ of } \delta \text{ disjoint linear subspaces of codimension 2 in the projective space } \mathbb{P}_k^{n+1}, \text{ such that, if } P = I(V) \text{ is the ideal in } K = K(n+2) \text{ of the forms vanishing on } V, \text{ then } P \text{ verifies the conditions (a), (b), and (c) in (4.1).}
\]

Proof of (4.1). It is enough to prove the case \( n = 1 \) (cf. (1.D(c))). Let \( P \) be a monomial ideal of height 2 in \( K(2) \) verifying the conditions (a), (b), and (c) of (3.3). Following a construction of R. Hartshorne (see [9] or [4]), we can fix \( x_1, \ldots, x_\sigma \in k \) (all different) and define \( L_i = x_i - x_j Z \in K(2)[Z] = K(3) \) (\( 1 \leq i \leq \alpha \)); similarly, we can fix \( y_1, \ldots, y_\beta \in k \) (all different) and define \( E_j = Y - y_j Z \in K(3) \) (\( 1 \leq j \leq \beta \)). If \( B \) is the monomial standard basis of \( P \) and \( X'Y' \in B \), we set \( F_{rs} = \prod_{h=1}^{r} \{ L_h, 1 \leq h \leq r \} \cdot \{ E_l, 1 \leq l \leq s \} \) and \( \tilde{B} = \{ F_{rs} | X'Y' \in B \} \). Let \( \tilde{P} = (\tilde{B}) \) (we say that this ideal is the canonical distraction of \( P \) defined by \( \{ x_1, \ldots, x_\sigma, y_1, \ldots, y_\beta \} \)): clearly \( \tilde{B} \) is a standard basis of \( \tilde{P} \), \( v_i(\tilde{P}) = v_i(P) \) (for each \( t \geq 0 \)) and \( \beta(\tilde{P}) = \beta \). From [4, Theorem 2.2] (or from a simple proof by induction on \( v(P) \)) it follows that \( \tilde{P} \) is a radical ideal; moreover, \( Z \) is \( K(3)/\tilde{P} \)-regular and \( K(3)/(\tilde{P}, Z) \cong K(2)/P \). It follows that \( \Delta H(K(3)/\tilde{P}, -) = H(K(3)/(\tilde{P}, Z), -) = H \) and this concludes the proof.

Notice that the hypothesis "\( k \) infinite" was only necessary in order to define \( \tilde{P} \). Thus, for a given \( (H, \alpha, \sigma) \), (4.1) holds with the weaker assumption \( \#(k) \geq \sigma \).

(b) A result of [6] (which characterizes the Hilbert functions of reduced \( k \)-algebras; see also [11, 13]) provides, in particular, a very simple construction of sets of points in \( \mathbb{P}_k^2 \) having a given Hilbert function. We sketch that construction in the next remark and afterwards prove a result concerning the ideal of the sets of points so constructed.

\[
\text{(4.3) Remark. Let } (H, \alpha, \sigma) \text{ be a given elementary } O\text{-sequence. For each } i \text{ such that } 0 \leq i \leq \alpha - 1, \text{ } (H_i, \alpha - i, \sigma_i) \text{ is the elementary } O\text{-sequence defined by } H_0 = H, \sigma_0 = \sigma, \text{ and}
\]

\[
H_i(t) - H(t + i) - i, \quad \text{if } 0 \leq i \leq \sigma_{i-1} - 1 - 2;
\]

\[
= 0, \quad \text{if } t \geq \sigma_{i-1} - 1.
\]

(Note: \( \sigma_i \leq \sigma_{i-1} - 1 \leq \sigma_{i-2} - 2 \leq \cdots \leq \sigma - i \).)
Let now $k$ be an infinite field and consider in $\mathbb{P}^2_k$ the following data:

(a) $\alpha$ lines $\mathcal{L} = \mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{\alpha-1}$ (represented, resp., by $L = L_0, L_1, \ldots, L_{\alpha-1} \in K(3)$); 

(b) $\alpha$ sets $V = V_0, V_1, \ldots, V_{\alpha-1}$, 

such that, if we set $X_i = \bigcup_j \{V_j, i \leq j \leq \alpha - 1\}$ (with $0 \leq i \leq \alpha - 1$) and $X_0 = X$, the following conditions are satisfied:

1. $V_i \subseteq L_i$ and $\#(V_i) = \sigma_i$ (with $0 \leq i \leq \alpha - 1$);
2. $\mathcal{L}_i \cap X_{i+1} = \emptyset$ (with $0 \leq i \leq \alpha - 2$).

We will say that a set $X \subseteq \mathbb{P}^2_k$ is an $H$-set if there exist $\alpha$ lines and $\alpha$ sets verifying the preceding conditions (in other words, $X$ is obtained by choosing $\sigma_{\alpha-1}$ points on a line $\mathcal{L}_{\alpha-1}$, $\sigma_{\alpha-2}$ points on a line $\mathcal{L}_{\alpha-2}$ missing the points previously chosen,...). It is proved in [6], by an inductive argument, that $AH(K(3)/J(X), \ldots) = H_i$ (with $0 \leq i \leq \alpha - 1$). In particular, $AH(K(3)/J(X), \ldots) = H$.

Now we prove that if $X$ is an $H$-set in $\mathbb{P}^2_k$ and $I(X)$ is the ideal of the forms in $K(3)$ vanishing on $X$, then the numbers $v_i(I(X))$ always reach the maximal values described in (2.1).

(4.4) Proposition. Let $k$ be an infinite field, $(H, \alpha, \sigma)$ be an elementary $O$-sequence, and $X$ be an $H$-set in $\mathbb{P}^2_k$. Then:

1. $\beta(I(X)) = \sigma$;
2. $\nu(I(X)) = \alpha + 1$.

Consequently, if $\lambda_i = H(\alpha, \sigma; t) - H(t)$ (with $t \geq 0$), then:

$v_i(I(X)) = \Delta \lambda_i$, if $\alpha < t < \sigma$;

$= \Delta \lambda_i + 1$, if $t = \sigma \neq \sigma$ or $t = \sigma \neq \alpha$;

$= \Delta \lambda_i + 2$, if $t = \alpha = \sigma$.

Proof. (1) From (4.3) and Bézout's theorem, $\sum_{t < \sigma} I(X)_t \subseteq LK$ (with $K = K(3)$). Hence $\sigma \leq \beta(I(X))$ and, obviously, $\beta(I(X)) \leq \sigma$.

(2) We proceed by induction on $\alpha$. If $\alpha = 1$, $I(X) = (L, F)$ with $F \in K_\sigma$ and $L \not\supseteq F$. Hence $\nu(I(X)) = 2$. Suppose $\alpha \geq 2$. Clearly $I(X) \not\supseteq LI(X_1)$ and $I(X)_t \setminus LI(X_1)$, for each $t < \sigma$. Hence $\lambda(I(X)_\sigma/LI(X_1)_{\sigma-1}) \geq 1$. By the induction hypothesis, $\nu(I(X_1)) = \alpha$. So it is sufficient to prove that

$\nu(I(X)) = \nu(I(X_1)) + \lambda(I(X)_\sigma/LI(X_1)_{\sigma-1})$ (*)&

and the conclusion follows from the fact that $\nu(I(X)) \leq \alpha + 1$ (cf. (2.3)). Now (*) follows from this more general claim (whose simple proof is left to the reader).
Claim. Let $I, J$ be homogeneous ideals in $K = K(n + 2)$ such that $I = J + I_o K$, with $\omega = \omega(I) \geq \omega(J)$. Then $v(I) = v(J) + \lambda(I_o/J_o)$.
(Note that $\omega(L(X)) = 1 + \sigma_1 \leq \omega(f(X)).$)

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