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Hammocks and the Algorithms of Zavadskiĭ

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1. INTRODUCTION

Hammocks have been considered by Brenner [3] in order to give a numerical criterion for a finite translation quiver to be the Auslander–Reiten quiver of some representation-finite algebra. Ringel and Vossieck [13] gave a combinatorial definition of left hammocks, which generalizes the concept of hammocks, in the sense of Brenner, as a translation quiver H and an additive function h on H (called the hammock function) satisfying some conditions. They also showed that a thin left hammock with finitely many projective vertices is just the preprojective component of the Auslander–Reiten quiver of the category of $\mathscr{S}$, where $\mathscr{S}$ is a finite partially ordered sets (abbreviated poset). An important role of posets in representation theory is played by two differentiation algorithms. One of the algorithms is due to Nazarova and Roiter [9] and it reduces a poset $\mathscr{S}$ with a maximal element $a \in \mathscr{S}$ to a new poset $\mathscr{S}' = {}_a\partial\mathscr{S}$ with same representation type. The second algorithm is due to Zavadskiĭ [15] and it reduces a poset $\mathscr{S}$ with a suitable pair (a, b) of elements a, b to a new poset $\mathscr{S}' = \partial_{(a,b)}\mathscr{S}$ with same representation type. Zavadskiĭ's algorithm is successfully used to give new proofs for characterizing posets of finite type [5] and for characterizing posets of wild type [10] in studying posets of finite growth [15]. In the paper [7], we discussed the relationship between hammocks and the algorithm of Nazarova and Roiter. The main purpose of the present paper is to construct some new left hammocks from a given one, and to show the relationship between these new left hammocks and the algorithm of Zavadskiĭ.

In Section 2, we recall some basic definitions and facts. Let H be a thin left hammock with hammock function h_H, let $p(a)$ a projective vertex of H different from the source, and let $q(b)$ an injective vertex of H different

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from the sink. In Section 3, we construct a new left hammock ${}_aH_b$ from the given one by using the pair of points $p(a)$ and $q(b)$. We determine its hammock function $h_{({}_aH_b)}$. It is shown that ${}_aH_b = {}_aH \cap H_b$, where ${}_aH$ and H_b are left hammocks induced from H by a point (see Section 2.5). In Section 4, we prove that the subquiver, denoted by $H/{}_aH_b$, consisting of all vertices x satisfying $h_H(x) - h_{({}_aH_b)}(x) \neq 0$, is an "almost" left hammock. If H is a thin left hammock with finitely many projective vertices, the relation between the left hammock ${}_aH_b^{\diamond}$ induced by the pair of points of H and the algorithm of Zavadskiĭ is as stated in Theorem 5.1. The proof of Theorem 5.1 will cover Sections 6 and 7. The corresponding results concerning $\ell(\mathcal{S})$, the category of $\mathcal{S}$-spaces, are also described.

Throughout this paper, all algebras are assumed to be finite-dimensional (associative) basic algebras with unit over an algebraically closed field and all modules are finitely generated right modules. We denote by A-mod the category of A-modules. The composition of two morphisms $f\colon M_1 \to M_2$ and $g\colon M_2 \to M_3$ is denoted by fg. All posets are assumed to be finite. We denote by N, N_1, and Z the set of natural numbers, positive integers, and integers, respectively. For all unexplained notation, we refer to [11] and [13].

2. PRELIMINARIES

2.1. *Left Hammocks and Hammocks*

Let $H = (H_0, H_1, \tau)$ be a proper translation quiver. We define inductively the full subquivers ${}^dH$ of H. First of all, ${}^{-1}H$ is the empty quiver, and z belongs to ${}^dH$ if and only if $z^- \subseteq {}^{d-1}H$. Also, ${}^\infty H = \bigcup_{d \in N} {}^dH$. Thus, for all $d \in N \cup \{\infty\}$, we see that ${}^dH$ is a predecessor closed subquiver, and we may consider it as a translation quiver, using the restriction of τ. Suppose H has a unique source ω and $H = {}^\infty H$. Then we define $h_H\colon H_0 \to Z$ inductively as follows. By abuse of notation, let $h_H(\tau x) = 0$ for x projective (note that, in this case, τx is not defined). Now, let $h_H(\omega) = 1$ and, for $x \neq \omega$, with h_H already defined on all proper predecessors of x, let $h_H(x) = \sum_{y \to x} h_H(y) - h_H(\tau x)$ (where the sum is taken over all arrows ending at x). With these preparations, we are able to recall the main definition: the translation quiver H is said to be a *left hammock* provided

(1) $H = {}^\infty H$;

(2) H has a unique source ω and $h_H(\omega) = 1$;

(3) h_H takes values in the set N_1 of positive integers,

(4) if q is an injective vertex, then $h_H(q) \geq \sum_{q \to y} h_H(y)$.

When H is a left hammock, the function h_H is said to be its *hammock function*.

A vertex x of H is called *thin* if $h_H(x) = 1$. A left hammock H is said to be *thin* provided $h_H(p) = 1$ for any projective vertex p of H. A left hammock H is called a *hammock* if $|H_0| < \infty$. A hammock is always thin and has a unique sink, say ω'.

2.2. *S-Spaces*

Fix some field k. Given a poset $\mathscr{S}$, an $\mathscr{S}$-*space* $V = (V_\omega; V_s)_{s \in \mathscr{S}}$ is given by a vector space V_ω over k and subspaces V_s of V_ω, for $s \in \mathscr{S}$, such that $V_s \subseteq V_t$ for $s \le t$. We call V_ω the *total space* of V, and define its k-dimension by $\underline{\dim}_\omega V = \dim_k V_\omega$. Given two $\mathscr{S}$-spaces V, W, a map $\psi: V \to W$ is given by a k-linear map $\psi_\omega: V_\omega \to W_\omega$ satisfying $\psi_\omega(V_s) \subseteq W_s$ for all $s \in \mathscr{S}$; the induced map $V_s \to W_s$ will be denoted by ψ_s. The posets we will consider are always assumed to be finite. We denote the category of $\mathscr{S}$-spaces V with $\dim_k V_\omega < \infty$ by $\ell(\mathscr{S})$. For convenience, we denote $\mathrm{Hom}_{\ell(\mathscr{S})}(V, W)$ for two $\mathscr{S}$-spaces V and W by $\mathrm{Hom}_{\mathscr{S}}(V, W)$. We denote by $\mathscr{S}^+$ the poset obtained from $\mathscr{S}$ by adjoining an element ω with $s < \omega$ for all $s \in \mathscr{S}$. Similarly we denote by $\mathscr{S}^-$ the poset obtained from $\mathscr{S}$ by adjoining an element ω' with $s > \omega'$ for all $s \in \mathscr{S}$. The *projective objects*, denoted $P_{\mathscr{S}}(s)$ with $s \in \mathscr{S}^+$, and the *injective objects*, $Q_{\mathscr{S}}(s)$ with $s \in \mathscr{S}^-$ are defined as follows. For all $t \in \mathscr{S}^+$,

$$P_{\mathscr{S}}(s)_t = \begin{cases} k & \text{for } t \ge s \\ 0 & \text{for } t \not\ge s \end{cases}$$

and

$$Q_{\mathscr{S}}(s)_t = \begin{cases} k & \text{for } t \not\le s \\ 0 & \text{for } t \le s. \end{cases}$$

For $t \in \mathscr{S}$ and $V \in \ell(\mathscr{S})$, we have $\dim_k \mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(t), V) = \dim_k V_t$ and $\dim_k \mathrm{Hom}_{\mathscr{S}}(V, Q_{\mathscr{S}}(t)) = \dim_k V_\omega - \dim_k V_t$. An $\mathscr{S}$-space V is *thin* if its total space V_ω is one dimensional. We denote by $\tau_{\mathscr{S}}$ the *Auslander–Reiten translation* in $\ell(\mathscr{S})$. It is well known that the Auslander–Reiten quiver of $\ell(\mathscr{S})$ always has a unique preprojective component, denoted by $\mathscr{P}_{\mathscr{S}}$, which is standard.

Given a Krull–Schmidt k-category Λ, let us define the full subcategories ${}_d\Lambda$. First of all, ${}_{-1}\Lambda$ contains only the zero object. Second, an indecomposable object X of Λ belongs to ${}_d\Lambda$ if and only if any indecomposable object Y of Λ with $\mathrm{rad}(Y, X) \ne 0$ belongs to ${}_{d-1}\Lambda$. Finally, ${}_\infty\Lambda = \bigcup_{d \in N} {}_d\Lambda$. Let $\mathscr{S}$ be a poset. We observe that ${}_\infty\ell(\mathscr{S})$ is just the full subcategory of $\ell(\mathscr{S})$ whose indecomposable objects occur in $\mathscr{P}_{\mathscr{S}}$. So ${}_\infty\ell(\mathscr{S}) \cong \mathrm{add}\, k(\mathscr{P}_{\mathscr{S}})$, where $k(\mathscr{P}_{\mathscr{S}})$ denote the mesh category for $\mathscr{P}_{\mathscr{S}}$.

There is a strong relationship between thin left hammocks and the representation theory of posets which is due to Ringel and Vossieck (see [13]) and is described as follows.

THEOREM 2.1. *Let $\mathscr{S}$ be a finite poset and let k be a field. Then the preprojective component $\mathscr{P}_{\mathscr{S}}$ of the Auslander–Reiten quiver of $\ell(\mathscr{S})$ is a thin left hammock with finitely many projective vertices. The hammock function on $\mathscr{P}_{\mathscr{S}}$ is* $\underline{\dim}_{\omega}$. *Conversely, given a thin left hammock H with n projective vertices, there exists a unique poset $\mathscr{S} := \mathscr{S}(H)$ with $n-1$ elements such that* $\operatorname{add} k(H) \cong {}_{\infty}\ell(\mathscr{S})$ *as categories and $H \cong \mathscr{P}_{\mathscr{S}}$ as translation quivers.*

From now on we will take any thin left hammock H as the preprojective components $\mathscr{P}_{\mathscr{S}}$ for $\mathscr{S} = \mathscr{S}(H)$. Accordingly we have a bijective map $p: \mathscr{S}^{+} \to \{\text{projective vertices of } H\}$, where $p(s)$ is the vertex corresponding to $P_{\mathscr{S}}(s)$. Let $(\mathscr{S}^{-})^{o}$ be the subset of $\mathscr{S}^{-}$ consisting of those elements s such that the injective object $Q_{\mathscr{S}}(s)$ occurs in ${}_{\infty}\ell(\mathscr{S})$. Then we have a bijective map $q: (\mathscr{S}^{-})^{o} \to \{\text{injective vertices of } H\}$, where $q(s)$ is the vertex corresponding to $Q_{\mathscr{S}}(s)$. In particular, we obtain $\operatorname{Hom}_{k(H)}(p(s), p(t)) \neq 0$ if and only if $s \geq t$ in $\mathscr{S}^{+}$ and $\operatorname{Hom}_{k(H)}(q(s), q(t)) \neq 0$ if and only if $s \geq t$ in $(\mathscr{S}^{-})^{o}$.

2.3. *Incidence Algebras and Socle-Projective Modules*

Let k be a field. Given a Krull–Schmidt k-category Λ, a Λ-*module* M is a finitely presented functor $\Lambda^{\mathrm{op}} \to k$-mod. We denote by Λ-mod the category of all Λ-modules and by Λ-spmod the full subcategory of Λ-mod generated by all modules $M \in \Lambda$-mod which have a projective socle. A module M in Λ-spmod is said to be *thin* if M has a simple socle. We will use the following easy result.

LEMMA 2.1. *Let Λ be a Krull–Schmidt k-category, $M, N, L \in$* Λ-spmod.

(1) *Assume that $0 \neq \psi \in \operatorname{Hom}_{\Lambda}(M, N)$ and M is thin. Then ψ is a monomorphism.*

(2) *Assume that $0 \neq \theta \in \operatorname{Hom}_{\Lambda}(M, N)$, $0 \neq \phi \in \operatorname{Hom}_{\Lambda}(N, L)$, and M, N are thin. Then $\theta\phi \neq 0$.*

Proof. Suppose that ψ is not a monomorphism, then $\operatorname{soc}(\ker(\lambda)) = \operatorname{soc} M$ since M is thin. As a consequence, $\operatorname{soc}(\operatorname{Im}(\psi)) \cong \operatorname{soc}(M/\ker(\psi))$ is not projective—a contradiction to the fact that Λ-spmod is closed under submodules. Thus (1) holds and (2) follows at once. ∎

Let $\mathscr{S}$ be a poset and let k be an algebraically closed field. By $A(\mathscr{S}) := k\mathscr{S}^{+}$ we mean the *k-incidence algebra* of the enlarged poset $\mathscr{S}^{+}$. Note that $P_{A}(\omega)$ is the unique simple projective $A(\mathscr{S})$-module. The following theorem is due to Ringel and Vossieck (see [13]).

THEOREM 2.2. *Let H be a left hammock with source ω and let k be a field. Let $\mathscr{P}(H,k)$ be the full additive subcategory of $k(H)$ whose indecomposable objects are just the projective vertices of H. Define the functor* $\mathbf{M}$: $k(H) \to \mathscr{P}(H,k)$-mod *by* $\mathbf{M}(x) = \mathrm{Hom}_{k(H)}(-,x)|\mathscr{P}(H,k)$. *Then*

(1) *there is a unique simple projective object in* $\mathscr{P}(H,k)$-mod, *namely*, $\mathbf{M}(\omega)$. *An object X of* $\mathscr{P}(H,k)$-mod *belongs to* $\mathscr{P}(H,k)$-spmod *if and only if its socle is generated by* $\mathbf{M}(\omega)$;

(2) ${}_\infty(\mathscr{P}(H,k)$-spmod) *has Auslander–Reiten sequences*;

(3) $\mathbf{M}$ *induces the equivalence* $k(H) \cong {}_\infty(\mathscr{P}(H,k)$-spmod) (*as categories*);

(4) $H \cong \Gamma_{\infty(\mathscr{P}(H,k)\text{-spmod})}$ (*as translation quivers*), *where* $\Gamma_{\infty(\mathscr{P}(H,k)\text{-spmod})}$ *is the Auslander–Reiten quiver of* ${}_\infty(\mathscr{P}(H,k)$-spmod).

For convenience, we put $\mathscr{F} := \mathscr{P}(H,k)$-spmod. Thus we write ${}_\infty(\mathscr{P}(H,k)$-spmod) as ${}_\infty\mathscr{F}$ and we write instead of $\mathrm{Hom}_{\mathscr{P}(H,k)\text{-spmod}}(X,Y)$ just $\mathrm{Hom}_{\infty\mathscr{F}}(X,Y)$. If the left hammock H has only finitely many projective vertices, $\mathscr{P}(H,k)$ is a finite category; therefore $\mathscr{P}(H,k)$-mod $\cong$ $A(H)$-mod for some finite-dimensional algebra $A(H)$ and $k(H) \cong {}_\infty(A$-spmod), $H \cong \Gamma_{\infty(A\text{-spmod})}$, where $A = A(H)$. We call $A(H)$ *the finite-dimensional algebra corresponding to* H. Note that if H is a thin left hammock with finitely many projective vertices, then $A(H)$ is just the incidence algebra of the poset $\mathscr{S}(H)$.

2.4. *Auslander–Reiten Translation in* $\ell(\mathscr{S})$

In order to describe the Auslander–Reiten translate in $\ell(\mathscr{S})$, Simson introduced the notion of prinjective modules (see [14]). Let $\mathscr{S}$ be a poset, k be a field, $A(\mathscr{S}) := k\mathscr{S}^+$ be the incidence algebra, and $k\mathscr{S} = A(\mathscr{S})/\mathrm{soc}(A(\mathscr{S}))$. As we know, the incidence algebra $A(\mathscr{S})$ is the one-point coextension of $k\mathscr{S}$ by $R := I_A(\omega)/\mathrm{soc}\, I_A(\omega)$. So we can identify the right A-module X with the triple $X = (X', X_\omega, \phi\colon X' \otimes_{k\mathscr{S}} R \to X_\omega)$, where X' is a right $k\mathscr{S}$-module and X_ω is a k-vector space. A right A-module $X = (X', X_\omega, \phi)$ is called *prinjective* if X' is a projective $k\mathscr{S}$-module. By $\mathrm{prin}(A(\mathscr{S}))$ we mean the full additive subcategory of $A(\mathscr{S})$-mod whose objects are prinjective modules; $\mathrm{prin}(A(\mathscr{S}))$ is closed under extension and kernels of epimorphisms. On the other hand, a module X in $A(\mathscr{S})$-mod will be identified with a system $X = (X_s; {}_t\phi_s)_{t \le s \le \omega}$, where X_s, $s \in \mathscr{S}^+$, are finite-dimensional k-vector spaces and ${}_t\phi_s\colon X_t \to X_s$, $t \le s$, are k-linear maps such that ${}_s\phi_s = \mathrm{id}$ for all $s \in \mathscr{S}^+$ and $({}_t\phi_s)({}_s\phi_u) = ({}_t\phi_u)$ for $t < s < u$. Now, we recall the functor Θ: $A(\mathscr{S})$-mod $\to \ell(\mathscr{S})$ defined by the formula $\Theta(X_{s,t}\phi_s) = (X_\omega, \mathrm{Im}({}_s\phi_\omega\colon X_s \to X_\omega))_{s \in \mathscr{S}}$.

Let $\mathscr{S}$ be a poset, k be a field, $A(\mathscr{S})$ be the incidence algebra, and $k\mathscr{S} = A(\mathscr{S})/\mathrm{soc}(A(\mathscr{S}))$. Given an $\mathscr{S}$-space V, we put $V^{\sim} = P(V)/\mathrm{soc}(\ker(\sigma))$, where $\sigma\colon P(V) \to V$ is the projective cover of V in

$\ell(\mathcal{S})$. Then $V^{\sim}$ is in prin$(A(\mathcal{S}))$, and $\tau_A(V^{\sim})$ is an $\mathcal{S}$-space, where τ_A is the Auslander–Reiten translate in A-mod. The following theorem is due to Simson (see [14]).

THEOREM 2.3. *The relative Auslander–Reiten translates in $\ell(\mathcal{S})$ are $\tau_{\mathcal{S}}^{-}(V) = \Theta\tau_A^{-}(V)$ and $\tau_{\mathcal{S}}(V) = \tau_A(V^{\sim})$.*

For a given poset $\mathcal{S}$ and $a \in \mathcal{S}$, set $a^{\vee} = \{x \in \mathcal{S} | x \geq a\}$ and $a_{\wedge} = \{x \in \mathcal{S} | x \leq a\}$. If $A \subseteq \mathcal{S}$, then $A^{\vee} = \bigcup_{a \in A} a^{\vee}$ and $A_{\wedge} = \bigcup_{a \in A} a_{\wedge}$. If $\{a_1, \ldots, a_r\}$, where $r \geq 1$, is a set of mutually incomparable points of the poset $\mathcal{S}$, we introduce a one-dimensional $\mathcal{S}$-space $P_S(a_1, \ldots, a_r)$ by setting $P_S(a_1, \ldots, a_r) = (U_\omega; U_s)_{s \in \mathcal{S}}$, where $U_\omega = U_x = k$ if $x \in \{a_1, \ldots, a_r\}^{\vee}$ and $U_x = 0$ otherwise.

In the case when X is a nonprojective $\mathcal{S}$-space and both X and τX are thin, then we call $X, \tau X$ a *pair of thin $\mathcal{S}$-spaces*. The following pairs of thin $\mathcal{S}$-spaces seem to be useful.

PROPOSITION 2.1. *Let $\mathcal{S}$ be a poset. Assume that a and b in $\mathcal{S}$ are incomparable. Then $\tau_{\mathcal{S}} P_{\mathcal{S}}(a, b) = P_{\mathcal{S}}(z_1, \ldots, z_r)$, where $\{z_1, \ldots, z_r] =$* min$(\mathcal{S} \setminus \{a, b\}_{\wedge})$.

Proof. It is clear that $P_{\mathcal{S}}(a) \oplus P_{\mathcal{S}}(b) \to P_{\mathcal{S}}(a, b)$ is the projective cover in $\ell(\mathcal{S})$. So $0 \to P_{\mathcal{S}}(\omega) \to P_{\mathcal{S}}(a) \oplus P_{\mathcal{S}}(b) \to P_{\mathcal{S}}(a, b)^{\sim} \to 0$ is a minimal projective resolution for $P_{\mathcal{S}}(a, b)^{\sim}$ in prin$(A(\mathcal{S}))$. We apply the Nakayama functor $D\mathrm{Hom}_A(-, A)$ to the sequence above, and by the definition of the Auslander–Reiten translation, we obtain the exact sequence $0 \to \tau_A(P(a, b)^{\sim}) \to I_A(\omega) \to I_A(a) \oplus I_A(b) \to 0$. By Theorem 2.3, we get the result. ∎

COROLLARY 2.1. *Let $\mathcal{S}$ be a poset. Assume that a and b in $\mathcal{S}$ are incomparable and that b is the unique maximal element of $\mathcal{S} \setminus a^{\vee}$. Then there exists an irreducible map $P_{\mathcal{S}}(a) \xrightarrow{\chi} P_{\mathcal{S}}(a, b)$.*

Proof. The assumption that b is the unique maximal element of $\mathcal{S} \setminus a^{\vee}$ implies $\mathcal{S} = b_{\wedge} \cup a^{\vee}$. So we have $P_{\mathcal{S}}(z_1, \ldots, z_r) = \mathrm{rad}\, P_{\mathcal{S}}(a)$, where $\{z_1, \ldots, z_r\} = \min(\mathcal{S} \setminus \{a, b\}_{\wedge})$. Thus there is an irreducible map $P_{\mathcal{S}}(z_1, \ldots, z_r) \to P_{\mathcal{S}}(a)$. This yields the existence of χ by Proposition 2.1. ∎

PROPOSITION 2.2. *Let $\mathcal{S}$ be a poset. Let $X \in \ell(\mathcal{S})$. Then both X and $\tau_{\mathcal{S}} X$ are thin if and only if $X = P_{\mathcal{S}}(s, t)$ for a pair of incomparable points s and t.*

Proof. One direction follows from Proposition 2.1. For the converse, we assume that $\oplus_{s \in \mathcal{S}} P_{\mathcal{S}}(s)^{d(s)} \to X$ is the projective cover for X in $\ell(\mathcal{S})$, where $d(s) \geq 0$. Then $0 \to \oplus_J P_{\mathcal{S}}(\omega) \to \oplus_{s \in \mathcal{S}} P_{\mathcal{S}}(s)^{d(s)} \to X^{\sim} \to 0$ is a minimal projective resolution for X in prin$(A(\mathcal{S}))$. Thus X thin implies

$|J| = \sum_{s \in \mathscr{S}} d(s) - 1$. Apply the Nakayama functor to the sequence above. By the definition of the Auslander–Reiten translation, we obtain the following exact sequence $0 \to \tau_A(X)^{\sim} \to \oplus_J I_A(\omega) \to \oplus_{s \in \mathscr{S}} I_A(s)^{d(s)} \to 0$. Since $\tau_{\mathscr{S}}(X) = \tau_A(X^{\sim})$ is thin, we see $|J| = 1$. This means $\sum_{s \in \mathscr{S}} d(s) = 2$. Thus the projective cover of $P_{\mathscr{S}}(a, b)$ is $P_{\mathscr{S}}(s) \oplus P_{\mathscr{S}}(t)$. Finally, X thin implies that s and t are incomparable.

2.5. *Hammocks Induced by a Point*

Let Λ be a Krull–Schmidt k-category and let Ξ be a class of objects of Λ. For $x, y \in \Lambda$, we denote by $\mathrm{Hom}_{\Lambda}(x, y)_{\Xi}$ the subspace of the all maps in $\mathrm{Hom}_{\Lambda}(x, y)$ which factor through some object of Ξ. In the paper [7], we obtained the following result.

THEOREM 2.4. *Let k be a field. Let H be a thin left hammock with source ω and let h_H be the hammock function of H. Assume that $p(a) \neq p(\omega)$ is a projective vertex of H and $q(a) \neq q(\omega')$ is an injective vertex of H. Then*

(1) ${}_aH = \{x \in H | \mathrm{Hom}_{k(H)}(p(a), x) \neq 0\}$ *is a left hammock with source $p(a)$. The hammock function on ${}_aH$ is* $h_{({}_aH)} = \dim_k \mathrm{Hom}_{k(H)}(p(a), -) = \dim_k \mathrm{Hom}_{k(H)}(p(\omega), -)_{\{p(a)\}}$.

(2) $H/{}_aH = \{x \in H | h_H(x) - h_{({}_aH)}(x) \neq 0\}$ *is a left hammock with source ω. The hammock function on $H/{}_aH$ is $h_{(H/{}_aH)} = h_H - h_{({}_aH)}$.*

(3) $H_a = \{x \in H | \mathrm{Hom}_{k(H)}(x, q(a)) \neq 0\}$ *is a hammock with source ω and since $q(a)$. The hammock function on H_a is* $h_{(H_a)} = \dim_k \mathrm{Hom}_{k(H)}(-, q(a))$.

(4) $H/H_a = \{x \in H | h_H(x) - h_{(H_a)}(x) \neq 0\}$ *is a left hammock with source $p(a)$. The hammock function on H/H_a is $h_{(H/H_a)} = h_H - h_{(H_a)}$.*

(5) *$H_a = H/{}_aH$ and ${}_aH = H/H_a$.*

(6) *Let $\mathscr{S}(H)$ be the poset corresponding to H. Then $\mathscr{S}({}_aH)$ ($\mathscr{S}(H_a)$, respectively) is obtained from $\mathscr{S}(H)$ by a finite sequence of differentiations with respect to maximal (minimal, respectively) elements in the sense of Nazarova and Roiter.*

3. HAMMOCKS INDUCED BY A PAIR OF POINTS

Let k be a field. Let H be a left hammock and let $k(H)$ be the mesh category of H. For a given projective vertex $p(a)$ of H, let ${}_a\mathscr{M}$ be the class of all objects x with $\mathrm{Hom}_{k(H)}(p(a), x) = 0$. For a given injective vertex $q(b)$ of H, let $\mathscr{M}_b$ be the class of all objects x with $\mathrm{Hom}_{k(H)}(x, q(b)) = 0$. There should be no confusion if we denote by ${}_a\mathscr{M}$ the class of all objects X with $\mathrm{Hom}_{\mathscr{F}}(P(a), X) = 0$ for a given projective object $P(a)$ of $\mathscr{F}$. Similarly, for a given injective object $Q(b)$ of $\mathscr{F}$, let $\mathscr{M}_b$ be the class of all objects X with $\mathrm{Hom}_{\mathscr{F}}(X, Q(b)) = 0$. Let $\mathscr{S}$ be a poset. We denote by ${}_a\mathscr{N}$ the class of all objects X with $\mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(a), X) = 0$ for a given projective object $P_{\mathscr{S}}(a)$ of $\ell(\mathscr{S})$, and let $\mathscr{N}_b$ be the class of all objects X with $\mathrm{Hom}_{\mathscr{S}}(X, Q_{\mathscr{S}}(b)) = 0$ for a given injective object $Q_{\mathscr{S}}(b)$ of $\ell(\mathscr{S})$.

LEMMA 3.1. *Let k be a field. Let H be a thin left hammock, $p(a) \neq p(\omega)$ be a projective vertex, and $q(b) \neq q(\omega')$ be an injective vertex of H. Assume that* $\mathrm{Hom}_{k(H)}(p(b), p(a)) = 0$. *Then we have*

(1) $({}_aH)_b = \{x \in {}_aH \mid \mathrm{Hom}_{k(H)}(x, q(b))/\mathrm{Hom}_{k(H)}(x, q(b))_{{}_a\mathscr{M}} \neq 0\}$ *is a hammock, and the hammock function is* $h_{(({}_aH)_b)} = \dim_k \mathrm{Hom}_{k(H)}(x, q(b)) - \dim_k \mathrm{Hom}_{k(H)}(x, q(b))_{{}_a\mathscr{M}}$.

(2) *Also,* ${}_a(H_b) = \{x \in H \mid \mathrm{Hom}_{k(H)}(p(a), x)/\mathrm{Hom}_{k(H)}(p(a), x)_{\mathscr{M}_b} \neq 0\}$ *is a hammock, and the hammock function is* $h_{({}_a(H_b))} = \dim_k \mathrm{Hom}_{k(H)}(p(a), x) - \dim_k \mathrm{Hom}_{k(H)}(p(a), x)_{\mathscr{M}_b}$.

Proof. We claim that $q(b) \in {}_aH$. Assume $q(b) \in {}^dH$. We consider the full subcategory $\mathscr{P}(d+2)$ of $\mathscr{P}(H, k)$ given by all projective vertices p with $p \in {}^{d+2}H$. We can consider $\mathscr{P}(d+2)$-modules as $\mathscr{P}(H, k)$-modules. Since $\mathscr{P}(d+2)$ is a finite category, there is a finite-dimensional algebra A with A-mod $\cong \mathscr{P}(d+2)$-mod and $k({}^{d+2}H) \cong {}_{d+2}(\mathscr{P}(H, k)\text{-spmod}) \cong {}_{d+2}(A\text{-spmod})$. We denote by $\mathbf{M}$ the corresponding equivalence functor $\mathbf{M}$: $k({}^{d+2}H) \cong {}_{d+2}(A\text{-spmod})$. We can write $\mathbf{M}(p(a)) = P_A(a)$, $\mathbf{M}(p(b)) = P_A(b)$, and $\mathbf{M}(q(b)) = Q_A(b)$. Note that $\mathrm{Hom}_{k(H)}(p(b), p(a)) = 0$ implies $\mathrm{Hom}_A(P_A(b), P_A(a)) = 0$. It follows that $\mathrm{Hom}_A(P_A(a), I_A(b)) = 0$, where $I_A(b)$ is the injective hull of the top of $P_A(b)$. By the definition of Auslander–Reiten translate, there is an exact sequence $0 \to \tau_A B_A(b) \to \oplus_J I_A(\omega) \to I_A(b) \to 0$. Applying $\mathrm{Hom}_A(P_A(a), -)$ to this sequence, we get $\mathrm{Hom}_A(P_A(a), \tau_A B_A(b)) \neq 0$, since $\mathrm{Hom}_A(P_A(a), I_A(\omega)) \neq 0$ and $\mathrm{Hom}_A(P_A(a), I_A(b)) = 0$. Thus, $\mathrm{Hom}_{k(H)}(p(a), q(b)) \neq 0$. Therefore $q(b) \in {}_aH$.

Of course, $q(b)$ is also an injective vertex of ${}_aH$. By Theorem 2.4, $({}_aH)_b = \{x \in {}_aH \mid \mathrm{Hom}_{k({}_aH)}(x, q(b)) \neq 0\}$ is a hammock with hammock function $h_{(({}_aH)_b)} = \dim_k \mathrm{Hom}_{k({}_aH)}(x, q(b))$. As we know, $k(H) \cong {}_\infty\mathscr{F}$, so $k({}_aH) \cong {}_\infty\mathscr{F}/{}_a\mathscr{M}$. Thus, we have $\mathrm{Hom}_{k({}_aH)}(x, q(b)) = \mathrm{Hom}_{{}_\infty\mathscr{F}/{}_a\mathscr{M}}(x, q(b)) = \mathrm{Hom}_{{}_\infty\mathscr{F}}(x, q(b))/\mathrm{Hom}_{{}_\infty\mathscr{F}}(x, q(b))_{{}_a\mathscr{M}} = \mathrm{Hom}_{k(H)}(x, q(b))/\mathrm{Hom}_{k(H)}(x, q(b))_{{}_a\mathscr{M}}$. Therefore we obtain (1). The proof of (2) is similar. ∎

Note that for H a thin left hammock with only finitely many projective vertices, $\mathrm{Hom}_{k(H)}(p(b), p(a)) = 0$ means that $a \nleq b$ in $\mathscr{S}(H)$.

LEMMA 3.2. *Let k be a field. Let H be a thin left hammock, let $p(a) \neq p(\omega)$ be a projective vertex, and let $q(b) \neq q(\omega')$ be an injective vertex of H. Assume that* $\mathrm{Hom}_{k(H)}(p(b), p(a)) = 0$. *Let* ${}_aH_b = \{x \in H \mid \mathrm{Hom}_{k(H)}(p(a), q(b))_{\{x\}} \neq 0\}$. *Then* ${}_aH_b = ({}_aH)_b = {}_a(H_b)$.

Proof. First, assume that there are $f \in \mathrm{Hom}_{k(H)}(p(a), x)$ and $g \in \mathrm{Hom}_{k(H)}(x, q(b))$ in $k(H)$ with $fg \neq 0$. We claim that $0 \neq f$ in $\mathrm{Hom}_{k(H)}(p(a), x)/\mathrm{Hom}_{k(H)}(p(a), x)_{\mathscr{M}_b}$. For, otherwise, $f \in \mathrm{Hom}_{k(H)}(p(a), x)_{\mathscr{M}_b}$ means that f factors through some object in $\mathscr{M}_b$, say z. We write $f = f_1 f_2$, where $f_1 \in \mathrm{Hom}_{k(H)}(p(a), z)$ and $f_2 \in \mathrm{Hom}_{k(H)}(z, x)$. Then $z \in \mathscr{M}_b$ implies $f_2 g = 0$ and $fg = 0$—a contradiction. Thus we have proved ${}_aH_b \subseteq ({}_aH)_b$.

Next, let $0 \neq f$ in $\mathrm{Hom}_{k(H)}(p(a), x)/\mathrm{Hom}_{k(H)}(p(a), x)_{\mathscr{M}_b}$. This implies that $0 \neq f \in \mathrm{Hom}_{k({}_a(H_b))}(p(a), x)$. So there exists $0 \neq g \in \mathrm{Hom}_{k({}_a(H_b))}(x, q(b))$ such that $fg \neq 0 \in k({}_a(H_b))$ (see [13], Corollary 5). This shows $fg \neq 0$ in $k(H)$, and therefore ${}_aH_b \supseteq ({}_aH)_b$.

The proof of ${}_aH_b = {}_a(H_b)$ is similar. ∎

THEOREM 3.1. *Let k be a field. Let H be a thin left hammock, let $p(a) \neq p(\omega)$ be a projective vertex, and let $q(b) \neq q(\omega')$ be an injective vertex of H. Assume that* $\mathrm{Hom}_{k(H)}(p(b), p(a)) = 0$. *Then* ${}_aH_b = \{x \in H \mid \mathrm{Hom}_{k(H)}(p(a), q(b))_{\{x\}} \neq 0\}$ *is a hammock with hammock function* $h_{({}_aH_b)} = \dim_k \mathrm{Hom}_{k(H)}(p(a), -) - \dim_k \mathrm{Hom}_{k(H)}(p(a), -)_{\mathscr{M}_b} = \dim_k \mathrm{Hom}_{k(H)}(-, q(b)) - \dim_k \mathrm{Hom}_{k(H)}(-, q(b))_{{}_a\mathscr{M}}$.

Proof. By Lemma 3.2 we know that ${}_aH_b = ({}_aH)_b = {}_a(H_b)$ is a hammock. Since the hammock function is uniquely determined, we have $h_{({}_aH_b)} = h_{(({}_aH)_b)} = h_{({}_a(H_b))}$. ∎

Remark. Note that if $\mathrm{Hom}_{k(H)}(p(b), p(a)) \neq 0$ and $a \neq b$, then $\mathrm{Hom}_{k(H)}(p(a), q(b)) = 0$. So ${}_aH_b = \emptyset$.

Remark. Let H be a hammock. According to Theorem 2.4, we can obtain the poset $\mathscr{S}({}_aH_b)$ corresponding to the hammock ${}_aH_b$ from the poset $\mathscr{S}(H)$ corresponding to the hammock H by a finite sequence of the algorithms of Nazarova and Roiter (see [7]).

THEOREM 3.2. *Let k be a field. Let H be a thin left hammock, let $p(a) \neq p(\omega)$ be a projective vertex, and let $q(b) \neq q(\omega')$ be an injective vertex of H. Assume that* $\mathrm{Hom}_{k(H)}(p(b), p(a)) = 0$. *Then* ${}_aH_b = {}_aH \cap H_b$.

In order to prove Theorem 3.2, we need some properties of $\mathscr{S}$-spaces. The following lemma is due to Zavadskiĭ (see [16]).

LEMMA 3.3. *Let $\mathscr{S}$ be a poset. Assume that $\{a_1, \ldots, a_t\}$, where $t \geq 1$, is a subset of $\mathscr{S}$ with $a_1, \ldots, a_t$ mutually incomparable. Then a morphism $\phi \in \mathrm{Hom}_{\mathscr{S}}(U, V)$ factors through a direct sum $(P_{\mathscr{S}}(a_1, \ldots, a_t))^m$ if and only if $\phi(U_\omega) \subseteq \bigcap_{i=1}^r V_{a_i}$ and $\phi(U_x) = 0$ for $x \in \mathscr{S} \setminus \{a_1, \ldots, a_t\}^\vee$.*

PROPOSITION 3.1. *Let $\mathscr{S}$ be a poset. Assume that a and b in $\mathscr{S}$ are incomparable. Then*

(1) $\phi \in \mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(a), X)_{\mathscr{N}_b}$ *if and only if* $\phi \in \mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(a), X)_{\{P_{\mathscr{S}}(a,b)\}}$.

(2) $\chi \in \mathrm{Hom}_{\mathscr{S}}(X, Q_{\mathscr{S}}(b))_{a\mathscr{N}}$ *if and only if* $\chi \in \mathrm{Hom}_{\mathscr{S}}(X, Q_{\mathscr{S}}(b))_{\{P_{\mathscr{S}}(z_1,\dots,z_r)\}}$, *where* $\{z_1, \dots, z_r\} = \min(\mathscr{S} \setminus \{a,b\}_\wedge)$.

Proof. (1) For $\phi \in \mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(a), X)_{\mathscr{N}_b}$, there is some $Y \in \mathscr{N}_b$ such that $\phi = \theta\psi$, where $\theta \in \mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(a), Y)$ and $\psi \in \mathrm{Hom}_{\mathscr{S}}(Y, X)$. Since $Y \in \mathscr{N}_b$, then $\mathrm{Hom}_{\mathscr{S}}(Y, Q_{\mathscr{S}}(b)) = 0$ and $Y_\omega = Y_b$. So $\theta(P_{\mathscr{S}}(a,b)_\omega) = \theta(P_{\mathscr{S}}(a,b)_a) \subseteq Y_a = Y_a \cup Y_\omega = Y_a \cap Y_b$, and $\theta(P_{\mathscr{S}}(a,b)_x) = 0$ for $x \in \mathscr{S} \setminus \{a,b\}^\vee$, since $P_{\mathscr{S}}(a,b)_x = 0$. Thus, according to Lemma 3.3 we see that θ factors through $P_{\mathscr{S}}(a,b)$ and ϕ factors through $P_{\mathscr{S}}(a,b)$. This means that $\phi \in \mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(a), x)_{\{P_{\mathscr{S}}(a,b)\}}$. The other implication is obvious, since $P_{\mathscr{S}}(a,b) \in \mathscr{N}_b$.

(2) The proof is similar to (1). ∎

The following consequence of the Proposition 3.1 will be useful.

COROLLARY 3.1. *Let k be a field. Let H be a thin left hammock with finitely many projective vertices, let $p(a) \neq p(\omega)$ be a projective vertex, and let $q(b) \neq q(\omega')$ be an injective vertex of H. Assume that* $\mathrm{Hom}_{k(H)}(p(b), p(a)) = 0$. *Then the following statements are equivalent for $x \in k(H)$.*

(1) $x \in {}_aH_b$;

(2) *there is a map* $\psi \in \mathrm{Hom}_{k(H)}(p(a), x)$ *which does not factor through* $p(a,b)$;

(3) *there is a map* $\phi \in \mathrm{Hom}_{k(H)}(x, q(b))$ *which does not factor through* $p(z_1, \dots, z_r)$, *where* $\{z_1, \dots, z_r\} = \min(\mathscr{S} \setminus \{a,b\}_\wedge)$.

Proof. It follows from Proposition 3.1 and Lemma 3.2. ∎

Proof of Theorem 3.2. It is easy to see that ${}_aH_b \subseteq {}_aH \cap H_b$. In order to show the other inclusion, assume that $q(b) \in {}^dH \setminus {}^{d-1}H$. We denote by $\mathscr{S}(d)$ the poset formed from all projective vertices p of H with $p \in {}^dH$. Then $a, b \in \mathscr{S}(d)$ and ${}_d\ell(\mathscr{S}) \cong k({}^dH)$. Now, assume that $x \in {}_aH \cap H_b$. Then there are $0 \neq f \in \mathrm{Hom}_{k(H)}(p(a), x)$ and $0 \neq g \in \mathrm{Hom}_{k(H)}(x, q(b))$. If $f \notin \mathrm{Hom}_{k(H)}(p(a), x)_{\mathscr{N}_b}$, then $x \in {}_aH_b$ by Corollary 3.1. If $g \notin \mathrm{Hom}_{k(H)}(x, q(b))_{a\mathscr{N}}$, then $x \in {}_aH_b$ by Corollary 3.1 again. Suppose $f \in \mathrm{Hom}_{k(H)}(p(a), x)_{\mathscr{N}_b}$ and $g \in \mathrm{Hom}_{k(H)}(x, q(b))_{a\mathscr{N}}$. From Proposition 3.1, we have $\mathrm{Hom}_{\mathscr{S}(d)}(P_{\mathscr{S}}(a,b), \mathbf{F}'(x)) \neq 0$ and $\mathrm{Hom}_{\mathscr{S}(d)}(\mathbf{F}'(x), P_{\mathscr{S}}(z_1, \dots, z_r)) \neq 0$, where $\mathbf{F}'$ is the functor $\mathbf{F}'\colon k({}^dH) \cong {}_d(\ell(\mathscr{S}(d)))$ and $\{z_1, \dots, z_r\} = \min(\mathscr{S} \setminus \{a,b\}_\wedge)$. This is impossible, since $\tau_{\mathscr{S}} P_{\mathscr{S}}(a,b) = P_{\mathscr{S}}(z_1, \dots, z_r)$ by Theorem 2.3 and since the preprojective component of $\ell(\mathscr{S})$ is directed. ∎

COROLLARY 3.2. *Let k be a field. Let $\mathscr{S}$ be a poset. Let $\mathscr{P}_{\mathscr{S}}$ be the preprojective component of the Auslander–Reiten quiver of $\ell(\mathscr{S})$, let $P_{\mathscr{S}}(a)$*

be a projective object in $\mathscr{P}_{\mathscr{S}}$ *different from* $P_{\mathscr{S}}(\omega)$, *and let* $Q_{\mathscr{S}}(b)$ *be an injective object on* $\mathscr{P}_{\mathscr{S}}$ *different from* $Q_{\mathscr{S}}(\omega')$. *Assume that* $\mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(b), P_{\mathscr{S}}(a)) = 0$. *Then* ${}_aH_b = \{X \in \mathscr{P}_{\mathscr{S}} | \mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(a), X) \neq 0$ *and* $\mathrm{Hom}_{\mathscr{S}}(X, Q_{\mathscr{S}}(b)) \neq 0\} = \{X \in \mathscr{P}_{\mathscr{S}} | \min\{\dim_k X_a, \dim_k X_\omega - \dim_k X_b\} \neq 0\}$ *is a hammock with hammock function*

$$
\begin{aligned}
h_{({}_aH_b)}(X) &= \begin{cases} \dim_k X_a & \mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(a,b), X) = 0 \\ \dim_k X_\omega - \dim_k X_b & \textit{otherwise} \end{cases} \\
&= \begin{cases} \dim_k X_\omega - \dim_k X_b & \mathrm{Hom}_{\mathscr{S}}(X, P_{\mathscr{S}}(z_1, \dots, z_r)) = 0 \\ \dim_k X_a & \textit{otherwise} \end{cases} \\
&= \min\{\dim_k X_a, \dim_k X_\omega - \dim_k X_b\},
\end{aligned}
$$

where $\{z_1, \dots, z_r\} = \min(\mathscr{S} \setminus \{a, b\}_\wedge)$.

Proof. If $\mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(a,b), X) = 0$, then $\mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(a), X)_{\mathscr{N}_b} = 0$. So $h_{({}_aH_b)}(X) = \dim_k \mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(a), X) = \dim_k X_a$ and $\dim_k \mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(a), X) = \dim_k \mathrm{Hom}_{\mathscr{S}}(X, Q_{\mathscr{S}}(b)) - \dim_k \mathrm{Hom}_{\mathscr{S}}(X, Q_{\mathscr{S}}(b))_{{}_a\mathscr{N}} \leq \dim_k \mathrm{Hom}_{\mathscr{S}}(X, Q_{\mathscr{S}}(b)) = \dim_k X_\omega - \dim_k X_b$. If $\mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(a, b), X) \neq 0$, then $\mathrm{Hom}_{\mathscr{S}}(X, P_{\mathscr{S}}(z_1, \dots, z_r)) = 0$ and $\mathrm{Hom}_{\mathscr{S}}(X, Q_{\mathscr{S}}(b))_{{}_a\mathscr{N}} = 0$. So $h_{({}_aH_b)}(X) = \dim_k \mathrm{Hom}_{\mathscr{S}}(X, Q_{\mathscr{S}}(b)) = \dim_k X_\omega - \dim_k X_b$ and $\dim_k \mathrm{Hom}_{\mathscr{S}}(X, Q_{\mathscr{S}}(b)) = \dim_k \mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(a), X) - \dim_k \mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(a), X)_{\mathscr{N}_b} \leq \dim_k \mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(a), X) = \dim_k X_a$. ∎

COROLLARY 3.3. *Let k be a field. Let* $\mathscr{S}$ *be a poset. Let* $U \in {}_\infty\ell(\mathscr{S})$. *Assume that* $\mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(a,b), U) = 0$ *and* $\mathrm{Hom}_{\mathscr{S}}(U, P_{\mathscr{S}}(z_1, \dots, z_r)) = 0$, *where* $\{z_1, \dots, z_r\} = \min(\mathscr{S} \setminus \{a, b\}_\wedge)$. *Then* $\dim_k U_\omega - \dim_k U_b = \dim_k U_a$.

PROPOSITION 3.2. *Let k be a field. Let H be a thin left hammock, let* $p(a), p(c)$ *be projective vertices of H different from* $p(\omega)$, *and let* $q(b), q(d)$ *be injective vertices of H different from* $q(\omega')$. *Assume that* $\mathrm{Hom}_{k(H)}(p(b), p(a)) = 0$, $\mathrm{Hom}_{k(H)}(p(d), p(a)) = 0$, $\mathrm{Hom}_{k(H)}(p(b), p(c)) = 0$, $\mathrm{Hom}_{k(H)}(p(c), p(a)) \neq 0$, *and* $\mathrm{Hom}_{k(H)}(p(b), p(d)) \neq 0$. *Then we have*

(1) ${}_aH_b \subseteq {}_aH_d$ *and* ${}_aH_b = ({}_aH_d)_b$;
(2) ${}_aH_b \subseteq {}_cH_b$ *and* ${}_aH_b = {}_a({}_cH_b)$;
(3) ${}_aH_b = {}_a({}_cH_d)_b$.

Proof. Let $x \in {}_aH_b$. Then there exist $f \in \mathrm{Hom}_{k(H)}(p(a), x)$ and $g \in \mathrm{Hom}_{k(H)}(x, q(b))$ such that $fg \neq 0$. So $0 \neq fgl$ by Lemma 2.1, where $0 \neq l \in \mathrm{Hom}_{k(H)}(q(b), q(d))$. This means $x \in {}_aH_d$ and $x \in ({}_aH_d)_b$. Therefore ${}_aH_b \subseteq {}_aH_d$ and ${}_aH_b \subseteq ({}_aH_d)_b$. On the other hand, let $f \in \mathrm{Hom}_{k({}_aH_b)}(p(a), x)$ and $g \in \mathrm{Hom}_{k({}_aH_b)}(x, q(b))$ with $fg \neq 0$. It follows that

$fg \neq 0$ in $k(H)$. Therefore ${}_aH_b \supseteq ({}_aH_d)_b$. The proof of (2) is similar. For (3), from Lemma 3.2 and (1) and (2), we have ${}_a({}_cH_d)_b = ({}_a({}_cH_d))_b = ({}_aH_d)_b = {}_aH_b$. ∎

4. "ALMOST" LEFT HAMMOCKS INDUCED BY A PAIR OF POINTS

Let k be a field. Let H be a thin left hammock with finitely many projective vertices, let $p(a) \neq p(\omega)$ be a projective vertex, and let $q(b) \neq q(\omega')$ be an injective vertex of H. Assume that $\mathrm{Hom}_{k(H)}(p(b), p(a)) = 0$, that is, $a \nleq b$ in $\mathscr{S}(H)$. In this section, we consider $H/{}_aH_b = \{x \in H \mid h_H(x) - h_{({}_aH_b)}(x) \neq 0\}$. Note that in the case when $a > b$, if $z \in p(b)^-$, then $z \in {}_aH_b$ and furthermore $p(b)$ is a source of $H/{}_aH_b$ and is different from ω. So we only consider the case when a and b are incomparable.

LEMMA 4.1. *Let H be a thin left hammock with finitely many projective vertices and let $\mathscr{S} := \mathscr{S}(H)$ be the poset corresponding to H. Let $p(a) \neq p(\omega)$ be a projective vertex and let $q(b) \neq q(\omega')$ be an injective vertex of H. Assume that a and b are incomparable in $\mathscr{S}$. Then $h_H(x) \geq h_{({}_aH_b)}(x)$ for $x \in H$, where we put $h_{({}_aH_b)}(x) = 0$ for $x \in H \setminus {}_aH_b$.*

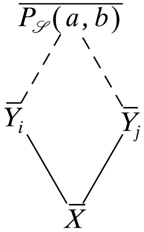
Proof. Since $\dim_k \mathrm{Hom}_{k(H)}(p(a), x) = \dim_k \mathrm{Hom}_{k(H)}(p(\omega), x)_{\{p(a)\}}$, according to [6, Lemma 3.1], we have $h_{({}_aH_b)}(x) = \dim_k \mathrm{Hom}_{k(H)}(p(a), x) - \dim_k \mathrm{Hom}_{k(H)}(p(a), x)_{\mathscr{M}_b} \leq \dim_k \mathrm{Hom}_{k(H)}(p(a), x) = \dim_k \mathrm{Hom}_{k(H)}(p(\omega), x)_{\{p(a)\}} \leq \dim_k \mathrm{Hom}_{k(H)}(p(\omega), x) = h_H(x)$. ∎

LEMMA 4.2. *Let H be a thin left hammock and let $\mathscr{S} := \mathscr{S}(H)$ be the poset corresponding to H. Let $p(a) \neq p(\omega)$ be a projective vertex and let $q(b) \neq q(\omega')$ be an injective vertex of H. Assume that a and b are incomparable in $\mathscr{S}$. Let $0 \to \tau_{\mathscr{S}} X \xrightarrow{\xi} \oplus_{i=1}^t Y_i \xrightarrow{\eta} X \to 0$ be an Auslander–Reiten sequence in ${}_\infty\ell(\mathscr{S})$. Then the following conditions are equivalent.*

(1) *$X, \tau_{\mathscr{S}} X \notin {}_aH_b$ and $Y_j \in {}_aH_b$ for some $j \in \{1, \dots, t\}$;*

(2) *$X = P_{\mathscr{S}}(a, b)$.*

Proof. (1) ⇒ (2): Note that $Y_j \in {}_aH_b$ for some j implies that there are $0 \neq \phi \in \mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(a), \oplus_{i=1}^t Y_i)$ and $0 \neq \psi \in \mathrm{Hom}_{\mathscr{S}}(\oplus_{i=1}^t Y_i, Q_{\mathscr{S}}(b))$. Now $X \notin {}_aH_b$ means $\xi\psi \neq 0$. So $\xi\psi$ factors through $P_{\mathscr{S}}(z_1, \dots, z_r)$, where $\{z_1, \dots, z_r\} = \min(\mathscr{S} \setminus \{a, b\}_\wedge)$. This follows from Corollary 3.1, since $\tau_{\mathscr{S}} X \notin {}_aH_b$. Thus we have $\mathrm{Hom}_{\mathscr{S}}(\tau_{\mathscr{S}} X, P_{\mathscr{S}}(z_1, \dots, z_r)) \neq 0$. Similarly, we know $\mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(a, b), X) \neq 0$. Now we get the sequence of maps $\tau_{\mathscr{S}} X \xrightarrow{\psi_1} P_{\mathscr{S}}(z_1, \dots, z_r) \xrightarrow{\psi_2} Z_1 \xrightarrow{\psi_3} P_{\mathscr{S}}(a, b) \xrightarrow{\psi_4} X$, where Z_1 is a summand of the module which occurs in the middle term of the Auslander–Reiten sequence ending in $P_{\mathscr{S}}(a, b)$, ψ_2 and ψ_3 are irreducible maps, and ψ_1 and ψ_4 are nonzero maps. Suppose that ψ_4 is not an isomorphism. Then ψ_1 is not an isomorphism either, since $\tau_{\mathcal{S}} P_{\mathcal{S}}(a, b) = P_{\mathcal{S}}(z_1, \dots, z_r)$. So ψ_4 factors through $\oplus_{i-1}^{t} Y_i$. This means that there is $0 \neq (\phi_1, \dots, \phi_t) \in \mathrm{Hom}_{\mathcal{S}}(P_{\mathcal{S}}(a, b), \oplus_{i-1}^{t} Y_i)$, and ψ_1 also factors through $\oplus_{i=1}^{t} Y_i$. This means that there is $0 \neq (\chi_1, \dots, \chi_t)' \in \mathrm{Hom}_{\mathcal{S}}(\oplus_{i=1}^{t} Y_i, P_{\mathcal{S}}(z_1, \dots, z_r))$. Consider the case when there is some i such that $\phi_i \neq 0$ and $\chi_i \neq 0$. We obtain a cycle sequence $Y_i \xrightarrow{\chi_i} P_{\mathcal{S}}(z_1, \dots, z_r) \xrightarrow{\psi_2} Z_1 \xrightarrow{\psi_3} P_{\mathcal{S}}(a, b) \xrightarrow{\phi_i} Y_i$—a contradiction to the fact that the preprojective component of the Auslander−Reiten quiver of $_{\infty}\ell(\mathcal{S})$ is directed. If the case above does not occur, we can choose $\phi_i \neq 0$ and $\chi_j \neq 0$, where $i \neq j$. Then we obtain a subgraph of the orbit graph of the preprojective component of the Auslander−Reiten quiver of $_{\infty}\ell(\mathcal{S})$ as follows



where a dotted line denotes the composition of some edges. This is a contradiction, because the orbit graph of the preprojective component of the Auslander−Reiten quiver of $_{\infty}\ell(\mathcal{S})$ is a tree. Note that obviously $P_{\mathcal{S}}(a, b) \neq \overline{X}$. Therefore, ψ_4 is an isomorphism, i.e., $X = P_{\mathcal{S}}(a, b)$.

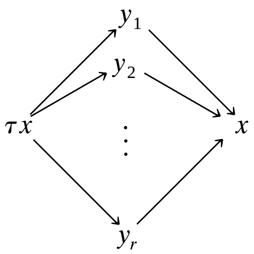
(2) ⇒ (1): Proposition 2.1 shows that $\tau_{\mathcal{S}} P_{\mathcal{S}}(a, b) = P_{\mathcal{S}}(z_1, \dots, z_r)$. Clearly, $P_{\mathcal{S}}(a, b) \notin H_b$ and $P_{\mathcal{S}}(z_1, \dots, z_r) \notin {}_aH$. So $P_{\mathcal{S}}(a, b), \tau_{\mathcal{S}} P_{\mathcal{S}}(a, b) \notin {}_aH_b$. Since both $P_{\mathcal{S}}(a, b)$ and $\tau_{\mathcal{S}} P_{\mathcal{S}}(a, b)$ are thin, we know $t \leq 2$. In case $t = 2, Y_1, Y_2$ both are thin. By Lemma 2.1 we know that $P_{\mathcal{S}}(z_1, \dots, z_r)$ is a $\mathcal{S}$-subspace of Y_i and Y_i is a $\mathcal{S}$-subspace of $P_{\mathcal{S}}(a, b)$, for $i = 1, 2$. So $P_{\mathcal{S}}(z_1, \dots, z_r)$ is a $\mathcal{S}$-subspace of $P_{\mathcal{S}}(a, b)$ and $\{z_1, \dots, z_r\} = \min(\{a, b\}^{\vee} \setminus \{a, b\})$. Thus, comparing $P_{\mathcal{S}}(z_1, \dots, z_r)_s, P_{\mathcal{S}}(a, b)_s$ with $(Y_i)_s$, for $i = 1, 2$ and $s \in \mathcal{S}^{+}$, we can obtain that $Y_1 = P_{\mathcal{S}}(a, u_1, \dots, u_s)$ and $Y_2 = P_{\mathcal{S}}(b, v_1, \dots, v_t)$, where $u_i \in \{z_1, \dots, z_r\}$, $u_i \notin a^{\vee}$, for $i = 1, \dots, s$, and $v_j \in \{z_1, \dots, z_r\}$, $v_j \notin b^{\vee}$, for $j = 1, \dots, t$. Therefore $Y_1 \in {}_aH_b$ and $Y_2 \notin {}_aH_b$. Consider now the case $t = 1$. Since $\mathrm{Hom}_{\mathcal{S}}(P_{\mathcal{S}}(a), P_{\mathcal{S}}(a, b)) \neq 0$ and $\mathrm{Hom}_{\mathcal{S}}(\tau_{\mathcal{S}} P_{\mathcal{S}}(a), Q_{\mathcal{S}}(b)) \neq 0$, we have $\mathrm{Hom}_{\mathcal{S}}(P_{\mathcal{S}}(a), Y_1) \neq 0$ and $\mathrm{Hom}_{\mathcal{S}}(Y, Q_{\mathcal{S}}(b)) \neq 0$. Thus $Y_1 \in {}_aH_b$ follows from Theorem 3.2. ∎

Let H be a left hammock with translation τ and let μ be a projective−injective vertex of H with $\mu^{+} = \{\varepsilon\}$. If $\mu^{-} = \{\tau\varepsilon\}$, then we call the subquiver $H \setminus \{\mu\}$, together with the restriction of τ on it, an *"almost" left hammock with respect to* ε. An "almost" left hammock $H \setminus \{\mu\}$ is called an *"almost" hammock*, if H is a hammock. If L is an "almost" left hammock obtained from some left hammock H with respect to ε, we write $H = L \cup \{\mu\}$ with $\mu^+ = \{\varepsilon\}$, and we call the vertex μ the *additional vertex*.

THEOREM 4.1. *Let H be a thin left hammock with finitely many projective vertices and let $\mathscr{S} := \mathscr{S}(H)$ be the poset corresponding to H. Let $p(a) \neq p(\omega)$ be a projective vertex and let $q(b) \neq q(\omega')$ be an injective vertex of H. Assume that a and b are incomparable in $\mathscr{S}$. Then $H/{}_aH_b = \{x \in H | h_H(x) - h_{({}_aH_b)}(x) \neq 0\}$ is an "almost" left hammock with respect to $p(a,b)$. For convenience, we denote by ${}_aH_b^\diamond$ the left hammock $(H/{}_aH_b) \cup \{\mu\}$, where $\mu^+ = \{p(a,b)\}$. Then the hammock function of ${}_aH_b^\diamond$ is*

$$h_{({}_aH_b^\diamond)}(x) = \begin{cases} h_H(x) - h_{({}_aH_b)}(x) & x \in H/{}_aH_b \\ 1 & x = \mu. \end{cases}$$

Proof. Consider a given vertex $x \in {}_aH_b$ different from $p(\omega)$. Let

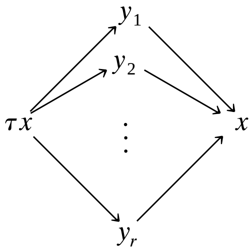


be the mesh in $k(H)$ (we put $\tau x = 0$ in case x is projective). We can observe combinatorially that the equality $h_{({}_aH_b)}(x) + h_{({}_aH_b)}(\tau x) = \sum_{y \to x} h_{({}_aH_b)}(y)$ holds (if $z \notin {}_aH_b$, let $h_{({}_aH_b)}(z) = 0$) except in the following cases: (i) $x = p(a)$; (ii) $x, \tau x \notin {}_aH_b$ and $y \in {}_aH_b$ for some $y \in x^-$.

Now, we check that ${}_aH_b^\diamond$ and $h_{({}_aH_b^\diamond)}$ satisfies the conditions of a left hammock.

(1) Clearly, ω is a source of $H/{}_aH_b$. Suppose there is another source z in $H/{}_aH_b$. We can suppose that $h_{({}_aH_b^\diamond)}(\tau z) = 0$, $\sum_{y \to z} h_{({}_aH_b^\diamond)}(y) = 0$, and $h_{({}_aH_b^\diamond)}(z) \neq 0$. Clearly, the case (i) and the case (ii) both do not occur. So $h_{({}_aH_b)}(z) + h_{({}_aH_b)}(\tau z) = \sum_{y \to z} h_{({}_aH_b)}(y)$. This together with $h_H(z) + h_H(\tau z) = \sum_{y \to z} h_H(y)$ implies $h_{({}_aH_b^\diamond)}(z) + h_{({}_aH_b^\diamond)}(\tau z) = \sum_{y \to z} h_{({}_aH_b^\diamond)}(y)$ and $h_{({}_aH_b^\diamond)}(z) = 0$—a contradiction.

(2) Let



be a mesh in $k(H)$ with $h_{({}_aH_b^\diamond)}(x) \neq 0$. This implies that the case (i) does not occur, since $h_H(p(a)) = h_{({}_aH_b)}(p(a))$. If the case (ii) does not occur, $h_{({}_aH_b)}(x) + h_{({}_aH_b)}(\tau x) = \sum_{y \to x} h_{({}_aH_b)}(y)$. This together with $h_H(x) + h_H(\tau x) = \sum_{y \to x} h_H(y)$ shows that $h_{({}_aH_b^\diamond)}(x) + h_{({}_aH_b^\diamond)}(\tau x) = \sum_{y \to x} h_{({}_aH_b^\diamond)}(y)$. In the case (ii), by Lemma 4.2 and Proposition 2.1 we know that $x = p(a,b)$ and $\tau x = p(z_1, \dots, z_r)$, where $\{z_1, \dots, z_r\} = \min(\mathscr{S} \setminus \{a,b\}_\wedge)$, and $\mathbf{F}(p(a,b)) = P_{\mathscr{S}}(a,b), \mathbf{F}(p(z_1, \dots, z_r)) = P_{\mathscr{S}}(z_1, \dots, z_r)$ under the equivalence functor $\mathbf{F}$: $k(H) \to {}_\infty\ell(\mathscr{S})$. So $h_H(x) = h_H(\tau x) = 1$, $\sum_{y \to X} h_H(y) = 2$, and $\sum_{y \to x} h_{({}_aH_b)}(y) = 1$. Thus, after adding an exceptional vertex μ with $\mu^+ = \{p(a,b)\}$ and $\mu^- = \{p(z_1, \dots, z_r)\}$, we have $h_{({}_aH_b^\diamond)}(x) + h_{({}_aH_b^\diamond)}(\tau x) = \sum_{y \to x} h_{({}_aH_b^\diamond)}(y)$.

(3) Assume that z is an injective vertex of ${}_aH_b^\diamond$. We have to prove that $h_{({}_aH_b^\diamond)}(z) \geq \sum_{z \to y} h_{({}_aH_b^\diamond)}(y)$. First, we consider the case when z is an injective vertex of H. It is clear that $|z^+| = 1$ and $h_H(z) = h_H(y_0)$ with $z^+ = \{y_0\}$. Now $z \in H/{}_aH_b$ implies $z \neq q(b)$, so $h_{({}_aH_b)}(z) = h_{({}_aH_b)}(y_0)$ and $h_{({}_aH_b^\diamond)}(z) = h_{({}_aH_b^\diamond)}(y_0)$. Next, in the case when z is not an injective vertex of H, we have the mesh $h_H(z) + h_H(\tau^- z) = \sum_{z \to y} h_H(y_i)$, $h_H(z) \neq h_{({}_aH_b)}(z)$, and $h_H(\tau^- z) = h_{({}_aH_b)}(\tau^- z)$. So the case (i) and (ii) both do not occur and $h_{({}_aH_b)}(z) + h_{({}_aH_b)}(\tau^- z) = \sum_{z \to y} h_{({}_aH_b)}(y)$. Thus, $h_{({}_aH_b^\diamond)}(z) = \sum_{z \to y} h_{({}_aH_b^\diamond)}(y)$. Finally, in case $z = \mu$, we have $h_{({}_aH_b^\diamond)}(z) = 1 = h_{({}_aH_b^\diamond)}(p(a,b))$. ∎

Remark. From Theorem 5.1 below, we know that the left hammock ${}_aH_b^\diamond$ corresponds to ${}_\infty\ell(\mathscr{S}')$ for some poset $\mathscr{S}'$. So ${}_aH_b^\diamond$ is a thin left hammock with finitely many projective vertices.

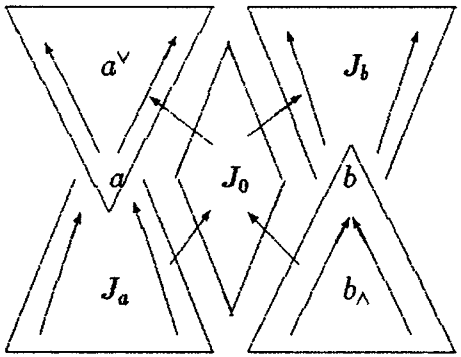
From Theorem 4.1 and Corollary 3.2, we have the following result:

COROLLARY 4.1. *Let k be a field. Let $\mathscr{S}$ be a poset and let $\mathscr{P}_{\mathscr{S}}$ be the preprojective component of the Auslander–Reiten quiver of $\ell(\mathscr{S})$. Let $P_{\mathscr{S}}(a)$ be a projective object in ${}_\infty\ell(\mathscr{S})$ different from $P_{\mathscr{S}}(\omega)$ and let $Q_{\mathscr{S}}(b)$ be an injective object in ${}_\infty\ell(\mathscr{S})$ different from $Q_{\mathscr{S}}(\omega')$. Assume that a and b are incomparable in $\mathscr{S}$. Then $H/{}_aH_b = \{X \in \mathscr{P}_{\mathscr{S}} | \max\{\dim_k X_b, \dim_k X_\omega - \dim_k X_a\} \neq 0\}$ is an "almost" left hammock with respect to $P_{\mathscr{S}}(a,b)$. We* *denote by* ${}_aH_b^{\diamond}$ *the left hammock* $(H/{}_aH_b) \cup \{\mu\}$ *with* $\mu^+ = \{P_{\mathscr{S}}(a,b)\}$. *Then the hammock function of* ${}_aH_b^{\diamond}$ *is*

$$h_{({}_aH_b^{\diamond})}(X) = \begin{cases} \max\{\dim_k X_b, \dim_k X_\omega - \dim_k X_a\} & X \in H/{}_aH_b \\ 1 & X = \mu. \end{cases}$$

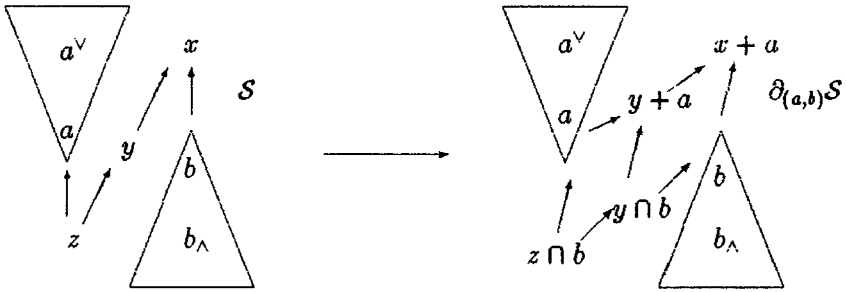
5. HAMMOCKS AND THE ALGORITHM OF ZAVADSKIĬ

First, we recall the algorithm of Zavadskiĭ. Let us fix some notation. Let $\mathscr{S}$ be a poset. We write $\mathscr{S} = A_1 + \cdots + A_n$ if $A_1 \cup \cdots \cup A_n = \mathscr{S}$ and $A_i \cap A_j = \varnothing$ for $i \neq j$ (note that the points from different A_i can be comparable). Let a pair of points a, b be incomparable. We put $\mathscr{S} = a^{\vee} + b_{\wedge} + J(a,b)$ and $J := J(a,b) = J_a + J_0 + J_b$, where $J_a = \{x \in J | x < a\}$ and $J_b = \{x \in J | x > b\}$ (see the diagram below). Then we have the following facts: points x, a, b are mutually incomparable for $x \in J_0$, points y, a are incomparable for $y \in J_b \cup J_0$, and z, b are incomparable for $z \in J_a \cup J_0$.



Let $\mathscr{S}$ be a poset. A pair of points (a,b) is called *suitable* (*for a stratification*) if a and b are incomparable, and $\mathscr{S} = a^{\vee} + b_{\wedge} + J$, where $J = \{z_1 < \cdots < z_n\}$. Following [17], we construct the (a,b)-*stratified poset* $\partial_{(a,b)}\mathscr{S}$ as follows: The points of $\partial_{(a,b)}\mathscr{S}$ consist of (1) x, for $x \in a^{\vee} \cup b_{\wedge}$; (2) $a + x$, for $x \in J_b \cup J_0$; (3) $b \cap x$, for $x \in J_a \cup J_0$. The order relation in $\partial_{(a,b)}\mathscr{S}$ is defined as follows: (1) we keep all relations in $\mathscr{S}$ between elements in $a^{\vee} \cup b_{\wedge}$; (2) we set $b \cap x < a + x$ for $x \in J_0$; (3) we set $a + x < a + y$, if $x < y$ in $J_b \cup J_0$; (4) we set $b \cap x < b \cap y$, if $x < y$ in $J_a \cup J_0$; (5) we set $a + x < y$, if $x < y$ for $x \in J_b \cup J_0$ and $y > a$; (6) we set $x < b \cap y$, if $x < y$ for $x \in J_a \cup J_0$ and $y < b$; (7) we add the relation

$a < a + x$ for $x \in J_b \cup J_0$, and $b \cap y < b$ for $y \in J_a \cap J_0$; (8) if x and y are such $x > y$ and $x < y$ under the relation above, then we identify x and y.



EXAMPLE.

Now, we can state the theorem concerning left hammocks and Zavadskiĭ stratification algorithms.

THEOREM 5.1. *Let H be a thin left hammock with finitely many projective vertices and let $\mathcal{S}(H)$ be the poset corresponding to H. Let $p(a)$ be a projective vertex different from $p(\omega)$ and let $q(b)$ be an injective vertex of H different from $q(\omega')$. Assume that a and b are incomparable in $\mathcal{S}(H)$. Denote by $\mathcal{S}({}_aH_b^\diamond)$ the poset corresponding to the left hammock ${}_aH_b^\diamond$. Then $\mathcal{S}({}_aH_b^\diamond)$ is obtained from $\mathcal{S}(H)$ as follows: there is a finite sequence of pairs of points $(c_1, d_1), (c_2, d_2), \dots, (c_l, d_l) = (a, b)$, and a finite sequence of posets $\mathcal{S}_1 = \mathcal{S}(H), \mathcal{S}_2, \dots, \mathcal{S}_l$ such that*

(1) *(c_i, d_i) is a suitable pair of points of $\mathcal{S}_i$, for $i = 1, \dots, l$;*

(2) *$\mathcal{S}_i = \partial_{(c_{i-1}, d_{i-1})}\mathcal{S}_{i-1}$ for $i = 2, \dots, l$, that is, $\mathcal{S}_i$ is the (c_{i-1}, d_{i-1})-stratified poset of $\mathcal{S}_{i-1}$;*

(3) *$\mathcal{S}_l = \mathcal{S}({}_aH_b^\diamond)$.*

The proof of this theorem will be covered in Sections 6 and 7.

We point out that Zavadskiĭ only considers the case $J_a = \emptyset = J_0$. Let us make some remarks here. In [15], Zavadskiĭ introduced the algorithm called "differentiation with respect to a pair of points." In [17], he used the two meticulous algorithms, which he called "stratification" and "replenishment," instead of the differentiation with respect to a pair of points. In the rest of this section, we will discuss the correspondence between two classes of left hammocks, in which replenishment will be completely explained.

The left hammock ${}_aH_b^\diamond$, by definition, has the property that it includes a projective–injective vertex μ with $\mu^+ = \{\varepsilon\}$ and $\mu^- = \{\tau\varepsilon\}$. The following proposition shows a bijection between the class of left hammocks with this property and the class of left hammocks including a projective vertex p and an injective vertex q with $p^- = q^+$.

PROPOSITION 5.1. *Let Φ be the set of pairs (H, μ), where H is a thin left hammock, and let μ be a projective–injective vertex of H with $\mu^+ = \{\varepsilon\}$ and $\mu^- = \{\tau\varepsilon\}$. Let Ψ be the set of triples (L, p, q), where L is a thin left hammock, p is a projective vertex, and q is an injective vertex of L with $p^- = q^+$. We define $\xi\colon \Phi \to \Psi$ by sending (H, μ) to (L, ε), where $L = H \setminus \{\mu\}$, and omitting the translation τ on ε in L. Then ξ is a bijective correspondence.*

Proof. Let $(H, \mu) \in \Phi$. We consider its translation subquiver $L \coloneqq H \setminus \{\mu\}$ and forget the translation τ on ε in L. Thus ε is a projective vertex and $\tau\varepsilon$ is an injective vertex in L. Since $\mu^+ = \{\varepsilon\}$ and $\mu^- = \{\tau\varepsilon\}$, we see that $h_H(\mu) = h_H(\varepsilon) = h_H(\tau\varepsilon) = 1$. So $|\varepsilon^-| = 2$, say, $\varepsilon^- = \{\mu, z\}$. Therefore, in $H \setminus \{\mu\}$, we have $h_H|_L(\varepsilon) = h_H|_L(z) = h_H|_L(\tau\varepsilon) = 1$. Hence L is a left hammock with hammock function $h_H|_L$, and L has a projective vertex ε and an injective vertex $\tau\varepsilon$ with $\varepsilon^- = \tau\varepsilon^+ = \{z\}$. Thus $(L, \varepsilon, \tau\varepsilon) \in \Psi$. On the other hand, for $(L, p, q) \in \Psi$, we know $|p^-| = |q^+| = 1$, say, $q^+ = \{z\}$. We construct a new left hammock H from L by adding an additional vertex μ with $\mu^+ = \{q\}$ and $\mu^- = \{p\}$, and define $\tau q = p$. It is easy to see that H is a left hammock with hammock function

$$h_H(x) = \begin{cases} h_L(x) & x \in L \\ 1 & x = \mu. \end{cases}$$

In this way, we define the map $\zeta\colon \Psi \to \Phi$ by sending (L, p, q) to (H, μ). Finally, it is obvious that $\xi\zeta = l_\Phi$ and $\zeta\xi = 1_\Psi$. ∎

PROPOSITION 5.2. *Let Φ' be the set of triples $(\mathscr{S}, a, b)$, where $\mathscr{S}$ is a poset and a and b are vertices in $\mathscr{S}$ with $\mathscr{S} = a^\vee + b_\wedge$. Let Ψ' be the set of triples $(\mathscr{S}, a, b)$, where $\mathscr{S}$ is a poset, and a and b are vertices in $\mathscr{S}$ with $a < b$ and with $\mathscr{S} = a^\vee \setminus \{b\} + b_\wedge$. We define $\xi'\colon \Phi' \to \Psi'$ by sending $(\mathscr{S}, a, b)$ to itself and adding the order relation $a < b$ in $\xi'(\mathscr{S}, a, b)$. Then ξ is a bijective correspondence.*

Proof. Define $\zeta'\colon \Psi' \to \Phi'$ by deleting the relation $a < b$ in $\zeta'(\mathscr{S}, a, b)$. Then we have $\xi'\zeta' = 1_{\Phi'}$ and $\zeta'\xi' = 1_{\Psi'}$. ∎

Following Zavadskiǐ [17], a pair of incomparable points (a, b) of a poset $\mathscr{S}$ is called *specific* if $\mathscr{S} = a^\vee + b_\wedge$. The new poset $\gamma_{(a,b)}\mathscr{S}$ obtained from $\mathscr{S}$ by adding the relation $a > b$ is called the *replenished poset*. We define the *replenishment functor* $\gamma\colon \ell(\mathscr{S}) \to \ell(\gamma_{(a,b)}\mathscr{S})$ by setting $\gamma(V)_x = V_x$, for

$x \neq b$, $\gamma(V)_b = V_a + V_b$, and $\gamma(\psi) = \psi$. The following result is owing to Zavadskiĭ and is presented in [17] as Theorem 2 and Corollary 2.

THEOREM 5.2. *The replenishment functor* γ: $\ell(\mathscr{S}) \to \ell(\gamma_{(a,b)}\mathscr{S})$ *induces an equivalence of factor categories*: $\ell(\mathscr{S})/\{P_{\mathscr{S}}(a), P_{\mathscr{S}}(a,b)\} \cong \ell(\gamma_{(a,b)}\mathscr{S})/\{P_{\gamma_{(a,b)}\mathscr{S}}(a)\}$ *and an equivalence of translation quivers* $\Gamma \setminus \{P_{\mathscr{S}}(a)\} \cong \Gamma'$, *where* Γ *and* Γ' *are the Auslander–Reiten quiver of the categories* $\ell(\mathscr{S})$ *and* $\ell(\gamma_{(a,b)}\mathscr{S})$.

It is easy to see that the left hammock H has a projective–injective vertex μ if and only if the poset $\mathscr{S}(H)$ has a specific pair of points (a,b) with $\mu = P_{\mathscr{S}}(a) = Q_{\mathscr{S}}(b)$. Now from Theorem 5.2 and Proposition 5.2, we have the following theorem.

THEOREM 5.3. *Let* H *be a thin left hammock with finitely many projective vertices and with a projective–injective vertex* μ. *Let* $\mathscr{S}(H)$ *be the poset corresponding to* H. *Assume that* $\xi(H,\mu) = (L,p,q)$ *in the sense of Proposition* 5.2. *Then* $\mathscr{S}(L)$ *is just the replenishment poset* $\gamma_{(a,b)}\mathscr{S}$ *for the specific pair* (a,b) *with* $\mu = P_{\mathscr{S}}(a) = Q_{\mathscr{S}}(b)$.

6. THE PROOF OF THEOREM 5.1 IN A SPECIAL CASE

Under the hypothesis of Theorem 5.1, we put $\mathscr{S}(H) = a^{\vee} + J + b_{\wedge}$ as in Section 5. If width$(J) = 1$, then (a,b) is a suitable pair and Zavadskiĭ's algorithm is valid. But in general, width$(J) > 1$. In this section we will prove Theorem 5.1 in the case width$(J) = 1$. We will establish the general case by induction in the next section. Now, we first consider the special case $J = \emptyset$.

PROPOSITION 6.1. *Let* H *be a thin left hammock with finitely many projective vertices and let* $\mathscr{S}(H)$ *be the poset corresponding to* H. *Let* $p(a)$ *be a projective vertex and let* $q(b)$ *be an injective vertex of* H. *Assume that* a *and* b *in* $\mathscr{S}(H)$ *are incomparable. Then the following conditions are equivalent.*

(1) (a,b) *is specific, that is,* $\mathscr{S}(H) = a^{\vee} + b_{\wedge}$;

(2) $|{}_aH_b| = 1$;

(3) ${}_aH_b = \{p(a)\} = \{q(b)\}$;

(4) ${}_aH_b^{\diamond} = H$.

Proof. It is obvious. ∎

Now we consider the case width$(J) = 1$.

Let $\mathscr{S}$ be a poset, let (a,b) be a suitable pair of points of $\mathscr{S}$, and let $\partial_{(a,b)}\mathscr{S}$ be the (a,b)-stratified poset. Following Zavadskiĭ, we define the *stratification functor* $\partial_{(a,b)}$: $\ell(\mathscr{S}) \to \ell(\partial_{(a,b)}\mathscr{S})$ by setting $(\partial_{(a,b)}(U))_\omega = U_\omega$ and $\partial_{(a,b)}(U)_x = U_x$ whenever $x \in a^{\vee} \cup b_{\wedge}$ and $\partial_{(a,b)}(U)_{a+x} = U_a + U_x$,

$\partial_{(a,b)}(U)_{b\cap x} = U_b \cap U_x$ for an $\mathscr{S}$-space U, and $\partial_{(a,b)}(\psi) = \psi$. The following theorem is owing to Zavadskiĭ (see [17]). Zavadskiĭ has considered only the case $J_a = \emptyset = J_b$. Although we allow $J_a \cup J_b \neq \emptyset$, the proof is the same.

THEOREM 6.1. *Let $\mathscr{S}$ be a poset. Assume that the points $a, b \in \mathscr{S}$ are incomparable. Assume that* $\operatorname{width}(J) = 1$, *and write $J = \{z_1, \dots, z_n\}$, where $z_1 \le \cdots \le z_n$. Then the functor $\partial_{(a,b)}$: $\ell(\mathscr{S}) \to \ell(\partial_{(a,b)}\mathscr{S})$ induces an equivalence of the factor categories $\partial_{(a,b)}$: $\ell(\mathscr{S})/\Omega \cong \ell(\partial_{(a,b)}\mathscr{S})/\Omega'$, where $\Omega = \{P_{\mathscr{S}}(a), P_{\mathscr{S}}(a, z_1), \dots, P_{\mathscr{S}}(a, z_n)\}$ and $\Omega' = \{P_{\partial_{(a,b)}\mathscr{S}}(a)\}$ (we put $P_{\mathscr{S}}(a, z_i) := P_{\mathscr{S}}(z_i)$ if $a > z_i$).*

Moreover, let $\Gamma_{\mathscr{S}}$ be the Auslander–Reiten quiver of $\ell(\mathscr{S})$ and let $\Gamma_{\partial_{(a,b)}\mathscr{S}}$ be the Auslander–Reiten quiver of $\ell(\partial_{(a,b)}\mathscr{S})$. Then $\Gamma_{\mathscr{S}} \setminus \{P_{\mathscr{S}}(a), P_{\mathscr{S}}(a, z_1), \dots, P_{\mathscr{S}}(a, z_n)\} \cong \Gamma_{\partial_{(a,b)}\mathscr{S}} \setminus \{P_{\partial_{(a,b)}\mathscr{S}}(a)\}$.

Remark. Observe that $P_{\partial_{(a,b)}\mathscr{S}}(a) = Q_{\partial_{(a,b)}\mathscr{S}}(b)$ and $P_{\partial_{(a,b)}\mathscr{S}}(a, b) = \partial_{(a,b)}(P_{\mathscr{S}}(a, b))$ in $\ell(\partial_{(a,b)}\mathscr{S})$.

LEMMA 6.1. *Let k be a field. Let H be a thin left hammock with finitely many projective vertices and let $\mathscr{S} := \mathscr{S}(H)$ be the poset corresponding to H. Let $p(a)$ be a projective vertex and let $q(b)$ be an injective vertex of H. Assume that a and b are incomparable in $\mathscr{S}$. Assume that* $\operatorname{width}(J(a, b)) = 1$, *and write $J(a, b) = \{z_1, \dots, z_n\}$, where $z_1 \le \cdots \le z_n$. Given $x \in k(H)$, assume that x corresponds to the object X in ${}_\infty\ell(\mathscr{S})$ under the equivalence $k(H) \cong {}_\infty\ell(\mathscr{S})$. Then the equality $h_H(x) = h_{({}_aH_b)}(x)$ holds if and only if $X \in \Omega$.*

Proof. Note that the objects in Ω occur on the preprojective component of the Auslander–Reiten quiver of $\ell(\mathscr{S})$, since $q(b) \in H$. For $X \in \Omega$, it is easy to see that $h_H(x) = 1$, $x \in {}_aH_b$, and $h_{({}_aH_b)}(x) \neq 0$. So $h_{({}_aH_b)}(x) = 1 = h_H(x)$ by Lemma 4.1.

On the other hand, $h_H(x) = h_{({}_aH_b)}(x)$ means $\dim_k \operatorname{Hom}_{k(H)}(\omega, x) = \dim_k \operatorname{Hom}_{k(H)}(p(a), x) - \dim_k \operatorname{Hom}_{k(H)}(p(a), x)_{\mathscr{M}_b} = \dim_k \operatorname{Hom}_{k(H)}(x, q(b)) - \dim_k \operatorname{Hom}_{k(H)}(x, q(b))_{{}_a\mathscr{M}}$. This implies that $\dim_k X = \dim_k X_a$ and $\dim_k X_b = 0$.

Let $\mathscr{S}' = \mathscr{S}(H) \setminus (a^\vee \cup b_\wedge)$; the order relation of $\mathscr{S}'$ follows from $\mathscr{S}(H)$. We define a functor $\mathbf{G}$: $\ell(\mathscr{S}') \to \ell(\mathscr{S})$ by setting $\mathbf{G}(U)_\omega = U_\omega$, $\mathbf{G}(U)_x = U$ whenever $x \in a^\vee$, $\mathbf{G}(U)_x = 0$ whenever $x \in b_\wedge$, and $\mathbf{G}(U)_x = U_x$ for $x \in \mathscr{S}(H) \setminus a^\vee \cup b_\wedge$, and $\mathbf{G}(\psi) = \psi$. Clearly, $\mathbf{G}$ is indeed a functor. Moreover, $\mathbf{G}$ induces an equivalence between $\ell(\mathscr{S}')$ and the full subcategory of $\ell(\mathscr{S})$ consisting of the objects V with $V_b = 0$ and $V_a = V_\omega$.

Since $\operatorname{width}(J) \le 1$, each indecomposable $\mathscr{S}'$-space V is thin. This implies that X is thin. Therefore $X \in \Omega$, since $X_b = 0$ and $X_a = X_\omega$. ∎

THEOREM 6.2. *Let H be a thin left hammock with finitely many projective vertices and let $\mathscr{S} := \mathscr{S}(H)$ be the poset corresponding to H. Let $p(a)$ be a projective vertex and let $q(b)$ be an injective vertex of H. Assume that (a,b) is a suitable pair of points in $\mathscr{S}(H)$ with* width $J(a,b) = 1$ *and that $\partial_{(a,b)}\mathscr{S}$ is the (a,b)-stratification of $\mathscr{S}(H)$. We denote by $\mathscr{S}({}_aH_b^\diamond)$ the poset corresponding to the left hammock ${}_aH_b^\diamond$, where ${}_aH_b^\diamond = H/{}_aH_b \cup \{\mu\}$ and $H/{}_aH_b = \{x \in H | h_H(x) - h_{({}_aH_b)}(x) \neq 0\}$. Then $\mathscr{S}({}_aH_b^\diamond) \cong \partial_{(a,b)}\mathscr{S}$.*

Proof. By definition, we know $k(H/{}_aH_b) \cong k(H)/\{x \in H | h_H(x) = h_{({}_aH_b)}(x)\}$. Note that the objects of Ω occur in ${}_\infty\ell(\mathscr{S})$. This together with Lemma 6.1 implies $k(H)/\{x \in H | h_H(x) = h_{({}_aH_b)}(x)\} \cong {}_\infty\ell(\mathscr{S})/\Omega$. Corollary 2.1 means $P_{\partial_{(a,b)}\mathscr{S}}(a)^+ = \{P_{\partial_{(a,b)}\mathscr{S}}(a,b)\}$. It follows that the objects of Ω' occur in ${}_\infty\ell(\partial_{(a,b)}\mathscr{S})$, since $\partial_{(a,b)}P_{\mathscr{S}}(a,b) = P_{\partial_{(a,b)}\mathscr{S}}(a,b)$. So by Theorem 6.1 we have ${}_\infty\ell(\mathscr{S})/\Omega \cong {}_\infty\ell(\partial_{(a,b)}\mathscr{S})/\Omega'$. Thus we obtain $k(H/{}_aH_b) \cong {}_\infty\ell(\partial_{(a,b)}\mathscr{S})/\Omega'$ and $H/{}_aH_b \cong \mathscr{P}_{\partial_{(a,b)}\mathscr{S}} \setminus \{P_{\partial_{(a,b)}\mathscr{S}}(a)\}$. Since $H/{}_aH_b$ is an "almost" left hammock with respect to $p(a,b)$, we see that $\mathscr{P}_{\partial_{(a,b)}\mathscr{S}} \setminus \{P_{\partial_{(a,b)}\mathscr{S}}(a)\}$ is an "almost" hammock with respect to $P_{\partial_{(a,b)}\mathscr{S}}(a,b)$. Note again that $\partial_{(a,b)}(P_{\mathscr{S}}(a,b)) = P_{\partial_{(a,b)}\mathscr{S}}(a,b)$. Thus we have ${}_aH_b^\diamond \cong \mathscr{P}_{\partial_{(a,b)}\mathscr{S}}$. Note that the projective objects of $\ell(\mathscr{S})$ occur in ${}_\infty\ell(\mathscr{S})$. This, together with the fact that $\ell(\mathscr{S})/\Omega \cong \ell(\partial_{(a,b)}\mathscr{S})/\Omega'$, implies that the projective objects of $\ell(\partial_{(a,b)}\mathscr{S})$ occur in ${}_\infty\ell(\partial_{(a,b)}\mathscr{S})$. Therefore $\mathscr{S}({}_aH_b^\diamond) \cong \partial_{(a,b)}\mathscr{S}$. ∎

7. THE PROOF OF THEOREM 5.1: THE INDUCTION PROCESS

In this section, we will prove Theorem 5.1 in the general case. First, we have the following lemma.

LEMMA 7.1. *Let $\mathscr{S}$ be a poset and let $\mathscr{P}_{\mathscr{S}}$ be the preprojective component of the Auslander–Reiten quiver of $\ell(\mathscr{S})$. Let b be a point in $\mathscr{S}$. Assume that there is a subset $\{y_1, y_2, y_3, y_4\}$ of $\mathscr{S}$ with mutually incomparable elements and let $y_4 \geq b$. Then $Q_{\mathscr{S}}(b)$ does not occur in $\mathscr{P}_{\mathscr{S}}$.*

Proof. Put $\mathscr{S}' = \{y_1, y_2, y_3, y_4\}^\vee$. Define a functor $\mathbf{G}$: $\ell(\mathscr{S}') \to \ell(\mathscr{S})$ by setting $(\mathbf{G}(U))_\omega = U_\omega$, $\mathbf{G}(U)_x = U_x$ for $x \in \{y_1, y_2, y_3, y_4\}^\vee$, and $\mathbf{G}(U)_x = 0$ for $x \in \mathscr{S} \setminus \{y_1, y_2, y_3, y_4\}^\vee$, and $\mathbf{G}(\psi) = \psi$. If $y_4 > b$, then clearly $\dim_k \mathbf{G}(U) \neq \dim_k \mathbf{G}(U)_b$ for each $U \in \ell(\mathscr{S}')$. So $\mathrm{Hom}_{\mathscr{S}}(\mathbf{G}(U), Q_{\mathscr{S}}(b)) \neq 0$ for each $U \in \ell(\mathscr{S}')$. Note that width$(\mathscr{S}') \geq 4$ implies that $\mathscr{S}'$ is infinite type. Thus $Q_{\mathscr{S}}(b)$ does not occur in $\mathscr{P}_{\mathscr{S}}$. If $y_4 = b$, we denote by $Q'_{\mathscr{S}'}(b)$ the injective object corresponding to b in $\ell(\mathscr{S}')$. Clearly, $\mathrm{Hom}_{\mathscr{S}}(\mathbf{G}(Q'_{\mathscr{S}'}(b)), Q_{\mathscr{S}}(b)) \neq 0$. Note that width$(\mathscr{S}') \geq 4$ implies that $\mathscr{S}'$

is of infinite type. Thus there are infinitely many $V \in \ell(\mathscr{S}')$ with $\mathrm{Hom}_{\mathscr{S}'}(V, Q_{\mathscr{S}'}(b)) \neq 0$ and $\mathrm{Hom}_{\mathscr{S}}(\mathbf{G}(V), Q_{\mathscr{S}}(b)) \neq 0$. So $Q_{\mathscr{S}}(b)$ does not occur in $\mathscr{P}_{\mathscr{S}}$. ∎

COROLLARY 7.1. *Let H be a thin left hammock with finitely many projective vertices and let $\mathscr{S} := \mathscr{S}(H)$ be the poset corresponding to H. Let $p(a)$ be a projective vertex and let $q(b)$ be an injective vertex of H. Assume that a and b are incomparable in $\mathscr{S}$. Let $\mathscr{S} = a^{\vee} + J(a,b) + b_{\wedge}$ and $J(a,b) = J_a + J_0 + J_b$ as above. Then* $\mathrm{width}(J(a,b)) \leq 3$, $\mathrm{width}(J_0) \leq 1$, $\mathrm{width}(J_a \cup J_0) \leq 2$, *and* $\mathrm{width}(J_0 \cup J_b) \leq 2$.

Proof. As we know, x, a, b are mutually incomparable for $x \in J_0$; also, y, a are incomparable for $y \in J_b \cup J_0$, and z, b are incomparable for $z \in J_a \cup J_0$. Suppose that $\mathrm{width}(J_0) \geq 2$ and say $x_1, x_2 \in J_0$ are incomparable. Then a, x_1, x_2, b are mutually incomparable—a contradiction to Lemma 7.1. Suppose that $\mathrm{width}(J_0 \cup J_a) \geq 3$ and that $x_1, x_2, x_3 \in J_0 \cup J_a$ are mutually incomparable. Then x_1, x_2, x_3, b are mutually incomparable—a contradiction to Lemma 7.1 again. Similarly, we can prove $\mathrm{width}(J_0 \cup J_b) \leq 2$. Now, we suppose that $\mathrm{width}(J(a,b)) \geq 4$ and that $x_1, x_2, x_3, x_4 \in J(a,b)$ are mutually incomparable. If b, x_i are incomparable for each i, $i = 1,2,3,4$, then $\{x_1, x_2, x_3, b\}$ is a subset of $\mathscr{S}$ with mutually incomparable elements—a contradiction to Lemma 7.1. If there is some $x_i \geq b$, then we also get a contradiction to Lemma 7.1 again. ∎

LEMMA 7.2. *Let H be a thin left hammock with finitely many projective vertices and let $\mathscr{S} := \mathscr{S}(H)$ be the poset corresponding to H. Let $p(a)$ be a projective vertex and let $q(b)$ be an injective vertex of H. Assume that a and b are incomparable in $\mathscr{S}$, and $\mathscr{S} = a^{\vee} + J(a,b) + b_{\wedge}$ and $J := J(a,b) = J_a + J_0 + J_b$ as before. Assume that* $\mathrm{width}(J(a,b)) = 2$. *Then either there exists $c \in J_a$ with* $\mathrm{width}(J(c,b)) = 1$ *or there exists $d \in J_b$ with* $\mathrm{width}(J(a,d)) = 1$, *where $\mathscr{S} = c^{\vee} + J(c,b) + b_{\wedge}$ and $\mathscr{S} = a^{\vee} + J(a,d) + d_{\wedge}$.*

Proof. Let $\Omega = \{x \in J_a |$ there is $y \in J$ such that x and y are incomparable$\}$.

If $\Omega \neq \emptyset$, we choose a minimal element of Ω, say c, such that first, c and y_c are incomparable for some $y_c \in J$ and, second, for $z < c$, the element z is comparable to each $x \in J$. We claim that $J \setminus c^{\vee}$ is linear. In fact, the first condition implies that $J \setminus c^{\vee} \neq \emptyset$. Now suppose that x_1, x_2 in $J \setminus c^{\vee}$ are incomparable. Then the second condition implies that c, x_1, x_2 are incomparable—a contradiction to the hypothesis. So $J \setminus c^{\vee}$ is linear, say $J \setminus c^{\vee} = \{z_1 \leq \cdots \leq z_r\}$. Clearly, c, b are incomparable and $\mathscr{S} = c^{\vee} + J(c,b) + b_{\wedge}$, where $J(c,b) = J \setminus c^{\vee}$ is linear.

If $\Omega = \emptyset$, then for each $x \in J_a$, x is comparable to each $y \in J$. Then we consider $\Omega' = \{x \in J_b |$ there is $y \in J$ such that x and y are incomparable}. Since $\operatorname{width}(J(a,b)) = 2$, we see that $\Omega' \neq \emptyset$. A discussion similar to the one above proves that there exists $d \in J_b$ with $\operatorname{width}(J(a,d)) = 1$. ∎

LEMMA 7.3. *Let H be a thin left hammock with finitely many projective vertices and let $\mathscr{S} := \mathscr{S}(H)$ be the poset corresponding to H. Let $p(a)$ be a projective vertex and let $q(b)$ be an injective vertex of H. Assume that a and b are incomparable in $\mathscr{S}$, and $\mathscr{S} = a^{\vee} + J(a,b) + b_{\wedge}$ and $J(a,b) = J_a + J_0 + J_b$ as before. Assume that* $\operatorname{width}(J(a,b)) = 3$. *Then there exists $c \in J_a$ and $d \in J_b$ with c, d incomparable such that* $\operatorname{width}(J(c,d)) = 1$.

Proof. Let $\Omega = \{x \in J_a | x, x_1$ and x_2 are mutually incomparable for some $x_1, x_2 \in J\}$. From Corollary 7.1, we have $\operatorname{width}(J_0 \cup J_b) \leq 2$. Together with $\operatorname{width}(J(a,b)) = 3$, this implies $\Omega \neq \emptyset$. So we can choose a minimal element, say c, such that first, c, x_{c1}, x_{c2} are mutually incomparable for some $x_{c1}, x_{c2} \in J$, and, second, for $y < c$, there does not exist a pair of points x_1, x_2 in J with y, x_1, x_2 mutually incomparable. We claim that $\operatorname{width}(J \setminus c^{\vee}) = 2$. In fact, the first condition implies $\operatorname{width}(J \setminus c^{\vee}) \geq 2$. Suppose that there are $x_1, x_2, x_3 \in J \setminus c^{\vee}$ mutually incomparable. Then the second condition implies that c, x_1, x_2, x_3 are mutually incomparable —a contradiction with the hypothesis. Now consider $\Omega' = \{x \in J_b \cap (J \setminus c^{\vee}) | x$ and y are incomparable for some $y \in J \setminus c^{\vee}\}$. Obviously, x_{c1} or x_{c2} are in Ω', since $\operatorname{width}(J_a \cup J_0) \leq 2$, so $\Omega' \neq \emptyset$. We can choose a maximal element of Ω', say d. A similar discussion to that above shows that $(J \setminus c^{\vee}) \setminus d_{\wedge}$ is linear, say $(J \setminus c^{\vee}) \setminus d_{\wedge} = \{z_1 < \cdots < z_r\}$. Note that c, d are incomparable, since $d \in J \setminus c^{\vee}$. Thus $\mathscr{S} = c^{\vee} + J(c,d) + d_{\wedge}$, where $J(c,d) = (J \setminus c^{\vee}) \setminus d_{\wedge}$. ∎

Let H be a thin left hammock with finitely many projective vertices and let $\mathscr{S} := \mathscr{S}(H)$ be the poset corresponding to H. Let $p(a)$ be a projective vertex and let $q(b)$ be an injective vertex of H. Assume that a and b are incomparable in $\mathscr{S}$. In the case when $\operatorname{width}(J(a,b)) \leq 3$, Lemmas 7.2 and 7.3 show that we can use the Zavadskiĭ 's algorithm for some suitable pair of points c and d with $c \leq a, d \geq b$. From Theorem 6.2 we obtain the thin left hammock of ${}_cH_d^{\diamond}$. Note that $c \leq a, d \geq b$ means that $a, b \in \delta_{(c,d)}\mathscr{S}$ and that a, b are incomparable in $\delta_{(c,d)}\mathscr{S}$. So we can consider the hammock ${}_a({}_cH_d^{\diamond})_b$. Now, we will consider the relation between the hammocks ${}_aH_b$ and ${}_a({}_cH_d^{\diamond})_b$, as well as between the "almost" hammocks $H/{}_aH_b$ and $({}_cH_d^{\diamond})/{}_a({}_cH_d^{\diamond})_b$. Since ${}_aH_b$, ${}_a({}_cH_d^{\diamond})_b$, $H/{}_aH_b$, and $({}_cH_d^{\diamond})/{}_a({}_cH_d^{\diamond})_b$ all are subquivers of H, we will not distinguish between the vertices in H and the vertices in these subquivers.

PROPOSITION 7.1. *Let H be a thin left hammock with finitely many projective vertices and let $\mathscr{S} := \mathscr{S}(H)$ be the poset corresponding to H. Let $p(a), p(c)$ be projective vertices and let $q(b), q(d)$ be injective vertices of H.*

Assume that a and b are incomparable, c and d are incomparable and $a \geq c, d \geq b$ *in* $\mathscr{S}$, *and* $J(c,d)$ *is linear, where* $\mathscr{S} = c^{\vee} + J(a,b) + d_{\wedge}$. *Then, as subsets of vertices of* H, *we have* ${}_aH_b = ({}_cH_d) \cup ({}_a({}_cH_d^{\diamond})_b \setminus \{\mu\})$.

Proof. First, $a \geq c, d \geq b$ means that $\mathrm{Hom}_{k(H)}(p(a), p(c)) \neq 0$ and $\mathrm{Hom}_{k(H)}(q(d), q(b)) \neq 0$. So ${}_cH_d = {}_c({}_aH_b)_d \subseteq {}_aH_b$ by Proposition 3.2.

Next, let $x \in {}_a({}_cH_d^{\diamond})_b \setminus \{\mu\}$. Then there are $f \in \mathrm{Hom}_{k({}_cH_d^{\diamond})}(p(a), x)$ and $g \in \mathrm{Hom}_{k({}_cH_d^{\diamond})}(x, q(b))$ with $fg \neq 0$. If neither f nor g factors through the additional vertex μ, then $fg \neq 0$ in $k(H/{}_cH_d)$, and, further, $fg \neq 0$ in $k(H)$. Thus we have $x \in {}_aH_b$. If f factors through μ, then f factors through $p(c,d)$, and g does not factor through μ. This means that there is $h \in \mathrm{Hom}_{k(H)}(p(c,d), x)$ with $hg \neq 0 \in \mathrm{Hom}_{k(H/{}_aH_d)}(p(c,d), q(b))_{\{x\}}$, and $hg \neq 0 \in \mathrm{Hom}_{k(H)}(p(c,d), q(b))_{\{x\}}$. We claim that g does not factor through $p(z_1, \ldots, z_r)$ in $k(H)$, where $\{z_1, \ldots, z_r\} = \min(\mathscr{S} \setminus \{a,b\}_{\wedge})$. In fact, if g factors through $p(z_1, \ldots, z_r)$, then $\mathrm{Hom}_{k(H)}(p(c,d), p(z_1, \ldots, z_r)) \neq 0$. This is impossible, since $c \leq a$ and $\mathrm{Hom}_{k(H)}(p(c), p(z_1, \ldots, z_r)) = 0$. Thus, by Corollary 3.1, we have $x \in {}_aH_b$. Similarly, if g factors through μ, then $f \in \mathrm{Hom}_{k(H)}(p(a), x)$ and f does not factor through $p(c,d)$. So $x \in {}_aH_b$ also. Thus, we have proven that ${}_aH_b \supseteq {}_a({}_cH_d^{\diamond})_b \setminus \{\mu\}$.

Finally, let $x \in {}_aH_b$. Let us assume that $x \notin {}_cH_d^{\diamond}$. Then $x = p(c, z_i)$, where $z_i \in J(c,d)$. Note that $x \in {}_aH_b$ means that there are $f \in \mathrm{Hom}_{k(H)}(p(a), x)$ and $g \in \mathrm{Hom}_{k(H)}(x, q(b))$ with $fg \neq 0$. Thus f factors through $p(c)$ and g factors through $q(d)$ by Lemma 3.3. This means $x \in {}_cH \cap H_d$, so $x \in {}_cH_d$ according to Theorem 3.2. In the case $x \in {}_cH_d^{\diamond}$, we have $x \in {}_a({}_cH_d^{\diamond})_b$, clearly. ∎

Proposition 7.2. *Let* H *be a thin left hammock with finitely many projective vertices and let* $\mathscr{S} := \mathscr{S}(H)$ *be the poset corresponding to* H. *Let* $p(a), p(c)$ *be projective vertices and* $q(b), q(d)$ *injective vertices of* H. *Assume that a and b are incomparable, that c and b are incomparable, that* $a \geq c$, $d \geq b$ *in* $\mathscr{S}$, *and that* $J(c,d)$ *is linear. Let* ${}_cH_d^{\diamond} = (H/{}_aH_b) \cup \{\mu\}$, *with* $\mu^+ = \{p(c,d)\}$. *Then we have* $h_{({}_aH_b)}(x) = h_{({}_cH_d)}(x) + h_{({}_a({}_cH_d^{\diamond})_b)}(x)$. (*Let* $h_{({}_cH_d)}(x) = 0$ *for* $x \in {}_aH_b \setminus {}_cH_d$ *and let* $h_{({}_a({}_cH_d^{\diamond})_b)}(x) = 0$ *for* $x \in {}_aH_b \setminus {}_a({}_cH_d^{\diamond})_b$.)

Proof. If $x \notin {}_cH_d^{\diamond}$, then $x = p(c, z_i)$ for some $z_i \in J(c,d)$. This implies $x \in {}_cH_d$ and $h_{({}_aH_b)}(x) = 1 = h_{H_{({}_cH_d)}}(x)$. Now we assume that $x \in {}_a({}_cH_d^{\diamond})_b \setminus \{\mu\}$.

First, we consider a vertex x with $\mathrm{Hom}_{k(H)}(p(c,d), x) = 0$. Let $h_{({}_aH_b)}(x) = n$ and let $f_1, \ldots, f_n$ be a basis of $\mathrm{Hom}_{k(H)}(p(a), x)/\mathrm{Hom}_{k(H)}(p(a), x)_{\mathscr{M}_b}$. Assume that $f_1, \ldots, f_t$ factor through $p(c)$ and $f_{t+1}, \ldots, f_n$ do not factor through $p(c)$. Let $f_i = lh_i$, $1 \leq i \leq t$, where l is a fixed nonzero map in $\mathrm{Hom}_{k(H)}(p(a), p(c))$ and $h_i \in$

$\mathrm{Hom}_{k(H)}(p(c), x)$. Then $h_1, \ldots, h_t$ are linearly independent, since $f_1, \ldots, f_t$ are linearly independent. Now $\mathrm{Hom}_{k(H)}(p(c), x)_{\mathscr{M}_d} = 0$ follows from $\mathrm{Hom}_{k(H)}(p(c, d), x) = 0$ by Proposition 3.1. So $h_1, \ldots, h_t$ are linearly independent in $\mathrm{Hom}_{k(H)}(p(c), x)/\mathrm{Hom}_{k(H)}(p(c), x)_{\mathscr{M}_d}$. Further $f_{t+1}, \ldots, f_n$ do not factor through add $\oplus_{z_i \in J(c, d)} p(c, z_i)$, since $f_{t+1}, \ldots, f_n$ do not factor through $p(c)$. So $f_{t+1}, \ldots, t_n$ are linearly independent in $k(H/{}_cH_d)$. Moreover, $f_{t+1}, \ldots, t_n$ are linearly independent in $k({}_cH_d^{\diamond})$, since $\mathrm{Hom}_{k(H)}(p(c, d), x) = 0$. Thus, we have shown $\dim_k \mathrm{Hom}_{k(H)}(p(a), x) - \dim_k \mathrm{Hom}_{k(H)}(p(a), x)_{\mathscr{M}_b} \le \dim_k \mathrm{Hom}_{k(H)}(p(c), x) - \dim_k \mathrm{Hom}_{k(H)}(p(c), x)_{\mathscr{M}_d} + \dim_k \mathrm{Hom}_{k({}_cH_d^{\diamond})}(p(a), x) - \dim_k \mathrm{Hom}_{k({}_cH_d^{\diamond})}(p(a), x)_{\mathscr{M}'_b}$, where $\mathscr{M}'_b$ is the objects class of all objects y in ${}_cH_d^{\diamond}$ with $\mathrm{Hom}_{k({}_cH_d^{\diamond})}(y, q(b)) = 0$. On the other hand, let $f_1, \ldots, f_r$ induce a basis of $\mathrm{Hom}_{k(H)}(p(c), x)/\mathrm{Hom}_{k(H)}(p(c), x)_{\mathscr{M}_d}$ and let $g_1, \ldots, g_s$ induce a basis of $\mathrm{Hom}_{k({}_cH_d^{\diamond})}(p(a), x)/\mathrm{Hom}_{k({}_cH_d^{\diamond})}(p(a), x)_{\mathscr{M}'_b}$. Note that $\mathrm{Hom}_{k(H)}(p(c, d), x) = 0$ implies that g_j is in $\mathrm{Hom}_{k(H)}(p(a), x)/\mathrm{Hom}_{k(H)}(p(a), x)_{\mathscr{M}_b}$, for $j = 1, \ldots, s$. We claim that $lf_1, \ldots, lf_r, g_1, \ldots, g_s$ are linearly independent, where l is a fixed nonzero map in $\mathrm{Hom}_{k(H)}(p(a), p(c))$. Consider $\sum_{i=1}^r k_i lf_i + \sum_{j=1}^s k'_j g_j = 0$. Since $\sum_{i=1}^r k_i lf_i$ factors through $p(c)$, we see that $\sum_{j=1}^s k'_j g_j = 0$. So $k_i = 0$ and $k'_j = 0$ for $1 \le i \le r$ and $1 \le j \le s$. Thus we have shown $\dim_k \mathrm{Hom}_{k(H)}(p(a), x) - \dim_k \mathrm{Hom}_{k(H)}(p(a), x)_{\mathscr{M}_b} \ge \dim_k \mathrm{Hom}_{k(H)}(c, x) - \dim_k \mathrm{Hom}_{k(H)}(c, x)_{\mathscr{M}_d} + \dim_k \mathrm{Hom}_{k({}_cH_d^{\diamond})}(p(a), x) - \dim_k \mathrm{Hom}_{k({}_cH_d^{\diamond})}(p(a), x)_{\mathscr{M}'_b}$.

Now we consider the case when $\mathrm{Hom}_{k(H)}(p(c, d), x) \ne 0$. We have $\mathrm{Hom}_{k(H)}(x, p(z_1, \ldots, z_r)) = 0$, where $\{z_1, \ldots, z_r\} = \min(\mathscr{S} \setminus \{c, d\}_{\wedge})$. Using a similar argument to that above, we have $\dim_k \mathrm{Hom}_{k(H)}(x, q(b)) - \dim_k \mathrm{Hom}_{k(H)}(x, q(b))_{a\mathscr{M}} = \dim_k \mathrm{Hom}_{k(H)}(x, q(d)) - \dim_k \mathrm{Hom}_{k(H)}(x, q(d))_{c\mathscr{M}} + \dim_k \mathrm{Hom}_{k({}_cH_d^{\diamond})}(x, q(b)) - \dim_k \mathrm{Hom}_{k({}_cH_d^{\diamond})}(x, q(b))_{a\mathscr{M}'}$, where ${}_a\mathscr{M}'$ is the object class of all objects y in $k({}_cH_d^{\diamond})$ with $\mathrm{Hom}_{k({}_cH_d^{\diamond})}(p(a), y) = 0$. Thus, the expected result follows from Theorem 3.1. ∎

From Propositions 7.1 and 7.2, we have the following result.

THEOREM 7.1. *Let H be a thin left hammock with finitely many projective vertices and let $\mathscr{S} := \mathscr{S}(H)$ be the poset corresponding to H. Let $p(a), p(c)$ be projective vertices and let $q(b), q(d)$ be injective vertices of H. Assume that a and b are incomparable, that c and d are incomparable, that $a \ge c$, $d \ge b$ in $\mathscr{S}$, and that $J(c, d)$ is linear. Let ${}_cH_d^{\diamond} = (H/{}_cH_d) \cup \{\mu\}$ with $\mu^+ = \{p(c, d)\}$. Then $H/{}_aH_b = ({}_cH_d^{\diamond})/{}_a({}_cH_d^{\diamond})_b$.*

Proof. We claim that $\mu \in {}_a({}_cH_d^{\diamond})_b$. In fact, $a \ge c, d \ge b$ in $\partial_{(c,d)}\mathscr{S}$ and $p(c) = q(d)$. So it follows that $\mu = p(c) = q(d)$ in $k({}_cH_d^{\diamond})$ and

$\mathrm{Hom}_{k({}_cH_d^\diamond)}(p(a), p(c)) \neq 0$, $\mathrm{Hom}_{k({}_cH_d^\diamond)}(q(d), q(b)) \neq 0$. Thus $\mu \in {}_a({}_cH_d^\diamond) \cap ({}_cH_d^\diamond)_b$ implies $\mu \in {}_a({}_cH_d^\diamond)_b$ by Theorem 3.2. Note that

$$h_{({}_cH_d^\diamond)}(x) = \begin{cases} h_H(x) - h_{H_{({}_cH_d)}}(x) & x \neq \mu \\ 1 & x = \mu. \end{cases}$$

So we have ${}_cH_d^\diamond / {}_a({}_cH_d^\diamond)_b = \{x \in {}_cH_d^\diamond \mid h_{({}_cH_d^\diamond)}(x) - h_{({}_a({}_cH_d^\diamond)_b)}(x) \neq 0\} = \{x \in H/{}_cH_d \mid (h_H(x) - h_{({}_cH_d)}(x)) - h_{({}_a({}_cH_d^\diamond)_b)}(x) \neq 0\} = \{x \in H \mid h_H(x) - (h_{({}_cH_d)}(x) + h_{({}_a({}_cH_d^\diamond)_b)}(x)) \neq 0\} = \{x \in H \mid h_H(x) - h_{({}_aH_b)}(x) \neq 0\}$. Note that the last equality holds by Proposition 7.2. Therefore ${}_cH_d^\diamond / {}_a({}_cH_d^\diamond)_b \cong H/{}_aH_b$. ∎

Proof of Theorem 5.1. Given a thin left hammock H with finitely many projective vertices, let $\mathcal{S} := \mathcal{S}(H)$ be the corresponding poset. If $J(a,b) = \emptyset$, we have $\mathcal{S}({}_aH_b^\diamond) = \mathcal{S}$ by Proposition 6.1. If $J(a,b) \geq 1$, by Lemmas 7.2 and 7.3, we can use Zavadskiĭ's stratification algorithm for a suitable pair of vertices (c_1, d_1) with $a \geq c_1$ and $d_1 \geq b$. Then we get the poset $\partial_{(c_1,d_1)}\mathcal{S}$ and the thin left hammock ${}_{c_1}H_{d_1}^\diamond$ with $\partial_{(c_1,d_1)}\mathcal{S} \cong \mathcal{S}({}_{c_1}H_{d_1}^\diamond)$ by Theorem 6.2. We also obtain ${}_a({}_{c_1}H_{d_1}^\diamond)_b$ as a subquiver of H. Note that if $a = c_1$, we have $\mu = p(a)$ in ${}_{c_1}H_{d_1}^\diamond$, and if $b = d_1$, we have $\mu = q(b)$ in ${}_{c_1}H_{d_1}^\diamond$. We point out that $\#\{x \in ({}_a({}_{c_1}H_{d_1}^\diamond)_b \setminus \{\mu\})\} < \#\{x \in {}_aH_b\}$. Now, if $\mathrm{width}(J'(a,b)) \geq 1$, where $\partial_{(c_1,d_1)}\mathcal{S} = a^\vee + J'(a,b) + b_\wedge$, we can use Zavadskiĭ's stratification algorithm again. Since ${}_aH_b$ is finite, after finitely many steps, say after l steps, this process will stop. So we obtain a sequence of suitable pairs of points $(c_1,d_1),(c_2,d_2),\ldots,(c_l,d_l) = (a,b)$, a sequence of left hammocks $H_1 = H, H_2, \ldots, H_l$, and a sequence of posets $\mathcal{S}_1 = \mathcal{S}(H), \mathcal{S}_2, \ldots, \mathcal{S}_l$ such that

(1) (c_i, d_i) is a suitable pair of points in $\mathcal{S}_i$;

(2) $\mathcal{S}_i = \partial_{(c_{i-1},d_{i-1})}\mathcal{S}_{i-1}$ for $i = 2,\ldots,l$, that is, $\mathcal{S}_i$ is the (c_{i-1},d_{i-1})-stratified poset;

(3) $H_i = {}_{c_i}(H_{i-1})_{d_i}^\diamond$ for $i = 2,\ldots,l$;

(4) ${}_a(H_l)_b^\diamond \setminus \{\mu\} = \emptyset$.

By Theorem 6.2,

(5) $\mathcal{S}_i \cong \mathcal{S}(H_i)$.

Now, ${}_a(H_l)_b^\diamond \setminus \{\mu\} = \emptyset$ means that $\mu = p(a) = q(b)$ in $\mathcal{S}_l$ and ${}_a(H_l)_b^\diamond \cong H_l$ follows from Proposition 6.1. Hence $H/{}_aH_b \cong H_1/{}_a(H_1)_b \cong H_2/{}_a(H_2)_b \cong \cdots \cong H_l/{}_a(H_l)_b$ by Theorem 7.1 again and again. Note that $\partial_{(c_i,d_i)}P_{\mathcal{S}_{i-1}}(a,b) = P_{\mathcal{S}_i}(a,b)$ for $i = 2,\ldots,l$. Therefore $\mathcal{S}({}_aH_b^\diamond) \cong \mathcal{S}({}_a(H_l)_b^\diamond) \cong \mathcal{S}(H_l) \cong \mathcal{S}_l$, this completes the proof. ∎

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REFERENCES

1. M. Auslander and I. Reiten, Representation theory of Artin algebras, III, *Comm. Algebra* **3** (1975), 269–310.
2. M. Auslander and S. O. Smalo, Almost split sequences in subcategories, *J. Algebra* **69** (1981), 426–454.
3. S. Brenner, A combinatorial characterization of finite Auslander–Reiten quivers, *in* "Representation Theory. I. Finite Dimensional Algebras," Lecture Notes in Mathematics, Vol. 1177, pp. 13–49, Springer-Verlag, Berlin/New York, 1986.
4. P. Gabriel, Représentations indécomposable des ensembles ordonnés, "Séminair P. Dubbeil, 26e, Algebre, pp. 13.1–4, Université de Paris-VI, Institut Henri Poincaré, 1972/1973.
5. O. Kerner, On partially ordered sets of finite type, *Comm. Algebra* **9** (1981), 783–809.
6. M. M. Klejner, Partially ordered sets of finite type, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* **28** (1972), 32–41; *J. Soviet Math.* **3** (1975), 607–615.
7. Y. Lin, Hammocks and the algorithm of Nazarova–Roiter, *J. London Math. Soc.*, to appear.
8. L. A. Nazarova, Partially ordered sets of infinite type, *Izv. Akad. Nauk SSSR Ser. Mat.* **39**(5) (1975), 963–991; *Math. USSR-Izv.* **9** (1975), 917–938.
9. L. A. Nazarova and A. V. Roiter, Representations of partially ordered sets, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* **28** (1972), 5–31; *J. Soviet Math.* **3** (1975), 585–606.
10. L. A. Nazarova and A. G. Zavadskiĭ, Partially ordered sets of tame type, "Matrix Problems," pp. 122–143, Akad. Nauk Ukrain. SSR Inst. Math., Kiev, 1977.
11. C. M. Ringel, "Tame Algebras and Integral Quadratic Forms," Lecture Notes in Mathematics, Vol. 1099, Springer-Verlag, Berlin/New York, 1984.
12. C. M. Ringel, Representation theory of finite-dimensional algebras, *in* "Representation Theory of Algebras," (P. Webb, Ed.), London Math. Soc. Lecture Note Series, Vol. 116, pp. 7–79, London Math. Soc., London, 1986.
13. C. M. Ringel and D. Vossieck, Hammocks, *Proc. London Math. Soc.* (*3*) **54** (1987), 216–246.
14. D. Simson, "Linear Representations of Partially Ordered Sets and Vector Space Categories," Gordon & Breach, New York, 1992.
15. A. G. Zavadskiĭ, Differentiation with respect to a pair of points, *in* "Matrix Problems," pp. 115–121, Akad. Nauk Ukrain. SSR Inst. Math., Kiev, 1977.
16. A. G. Zavadskiĭ, Differentiation algorithm and classification of representations, *Math. USSR-Izv.* **39**(2) (1992), 975–1012.
17. A. G. Zavadskiĭ, The Auslander–Reiten quiver for posets of finite growth, "Topics in Algebra," Banach Center Publ. Vol. 26, 569–587, PWN, Warsaw.