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# On Counting the *k*-face Cells of Cyclic Arrangements

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In this paper, we compute the exact number of $k$-face cells of the *cyclic arrangements* which are the dual to the well-known cyclic polytopes. The proof uses the combinatorial interpretation of arrangements in terms of oriented matroids.



## 1. INTRODUCTION

A *projective d-arrangement of n hyperplanes* $H(d,n)$ is a finite collection of hyperplanes in the real projective space $P^d$ such that no point belongs to every hyperplane of $H(d,n)$. Any arrangement $H(d,n)$ decomposes $P^d$ into a $d$-dimensional cell complex $K$. We may call *cells* of $H(d,n)$ the $d$-cells of $K$, and *facets* of $H(d,n)$ the $(d-1)$-cells of $K$. Clearly any cell of $H(d,n)$ has at least (resp. at most) $d+1$ (resp. $n$) facets. We shall denote by $f_p[H(d,n)]$ the number of $d$-cells of $H(d,n)$ having exactly $p$ facets, $d+1 \leq p \leq n$.

The *cyclic polytope* of dimension $d$ with $n$ vertices $C_d(t_1,\ldots,t_n)$ was discovered by Carathéodory [3, 4] and has been rediscovered many times; it is usually defined as the convex hull in the Euclidean space $R^d$, $d \geq 2$, of $n$, $n \geq d+1$, different points $x(t_1),\ldots,x(t_n)$ of the *moment curve* $x : R \to R^d$, $t \to (t, t^2, \ldots, t^d)$. Cyclic polytopes, and simplicial neighbourly polytopes, in general, play an important role in the combinatorial convex geometry due to their connection with certain extremal problems. For example, the upper bound theorem established by McMullen [7, 8], says that the number of $j$-dimensional faces of $d$-polytope with $n$ vertices is maximized by $C_d(t_1,\ldots,t_n)$.

Here, we focus our attention to *cyclic arrangements*, $A(d,n)$, defined as the dual to cyclic polytopes $C_d(t_1,\ldots,t_n)$. As for cyclic polytopes, cyclic arrangements also have extremal properties. For instance, Shannon [12] has introduced cyclic arrangements $A(d,n)$ as examples of projective arrangements with a minimum number of cells with $(d+1)$-facets. In this paper, we give an explicit formula to compute $f_p[A(d,n)]$ for each $d+1 \leq p \leq n$.

THEOREM 1.1. *Let $d, n, p$ be positive integers such that $d+1 \leq p \leq n$. Then*

$$f_p[A(d,n)] = \begin{cases} \sum_{i=0}^{d-2}\binom{n-1}{i} + \binom{d}{n-d} + \binom{d-1}{n-d-1} & \textit{if } p = n, \\ \binom{n-d-1}{n-p}\binom{d}{p-d} + \binom{n-d}{n-p}\binom{d-1}{p-d-1} & \textit{if } p < n. \end{cases}$$

Our proof uses the combinatorial interpretation of the cyclic arrangements in terms of oriented matroid. Indeed, cyclic arrangements of $n$ hyperplanes in $P^d$ are combinatorially equivalent to *alternating oriented matroids* of rank $r = d+1$ on $n$ elements, by the representation of Folkman and Lawrence [6] (for the basic notation of oriented matroid theory, we refer the reader to [1]). The (uniform) *alternating oriented matroid* $\mathcal{A}(r,n)$ of rank $r$ on $n$ elements is defined as follows, see [1]: let $E$ denote an $n$-element set with $n \geq r$, together with a total order $<$. The signed circuits of $\mathcal{A}(r,n)$ are the subsets $C = \{e_1,\ldots,e_{r+1}\}$, $e_1 < e_2 < \cdots < e_{r+1}$, of $E$ with the signature $C^+ = \{e_i, i \text{ odd}\}$ and $C^- = \{e_i, i \text{ even}\}$, and their negatives.

In Section 2, we give general results concerning the acyclic reorientations together with their corresponding interior elements of the alternating oriented matroid that play an important role in the main result.

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In Section 3, we prove Theorem 1.1. Finally, in Section 4, we apply Theorem 1.1 to give straightforward proofs for some known results and to improve an upper bound on the number of *complete* cells in $A(d, n)$.

## 2. INTERIOR ELEMENTS OF $\mathcal{A}(r, n)$

Let $\mathcal{M}(r, n)$ be an oriented matroid of rank $r$ with $n$ elements. Let $T \in \{+, -\}^n$. We call $T$ a *sign n-vector* and denote by $T(i)$ the sign of element $i$ in $T$. We denote by $\mathcal{M}_T(r, n)$ the oriented matroid obtained from $\mathcal{M}$ by *reorienting* element $i$ if and only if $T(i) = -$. Notice that $T$ partitions $\{1, \ldots, n\}$ into *signed intervals* $I_1, \ldots, I_m$, $1 \le m \le n$, where $I_j$ denote a maximal set of consecutive elements having the same sign. We denote by $|I_j|$ the number of elements in $I_j$ and by $d(T)$ the number of intervals in $T$.

OBSERVATIONS. Let $T$ be a sign $n$-vector and $k \in \{1, \ldots, n\}$. Let $T'$ be a sign $n$-vector such that $T'(j) = T(j)$ for all $j \ne k$ and $T'(k) = -T(k)$.
(A) If $2 \le k \le n - 1$ then:

(i) $d(T') = d(T) + 2$ if and only if $T(k) = T(k + 1) = T(k - 1)$,
(ii) $d(T') = d(T) - 2$ if and only if $T(k) = -T(k + 1) = -T(k - 1)$ and
(iii) $d(T') = d(T)$ if and only if $T(k + 1) = -T(k - 1)$.

(B) If $k = 1$ (resp. $k = n$) then:

(i) $d(T') = d(T) + 1$ if and only if $T(1) = T(2)$ (resp. $T(n) = T(n - 1)$),
(ii) $d(T') = d(T) - 1$ if and only if $T(1) = -T(2)$ (resp. $T(n) = -T(n - 1)$).

Recall that an oriented matroid $\mathcal{M} = (E, \mathcal{C})$ is *acyclic* if it does not contain positive circuits (otherwise, $\mathcal{M}$ is called *cyclic*).

LEMMA 2.1. *Let $T$ be a sign $n$-vector. Then $\mathcal{A}_T(r, n)$ is acyclic if and only if $d(T) \le r$.*

PROOF. We shall show that $\mathcal{A}_T(r, n)$ is cyclic if and only if $d(T) > r$. Suppose that $\mathcal{A}_T(r, n)$ is cyclic. Let $C = (e_1, \ldots, e_{r+1})$ be a positive circuit in $\mathcal{A}_T(r, n)$. Since $C$ was alternating in $\mathcal{A}(r, n)$ then this implies that $T(e_i) = C(e_i)$ or $-C(e_i)$ where $C(i)$ denote the sign of element $i$ in $C$. Therefore, $T$ must have at least $r + 1$ intervals.

Let $T$ be a sign $n$-vector having as intervals $I_1, \ldots, I_l$ with $l > r$. Let $C = (e_1, \ldots, e_{r+1})$ be a circuit in $\mathcal{A}(r, n)$ such that $e_i \in I_i$ for each $i = 1, \ldots, r + 1$. Since $T(e_i) = -T(e_{i+1})$, $i = 1, \ldots, r$ and the elements of $C$ are signed alternatively then $C$ is a positive circuit in $\mathcal{A}_T(r, n)$. $\square$

We say that an element $e \in E$ of an uniform oriented acyclic matroid is *interior* if there exists a signed circuit $C = (C^+, C^-)$ with $C^- = \{e\}$. It is equivalent to define the interior elements as the elements whose reorientation give a cyclic matroid.

LEMMA 2.2. *Let $T$ be a sign $n$-vector such that $\mathcal{A}_T(r, n)$ is acyclic. Then (a) $2 \le i \le n-1$ is an interior element in $\mathcal{A}_T(r, n)$ if and only if $d(T) = r$ or $r - 1$, and $T(i) = T(i + 1) = T(i - 1)$ and (b) $1$ (resp. $n$) is an interior element in $\mathcal{A}_T(r, n)$ if and only if $d(T) = r$ and $T(1) = T(2)$ (resp. $T(n) = T(n - 1)$).*

PROOF. Let $T$ be a sign $n$-vector and $k \in \{1, \ldots, n\}$. Let $T'$ be the sign $n$-vector such that $T'(j) = T(j)$ for all $j \ne k$ and $T'(k) = -T(k)$. We know that $k$ is an interior element if and only if $\mathcal{A}_T(r, n)$ is acyclic and $\mathcal{A}_{T'}(r, n)$ is cyclic. Equivalently, $k$ is an interior element if and only if $d(T) \le r$ and $d(T') \ge r + 1$. Hence,

(a) by observation A($i$), $2 \leq k \leq n-1$ is an interior element if and only if $d(T) \leq r$, $d(T') \geq r+1$ and $d(T') = d(T)+2$ or equivalently if and only if $d(T) = r$ or $r-1$ and $T(k) = T(k+1) = T(k-1)$.

(b) By observation B($i$), 1 (resp. $n$) is an interior element if and only if $d(T) \leq r$, $d(T') \geq r+1$ and $d(T') = d(T)+1$ or equivalently if and only if $d(T) = r$ and $T(1) = T(2)$ (resp. $T(n) = T(n-1)$). □

## 3. THE FORMULA

Let $H(d,n) = \{h_i\}_{1 \leq i \leq n}$ be an arrangement of hyperplanes and $\mathcal{M}_{H(d,n)}$ its corresponding oriented matroid. We denote by $e_i$ the element of $\mathcal{M}_{H(d,n)}$ corresponding to hyperplane $h_i$. It is well known [6] that an acyclic reorientation of $\mathcal{M}_{H(d,n)}$ having $\{e_{i_1}, \ldots, e_{i_l}\}$, $l \leq n$ as interior elements corresponds to a cell in $H(d,n)$ which is boarded by hyperplanes $h_j \notin \{h_{i_1}, \ldots, h_{i_l}\}$.

Hence, $f_p[A(d,n)]$ can be computed by counting all sign $n$-vectors $T$ such that $\mathcal{A}_T(r,n)$ has exactly $p$ non-interior elements with $d+1 \leq p \leq n$.

PROPOSITION 3.1. *Let $T$ be a sign $n$-vector with intervals $I_1, \ldots, I_{r-1}$. Then $I_j$ contains exactly $|I_j| - 2$ interior elements in $\mathcal{A}_T(r,n)$ if $|I_j| \geq 3$ and no interior element if $|I_j| \leq 2$. Moreover, if $L = |\{I_j \| I_j| \geq 2\}|$ then there are exactly $r-1+L$ non-interior elements.*

PROOF. We have two cases.

(a) $2 \leq i \leq r-1$. Let $I_i$ be an interval of $T$ and suppose that $e \in I_i$. By Lemma 2.2 (a), $e$ is an interior element if and only if $T(e) = T(e-1) = T(e+1)$, in other words if and only if $e-1, e+1 \in I_i$. That is, $e$ is an interior element if and only if $e$ is not an end of $I_i$. So, if $|I_i| \geq 3$ then $I_i$ contains $|I_i| - 2$ interior elements and if $|I_i| \leq 2$ then $I_i$ contains no interior elements.

(b) $i = 1$ or $r-1$. Assume that $e \in I_1$ (the case for $i = r-1$ is analogous). By Lemma 2.2 (b), 1 is not an interior element. So, $e$ is an interior element if and only if $e-1, e+1 \in I_1$ and $e \neq 1$. That is, $e$ is an interior element if and only if $e$ is not an end of $I_1$.

Finally, there are $L$ intervals of length at least 2. So there are $r-1-L$ of length 1 and the number of non-interior elements is $r-1-L+2L = r-1+L$. □

PROPOSITION 3.2. *Let $T$ be a sign $n$-vector with intervals $I_1, \ldots, I_r$. Then (a) for each $2 \leq j \leq r-1$, $I_j$ contains exactly $|I_j| - 2$ interior elements in $\mathcal{A}_T(r,n)$ if $|I_j| \geq 3$ and no interior element if $|I_j| \leq 2$ and (b) $I_1$ (resp. $I_r$) contains $|I_1| - 1$ (resp. $|I_r| - 1$) interior elements in $\mathcal{A}_T(r,n)$. Moreover, if $L = |\{I_j \| I_j| \geq 2,\ 2 \leq j \leq r-1\}|$ then there are exactly $r+L$ non-interior elements.*

PROOF. Part (a) is similar to Proposition 3.1 (a). For part (b) assume that $e \in I_1$ (the case for $i = r$ is analogous). Suppose that $|I_1| > 1$ (if $|I_1| = 1$ then $I_1$ contains no interior elements). Since $T(1) = T(2)$ then by Lemma 2.2(b) 1 is an interior element. Now, $e \neq 1$ is an interior element if and only if $e-1, e+1 \in I_1$. Hence, the right-end of $I_1$ is the only not interior element. Therefore, $I_1$ contains $|I_1| - 1$ interior elements.

Finally, the first (and the last) interval has 1 non-interior element and there are exactly $L$ intervals of length at least 2 (other than the first and last intervals). So, there are $r-2-L$ intervals of length 1 and the number of non-interior elements is $2+r-2-L+2L = r+L$. □

PROOF OF THEOREM 1.1. Let $h_k(n, r, p)$ be the number of sign $n$-vectors $T$ having $1 \leq k \leq r$ intervals such that $\mathcal{A}_T$ has exactly $p$ non-interior elements. We shall compute $h_k(n, r, p)$ for each $1 \leq k \leq r$ since $f_p[A(d, n)] = \sum_{k=1}^{r} h_k(n, r, p)$. We have three cases.

CASE (I). $k = r$. By Proposition 3.2 we know that $p = r + L$. Thus, we must count all sign $n$-vectors $T = I_1, \ldots, I_r$ having $L$ intervals, say $I_{i_1}, \ldots, I_{i_L}$ with $2 \leq i_j \leq r - 1$, of size at least 2. In other words, we have to find all the solutions of

$$
\begin{array}{ll}
(*) & x_1 + \cdots + x_r = n \ \text{ such that} \\
 & \text{(a) } x_1, x_r \geq 1, \\
 & \text{(b) } x_{i_j} \geq 2 \text{ with } 2 \leq i_1 \leq \cdots \leq i_L \leq r - 1 \\
 & \text{(c) } x_{i'} = 1 \ \text{ with } \ 2 \leq i' \leq r - 1, i' \neq i_j.
\end{array}
$$

By setting $x_{i_j} - 1 = y_{i_j}$ we have that number of solutions of $(*)$ is $\binom{r-2}{L}$ (number of choices for (b)) times the number of solutions of $(**)$

$$(**)\ x_1 + y_{i_1} + \cdots + y_{i_L} + x_r = n - (r - 2 - L) - L \text{ with } x_1, x_r, y_{i_1}, \ldots, y_{i_L} \geq 1.$$

Since the number of solutions of $(**)$ is equal to $\binom{n-r+2-1}{L+2-1}$ then $h_r(n, r, p) = \binom{r-2}{L}\binom{n-r+2-1}{L+1}$. Notice that by setting $r = d + 1$ and $L = p - r$ we have that
$h_r(n, r, p) = \binom{r-2}{p-r}\binom{n-r+1}{p-r+1} = \binom{d-1}{p-d-1}\binom{n-d}{p-d} = \binom{d-1}{p-d-1}\binom{n-d}{n-p}$.

CASE (II). $k = r - 1$. By Proposition 3.1 we know that $p = r - 1 + L$. Thus, we must count all sign $n$-vectors $T = I_1, \ldots, I_{r-1}$ having $L$ intervals of size at least 2. By similar arguments as in Case (I) we have that $h_{r-1}(n, r, p) = \binom{r-1}{L}\binom{n-(r-1)-1}{L-1}$. Notice that by setting $r = d + 1$ and $L = p - r + 1$ we have that
$h_{r-1}(n, r, p) = \binom{r-1}{p-r+1}\binom{n-r}{p-r+1-1} = \binom{d}{p-d}\binom{n-d-1}{p-d-1} = \binom{d}{p-d}\binom{n-d-1}{n-p}$.

CASE (III). If $1 \leq k < r - 1$ then by Lemma 2.2 we have that

$$h_k(n, r, p) = \begin{cases} 0 & \text{if } p < n, \\ \binom{n-1}{k-1} & \text{if } p = n. \end{cases}$$

Hence,

$$f_p[A(d, n)] = \begin{cases} \sum_{i=0}^{d-2}\binom{n-1}{i} + \binom{d}{n-d} + \binom{d-1}{n-d-1} & \text{if } p = n, \\ \binom{n-d-1}{n-p}\binom{d}{p-d} + \binom{n-d}{n-p}\binom{d-1}{p-d-1} & \text{if } p < n. \end{cases} \qquad \square$$

## 4. SOME APPLICATIONS

Let us have a closer look at the extremals cells of the $A(d, n)$, that is, *simplices* (i.e., $(d+1)$-facet cells) and *complete cells* (i.e., $n$-facets cells).

Shannon [12] has proven that $A(d, n)$ has exactly $n$ simplices. This can easily be verified by checking that $f_{d+1}[A(d, n)] = n$. Moreover, Shannon gave the set of simplices explicitly.

THEOREM 4.1 ([12]). *Let $n, d$ be integers with $n \geq d + 1$. Then $A(d, n)$, has exactly $n$ simplices given by the set of $d + 1$ consecutive elements in the set $\{1, \ldots, n\}$ in cyclic order, that is,* $(1, 2, \ldots, d + 1), (2, 3, \ldots, d + 2), \ldots, (n, 1, \ldots, d)$.

Shannon's proof uses long geometric arguments. We propose a purely combinatorial proof. By Theorem 1.1, $A(d, n)$ has exactly $n$ simplices. Now, consider the following two sets of sign $n$-vectors given as a set of intervals (recall that the set of elements of two consecutive intervals has opposite sign). For each $i = 2, \dots, n - r + 2$,

$P_i = [1, \dots, i - 1], \underbrace{[i], \dots, [i + r - 3]}_{r-2}, [i + r - 2, \dots, n]$ and

$P_{n-r+3} = [1, \dots, n - r + 2], \underbrace{[n - r + 3], \dots, [n]}_{r-2}$. And, for each $j = 2, \dots r - 2$,

$Q_j = [1], [2], \dots, [j - 1], [j, \dots, j + n - r + 1], \underbrace{[j + n - r + 2], \dots, [n]}_{r-j+1}$

and $Q_{r-1} = [1], [2], \dots, [r - 2], [r - 1, \dots, n]$.

We claim that each of the signed vectors $P_i$ and $Q_j$ correspond to a simplex in $A(d, n)$. We leave the proof of this claim to the reader, as an easy combinatorial exercise.

Roudneff [9] has shown that the number of complete cells of $A(d, n)$ is at least $\sum_{i=0}^{d-2} \binom{n-1}{i}$ and that this is tight for all $n \geq 2d + 1$ (see also [2] where an asymptotically tight upper bound on the number of complete cells in arrangements is given). Theorem 1.1 gives the exact number of complete cells of $A(d, n)$ for any $n$.

Finally, we mention the following result due to Grünbaum [5] (see also [10]).

THEOREM 4.2 ([5, P. 29]). *Let $L(2, n)$, $n \geq 5$ be an arrangement of $n$ lines. Then $f_4[L(2, n)] \leq n(n - 3)/2$. Moreover, for each $n \geq 5$, there exists (up to isomorphism) a unique simple arrangement $L'$ of $n$ lines satisfying $f_4[L'(2, n)] = n(n - 3)/2$.*

Although no proof was given, Grünbaum certainly had in mind $A(2, n)$, $n \geq 5$, for the second part of the above theorem. This can easily be verified since, by Theorem 1.1, $f_4[A(2, n)] = n(n - 3)/2$.

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