The purpose of this paper is to compute the explicit form of the relative invariants and their \( b \)-functions of a certain important prehomogeneous vector space appearing in the study of a cuspidal character sheaf of the exceptional group \( E_7 \). For every simple algebraic group excepting the type \( E_7 \), the \( b \)-functions of this kind of prehomogeneous vector space have been determined.

Consider a unipotent conjugacy class of a complex simple algebraic group \( \tilde{G} \) whose closure is the support of a cuspidal character sheaf. The corresponding nilpotent class gives a prehomogeneous vector space via the Dynkin–Kostant theory, whose \( b \)-functions are important in the study of character sheaves. These \( b \)-functions have been explicitly determined by Sato [9] when \( \tilde{G} \) is of classical type; by Shintani [13] for \( 2 \Rightarrow 0 \); by Kimura and Muro [8] for \( 02 \Rightarrow 00 \); by Gyoja [6] for \( 202^0 \); and by Ozeki and Yano [14] for \( 002^000 \). In this paper we determine the relative invariants and their \( b \)-functions for the last remaining one, \( 002^002 \).

The prehomogeneous vector space for \( 002^002 \) is given by the following \((G, V)\) as an explicit form. Let \( G = GL_3(\mathbb{C}) \times GL_3(\mathbb{C}) \times GL_2(\mathbb{C}) \) and let
$V = M_3(\mathbb{C}) + M_3(\mathbb{C}) + \mathbb{C}^3$. Define the action of $G$ on $V$ by

$$g v = \left( (g_1 X Y^{-1}, g_1 Y X^{-1})' g_2, g_2 p \right)$$

for $g = (g_1, g_2, g_3) \in G$ and $v = (X, Y, p) \in V$. Then $(G, V)$ is a prehomogeneous vector space which has two irreducible relative invariants. Let

$$f_1(v) = \text{the discriminant of the cubic form } \det(X \xi + Y \eta)$$

in the variables $(\xi, \eta) \in \mathbb{C}^2$. Then we can easily see that $f_1$ is an irreducible relative invariant.

The main results of this paper are the following. We construct the other irreducible relative invariant, say $f_2$, in Theorem 1 of Section 2. Then the $b$-function of the relative invariant $f^m = f_1^m f_2^m$ ($m_1, m_2 \in \mathbb{Z}_{\geq 0}$) is given by

$$b_{f^m}(s) = \left( \prod_{\nu=0}^{m_1-1} (m_1 s + 1 + \nu)^2 \left( m_1 s + \frac{5}{6} + \nu \right) \left( m_1 s + \frac{7}{6} + \nu \right) \right) \times \left( \prod_{\nu=0}^{m_2-1} (m_2 s + 1 + \nu)^3 \right) \times \left( \prod_{\nu=0}^{m_1+m_2-1} \left( (m_1 + m_2) s + \frac{3}{2} + \nu \right)^2 \left( (m_1 + m_2) s + \frac{4}{3} + \nu \right) \left( (m_1 + m_2) s + \frac{5}{3} + \nu \right) \right) \times \left( \prod_{\nu=0}^{2m_1+m_2-1} \left( (2m_1 + m_2) s + 2 + \nu \right)^3 \right) \times \text{constant.}$$

See Theorem 3 of Section 4.

The plan of this paper is the following. In Section 1 we review some fundamental results on prehomogeneous vector spaces. In Section 2 we give an explicit form of $f_2$, the unknown irreducible relative invariant. In the remaining sections, we determine the explicit form of the $b$-function of $f^m$.

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1. PRELIMINARIES

In this section we review fundamental notions about the theory of prehomogeneous vector spaces and give some lemmas used in the later sections.

1.1. Prehomogeneous Vector Spaces

Let $G$ be a connected reductive group defined over the complex number field $\mathbb{C}$ and let $\rho: G \to GL(V)$ be a finite-dimensional rational representation. The triple $(G, \rho, V)$ (or simply the pair $(G, V)$) is called a prehomogeneous vector space if $V$ has an open dense $G$-orbit, say $O_0 = Gv_0$. Let $f$ be a nonzero polynomial function on $V$ and let $\phi \in \text{Hom}(G, \mathbb{C}^\times)$. We call $f$ a relative invariant whose character is $\phi$ if $f(gv) = \phi(g)f(v)$ for all $g \in G$ and $v \in V$. Then $(G, V)$ and $f$ have the properties given in Lemmas 1.1.1–1.1.8. The proofs are found in the literature assigned there.

**Lemma 1.1.1** [11, Proposition 3 in Section 4]. Let $f_1$ and $f_2$ be relative invariants which have the same character. Then $f_1$ is a constant multiple of $f_2$.

**Lemma 1.1.2** [11, Proposition 5 in Section 4]. Let $S_1, \ldots, S_l$ be the irreducible components of $V \setminus O_0$ with codimension one and suppose each $S_i$ is the zero of some irreducible polynomial $f_i$, namely $S_i = \{v \in V; f_i(v) = 0\}$. Then $f_1, \ldots, f_l$ are algebraically independent relative invariants. Every relative invariant $f$ is of the form $f = cf_1^{m_1} \cdots f_l^{m_l}$ ($c \in \mathbb{C}, m_i \in \mathbb{Z}$).

We call the polynomials $f_1, \ldots, f_l$ the fundamental system of relative invariants.

**Lemma 1.1.3** [11, Proposition 19 in Section 4]. Let $X^*(G, V)$ be the totality of the characters associated to relative invariants of $(G, V)$ and let $G_{v_0}$ be the isotropy group at the point $v_0$ of the open orbit. Then

$$X^*(G, V) = \{ \phi \in \text{Hom}(G, \mathbb{C}^\times); \phi|_{G_{v_0}} = 1 \}.$$

**Lemma 1.1.4** [2, Lemma 1.5]. Let $\rho^\vee: G \to GL(V^\vee)$ be the contragradient representation of $\rho$. Then the triple $(G, \rho^\vee, V^\vee)$ is a prehomogeneous vector space. It has a relative invariant of degree $d := \text{deg } f$, say $f^\vee$, whose character is $\phi^{-1}$.

**Lemma 1.1.5** [2, Lemmas 1.6 and 1.7]. There exists a polynomial $b_f(s) = b_0s^d + b_1s^{d-1} + \cdots + b_d$ with $b_0 \neq 0$ such that

$$f^\vee(\text{grad}_sf)(x)^{d+1} = b_f(s)f(x)^d,$$

$$f(\text{grad}_yf^\vee)(y)^{d+1} = b_f(s)f^\vee(y)^d,$$
for all \( s \in \mathbb{Z}_{\geq 0} \). Here we put
\[
\operatorname{grad}_x := \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \quad \text{and} \quad \operatorname{grad}_y := \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}.
\]
We call \( b_i \) the \( b \)-function of \( f \). We put \( a_f := b_0 \). We have the formulas for calculating the values of \( b_0 \) and \( b_1 \) as the following lemma.

**LEMMA 1.1.6.** When
\[
f^\vee (\operatorname{grad}) = \sum_{i_1, \ldots, i_d} c_{i_1, \ldots, i_d} \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_d}},
\]
put
\[
D_f(x) := \sum_{i_1, \ldots, i_d} c_{i_1, \ldots, i_d} \sum_{1 \leq j < k \leq d} f_{x_{i_j}} \cdots \hat{f}_{x_{i_k}} \cdots f_{x_{i_d}},
\]
Here \( ^\wedge \) denotes that the element is removed from the formula. Then
\[
a_f = b_0 = f^\vee (\operatorname{grad} \log f(v)) f(v),
\]
\[
b_1 = - \frac{d(d - 3)}{2} b_0 + D_f(v) f(v)^{2-d},
\]
for \( v \in V_f := \{ v \in V; f(v) \neq 0 \} \).

**Proof.** The first claim is Lemma 1.8 (1) in [2]. The proof of the second claim is given similarly.

The following fact about the \( b \)-functions is well known.

**LEMMA 1.1.7** [7, Corollary 5.2]. Let
\[
b_f(s) = b_0 \prod_{j=1}^{d} (s + \alpha_j).
\]
Then each \( \alpha_j \) is a positive rational number.

We define the exponential \( b \)-function of \( f \) by
\[
b_f^{\exp}(t) = \prod_{j=1}^{d} (t - e^{2\pi \sqrt{-1} \alpha_j}).
\]
The exponential \( b \)-functions have the following property.

**LEMMA 1.1.8** [4, Section 7.3]. The exponential \( b \)-function \( b_f^{\exp}(t) \) is a product of some cyclotomic polynomials.

1.2. **Exact Prehomogeneous Vector Spaces**

We say that a prehomogeneous vector space \((G, V)\) is exact if \( \dim G = \dim V \). We assume that \((G, V)\) is an exact prehomogeneous vector space in this subsection. Then we have the following lemmas.
LEMMA 1.2.1 [1, Remark 6.3.1]. Fix linear bases of $\mathfrak{g} := \text{Lie}(G)$ and of $V$: Put $f_0(v) := \det(g \rightarrow V; A \rightarrow Av)$ for $v \in V$ and $\phi(g) := \det(V \rightarrow V; g \rightarrow g\cdot v)$ for $g \in G$. Then $f_0$ is a relative invariant whose character is $\phi_0$, and $V \setminus f_0^{-1}(0) = O_0$.

Then the following equality about the $b$-function of the relative invariant $f_0$ holds.

LEMMA 1.2.2 [1, Remark 6.3.1]. We have

$$b_{f_0}(s) = (-1)^d b_{f_0}(-s - 2)$$

for any $s$.

1.3. Contraction

We explain here that the exponential $b$-function is invariant under the operator called the contraction of prehomogeneous vector space. Let $(G, V)$ be a prehomogeneous vector space and let $f$ be its relative invariant. Let $V_f = \{v \in V; f(v) \neq 0\}$ and let $O_f$ be the unique $G$-orbit which is closed in $V_f$. Take a point $v_f \in O_f$, and put $G_{v_f} := \{g \in G; g v_f = v_f\}$. Let $T_f$ be a maximal torus of $G_{v_f}$, and put $N_f(T_f) := \{g \in G; g T_f g^{-1} = T_f\}$. Let $G^{(f)} := N_f(T_f)/T_f$ and let $V^{(f)} := V T_f = \{v \in V; tv = v \ (t \in T_f)\}$.

LEMMA 1.3.1 [3, Theorem A]. $(G^{(f)}, V^{(f)})$ is a prehomogeneous vector space. Moreover, it has a relative invariant $f' := f|_{V^{(f)}}$.

We call the transformation $(G, V) \rightarrow (G^{(f)}, V^{(f)})$ the contraction with respect to the relative invariant $f$. The following lemma was posed as a conjecture in [3], and its proof was given in [6, Section 11].

LEMMA 1.3.2. We have

$$b_f^\exp = b_f'^\exp.$$  

Namely, the two exponential $b$-functions coincide with each other when one is the contraction of the other.

1.4. $a$-Functions and $b$-Functions

We give here the definitions of $a$-functions and $b$-functions and some properties of them. The references of this subsection are [6, Section 8] and [12, Sections 3, 4].

Let $f_1, \ldots, f_l$ be the fundamental system of the irreducible relative invariants of a prehomogeneous vector space $(G, V)$ and let $f_1^{\vee}, \ldots, f_l^{\vee}$ be the irreducible relative invariants of $(G, V^\vee)$ such that the characters of $f_j$ and $f_j^{\vee}$ are the inverse of each other. We put $f := (f_1, \ldots, f_l)$ and $f^{\vee} := (f_1^{\vee}, \ldots, f_l^{\vee})$ and $V_{f} := \cap_{i=1}^{l} f_i$. For each $l$-tuple $s = (s_1, \ldots, s_l)$, we put $f^s := \prod_{i=1}^{l} f_i^{s_i}$ and $f^{\vee,s} := \prod_{i=1}^{l} f_i^{s_i \vee}$.  


First we define the $a$-functions and give their properties.

**Lemma 1.4.1.** For any $l$-tuple $m = (m_1, \ldots, m_l)$, we have
\[
 f^m(v) f^{-m} (\text{grad} \log f^2(v)) = a_m(s),
\]
for all $v \in V$, with some nonzero homogeneous polynomial $a_m(s)$ which is independent of $v$.

We call $a_m(s)$ the $a$-functions of $f$. When $m = e_i := (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 appears only at the $i$th place, we write $a_i(s)$ instead of $a_m(s)$ for an abbreviation. We can easily see that $a_m = \prod_{j=1}^l a_{m_j}$ by definition. The following lemmas about the structures of $a$-functions $a_m(s)$ are stated in [6, Section 8].

**Lemma 1.4.2.** The a-function $a_m(s)$ is expressed as
\[
 a_m(s) = A^2 \prod_{j=1}^N \left( \frac{\gamma_j(s)}{\gamma_j(m)} \right)^{\mu_j}.
\]
Here $A^2 := \prod_{j=1}^l A_j^j$ with $A_j \in \mathbb{C}$, $N \in \mathbb{Z}_{>0}$, and $\mu_j \in \mathbb{Z}_{\geq 0}$, while each $\gamma_j(s)$ is a $\mathbb{Z}$-linear function $\sum_{i=1}^{l_j} \gamma_{ij}s_i$ with $\gamma_{ij} \in \mathbb{Z}_{\geq 0}$ and $\text{GCD}(\gamma_{ij}, \ldots, \gamma_{ij}) = 1$.

We have the following lemma by Lemma 1.4.2.

**Lemma 1.4.3.** The leading coefficient $a_{m^\forall}$ of the $b$-function $b_{m^\forall}(s)$ of the relative invariant $f^m$ ($m \in (\mathbb{Z}_{\geq 0})^l$) is given by
\[
 a_{m^\forall} = A^2 \prod_{j=1}^N \left( \frac{\gamma_j(m)}{\gamma_j(s)} \right)^{\mu_j}.
\]

Next we define the $b$-functions and give their properties.

**Lemma 1.4.4.** For any $l$-tuple $m = (m_1, \ldots, m_l) \in (\mathbb{Z}_{\geq 0})^l$, we have
\[
 f^{-m}(\text{grad}) f^{m^\forall} = b_m(s) f^2
\]
with some nonzero polynomial $b_m(s)$.

These polynomials $b_m(s)$ are called the $b$-functions of $f$. We write $b_i(s)$ instead of $b_m(s)$ for an abbreviation. Let $a$-functions $a_m(s)$ be as in Lemma 1.4.2. The following lemmas about the structures of $b_i(s)$ and $b_m(s)$ are stated in [6, Section 8].

**Lemma 1.4.5.** The $b$-function $b_i(s)$ is expressed as
\[
 b_i(s) = A_i \prod_{j=1}^N \prod_{\nu=0}^{\gamma_j(m_i) - 1} \prod_{r=1}^\mu_j \left( \gamma_j(s) + \alpha_{j,r} + \nu \right),
\]
with some $\alpha_{j,r} \in \mathbb{Q}_{>0}$.
LEMMA 1.4.6. The b-function \( b_m(s) \) is expressed as
\[
b_m(s) = A^2 \prod_{j=1}^{N} \prod_{\nu=0}^{\gamma(m)-1} \beta_{j}(\gamma(s) + \nu),
\]
where
\[
\beta_{j}(u) = \prod_{r=1}^{\mu_j} \left( u + \alpha_{j,r} \right).
\]

Lemma 1.4.6 implies the following lemma.

LEMMA 1.4.7. The b-function \( b_{\vec{m}}(s) \) of the relative invariant \( f_{\vec{m}}^s \) \((\vec{m} \in (\mathbb{Z}_{\geq 0})^r)\) is given by
\[
b_{\vec{m}}(s) = A^2 \prod_{j=1}^{N} \prod_{\nu=0}^{\gamma(\vec{m})-1} \beta_{j}(\gamma(\vec{m})s + \nu)
\]
with the same \( \beta_{j} \) as in Lemma 1.4.6.

Let b-functions \( b_m(s) \) be as in Lemma 1.4.6. For each \( j \), put
\[
\beta_{j}^{\text{exp}}(t) := \prod_{r=1}^{\mu_j} \left( t - e^{2\pi \sqrt{-1} a_{j,r}} \right).
\]
The following property of \( \beta_{j}^{\text{exp}}(t) \) is stated in [6, Section 8].

LEMMA 1.4.8. For each \( j \), \( \beta_{j}^{\text{exp}}(t) \) is product of cyclotomic polynomials.

2. IRREDUCIBLE RELATIVE INVARIANTS

In this section, we show that the pair \((G,V)\) defined below is a prehomogeneous vector space and construct its irreducible relative invariants. Let \( G = GL_3(\mathbb{C}) \times GL_3(\mathbb{C}) \times GL_2(\mathbb{C}) \) and let \( V = M_3(\mathbb{C}) + M_3(\mathbb{C}) + \mathbb{C}^3 \). Define the action \( \rho \) of \( G \) on \( V \) by
\[
gv = \left( (g_1 X g_2^{-1}, g_1 Y g_2^{-1}) g_3, g_2 p \right)
\]
for \( g = (g_1, g_2, g_3) \in G \) and \( v = (X, Y, p) \in V \). Then the action of the Lie algebra \( \mathfrak{g} = \mathfrak{gl}_3(\mathbb{C}) + \mathfrak{gl}_3(\mathbb{C}) + \mathfrak{gl}_2(\mathbb{C}) \) is given by
\[
Av = (A_1 X - XA_2 + c_{11} X + c_{12} Y, A_1 Y - YA_2 + c_{21} X + c_{22} Y, A_2 p)
\]
for
\[ A = \left( A_1, A_2, \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \right) \in g. \]

2.1. Prehomogeneity

We show that \((G, V)\) is a prehomogeneous vector space. Let
\[ v_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 \end{pmatrix}. \]

Then the isotropy group at \( v_0 \) is given by
\[ G_{v_0} = \begin{cases} \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } a \neq 0, \\
\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, a^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{if } a = 0. \end{cases} \]

Since \( \dim G = 22, \dim V = 21, \) and \( \dim G_{v_0} = 21, \) \((G, V)\) is prehomogeneous and \( v_0 \) is a point in the open dense \( G\)-orbit \( O_0. \) We have \( G_{v_0}/\ker \rho = \mathbb{P}^2. \)

Let \( G' = GL_3(\mathbb{C}) \times GL_3(\mathbb{C}) \times SL_3(\mathbb{C}) \) and let \( \rho' = \rho|_{G'}. \) Then \((G', \rho', V)\) is an exact prehomogeneous vector space. For the sake of convenience we consider the action of the larger group \( G, \) although \( G' \) is enough for our purpose.
2.2. Irreducible Relative Invariants and Their Degrees

The prehomogeneous vector space \((G, V)\) has two nontrivial irreducible relative invariants. This follows from the fact that \(X^*(G, V)\) in Lemma 1.1.3 is two-dimensional. One irreducible relative invariant is

\[ f_1(v) := \text{the discriminant of the cubic form } \det(X\xi + Y\eta) \]

in the variables \((\xi, \eta) \in \mathbb{C}^2\). Here the discriminant of the cubic form \(a_0\xi^3 + a_1\xi^2\eta + a_2\xi\eta^2 + a_3\eta^3\) is given by \(a_1^2a_2^2 + 18a_0a_1a_2a_3 - 4a_0a_2^2 - 4a_1^2a_3 - 27a_0^2a_3^2\). Then \(f_1\) is an irreducible relative invariant whose character is

\[ \phi_1(g) := (\det g_1)^4(\det g_2)^{-4}(\det g_3)^6 \quad \text{for } g = (g_1, g_2, g_3) \in G. \]

Next we proceed to the construction of the other irreducible relative invariants, which we denote by \(f_2\). First we determine its character. Since \((G', V)\) is exact, we can consider the relative invariant \(f_0\) constructed in Lemma 1.2.1. Its character is

\[ \phi_0(g) = (\det g_1)^6(\det g_2)^{-5}(\det g_3)^9 \quad \text{for } g = (g_1, g_2, g_3) \in G. \]

Comparing the zeros of \(f_i\), we see that \(f_0 = cf_1f_2^c\) for some \(c \in \mathbb{C}^\times\). Hence we have that the character of \(f_2\) is

\[ \phi_2(g) := (\det g_1)^3(\det g_2)^{-1}(\det g_3)^3 \quad \text{for } g = (g_1, g_2, g_3) \in G. \]

Next we consider the degree of the relative invariant \(f_2\). We can easily see that \(f_0\) is of degree 9 with respect to \(X\), of degree 9 with respect to \(Y\), and of degree 3 with respect to \(p\), and that \(f_1\) is of degree 6 with respect to \(X\) and of degree 6 with respect to \(Y\). Therefore we see that \(f_2\) is of degree 3 with respect to \(X\), of degree 3 with respect to \(Y\), and of degree 3 with respect to \(p\).

2.3. Construction of \(f_2\)

We will construct the irreducible relative invariant \(f_2\) explicitly in this subsection. We write \(f\) for \(f_2\) and \(\phi\) for \(\phi_2\) throughout this subsection.

Consider the \(G\)-action on the polynomial ring \(\mathbb{C}[V]\). Since \(f\) is a relative invariant, we have \((gf)(gv) = f(v) = \phi(g)^{-1}f(gv)\) for any \(g \in G\) and \(v \in V\). Hence we have only to find a polynomial \(f\) on \(V\) such that \(gf = \phi(g)^{-1}f\) for any \(g \in G\).

As we have seen at the end of Subsection 2.2, \(f\) is a linear combination of monomials of the form

\[ x_i x_j x_i x_j x_k x_j x_i x_j y_k y_k y_k y_k y_k y_k y_k y_k y_k y_k p_n p_n p_n. \]
We denote the monomial $x_{i_1}x_{i_2}x_{i_3}y_{k_1}y_{k_2}y_{k_3}$ by

$$\begin{pmatrix} i_1i_2i_3 & k_1k_2k_3 \\ j_1j_2j_3 & l_1l_2l_3, n_1n_2n_3 \end{pmatrix}.$$ 

Then $f$ is expressed as

$$f = \sum_{i,j,k,l,n} \alpha_{i,j,k,l,n} \begin{pmatrix} i_1i_2i_3 & k_1k_2k_3 \\ j_1j_2j_3 & l_1l_2l_3, n_1n_2n_3 \end{pmatrix}$$

with coefficients $\alpha_{i,j,k,l,n} \in \mathbb{C}$.

Let

$$g = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}, \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \in G.$$ 

Then

$$g \begin{pmatrix} i_1i_2i_3 & k_1k_2k_3 \\ j_1j_2j_3, l_1l_2l_3, n_1n_2n_3 \end{pmatrix}$$

is a scalar multiple of

$$\begin{pmatrix} i_1i_2i_3 & k_1k_2k_3 \\ j_1j_2j_3, l_1l_2l_3, n_1n_2n_3 \end{pmatrix}.$$ 

On the other hand, since we have $gf = \phi(g)^{-1}f$, this scalar must coincide with $\phi(g)^{-1}$. Thus we get

$$\{i_1, i_2, i_3, k_1, k_2, k_3\} = \{1, 1, 2, 2, 3, 3\},$$

$$\{j_1, j_2, j_3, l_1, l_2, l_3\} = \{n_1, n_2, n_3, 1, 2, 3\}.$$ 

We see that only the monomials satisfying (3) appear in the sum (2).

Next consider the symmetric groups $\Sigma_3 \subset GL_3(\mathbb{C})$ and $\Sigma_2 \subset GL_2(\mathbb{C})$ which permute the bases. We denote by $(ij)$ the transposition of $i$ and $j$.

We consider the action of $\Sigma := \Sigma_3 \times \Sigma_3 \times \Sigma_2$ on $f$. The action of $\Sigma$...
on monomials satisfying (1) and (3) is given by
\[
(\sigma_1, \sigma_2, 1) \begin{pmatrix} i_1 i_2 i_3 & k_1 k_2 k_3 \\ j_1 j_2 j_3 & l_1 l_2 l_3 \end{pmatrix}, n_1 n_2 n_3
\]
\[
= \left( \begin{array}{c}
\sigma_1(i_1) \sigma_1(i_2) \sigma_1(i_3) \\
\sigma_2(j_1) \sigma_2(j_2) \sigma_2(j_3) \\
\sigma_2(l_1) \sigma_2(l_2) \sigma_2(l_3) \\
\sigma_2(n_1) \sigma_2(n_2) \sigma_2(n_3)
\end{array} \right),
\]
\[
(1, 1, (12)) \begin{pmatrix} i_1 i_2 i_3 & k_1 k_2 k_3 \\ j_1 j_2 j_3 & l_1 l_2 l_3 \end{pmatrix}, n_1 n_2 n_3)
\]
\[
= \left( \begin{array}{c}
k_1 k_2 k_3 \\
1 i_1 i_3 \\
l_1 l_2 l_3, n_1 n_2 n_3
\end{array} \right).
\]
Since \(\phi((\sigma_1, \sigma_2, \sigma_3))^{-1} = \text{sgn}(\sigma_2 \sigma_3)\), the coefficient of \(\sigma m\) in \(f\) is
\[
\text{sgn}(\sigma_2 \sigma_3) \text{ times } \text{coefficient of } m \text{ for any monomial } m \text{ satisfying (1) and (3).}
\]
Hence we can express \(f\) as
\[
f = \sum_i (\alpha_i \sum' \pm m_i)
\]
with \(\alpha_i \in \mathbb{C}\), where \(\{m_i\}\) is a system of representatives of the \(\Xi\)-action on
the totality of the monomials satisfying (1) and (3). Here \(\sum' \pm m_i\) means the sum of the distinct terms of
\[
\sum_{\sigma_1, \sigma_2 \in \Xi_1, \sigma_3 \in \Xi_2} \text{sgn}(\sigma_2 \sigma_3)(\sigma_1, \sigma_2, \sigma_3)m_i.
\]
We find the representatives \(m_i\) in Table I.

The isotropy subgroups of \(\Xi\) at \(\{m_i, -m_i\}\) are isomorphic to groups in
the third entries of Table I. If the isotropy subgroup at \(\{m_i, -m_i\}\) contains
an element \((\sigma_1, \sigma_2, \sigma_3)\) such that \(\text{sgn}(\sigma_2 \sigma_3) = -1\), the sum \(\sum' \pm m_i\) equals
0. Such monomials are \(m_1, m_3, m_{33}, m_{34}, \text{ and } m_{35}\). Hence we may take
their coefficients to be 0.

Finally we determine the remaining coefficients. The values of the coefficients \(\alpha_i\) in (4) are given in the second entries of Table I. We can carry out the actual calculation based on the following trick.

We put \(A_{ij} := (E_{ij}, 0, 0), B_{ij} := (0, E_{ij}, 0), \text{ and } C_{ij} := (0, 0, E_{ij})\) as elements in \(g\), where \(E_{ij}\) is the matrix whose \((i, j)\)-element is 1 and other elements are 0. Since \(gf = \phi(g)^{-1} f\), \(f\) is annihilated by all \(A_{ij}, B_{ij}, \text{ and } C_{ij} (i \neq j)\). Consider the equation \(A_{31} f = 0\). Let \(m\) be a monomial satisfying (1) and (3). Since \(A_{31}\) replaces the first index 3 of \(x\) and \(y\) by the index 1, \(A_{31} m\) is the sum of the monomials satisfying (1) and the conditions
\[
\{i_1, i_2, i_3, k_1, k_2, k_3\} = \{1, 1, 1, 2, 2, 3\},
\]
\[
\{j_1, j_2, j_3, l_1, l_2, l_3\} = \{n_1, n_2, n_3, 1, 2, 3\}.
\]
Hence \(A_{31} f\) is also the sum of such monomials. Taking any monomial satisfying (1) and (5) and considering its coefficient in \(A_{31} f\), the equality \(A_{31} f = 0\) implies some linear relations among the coefficients \(\alpha_i\) in (4). In
### Table 1

<table>
<thead>
<tr>
<th>Representative</th>
<th>Coefficient</th>
<th>Isotropy group</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_1 = \begin{pmatrix} 112 &amp; 233 \ 111 &amp; 123 \ 11 \end{pmatrix} )</td>
<td>( \alpha_1 = 0 )</td>
<td>( \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>( m_2 = \begin{pmatrix} 112 &amp; 323 \ 111 &amp; 123 \ 11 \end{pmatrix} )</td>
<td>( \alpha_2 = 0 )</td>
<td>1</td>
</tr>
<tr>
<td>( m_3 = \begin{pmatrix} 123 &amp; 123 \ 111 &amp; 123 \ 11 \end{pmatrix} )</td>
<td>( \alpha_3 = 0 )</td>
<td>( \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>( m_4 = \begin{pmatrix} 112 &amp; 233 \ 112 &amp; 113 \ 111 \end{pmatrix} )</td>
<td>( \alpha_4 = 1 )</td>
<td>1</td>
</tr>
<tr>
<td>( m_5 = \begin{pmatrix} 112 &amp; 332 \ 112 &amp; 113 \ 11 \end{pmatrix} )</td>
<td>( \alpha_5 = -1 )</td>
<td>( \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>( m_6 = \begin{pmatrix} 121 &amp; 233 \ 112 &amp; 113 \ 111 \end{pmatrix} )</td>
<td>( \alpha_6 = -1 )</td>
<td>( \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>( m_7 = \begin{pmatrix} 123 &amp; 123 \ 112 &amp; 113 \ 111 \end{pmatrix} )</td>
<td>( \alpha_7 = 2 )</td>
<td>( \mathbb{Z}_2 \times \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>( m_8 = \begin{pmatrix} 123 &amp; 132 \ 111 &amp; 113 \ 111 \end{pmatrix} )</td>
<td>( \alpha_8 = -1 )</td>
<td>( \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>( m_9 = \begin{pmatrix} 112 &amp; 233 \ 111 &amp; 223 \ 112 \end{pmatrix} )</td>
<td>( \alpha_9 = 0 )</td>
<td>1</td>
</tr>
<tr>
<td>( m_{10} = \begin{pmatrix} 112 &amp; 332 \ 111 &amp; 223 \ 112 \end{pmatrix} )</td>
<td>( \alpha_{10} = 0 )</td>
<td>1</td>
</tr>
<tr>
<td>( m_{11} = \begin{pmatrix} 123 &amp; 123 \ 111 &amp; 223 \ 112 \end{pmatrix} )</td>
<td>( \alpha_{11} = 0 )</td>
<td>( \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>( m_{12} = \begin{pmatrix} 112 &amp; 233 \ 112 &amp; 123 \ 112 \end{pmatrix} )</td>
<td>( \alpha_{12} = 1 )</td>
<td>1</td>
</tr>
<tr>
<td>( m_{13} = \begin{pmatrix} 112 &amp; 323 \ 112 &amp; 123 \ 112 \end{pmatrix} )</td>
<td>( \alpha_{13} = 1 )</td>
<td>1</td>
</tr>
<tr>
<td>( m_{14} = \begin{pmatrix} 112 &amp; 332 \ 112 &amp; 123 \ 112 \end{pmatrix} )</td>
<td>( \alpha_{14} = -2 )</td>
<td>1</td>
</tr>
<tr>
<td>( m_{15} = \begin{pmatrix} 121 &amp; 233 \ 112 &amp; 123 \ 112 \end{pmatrix} )</td>
<td>( \alpha_{15} = -1 )</td>
<td>1</td>
</tr>
<tr>
<td>( m_{16} = \begin{pmatrix} 121 &amp; 323 \ 112 &amp; 123 \ 112 \end{pmatrix} )</td>
<td>( \alpha_{16} = -1 )</td>
<td>1</td>
</tr>
<tr>
<td>( m_{17} = \begin{pmatrix} 121 &amp; 332 \ 112 &amp; 123 \ 112 \end{pmatrix} )</td>
<td>( \alpha_{17} = 2 )</td>
<td>1</td>
</tr>
<tr>
<td>( m_{18} = \begin{pmatrix} 123 &amp; 123 \ 112 &amp; 123 \ 112 \end{pmatrix} )</td>
<td>( \alpha_{18} = 2 )</td>
<td>1</td>
</tr>
<tr>
<td>( m_{19} = \begin{pmatrix} 123 &amp; 132 \ 112 &amp; 123 \ 112 \end{pmatrix} )</td>
<td>( \alpha_{19} = -1 )</td>
<td>1</td>
</tr>
<tr>
<td>( m_{20} = \begin{pmatrix} 123 &amp; 312 \ 112 &amp; 123 \ 112 \end{pmatrix} )</td>
<td>( \alpha_{20} = -1 )</td>
<td>1</td>
</tr>
<tr>
<td>( m_{21} = \begin{pmatrix} 112 &amp; 233 \ 113 &amp; 122 \ 112 \end{pmatrix} )</td>
<td>( \alpha_{21} = -1 )</td>
<td>1</td>
</tr>
<tr>
<td>( m_{22} = \begin{pmatrix} 112 &amp; 323 \ 113 &amp; 122 \ 112 \end{pmatrix} )</td>
<td>( \alpha_{22} = 1 )</td>
<td>1</td>
</tr>
</tbody>
</table>
TABLE I—Continued

<table>
<thead>
<tr>
<th>Representative</th>
<th>Coefficient</th>
<th>Isotropy group</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_{21} = \left( \begin{array}{c} 121 \ 113, 122, 112 \end{array} \right)$</td>
<td>$\alpha_{21} = 1$</td>
<td>1</td>
</tr>
<tr>
<td>$m_{24} = \left( \begin{array}{c} 121 \ 113, 122, 112 \end{array} \right)$</td>
<td>$\alpha_{24} = -1$</td>
<td>1</td>
</tr>
<tr>
<td>$m_{25} = \left( \begin{array}{c} 123 \ 113, 122, 112 \end{array} \right)$</td>
<td>$\alpha_{25} = -1$</td>
<td>1</td>
</tr>
<tr>
<td>$m_{26} = \left( \begin{array}{c} 123 \ 113, 122, 112 \end{array} \right)$</td>
<td>$\alpha_{26} = 2$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>$m_{27} = \left( \begin{array}{c} 112 \ 233, 122, 123 \end{array} \right)$</td>
<td>$\alpha_{27} = 1$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>$m_{28} = \left( \begin{array}{c} 112 \ 233, 122, 123 \end{array} \right)$</td>
<td>$\alpha_{28} = -1$</td>
<td>1</td>
</tr>
<tr>
<td>$m_{29} = \left( \begin{array}{c} 121 \ 112, 233, 123 \end{array} \right)$</td>
<td>$\alpha_{29} = 1$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>$m_{30} = \left( \begin{array}{c} 123 \ 112, 233, 123 \end{array} \right)$</td>
<td>$\alpha_{30} = 1$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>$m_{31} = \left( \begin{array}{c} 123 \ 112, 233, 123 \end{array} \right)$</td>
<td>$\alpha_{31} = -2$</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2$</td>
</tr>
<tr>
<td>$m_{32} = \left( \begin{array}{c} 112 \ 233, 123, 123 \end{array} \right)$</td>
<td>$\alpha_{32} = -3$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>$m_{33} = \left( \begin{array}{c} 112 \ 232, 123, 123 \end{array} \right)$</td>
<td>$\alpha_{33} = 0$</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2$</td>
</tr>
<tr>
<td>$m_{34} = \left( \begin{array}{c} 123 \ 123, 123, 123 \end{array} \right)$</td>
<td>$\alpha_{34} = 0$</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2$</td>
</tr>
<tr>
<td>$m_{35} = \left( \begin{array}{c} 123 \ 123, 123, 123 \end{array} \right)$</td>
<td>$\alpha_{35} = 0$</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2$</td>
</tr>
<tr>
<td>$m_{36} = \left( \begin{array}{c} 123 \ 123, 231, 123 \end{array} \right)$</td>
<td>$\alpha_{36} = -3$</td>
<td>$\mathbb{Z}_3$</td>
</tr>
</tbody>
</table>

This way, we can determine all the coefficients $\alpha_i$ by taking them to satisfy all the equations $A_{ij}f = 0$, $B_{ij}f = 0$, and $C_{ij}f = 0$ ($i \neq j$). After all this we have the following theorem.

**Theorem 1.** The irreducible relative invariant $f_2$ of the prehomogeneous vector space $(G, V)$ is explicitly given by

$$f_2 = \sum_{i=1}^{36} \left( \alpha_i \sum' \pm m_i \right).$$

Here monomials $m_i$ and coefficients $\alpha_i$ are listed in Table I, and $\sum' \pm m_i$ is the sum of the distinct terms of
3. THE $b$-FUNCTIONS OF IRREDUCIBLE RELATIVE INVARIANTS

In this section we determine the $b$-functions of $f_1$ and $f_2$. The $b$-function of $f_1$ has already been known to be

$$b_f(s) = (s + 1)^4 \left( s + \frac{3}{2} \right)^4 \left( s + \frac{4}{3} \right) \left( s + \frac{5}{3} \right) \left( s + \frac{5}{6} \right) \left( s + \frac{7}{6} \right).$$

(See [8].) Then the remaining problem is the calculation of the $b$-function of $f_2$. We write $f$ for $f_2$ and $b$ for the $b$-function of $f_2$ throughout this section.

In order to determine the $b$-function $b_f$, first we calculate the exponential $b$-function of $f_1$ in Subsection 3.1, the values of $b_0$ and $b_1$ in Subsection 3.2, and the value of $b_0$ in Subsection 3.3. Using these data, we determine the $b$-function $b_f$ in Subsection 3.4.

3.1. Calculation of the Exponential $b$-Function of $f_2$

Considering the contraction defined in Subsection 1.3, we calculate the exponential $b$-function of $f_1$ in this subsection.

First we find the $G$-orbit $O_f$ which is closed in $V_f$ and a point $v_f \in O_f$. Since the $G$-orbit in $V_f$ of the minimal dimension is closed in $V_f$, it is sufficient to find such a $G$-orbit in $V_f$. The orbit decomposition of the prehomogeneous vector space $(SL_3 \times SL_3 \times GL_2, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(3) \otimes V(3) \otimes V(2))$ is known as in [8]. We show this in Table II. For each orbit $O$ in Table II, we see that $O \times V(3) := \{(v, p); v \in O, p \in V(3)\}$ is a union of several $G$-orbits of $(G, V)$. Hence the $G$-orbit decomposition of $(G, V)$ is given by decomposing these 18 mutually disjoint $G$-invariant sets $O \times V(3)$. The $G$-orbits contained in $V_f$ arise from only (a), (b), (c), and (f) in Table II.

Let

$$v' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

be the orbit representative (f) in Table II. We consider the dimension of the isotropy algebra $g_v$ at

$$v = \begin{pmatrix} v' \end{pmatrix}, \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \in V_f.$$
Note that \( v \in V_f \) requires \( p_1 \neq 0 \). Then \( \dim \mathfrak{g}_v \) is 5 which is independent of \( p_2 \) and \( p_3 \). Hence the dimension of the \( G \)-orbit \( Gv \) is 17. Moreover, the dimensions of all the \( G \)-orbits in \( V_f \) arising from (a), (b), and (c) in Table II are greater than 17. Hence the minimal dimension of the \( G \)-orbits contained in \( V_f \) is 17. We may put

\[
v_f := \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Then

\[
\mathfrak{g}_{v_f} = \left\{ \begin{pmatrix}
a_{11} & 0 & 0 \\
0 & a_{22} & a_{23} \\
0 & a_{32} & a_{33}
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 \\
0 & a_{11} - a_{22} & -a_{32} \\
0 & -a_{32} & a_{11} - a_{33}
\end{pmatrix}, \begin{pmatrix}
-a_{33} & -a_{23} \\
-a_{32} & -a_{22}
\end{pmatrix} \right\}
\]

and \( O_f = Gv_f \).
Next we take the maximal torus \( T_f \) of the isotropy group \( G_f \) given by
\[
T_f = \left\{ \begin{pmatrix}
    a_1 & 0 & 0 \\
    0 & a_2 & 0 \\
    0 & 0 & a_3
\end{pmatrix} \left| \begin{pmatrix}
    1 & 0 & 0 \\
    0 & a_1 & 0 \\
    0 & 0 & a_3
\end{pmatrix}, \left| \begin{pmatrix}
    1 & 0 \\
    0 & a_3
\end{pmatrix} \right|; a_1a_2a_3 \neq 0 \right\}.
\]

Then we can determine the normalizer \( N_G(T_f) \) and the fixed point set \( V^{T_f} \) as
\[
N_G(T_f) = \left\{ \begin{pmatrix}
    a_{11} & 0 & 0 \\
    0 & a_{22} & 0 \\
    0 & 0 & a_{33}
\end{pmatrix}, \begin{pmatrix}
    b_{11} & 0 & 0 \\
    0 & b_{22} & 0 \\
    0 & 0 & b_{33}
\end{pmatrix}, \begin{pmatrix}
    c_{11} & 0 \\
    0 & c_{22}
\end{pmatrix} \right\},
\]
\[
V^{T_f} = \left\{ \begin{pmatrix}
    0 & 0 & x_{13} \\
    0 & 0 & 0 \\
    x_{31} & 0 & 0
\end{pmatrix}, \begin{pmatrix}
    0 & y_{12} & 0 \\
    y_{21} & 0 & 0 \\
    0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
    p_1 \\
    0 \\
    0
\end{pmatrix} \right\}.
\]

Hence we get \( f' := f|_{V^{T_f}} \) as
\[
f'(v) = x_{13}x_{31}^2y_{12}y_{21}^2p_1^3
\]
for
\[
v = \begin{pmatrix}
    0 & 0 & x_{13} \\
    0 & 0 & 0 \\
    x_{31} & 0 & 0
\end{pmatrix}, \begin{pmatrix}
    0 & y_{12} & 0 \\
    y_{21} & 0 & 0 \\
    0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
    p_1 \\
    0 \\
    0
\end{pmatrix} \in V^{T_f}.
\]

Finally, we calculate \( b_f^{\text{exp}} \). We have
\[
b_f(s) = (s + 1)^5\left( s + \frac{1}{2} \right)^2\left( s + \frac{1}{3} \right)^2\left( s + \frac{2}{3} \right) \times \text{constant}.
\]

Then the exponential \( b \)-function is given by
\[
b_f^{\text{exp}}(t) = b_f^{\text{exp}}(t) = (t^3 - 1)(t^2 - 1)^2(t - 1)^2.
\]
3.2. The Values of $b_0$ and $b_1$

In this subsection we calculate the values of $b_0$ and $b_1$ using Lemma 1.1.6 by putting

$$v_1 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Put

$$\tilde{m} := \begin{pmatrix} 331 & 221 & 111 \\ 113 & 112 & 111 \end{pmatrix}.$$

As we have seen in Subsection 3.1, $v_1$ belongs to $O_f$. From Table I, we see that $\tilde{m}(v_1) \neq 0$ and that $m(v_1) = 0$ for any other monomial $m$ in $f$. Hence we get $f(v_1) = 1$.

Considering the first-order partial derivatives of $f$, we have

$$\nabla \log f(v_1) := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}.$$

Thus we get the value of $b_0$:

$$b_0 = f^{-1}(\nabla \log f)(v_1) f(v_1) = 2^43^3.$$

Next we calculate $b_1$. Find all the monomials $m$ and all the variables $x$ and $y$ such that $\alpha_m m_{xy}(v_1) \neq 0$, where $\alpha_m$ is the coefficient of the monomial $m$ in $f$. If $(m, x, y)$ and $(m', x', y')$ are such triples and $m \neq m'$, then we can easily show that $(x, y) \neq (x', y')$. Hence we get $f_{x,y}(v_1) = \alpha_m m_{xy}(v_1)$ for such $m$, $x$, and $y$. These data are given in Table III.

For any monomial $m = x_{i_1} \cdots x_{i_d}$ in $f$, let $D_m(v_1)$ be the sum

$$\sum_{1 \leq j \leq i \leq d} f_{x_{i_1}, \cdots, x_{i_{j-1}}, x_{i_{j}}, \cdots, x_{i_d}} f_{x_{i_{j}}, x_{i_{j+1}}, \cdots, x_{i_d}} = D_n(v_1)$$

in $D_n(v_1)$ of Lemma 1.1.6. For the purpose of calculating $D_n(v_1)$, we consider $\alpha_m D_m(v_1)$ for each monomial $m$ in $f$.

1. Let $m$ be a monomial which does not appear in Table III. Since $m_{xy} = 0$ for any variables $x$ and $y$, we get $D_m(v_1) = 0$.

2. Let $m$ be a monomial which appears in Table III and $m \neq \tilde{m}$. Let $x$ and $y$ be variables such that $m_{xy}(v_1) \neq 0$. Since $m_{x', y'} = 0$ for any pair of variables $(x', y') \neq (x, y)$, $D_m(v_1)$ equals the nonzero term containing $f_{x,y}$. These values are listed in the fourth entries of Table III.

3. Let $m = \tilde{m} = (331, 221, 111)$. We get $\alpha_m D_m(v_1) = 2^43^3 \times 34$ by calculating actually.
TABLE III

<table>
<thead>
<tr>
<th>(m, x, y)</th>
<th>α_m</th>
<th>f_y(v_1)</th>
<th>D_m(v_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (331, 221, 113^1, 112, \ldots, x_{31}, x_{32}) )</td>
<td>1</td>
<td>( f_{x_{31} x_{32}}(v_1) = 2 )</td>
<td></td>
</tr>
<tr>
<td>( (331, 221, 113^1, 112, \ldots, x_{31}, x_{33}) )</td>
<td>1</td>
<td>( f_{x_{31} x_{33}}(v_1) = 2 )</td>
<td></td>
</tr>
<tr>
<td>( (331, 221, 113^1, 112, \ldots, x_{31}, y_{21}) )</td>
<td>1</td>
<td>( f_{x_{31} y_{21}}(v_1) = 4 )</td>
<td></td>
</tr>
<tr>
<td>( (331, 221, 113^1, 112, \ldots, x_{31}, y_{12}) )</td>
<td>1</td>
<td>( f_{x_{31} y_{12}}(v_1) = 2 )</td>
<td></td>
</tr>
<tr>
<td>( (331, 221, 113^1, 112, \ldots, x_{31}, p_1) )</td>
<td>1</td>
<td>( f_{x_{31} p_1}(v_1) = 6 )</td>
<td></td>
</tr>
<tr>
<td>( (331, 221, 113^1, 112, \ldots, x_{32}, y_{21}) )</td>
<td>1</td>
<td>( f_{x_{32} y_{21}}(v_1) = 2 )</td>
<td></td>
</tr>
<tr>
<td>( (331, 221, 113^1, 112, \ldots, x_{32}, y_{12}) )</td>
<td>1</td>
<td>( f_{x_{32} y_{12}}(v_1) = 1 )</td>
<td></td>
</tr>
<tr>
<td>( (331, 221, 113^1, 112, \ldots, x_{32}, p_1) )</td>
<td>1</td>
<td>( f_{x_{32} p_1}(v_1) = 3 )</td>
<td></td>
</tr>
<tr>
<td>( (331, 221, 113^1, 112, \ldots, y_{21}, y_{21}) )</td>
<td>1</td>
<td>( f_{y_{21} y_{21}}(v_1) = 2 )</td>
<td></td>
</tr>
<tr>
<td>( (331, 221, 113^1, 112, \ldots, y_{21}, y_{12}) )</td>
<td>1</td>
<td>( f_{y_{21} y_{12}}(v_1) = 2 )</td>
<td></td>
</tr>
<tr>
<td>( (331, 221, 113^1, 112, \ldots, y_{21}, p_1) )</td>
<td>1</td>
<td>( f_{y_{21} p_1}(v_1) = 6 )</td>
<td></td>
</tr>
<tr>
<td>( (331, 221, 113^1, 112, \ldots, y_{12}, p_1) )</td>
<td>1</td>
<td>( f_{y_{12} p_1}(v_1) = 3 )</td>
<td></td>
</tr>
<tr>
<td>( (331, 221, 113^1, 112, \ldots, p_1, p_1) )</td>
<td>1</td>
<td>( f_{p_1 p_1}(v_1) = 6 )</td>
<td></td>
</tr>
<tr>
<td>( (133, 221, 113^1, 112, \ldots, x_{11}, x_{33}) )</td>
<td>-1</td>
<td>( f_{x_{11} x_{33}}(v_1) = -1 )</td>
<td>( 2^3 3^3 )</td>
</tr>
<tr>
<td>( (331, 221, 113^1, 112, \ldots, x_{11}, y_{32}) )</td>
<td>-1</td>
<td>( f_{x_{11} y_{33}}(v_1) = -1 )</td>
<td>( 2^3 3^3 )</td>
</tr>
</tbody>
</table>

Since we have \( D_f(v_1) = \sum_m \alpha_m D_m(v_1) \), we get \( D_f(v_1) = 2^4 3^3 \times 40 \). Then

\[
b_1 = -27b_0 + D_f(v_1)f(v_1)^{-7} = 2^4 3^3 13.
\]

Thus \( b_1 \) has been determined.

3.3. The Value of \( b_9 \)

We have \( b_9 = b_f(0) = \varphi(\text{grad}_x)f(x) \) by definition. Hence we can calculate this value by Table I. By carrying out the calculation, we obtain

\[
b_9 = 2^6 3^3 5.
\]
3.4. Determination of the $b$-Function of $f_2$

Let $-\alpha_i$ ($i = 1, \ldots, 9$) be the roots of the $b$-function $b_f(s)$ as in Subsection 1.1. Then we see that each $\alpha_i$ is a positive rational number by Lemma 1.1.7. Moreover, we have the following facts about $\alpha_i$ ($i = 1, \ldots, 9$) from the previous arguments.

(1) Since $b_f^{exp}(t) = (t^3 - 1)(t^2 - 1)^2(t - 1)^2$, we may assume that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$, $\alpha_6 = \alpha_7 = \frac{1}{2}$, $\alpha_8 = \frac{1}{3}$, and $\alpha_9 = \frac{5}{6}$ (mod $\mathbb{Z}$).

(2) We get $\Sigma \alpha_i = b_1/b_0 = 13$ from $b_0 = 2^43^3$ and $b_1 = 2^43^313$.

(3) We get $\Pi \alpha_i = b_0/b_0 = 2^5$ from $b_0 = 2^43^3$ and $b_0 = 2^63^5$.

Then we can find positive rational numbers $\alpha_i$ ($i = 1, \ldots, 9$) satisfying (1)--(3) by a simple consideration. Namely we have that $\{\alpha_i \mid 1 \leq i \leq 5\} = \{1, 1, 1, 2, 2\}$, $\alpha_6 = \alpha_7 = \frac{1}{2}$, $\alpha_8 = \frac{1}{3}$, and $\alpha_9 = \frac{5}{6}$. Thus we have the explicit form of the $b$-function of $f_2$ in the following theorem.

**Theorem 2.** The $b$-function of the irreducible relative invariant $f_2$ of Theorem 1 is given by

$$b_{f_2}(s) = (s + 1)^3(s + 2)^2\left(s + \frac{3}{2}\right)^2\left(s + \frac{4}{3}\right)\left(s + \frac{5}{3}\right).$$
4. THE b-FUNCTIONS OF RELATIVE INVARIANTS

Let $f_1$ and $f_2$ be the irreducible relative invariants constructed in Section 2. We define $f^m$, $a_m$, and $b_m$ as in Subsection 1.4. In this section we determine the $b$-function $b_{f^m}$ of the relative invariant $f^m = f_1^{m_1} f_2^{m_2}$ ($m_1, m_2 \in \mathbb{Z}_{\geq 0}$).

4.1. a-Functions

In this subsection we determine the leading coefficients $a_{f^m}$. Let

$$v_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} .$$

We calculate $a_1(s)$ and $a_2(s)$. We see that

$$\text{grad } \log f_1(v_0) = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} , \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ,$$

$$\text{grad } \log f_2(v_0) = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} , \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} , \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} ,$$

after some explicit calculation. Then we can calculate $a_1(s)$ as

$$a_1(s) = f_1^\nu (\text{grad } \log f_2(v_0)) f_1(v_0)$$

$$= f_1^\nu (s_1 \text{ grad } \log f_1(v_0) + s_2 \text{ grad } \log f_2(v_0)) f_1(v_0)$$

$$= 2^3 3^3 s_1^3 (s_1 + s_2)^4 (2s_1 + s_2)^4.$$  

Since $a_1(s)$ is as above, $a_m(n) = a_1^{m_1}(m) a_2^{m_2}(m)$ is a product of powers of $m_1^{m_1}$ and $(m_1 + m_2)^{m_2} (m_1 + m_2)^{m_2}$ by Lemma 1.4.3, and then $a_2(s)$ is a product of powers of $s_2$ and $(s_1 + s_2)^4 (2s_1 + s_2)^4$. Hence it is expressed as $a_2(s) = A_2 s_2^3 (s_1 + s_2)^4 (2s_1 + s_2)^4$ with some $A_2 \in \mathbb{C}$. Taking $s = \epsilon_2$, we get $a_2(\epsilon_2) = A_2$. Since $a_2(\epsilon_2) = a_{f_2}$ by definition, we get $A_2 = 2^4 3^3$. Hence we have

$$a_1(s) = 2^3 3^3 s_1^3 (s_1 + s_2)^4 (2s_1 + s_2)^4,$$

$$a_2(s) = 2^4 3^3 s_2^3 (s_1 + s_2)^4 (2s_1 + s_2)^2.$$

Thus we see that the leading coefficient of the $b$-function $b_{f^m}(s)$ is given by

$$a_{f^m} = 2^{8m_1 + 4m_2} 3^{6m_1 + 3m_2} (m_1^{m_1})^4 (m_2^{m_2})^3$$

$$\times \left\{ (m_1 + m_2)^{m_1 + m_2} \right\}^4 \left\{ (2m_1 + m_2)^{2m_1 + m_2} \right\}^2.$$
4.2. \( b\)-Functions

In this subsection we determine the \( b\)-functions \( b_{\gamma_a} \).

Since \( a_1(s) \) and \( a_2(s) \) are given as in the preceding subsection, \( b_1(s) \) and \( b_2(s) \) are expressed as

\[
b_1(s) = 2^3 3^6 \left( \prod_{r=1}^{4} (s + \alpha_{1,r}) \right) \left( \prod_{r=1}^{4} (s + s_2 + \alpha_{3,r}) \right) \times \left( \prod_{r=1}^{2} (2s_1 + s_2 + \alpha_{4,r})(2s_1 + s_2 + \alpha_{4,r} + 1) \right),
\]

\[
b_2(s) = 2^4 3^3 \left( \prod_{r=1}^{3} (s + \alpha_{2,r}) \right) \left( \prod_{r=1}^{4} (s + s_2 + \alpha_{3,r}) \right) \times \left( \prod_{r=1}^{2} (2s_1 + s_2 + \alpha_{4,r}) \right),
\]

with \( \alpha_{i,r} \in \mathbb{Q}_{>0} \) by Lemma 1.4.5.

Since \( b_i(s \epsilon_1) = b_i(s) \) by definition, we get

\[
\left( \prod_{r=1}^{4} (s + \alpha_{1,r}) \right) \left( \prod_{r=1}^{4} (s + \alpha_{3,r}) \right) \left( \prod_{r=1}^{2} \left( s + \frac{\alpha_{4,r}}{2} \right) \left( s + \frac{\alpha_{4,r} + 1}{2} \right) \right) = (s + 1)^4 \left( s + \frac{3}{2} \right)^4 \left( s + \frac{4}{3} \right)^4 \left( s + \frac{5}{3} \right)^4 \left( s + \frac{7}{6} \right).
\]

Since \( b_2(s \epsilon_2) = b_2(s) \), we get

\[
\left( \prod_{r=1}^{3} (s + \alpha_{2,r}) \right) \left( \prod_{r=1}^{4} (s + \alpha_{3,r}) \right) \left( \prod_{r=1}^{2} (s + \alpha_{4,r}) \right) = (s + 1)^3 (s + 2)^2 \left( s + \frac{3}{2} \right)^2 \left( s + \frac{4}{3} \right)^2 \left( s + \frac{5}{3} \right).
\]

Comparing the roots of both sides, we get

\[
\begin{align*}
\{ \alpha_{1,r} \mid 1 \leq r \leq 4 \}, \quad \alpha_{3,r} \mid 1 \leq r \leq 4 \}, \quad \frac{\alpha_{4,r}}{2}, \quad \frac{\alpha_{4,r} + 1}{2} \mid r = 1, 2 \}, \\
= \left\{ 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 3, 3, 3, 3, 4, 4, 5, 5, 5, 7 \right\},
\end{align*}
\]

(1)

\[
\{ \alpha_{2,r} \mid 1 \leq r \leq 3 \}, \quad \alpha_{3,r} \mid 1 \leq r \leq 4 \}, \quad \alpha_{4,r} \mid r = 1, 2 \}, \\
= \left\{ 1, 1, 1, 2, 2, 3, 3, 3, 4, 4, 5 \right\}.
\]

(2)
If $\alpha = \alpha_{4,1}$ or $\alpha_{4,2}$, then $\alpha$ belongs to (2) and that $(\alpha/2, (\alpha + 1)/2)$ is contained in (1). Such an $\alpha$ is in $\{2, 2, 7\}$. Since

$$\beta_{\gamma_{\pm}}(t) = (t - e^{2\pi i a_{4,1}})(t - e^{2\pi i a_{4,2}})$$

is the product of some cyclotomic polynomials by Lemma 1.4.8, we get $\alpha_{4,1} = \alpha_{4,2} = 2$. Then (1) and (2) yield that

$$\{\alpha_{1,r} (1 \leq r \leq 4), \alpha_{3,r} (1 \leq r \leq 4)\} = \left\{1, 1, \frac{3}{2}, \frac{3}{2}, \frac{4}{3}, \frac{4}{3}, \frac{5}{6}, \frac{5}{6}\right\}, \quad (3)$$

$$\{\alpha_{2,r} (1 \leq r \leq 3), \alpha_{3,r} (1 \leq r \leq 4)\} = \left\{1, 1, \frac{3}{2}, \frac{3}{2}, \frac{4}{3}, \frac{4}{3}\right\}. \quad (4)$$

Note that $\frac{5}{6}$ and $\frac{7}{6}$ belong to the set (1) but not to the set (2). Then we have $\left\{\frac{5}{6}, \frac{7}{6}\right\} \subset \{\alpha_{1,r} (1 \leq r \leq 4)\}$. Hence we may assume $\alpha_{1,3} = \frac{5}{6}$ and $\alpha_{1,4} = \frac{7}{6}$. We may assume that $\alpha_{2,3} = 1$, $\alpha_{2,1} = \alpha_{1,1}$, and $\alpha_{2,2} = \alpha_{1,2}$ similarly. Then (3) and (4) yield that

$$\{\alpha_{1,1}, \alpha_{1,2}, \alpha_{3,1}, \alpha_{3,2}, \alpha_{3,3}, \alpha_{3,4}\} = \left\{1, 1, \frac{3}{2}, \frac{3}{2}, \frac{4}{3}, \frac{4}{3}\right\}. \quad (5)$$

Let $f_0 = f_1 f_2$. Then we have

$$b_{f_0}(s) = 2^{12} 3^9 (s + \alpha_{1,1})^2 (s + \alpha_{1,2})^2 \left(s + \frac{5}{6}\right) \left(s + \frac{7}{6}\right) \times \prod_{r=1}^{4} (2s + \alpha_{3,r}) (2s + \alpha_{3,r} + 1) (3s + 2)^2 (3s + 3)^2 (3s + 4)^2$$

by Lemma 1.4.7. The absolute values of the roots of $b_{f_0}(s)$ are

$$\left\{\alpha_{1,1}, \alpha_{1,1}, \frac{\alpha_{3,r} + 1}{2}, \frac{\alpha_{3,r}}{2}, (1 \leq r \leq 4), \frac{2}{3}, \frac{2}{3}, 1, 1, \frac{4}{3}, \frac{4}{3}, \frac{5}{6}, \frac{5}{6}\right\}.$$ 

We have that $b_{f_0}(s) = -b_{f_0}(-s - 2)$ by Lemma 1.2.2. If $-\alpha$ is a root of $b_{f_0}(s)$, then we have $0 = b_{f_0}(-\alpha) = b_{f_0}(\alpha + 2)$. Thus $\alpha - 2$ is also a root of $b_{f_0}(s)$. Therefore all roots are symmetric with respect to $-1$. Since the sum of them must be $-21$, we get

$$2\alpha_{1,1} + 2\alpha_{1,2} + \sum_{r=1}^{4} \alpha_{3,r} = 10. \quad (6)$$
By Lemma 1.4.8, we have that \( \alpha_{1,r} \) \( (r = 1, 2) \) and \( \alpha_{3,r} \) \( (1 \leq r \leq 4) \) satisfying (5) and (6) are the following,

\[
\alpha_{1,1} = \alpha_{1,2} = 1 \quad \text{and} \quad \{ \alpha_{3,1}, \alpha_{3,2}, \alpha_{3,3}, \alpha_{3,4} \} = \left\{ \frac{3}{2}, \frac{3}{2}, \frac{4}{3}, \frac{5}{3} \right\}.
\]

Hence the \( b \)-functions are given by

\[
b_1(s) = 2^6 3^6 (s_1 + 1)^2 \left( s_1 + \frac{5}{6} \right) \left( s_1 + \frac{7}{6} \right) \left( s_1 + s_2 + \frac{3}{2} \right) \left( s_1 + s_2 + \frac{4}{3} \right) \\
\times \left( s_1 + s_2 + \frac{5}{3} \right) (2s_1 + s_2 + 2)^2 (2s_1 + s_2 + 3)^2,
\]

\[
b_2(s) = 2^4 3^4 (s_2 + 1)^3 \left( s_1 + s_2 + \frac{3}{2} \right)^2 \left( s_1 + s_2 + \frac{4}{3} \right) \left( s_1 + s_2 + \frac{5}{3} \right) \\
\times (2s_1 + s_2 + 2)^2.
\]

Therefore we get the main result of this paper.

**Theorem 3.** Let \( f_1 \) and \( f_2 \) be the irreducible relative invariants constructed in Section 2. Then the \( b \)-function of the relative invariant \( f_1^{m_1} f_2^{m_2} \) \( (m_1, m_2 \in \mathbb{Z}_{>0}) \) is given by

\[
b_{f_1^{m_1} f_2^{m_2}}(s) = \left\{ \prod_{\nu=0}^{m_1-1} (m_1 s + 1 + \nu)^2 \left( m_1 s + \frac{5}{6} + \nu \right) \left( m_1 s + \frac{7}{6} + \nu \right) \right\} \\
\times \left\{ \prod_{\nu=0}^{m_2-1} (m_2 s + 1 + \nu)^3 \right\} \\
\times \left\{ \prod_{\nu=0}^{m_1+m_2-1} \left( (m_1 + m_2) s + \frac{3}{2} + \nu \right)^2 \right\} \\
\times \left\{ \prod_{\nu=0}^{2m_1+m_2-1} \left( (2m_1 + m_2) s + 2 + \nu \right)^2 \right\} \\
\times \text{constant}.
\]
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