Computers and Mathematics with Applications 57 (2009) 248–265

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

# Dynamic behaviors of the impulsive periodic multi-species predator–prey system

Mengxin He, Fengde Chen *

*College of Mathematics and Computer Science, Fuzhou University, Fuzhou, Fujian 350002, PR China*

ARTICLE INFO

*Article history:*
Received 18 July 2007
Accepted 10 September 2008

*Keywords:*
Predator–prey
Impulsive
Extinction
Permanence
Global attractivity

ABSTRACT

The dynamic behaviors of an impulsive periodic predator–prey model with $n$-preys and $m$-predators are studied in this paper. By constructing a suitable Lyapunov function and using the Comparison theorem of impulsive differential equation, sufficient conditions which ensure the permanence and global attractivity of the system are obtained. At the same time, a set of criteria which guarantee that some species in the system are permanent and globally attractive while the remaining species are driven to extinction is obtained. Our results show that, for the multi-species predator–prey community, impulsivity is one of the important reasons that can change the long time behaviors of species.


## 1. Introduction

A traditional periodic predator–prey system with periodic coefficients has been studied extensively (see [1–5]). In [6], Yang and Xu considered the following periodic Lotka–Volterra system of differential equations

$$\begin{cases} x_i'(t) = x_i(t)\left[b_i(t) - \sum_{k=1}^{n} a_{ik}(t)x_k(t) - \sum_{k=1}^{m} c_{ik}(t)y_k(t)\right], & i = 1, 2, \dots, n, \\ y_j'(t) = y_j(t)\left[-r_j(t) + \sum_{k=1}^{n} d_{jk}(t)x_k(t) - \sum_{k=1}^{m} e_{jk}(t)y_k(t)\right], & j = 1, 2, \dots, m, \end{cases} \tag{1.1}$$

where $x_i(t)$ denotes the density of the prey species $x_i$ at time $t$, $y_j(t)$ denotes the density of the predator species $y_j$ at time $t$; $b_i(t), r_j(t), a_{ik}(t), c_{il}(t), d_{jk}(t)$ and $e_{jl}(t) (i, k = 1, \dots, n; j, l = 1, \dots, m)$ are continuous periodic functions defined on $[0, +\infty)$ with a common period $T > 0$; $r_j(t), a_{ik}(t), c_{il}(t), d_{jk}(t)$ and $e_{jl}(t)$ are nonnegative; $a_{ii}(t), e_{jj}(t)$ are strictly positive. Under the assumption that $b_i(t)$ are positive periodic functions, they studied the existence, uniqueness and stability of the periodic solution of system (1.1). Recently, under the assumption that $b_i(t)$ may be negative but $\int_0^T b_i(t)\mathrm{d}t > 0$, that is, the intrinsic growth rate of the prey species may be negative while the total intrinsic growth rate in a period is positive, Zhao and Chen [7] gave sufficient conditions for the existence and global asymptotic stability of system (1.1). Xia et al. [8] further generalized the above system to the almost periodic case; by constructing a suitable Lyapunov function, sufficient conditions for the existence and global asymptotic stability of an almost periodic solution to system (1.1) is obtained. Recently, Zhao and Jiang [9] and Zhao, Jiang and Lazer [10] further considered the general nonautonomous case of system (1.1). Average conditions are obtained for the permanence and global attractivity of the system. The results of Zhao and Jiang [9] generalized the main results of Yang and Xu [6] and Zhao and Chen [7]. For more works in this direction, one could refer to [11–19] and the references cited therein.

* Corresponding author.
*E-mail addresses:* hemx_206@126.com (M. He), fdchen@fzu.edu.cn, fdchen@263.net (F. Chen).

0898-1221/$ – see front matter 
doi:10.1016/j.camwa.2008.09.041

However, the ecological system is often deeply perturbed by activities of human exploitation such as planting and harvesting etc., which are not suitable to be considered continually. To accurately describe the system, one needs to use the impulsive differential equations. The theory of impulsive differential equations is now being recognized as having much richer content than the corresponding theory of differential equations without impulses, also it represents a more natural framework for mathematical modeling of many real world phenomena [20]. In recent years, impulsive differential equations have been intensively researched, see [20,21] and the references cited therein. Some impulsive equations have been recently introduced in population dynamics in relation to: population ecology [22–31] and chemotherapeutic treatment of disease [19,32]. Tang and Chen [29] investigated the persistence and existence of at least one strictly positive periodic solution of periodic Lotka–Volterra predator–prey system with impulses. Ahmad and Stamova [22] studied the $N$-dimensional Lotka–Volterra system with fixed moments of impulsive perturbations. However, to the best of the author's knowledge, to this day, still no scholar has investigated the dynamic behaviors of the periodic multi-species predator–prey system with impulses.

The main purpose of this paper is to study the following $T$-periodic multi-species predator–prey system with impulses:

$$\begin{cases} x_i'(t) = x_i(t)\left[b_i(t) - \sum_{l=1}^{n} a_{il}(t)x_l(t) - \sum_{l=1}^{m} c_{il}(t)y_l(t)\right], \\ y_j'(t) = y_j(t)\left[-r_j(t) + \sum_{l=1}^{n} d_{jl}(t)x_l(t) - \sum_{l=1}^{m} e_{jl}(t)y_l(t)\right], \quad t \neq \tau_k, \\ x_i(\tau_k^+) = (1+h_{ik})x_i(\tau_k), \\ y_j(\tau_k^+) = (1+g_{jk})y_j(\tau_k), \quad t = \tau_k,\ k = 1, 2, \ldots, \end{cases} \tag{1.2}$$

where $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$; $x_i(t)$ denotes the density of the prey species $X_i$ at time $t$; $y_j(t)$ denotes the density of the predator species $Y_j$ at time $t$; $b_i(t), r_j(t), a_{ik}(t), c_{il}(t), d_{jk}(t)$, and $e_{jl}(t)$ $(i, k = 1, \ldots, n; j, l = 1, \ldots, m)$ are continuous periodic functions defined on $[0, +\infty)$ with a common period $T > 0$; $r_j(t), a_{ik}(t), c_{il}(t), d_{jk}(t)$ and $e_{jl}(t)$ are nonnegative; $a_{ii}(t), e_{jj}(t)$ are strictly positive; $\tau_k \to +\infty (t \to +\infty)$ and $0 = \tau_0 < \tau_1 < \tau_2 < \tau_3 < \cdots < \tau_k < \tau_{k+1} < \cdots$. Assume that $h_{ik}, g_{jk},\ i = 1, 2, \ldots, n, j = 1, 2, \ldots, m, k = 1, 2, \ldots,$ are constants and there exists an integer $q > 0$ such that

$$h_{i(k+q)} = h_{ik}, \qquad g_{j(k+q)} = g_{jk}, \qquad \tau_{k+q} = \tau_k + T.$$

In this paper, the growth rate $b_i(t)$ of species is not necessarily positive, and this is realistic since the environment fluctuates randomly (e.g. seasonal effect of weather condition, temperature, mating habits and food supplies) and in some bad conditions $b_i(t)$ may be negative. So we assume that:

$$m[b_i] = \frac{1}{T}\int_0^T b_i(t)\mathrm{d}t > 0, \quad i = 1, 2, \ldots, n,$$

and natural biological meaning:

$$1 + h_{ik} > 0, \qquad 1 + g_{jk} > 0 \quad (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m;\ k = 1, 2, \ldots).$$

**Definition 1.1.** System (1.2) is said to be permanent, if for any positive solution $F(t) = (X(t), Y(t))^T = (x_1(t), x_2(t), \ldots, x_n(t), y_1(t), y_2(t), \ldots, y_m(t))^T$ of (1.2), there exist positive constants $\lambda_i^1,\ \lambda_j^2,\ \theta_i^1$ and $\theta_j^2$ such that

$$\lambda_i^1 \le \liminf_{t\to+\infty} x_i(t) \le \limsup_{t\to+\infty} x_i(t) \le \theta_i^1, \quad i = 1, 2, \ldots, n;$$

$$\lambda_j^2 \le \liminf_{t\to+\infty} y_j(t) \le \limsup_{t\to+\infty} y_j(t) \le \theta_j^2, \quad j = 1, 2, \ldots, m.$$

**Definition 1.2.** System (1.2) is said to be globally attractive if any two positive solutions $F(t) = (X(t), Y(t))^T = (x_1(t), x_2(t), \ldots, x_n(t), y_1(t), y_2(t), \ldots, y_m(t))^T$ and $W(t) = (U(t), V(t))^T = (u_1(t), u_2(t), \ldots, u_n(t), v_1(t), v_2(t), \ldots, v_m(t))^T$ of system (1.2) satisfy

$$\lim_{t\to+\infty} |x_i(t) - u_i(t)| = 0, \quad i = 1, 2, \ldots, n;$$

$$\lim_{t\to+\infty} |y_j(t) - v_j(t)| = 0, \quad j = 1, 2, \ldots, m.$$

**Definition 1.3.** For any $(t, F(t)) \in [\tau_{k-1}, \tau_k) \times \Re_+^{n+m}$, the right-hand derivative $\mathrm{D}^+V(t, F(t))$ along the solution $F(t, F_0)$ of system (1.2) is defined by

$$\mathrm{D}^+V(t, F(t)) = \liminf_{h\to 0^+} \frac{1}{h}[V(t+h, F(t+h)) - V(t, F(t))].$$

By the basic theories of impulsive differential equations in [20,21], system (1.2) has a unique solution $F(t) = F(t, F_0) \in PC([0, +\infty), R^{n+m})$ and $PC([0, +\infty), R^{n+m}) = \{\phi : [0, +\infty) \to R^{n+m}, \phi$ is continuous for $t \neq \tau_k$, $\phi(\tau_k^-)$ and $\phi(\tau_k^+)$ exist and $\phi(\tau_k^-) = \phi(\tau_k),\ k = 1, 2, \ldots\}$ for each initial value $F(0) = F_0 \in R^{n+m}$.

The organization of this paper is as follows: In Section 2, necessary preliminaries are presented. The dynamic behaviors such as the permanence, extinction and the global attractivity of the system are investigated in Section 3; Three examples together with their numerical simulations are presented in Section 4 to illustrate the feasibility of main results and the effect of impulses on the dynamic behaviors of the system.

## 2. Preliminaries

Now let us state several lemmas which will be useful in proving the main results.

Firstly, we introduce an important Comparison theorem on impulsive differential equation [21].

**Lemma 2.1.** *Assume that $m \in PC[R_+, R]$ with points of discontinuity at $t = t_k$ and is left continuous at $t = t_k, k = 1, 2, \ldots$, and*

$$\begin{cases} D\_m(t) \le g(t, m(t)), & t \neq t_k,\ k = 1, 2, \ldots, \\ m(t_k^+) \le \phi_k(m(t_k)), & t = t_k,\ k = 1, 2, \ldots, \end{cases} \tag{2.1}$$

*where $g \in C[R_+ \times R_+, R],\ \phi_k \in C[R, R]$ and $\phi_k(u)$ is nondecreasing in $u$ for each $k = 1, 2, \ldots$. Let $r(t)$ be the maximal solution of the scalar impulsive differential equation*

$$\begin{cases} \dot{u} = g(t, u), & t \neq t_k,\ k = 1, 2, \ldots, \\ u(t_k^+) = \phi_k(u(t_k)) \ge 0, & t = t_k,\ t_k > t_0,\ k = 1, 2, \ldots, \\ u(t_0^+) = u_0, \end{cases} \tag{2.2}$$

*existing on $[t_0, \infty)$, then $m(t_0^+) \le u_0$ implies $m(t) \le r(t), t \ge t_0$.*

**Remark 2.1.** In Lemma 2.1, assume that inequality (2.1) reversed. Let $p(t)$ be the minimal solution of (2.2) existing on $(t_0, +\infty)$. Then, $m(t_0^+) \ge u_0$ implies $m(t) \ge p(t),\ t \ge t_0$.

**Lemma 2.2.** *Let $F(t) = (X(t), Y(t))^T = (x_1(t), x_2(t), \ldots, x_n(t), y_1(t), y_2(t), \ldots, y_m(t))^T$ be any solution of system (1.2) such that $x_i(0^+) > 0$ and $y_j(0^+) > 0$ $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$, then $x_i(t) > 0$ and $y_j(t) > 0$ for all $t \ge 0$.*

**Proof.** From the $i$th equation of (1.2), one has

$$x_i'(t) = P_i(t)x_i(t), \quad t \neq \tau_k,\ i = 1, 2, \ldots, n,$$

where

$$P_i(t) = b_i(t) - \sum_{l=1}^{n} a_{il}(t)x_l(t) - \sum_{l=1}^{m} c_{il}(t)y_l(t), \quad i = 1, 2, \ldots, n.$$

Thus we have

$$x_i(t) = \prod_{0<\tau_k<t} (1 + h_{ik})x_i(0^+) \exp\left(\int_0^t P_i(s)\mathrm{d}s\right) > 0,$$

because of $x_i(0^+) > 0$.

From the $(n + j)$th equation of (1.2), one has

$$y_j'(t) = Q_j(t)y_j(t), \quad t \neq \tau_k,\ j = 1, 2, \ldots, m,$$

where

$$Q_j(t) = -r_j(t) + \sum_{l=1}^{n} d_{jl}(t)x_l(t) - \sum_{l=1}^{m} e_{jl}(t)y_l(t), \quad j = 1, 2, \ldots, m.$$

Thus we have

$$y_j(t) = \prod_{0<\tau_k<t} (1 + g_{jk})y_j(0^+) \exp\left(\int_0^t Q_i(s)\mathrm{d}s\right) > 0,$$

because of $y_j(0^+) > 0$. This completes the proof of Lemma 2.2. □

Consider the periodic logistic equation with impulses

$$\begin{cases} x'(t) = x(t)\left(b(t) - a(t)(x(t))\right), & t \neq \tau_k, \\ x(\tau_k^+) = (1 + h_k)x(\tau_k), & t = \tau_k,\ k = 1, 2, \ldots, \end{cases} \tag{2.3}$$

where $a(t),\ b(t)$ are continuous $T$-periodic functions with $a(t) > 0,\ m[b] > 0$ and $h_{k+q} = h_k,\ \tau_{k+q} = \tau_k + T$.

**Lemma 2.3** (*[27]*). (1)*: If*

$$\sum_{k=1}^{q} \ln(1 + h_k) + Tm[b] > 0, \tag{2.4}$$

*then system (2.3) has a unique $T$-periodic solution $x^*(t)$, and $x^*(t)$ is globally asymptotically stable in the sense that*

$$\lim_{t\to+\infty} |x(t) - x^*(t)| = 0,$$

*where $x(t)$ is any solution of system (2.3) with initial value $x(0^+) > 0$.*

(2)*: If*

$$\sum_{k=1}^{q} \ln(1 + h_k) + Tm[b] < 0, \tag{2.5}$$

*then*

$$\lim_{t\to\infty} x(t) = 0,$$

*where $x(t)$ is any solution of system (2.3) with initial value $x(0^+) > 0$.*

From Theorem 3.1 in [29], we can get the following lemma.

**Lemma 2.4.** *If $x^*(t)$ is the unique $T$-periodic positive solution of system (2.3), and $\tilde{x^*}(t)$ is the unique $T$-periodic positive solution of the following system*

$$\begin{cases} x'(t) = x(t)\left(\tilde{b}(t) - a(t)(x(t))\right), & t \neq \tau_k, \\ x(\tau_k^+) = (1 + h_k)x(\tau_k), & t = \tau_k,\ k = 1, 2, \ldots, \end{cases}$$

*where $\tilde{b}$ is the continuous $T$-periodic function with $m[\tilde{b}] > 0$ and $\lim_{t\to+\infty} \widetilde{b}(t) = b(t)$. Then*

$$\lim_{t\to+\infty} |\tilde{x^*}(t) - x^*(t)| = 0.$$

(I) Consider the periodic logistic equation with impulses

$$\begin{cases} x_i'(t) = x_i(t)\left(b_i(t) - a_{ii}(t)x_i(t)\right), & t \neq \tau_k, \\ x_i(\tau_k^+) = (1 + h_{ik})x_i(\tau_k), & t = \tau_k,\ k = 1, 2, \ldots, \end{cases} \tag{2.6}$$

where $i = 1, 2, \ldots, n$.

If $\sum_{k=1}^{q} \ln(1 + h_{ik}) + Tm[b_i] > 0$ holds, it follows from Lemma 2.3 that system (2.6) has a unique $T$-periodic solution $X_i^*(t)$, which is globally asymptotically stable in the sense that

$$\lim_{t\to+\infty} |x_i(t) - X_i^*(t)| = 0,$$

where $x_i(t)$ is any solution of system (2.6) with initial value $x_i(0^+) > 0$.

(II) Consider the periodic logistic equation with impulses

$$\begin{cases} y_j'(t) = y_j(t)\left(-r_j(t) + \sum_{l=1}^{n} d_{jl}(t)X_l^*(t) - e_{jj}(t)y_j(t)\right), & t \neq \tau_k, \\ y_j(\tau_k^+) = (1 + g_{jk})y_j(\tau_k), & t = \tau_k,\ k = 1, 2, \ldots, \end{cases} \tag{2.7}$$

where $j = 1, 2, \ldots, m$.

If

$$\sum_{k=1}^{q} \ln(1 + g_{jk}) + Tm\left[-r_j(t) + \sum_{l=1}^{n} d_{jl}(t)X_l^*(t)\right] > 0$$

holds, it follows from Lemma 2.3 that system (2.7) has a unique $T$-periodic solution $Y_j^*(t)$, which is globally asymptotically stable in the sense that

$$\lim_{t\to+\infty} |y_j(t) - Y_j^*(t)| = 0,$$

where $y_j(t)$ is any solution of system (2.7) with initial value $y_j(0^+) > 0$.

(III) Consider the periodic logistic equation with impulses

$$\begin{cases} x_i'(t) = x_i(t)\left(b_i(t) - \sum_{l=1}^{m} c_{il}(t)Y_l^*(t) - \sum_{l=1,l\neq i}^{n} a_{il}(t)X_l^*(t) - a_{ii}(t)x_i(t)\right), & t \neq \tau_k, \\ x_i(\tau_k^+) = (1+h_{ik})x_i(\tau_k), & t = \tau_k,\ k = 1, 2, \dots, \end{cases} \tag{2.8}$$

where $i = 1, 2, \dots, n$.

If

$$\sum_{k=1}^{q} \ln(1+h_{ik}) + Tm\left[b_i(t) - \sum_{l=1}^{m} c_{il}(t)Y_l^*(t) - \sum_{l=1,l\neq i}^{n} a_{il}(t)X_l^*(t)\right] > 0$$

holds, it follows from Lemma 2.3 that system (2.8) has a unique $T$-periodic solution $X_{*i}(t)$, which is globally asymptotically stable in the sense that

$$\lim_{t\to+\infty} |x_i(t) - X_{*i}(t)| = 0,$$

where $x_i(t)$ is any solution of system (2.8) with initial value $x_i(0^+) > 0$.

## 3. Main results

In this section, we present out our main results for system (1.2). Let

$$\alpha_i = \sum_{k=1}^{q} \ln(1+h_{ik}) + Tm[b_i], \quad i = 1, 2, \dots, n;$$

$$\beta_j(n) = \sum_{k=1}^{q} \ln(1+g_{jk}) + Tm\left[-r_j(t) + \sum_{l=1}^{n} d_{jl}(t)X_l^*(t)\right], \quad j = 1, 2, \dots, m.$$

For integers $n_1$ and $m_1$ such that $1 \le n_1 \le n$ and $1 \le m_1 \le m$, assume that

$$\alpha_i > 0, \qquad \beta_j(n) > 0, \quad i = 1, 2, \dots, n;\ j = 1, 2, \dots, m, \tag{H$_1$}$$

and

$$\begin{cases} \alpha_i > 0, & \alpha_p < 0, \quad i = 1, 2, \dots, n_1;\ p = n_1 + 1, \dots, n; \\ \beta_j(n_1) > 0, & \beta_q(n_1) < 0, \quad j = 1, 2, \dots, m_1;\ q = m_1 + 1, \dots, m. \end{cases} \tag{H$_2$}$$

Under condition (H$_1$), we study the permanence and global attractivity of system (1.2).

**Theorem 3.1.** *Assume that* (H$_1$) *holds and* $F(t) = (X(t), Y(t))^T = (x_1(t), x_2(t), \dots, x_n(t), y_1(t), y_2(t), \dots, y_m(t))^T$ *is any solution of system* (1.2) *with* $x_i(0^+) > 0$ *and* $y_j(0^+) > 0\ (i = 1, 2, \dots, n; j = 1, 2, \dots, m)$, *then there exist constants* $\theta_i^1 > 0,\ \theta_j^2 > 0,\ (i = 1, 2, \dots, n; j = 1, 2, \dots, m)$ *such that*

$$\limsup_{t\to+\infty} x_i(t) \le \theta_i^1, \quad i = 1, 2, \dots, n;$$

$$\limsup_{t\to+\infty} y_j(t) \le \theta_j^2, \quad j = 1, 2, \dots, m.$$

**Proof.** From system (1.2), we obtain

$$\begin{cases} x_i'(t) \le x_i(t)\left(b_i(t) - a_{ii}(t)x_i(t)\right), & t \neq \tau_k,\ i = 1, 2, \dots, n, \\ x_i(\tau_k^+) = (1+h_{ik})x_i(\tau_k), & t = \tau_k,\ k = 1, 2, \dots. \end{cases} \tag{3.1}$$

Consider the following system

$$\begin{cases} w_i'(t) = w_i(t)\left(b_i(t) - a_{ii}(t)w_i(t)\right), & t \neq \tau_k,\ i = 1, 2, \dots, n, \\ w_i(\tau_k^+) = (1+h_{ik})w_i(\tau_k), & t = \tau_k,\ k = 1, 2, \dots. \end{cases} \tag{3.2}$$

By Lemma 2.1, we have $x_i(t) \le w_i(t)$, where $w_i(t)$ is the solution of (3.2) with $w_i(0^+) = x_i(0^+)$, $i = 1, 2 \dots, n$. By $\alpha_i > 0$, from Lemma 2.3, system (3.2) admits a unique $T$-periodic solution $X_i^*(t)$, which is globally asymptotically stable.

For any positive constant $\varepsilon > 0$, there exists a $T_{i1} > 0$ $(i = 1, 2, \dots, n)$ such that for $t > T_{i1}$

$$|w_i(t) - X_i^*(t)| < \varepsilon.$$

Let $\theta_i^1 = \sup \left\{ X_i^*(t) \middle| t \in [0, T] \right\}$, $i = 1, 2, \dots, n$, then it follows that

$$x_i(t) \le w_i(t) < X_i^*(t) + \varepsilon \le \theta_i^1 + \varepsilon, \quad \text{for } t > T_{i1},\ i = 1, 2, \dots, n.$$

Setting $\varepsilon \to 0$, we obtain

$$\limsup_{t\to+\infty} x_i(t) \le \theta_i^1, \quad i = 1, 2, \dots, n.$$

By condition $(H_1)$, one has

$$\sum_{k=1}^{q} \ln(1 + g_{jk}) + Tm \left[ -r_j(t) + \sum_{l=1}^{n} d_{jl}(t)(X_l^*(t) + \varepsilon) \right] > 0. \tag{3.3}$$

Let $T_1 = \max\{T_{i1} | i = 1, 2, \dots, n\}$. From system (1.2), for $t > T_1$ and $t \ne \tau_k$ $(k = 1, 2, \dots)$, we obtain

$$y_j'(t) \le y_j(t) \left( -r_j(t) + \sum_{l=1}^{n} d_{jl}(t)(X_l^*(t) + \varepsilon) - e_{jj}(t) y_j(t) \right), \quad j = 1, 2, \dots, m.$$

Consider the following system

$$\begin{cases} w_j'(t) = w_j(t) \left( -r_j(t) + \sum_{l=1}^{n} d_{jl}(t)(X_l^*(t) + \varepsilon) - e_{jj}(t) w_j(t) \right), & t \ne \tau_k, \\ w_j(\tau_k^+) = (1 + g_{jk}) w_j(\tau_k), \qquad t = \tau_k, & k = 1, 2, \dots,\ j = 1, 2, \dots, m. \end{cases} \tag{3.4}$$

By Lemma 2.1, for all $t > T_1$, we have $y_j(t) \le w_j(t)$, where $w_j(t)$ is the solution of (3.4) with $w_j(T_1^+) = y_j(T_1^+)$, $j = 1, 2 \dots, m$. By (3.3) and Lemma 2.3, system (3.4) admits a unique $T$-periodic solution $Y_{j\varepsilon}^*(t)$, which is globally asymptotically stable. Setting $\varepsilon \to 0$, from Lemma 2.4, we have

$$Y_{j\varepsilon}^*(t) \to Y_j^*(t).$$

Let $\theta_j^2 = \sup \left\{ Y_j^*(t) \middle| t \in [0, T] \right\}$, $j = 1, 2, \dots, m$, and for any positive constant $\varepsilon_0 > 0$, there exists a $T_{j2} > T_1$ $(j = 1, 2, \dots, m)$ such that

$$y_j(t) \le w_j(t) < Y_j^*(t) + \varepsilon_0 \le \theta_j^2 + \varepsilon_0, \quad \text{for } t > T_{j2},\ j = 1, 2, \dots, m.$$

Setting $\varepsilon_0 \to 0$, we obtain

$$\limsup_{t\to+\infty} y_j(t) \le \theta_j^2, \quad j = 1, 2, \dots, m.$$

The proof is completed. □

**Theorem 3.2.** *Assume that*

$$\sum_{k=1}^{q} \ln(1 + h_{ik}) + Tm \left[ b_i(t) - \sum_{l=1}^{m} c_{il}(t) Y_l^*(t) - \sum_{l=1, l\ne i}^{n} a_{il}(t) X_l^*(t) \right] > 0, \quad i = 1, 2, \dots, n; \tag{A$_1$}$$

$$\sum_{k=1}^{q} \ln(1 + g_{jk}) + Tm \left[ -r_j(t) + \sum_{l=1}^{n} d_{jl}(t) X_{*l}(t) - \sum_{l=1, l\ne j}^{m} e_{jl}(t) Y_l^*(t) \right] > 0, \quad j = 1, 2, \dots, m, \tag{A$_2$}$$

*where $X_i^*(t)$, $Y_j^*(t)$ and $X_{*i}(t)$ are the unique $T$-positive solutions of systems (2.6), (2.7) and (2.8), respectively, then the species $x_i$, $y_j$ $(i = 1, 2, \dots, n,\ j = 1, 2, \dots, m)$ are permanent, that is, for any positive solution $F(t) = (X(t), Y(t))^T = (x_1(t), x_2(t), \dots, x_n(t), y_1(t), y_2(t), \dots, y_m(t))^T$ of (1.2), there exist positive constants $\lambda_i^1$, $\lambda_j^2$, $\theta_i^1$ and $\theta_j^2$ such that*

$$\lambda_i^1 \le \liminf_{t\to+\infty} x_i(t) \le \limsup_{t\to+\infty} x_i(t) \le \theta_i^1, \quad i = 1, 2, \dots, n;$$

$$\lambda_j^2 \le \liminf_{t\to+\infty} y_j(t) \le \limsup_{t\to+\infty} y_j(t) \le \theta_j^2, \quad j = 1, 2, \dots, m.$$

**Proof.** It is not difficult to show that $(A_1)$ and $(A_2)$ imply $(H_1)$. Hence, it follows from Theorem 3.1 that there exist constants $\theta_i^1 > 0,\ \theta_j^2 > 0\ (i = 1, 2, \dots, n; j = 1, 2, \dots, m)$ such that

$$\limsup_{t\to+\infty} x_i(t) \le \theta_i^1, \quad i = 1, 2, \dots, n;$$
$$\limsup_{t\to+\infty} y_j(t) \le \theta_j^2, \quad j = 1, 2, \dots, m.$$

By the boundedness of $a_{ik}(t)$ and $c_{il}(t)$ $(i, k = 1, \dots, n; l = 1, \dots, m)$ and the conditions $(A_1)$ and $(A_2)$, for $\varepsilon > 0$ small enough, there exists a $t_1 > \max\{T_{j2} | j = 1, 2, \dots, m\}$, such that for $t > t_1$

$$x_i(t) < X_i^*(t) + \varepsilon, \qquad y_j(t) < Y_j^*(t) + \varepsilon; \tag{3.5}$$

$$\sum_{k=1}^{q} \ln(1 + h_{ik}) + Tm\left[b_i(t) - \sum_{l=1}^{m} c_{il}(t)(Y_l^*(t) + \varepsilon) - \sum_{l=1,l\ne i}^{n} a_{il}(t)(X_l^*(t) + \varepsilon)\right] > 0; \tag{3.6}$$

$$\sum_{k=1}^{q} \ln(1 + g_{jk}) + Tm\left[-r_j(t) + \sum_{l=1}^{n} d_{jl}(t)X_{*l}(t) - \sum_{l=1,l\ne j}^{m} e_{jl}(t)(Y_l^*(t) + \varepsilon)\right] > 0, \tag{3.7}$$

where $i = 1, 2, \dots, n;\ j = 1, 2, \dots, m$ and $T_{j2}$ $(j = 1, 2, \dots, m)$ are defined in Theorem 3.1.

From system (1.2) and inequality (3.5), for $t > t_1$ and $t \ne \tau_k$, we obtain

$$x_i'(t) \ge x_i(t)\left(b_i(t) - \sum_{l=1}^{m} c_{il}(t)(Y_l^*(t) + \varepsilon) - \sum_{l=1,l\ne i}^{n} a_{il}(t)(X_l^*(t) + \varepsilon) - a_{ii}(t)x_i(t)\right).$$

Consider the following system

$$\begin{cases} w_i'(t) = w_i(t)\left(b_i(t) - \sum_{l=1}^{m} c_{il}(t)(Y_l^*(t) + \varepsilon) - \sum_{l=1,l\ne i}^{n} a_{il}(t)(X_l^*(t) + \varepsilon)\right. \\ \qquad \left. -a_{ii}(t)w_i(t)\right), \quad t \ne \tau_k,\ i = 1, 2, \dots, n, \\ w_i(\tau_k^+) = (1 + h_{ik})w_i(\tau_k), \quad t = \tau_k,\ k = 1, 2, \dots. \end{cases} \tag{3.8}$$

By Lemma 2.1, for all $t > t_1$, we have $x_i(t) \ge w_i(t)$, where $w_i(t)$ is the solution of (3.8) with $w_i(t_1^+) = x_i(t_1^+)$, $i = 1, 2 \dots, n$. From Lemma 2.3 and inequality (3.6), system (3.8) admits a unique $T$-periodic solution $X_{*i\varepsilon}(t)$, which is globally asymptotically stable. Setting $\varepsilon \to 0$, from Lemma 2.4, we have

$$X_{*i\varepsilon}(t) \to X_{*i}(t).$$

Let $\lambda_i^1 = \inf\{X_{*i}(t) |\, t \in [0, T]\}$, $i = 1, 2, \dots, n$ and for a positive constant $\varepsilon_1 < \frac{1}{2}\min\{\lambda_i^1 | i = 1, 2, \dots, n\}$ small enough, by the boundedness of $d_{jk}(t)$ and $e_{jl}(t)$ $(k = 1, \dots, n; j, l = 1, \dots, m)$ and inequality (3.7), there exists a $t_{i2} > t_1$ $(i = 1, 2, \dots, n)$ such that

$$x_i(t) \ge w_i(t) > X_{*i}(t) - \varepsilon_1 \ge \lambda_i^1 - \varepsilon_1, \quad \text{for } t > t_{i2},\ i = 1, 2, \dots, n,$$

and

$$\sum_{k=1}^{q} \ln(1 + g_{jk}) + Tm\left[-r_j(t) + \sum_{l=1}^{n} d_{jl}(t)(X_{*l}(t) - \varepsilon_1) - \sum_{l=1,l\ne j}^{m} e_{jl}(t)(Y_l^*(t) + \varepsilon)\right] > 0, \tag{3.9}$$

where $j = 1, 2, \dots, m$.

Setting $\varepsilon_1 \to 0$, we obtain

$$\liminf_{t\to+\infty} x_i(t) \ge \lambda_i^1, \quad i = 1, 2, \dots, n.$$

Let $t_2 = \max\{t_{i2} | i = 1, 2, \dots, n\}$. From system (1.2), for $t > t_2$ and $t \ne \tau_k$, we obtain

$$y_j'(t) \ge y_j(t)\left(-r_j(t) + \sum_{l=1}^{n} d_{jl}(t)(X_{*l}(t) - \varepsilon_1) - \sum_{l=1,l\ne j}^{m} e_{jl}(t)(Y_l^*(t) + \varepsilon) - e_{jj}(t)y_j(t)\right).$$

Consider the following system

$$\begin{cases} w_j'(t) = w_j(t)\left(-r_j(t) + \sum_{l=1}^{n} d_{jl}(t)(X_{*l}(t) - \varepsilon_1) - \sum_{l=1,l\ne j}^{m} e_{jl}(t)(Y_l^*(t) + \varepsilon) - e_{jj}(t)w_j(t)\right), \\ \quad t \ne \tau_k,\ j = 1, 2, \dots, m, \\ w_j(\tau_k^+) = (1 + g_{jk})w_j(\tau_k), \quad t = \tau_k,\ k = 1, 2, \dots. \end{cases} \tag{3.10}$$

By Lemma 2.1, for all $t > t_2$, we have $y_j(t) \geq w_j(t)$, where $w_j(t)$ is the solution of (3.10) with $w_j(t_2^+) = y_j(t_2^+)$, $j = 1, 2 \ldots, m$. Again from Lemma 2.3 and inequality (3.9), then system (3.10) admits a unique $T$-periodic solution $Y_{*j(\varepsilon,\varepsilon_1)}(t)$, which is globally asymptotically stable. Setting $\varepsilon,\ \varepsilon_1 \to 0$, from Lemma 2.4, we have

$$Y_{*j(\varepsilon,\varepsilon_1)}(t) \to Y_{*j}(t).$$

Let $\lambda_j^2 = \inf\left\{Y_{*j}(t) | t \in [0, T]\right\}$, $j = 1, 2, \ldots, m$ and for any positive constant $\varepsilon_2 < \frac{1}{2}\min\{\lambda_j^2 | j = 1, 2, \ldots, m\}$, there exists a $t_{j3} > t_2$ $(j = 1, 2, \ldots, m)$ such that

$$y_j(t) \geq w_j(t) > Y_{*j}(t) - \varepsilon_2 \geq \lambda_j^2 - \varepsilon_2, \quad \text{for } t > t_{j3},\ j = 1, 2, \ldots, m.$$

Setting $\varepsilon_2 \to 0$, we obtain

$$\liminf_{t\to+\infty} y_j(t) \geq \lambda_j^2, \quad j = 1, 2, \ldots, m.$$

The proof is completed. □

Now let us consider the global attractivity of $x_i(t),\ y_j(t)$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$ of system (1.2), and we obtain the following result.

**Theorem 3.3.** *Assume that* $(A_1)$ *and* $(A_2)$ *hold. Assume further that there exist positive constants* $\rho_i$ $(i = 1, 2, \ldots, n)$ *and* $\mu_j$ $(j = 1, 2, \ldots, m)$ *such that*

$$\rho_i a_{ii}(t) > \sum_{l=1, l\neq i}^{n} \rho_l a_{li}(t) + \sum_{l=1}^{m} \mu_l d_{li}(t); \qquad (A_3)$$

$$\mu_j e_{jj}(t) > \sum_{l=1, l\neq j}^{m} \mu_l e_{lj}(t) + \sum_{l=1}^{n} \rho_l c_{lj}(t), \qquad (A_4)$$

*then the species* $x_i, y_j$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$ *are globally attractive, that is, for any positive solutions* $F(t) = (X(t), Y(t))^T = (x_1(t), x_2(t), \ldots, x_n(t), y_1(t), y_2(t), \ldots, y_m(t))^T$ *and* $W(t) = (U(t), V(t))^T = (u_1(t), u_2(t), \ldots, u_n(t), v_1(t), v_2(t), \ldots, v_m(t))^T$ *of* (1.2), *one has*

$$\lim_{t\to+\infty} |x_i(t) - u_i(t)| = 0, \quad i = 1, 2, \ldots, n;$$
$$\lim_{t\to+\infty} |y_j(t) - v_j(t)| = 0, \quad j = 1, 2, \ldots, m.$$

**Proof.** Let $F(t) = (X(t), Y(t))^T = (x_1(t), x_2(t), \ldots, x_n(t), y_1(t), y_2(t), \ldots, y_m(t))^T$ and $W(t) = (U(t), V(t))^T = (u_1(t), u_2(t), \ldots, u_n(t), v_1(t), v_2(t), \ldots, v_m(t))^T$ be any positive solutions of system (1.2). For any positive $\varepsilon < \frac{1}{2}\min\{\lambda_i^1, \lambda_j^2 \,|\, i = 1, \ldots, n;\ j = 1, \ldots, m\}$, from Theorem 3.2, it immediately follows that there exists a large enough $T_0 > 0$ such that for all $t > T_0$, one has

$$\begin{aligned} &\lambda_i^1 - \varepsilon \leq x_i(t), \qquad u_i(t) \leq \theta_i^1 + \varepsilon, \quad i = 1, 2, \ldots, n;\\ &\lambda_j^2 - \varepsilon \leq y_j(t), \qquad v_j(t) \leq \theta_j^2 + \varepsilon, \quad j = 1, 2, \ldots, m. \end{aligned} \qquad (3.11)$$

Set

$$V(t) = V_1(t) + V_2(t),$$

where

$$V_1(t) = \sum_{i=1}^{n} \rho_i |\ln u_i(t) - \ln x_i(t)|;$$

$$V_2(t) = \sum_{j=1}^{m} \mu_j |\ln v_j(t) - \ln y_j(t)|.$$

For $t \geq 0$, and $t \neq \tau_k,\ k = 1, 2, \ldots$, calculating the upper right derivatives of $V_1(t)$ and $V_2(t)$, respectively, we have

$$\begin{aligned}
\mathrm{D}^+V_1(t) &= \sum_{i=1}^{n} \rho_i \left( \frac{u_i'(t)}{u_i(t)} - \frac{x_i'(t)}{x_i(t)} \right) \operatorname{sgn}(u_i(t) - x_i(t)) \\
&= \sum_{i=1}^{n} \rho_i \left[ \sum_{l=1}^{n} a_{il}(t)(x_l(t) - u_l(t)) + \sum_{l=1}^{m} c_{il}(t)(y_l(t) - v_l(t)) \right] \operatorname{sgn}(u_i(t) - x_i(t)) \\
&\leq \sum_{i=1}^{n} \left\{ \left[ -\rho_i a_{ii}(t) + \sum_{l=1, l\neq i}^{n} \rho_l a_{li}(t) \right] |x_i(t) - u_i(t)| + \sum_{j=1}^{m} \rho_i c_{ij}(t) |y_j(t) - v_j(t)| \right\};\\
\mathrm{D}^+V_2(t) &= \sum_{j=1}^{m} \mu_j \left( \frac{v_j'(t)}{v_j(t)} - \frac{y_j'(t)}{y_j(t)} \right) \operatorname{sgn}(v_j(t) - y_j(t)) \\
&= \sum_{j=1}^{m} \mu_j \left[ \sum_{l=1}^{n} d_{jl}(t)(u_l(t) - x_l(t)) - \sum_{l=1}^{m} e_{jl}(t)(v_l(t) - y_l(t)) \right] \operatorname{sgn}(v_j(t) - y_j(t)) \\
&\leq \sum_{j=1}^{m} \left\{ \left[ -\mu_j e_{jj}(t) + \sum_{l=1, l\neq j}^{m} \mu_l e_{lj}(t) \right] \left| v_j(t) - y_j(t) \right| + \sum_{i=1}^{n} \mu_j d_{ji}(t) \left| u_i(t) - x_i(t) \right| \right\}.
\end{aligned}$$

By the Mean Value Theorem and (3.11), for any closed interval contained in $t \in (\tau_k, \tau_{k+1}],\ k = p, p+1, \ldots$ and $\tau_p > T_0$, it follows that

$$\begin{aligned}
&\frac{1}{\theta_i^1 + \varepsilon} |x_i(t) - u_i(t)| \leq |\ln x_i(t) - \ln u_i(t)| \leq \frac{1}{\lambda_i^1 - \varepsilon} |x_i(t) - u_i(t)|, \quad i = 1, 2, \ldots, n; \\
&\frac{1}{\theta_j^2 + \varepsilon} |y_j(t) - v_j(t)| \leq |\ln y_j(t) - \ln v_j(t)| \leq \frac{1}{\lambda_j^2 - \varepsilon} |y_j(t) - v_j(t)|, \quad j = 1, 2, \ldots, m.
\end{aligned} \tag{3.12}$$

It follows from conditions (A$_3$) and (A$_4$) that there exists a positive constant $\delta$ such that

$$\begin{aligned}
&\rho_i a_{ii}(t) > \sum_{l=1, l\neq i}^{n} \rho_l a_{li}(t) + \sum_{l=1}^{m} \mu_l d_{li}(t) + \delta; \\
&\mu_j e_{jj}(t) > \sum_{l=1, l\neq j}^{m} \mu_l e_{lj}(t) + \sum_{l=1}^{n} \rho_l c_{lj}(t) + \delta.
\end{aligned}$$

Therefore, for $t \in (\tau_k, \tau_{k+1}],\ k = p, p+1, \ldots$ and $\tau_p > T_0$, from (3.12), one has

$$\begin{aligned}
\mathrm{D}^+V(t) &\leq \sum_{i=1}^{n} \left[ -\rho_i a_{ii}(t) + \sum_{l=1, l\neq i}^{n} \rho_l a_{li}(t) + \sum_{l=1}^{m} \mu_l d_{li}(t) \right] + \sum_{j=1}^{m} \left[ -\mu_j e_{jj}(t) + \sum_{l=1, l\neq j}^{m} \mu_l e_{lj}(t) + \sum_{l=1}^{n} \rho_l c_{lj}(t) \right] \\
&\leq -\delta \left[ \sum_{i=1}^{n} |x_i(t) - u_i(t)| + \sum_{j=1}^{m} |v_j(t) - y_j(t)| \right] \\
&\leq -\delta \left[ \sum_{i=1}^{n} \frac{\lambda_i^1 - \varepsilon}{\rho_i} \rho_i |\ln x_i(t) - \ln u_i(t)| + \sum_{j=1}^{m} \frac{\lambda_j^2 - \varepsilon}{\mu_j} \mu_j |\ln v_j(t) - \ln y_j(t)| \right] \\
&\leq -\phi_\varepsilon V(t),
\end{aligned} \tag{3.13}$$

where

$$\phi_\varepsilon = \delta \min \left\{ \frac{\lambda_j^2 - \varepsilon}{\mu_j}, \frac{\lambda_i^1 - \varepsilon}{\rho_i} \,\middle|\, i = 1, \ldots, n;\ j = 1, \ldots, m \right\}.$$

For $t = \tau_k,\ k = 1, 2, \ldots$, we have

$$\begin{aligned}
V(\tau_k^+) &= \sum_{i=1}^{n} \rho_i |\ln u_i(\tau_k^+) - \ln x_i(\tau_k^+)| + \sum_{i=1}^{m} \mu_j |\ln v_j(\tau_k^+) - \ln y_j(\tau_k^+)| \\
&= \sum_{i=1}^{n} \rho_i |\ln\left[(1 + h_{ik}) u_i(\tau_k)\right] - \ln[(1 + h_{ik}) x_i(\tau_k)]|
\end{aligned}$$

$$+\sum_{j=1}^{m}\mu_j|\ln[(1+g_{jk})v_j(\tau_k)]-\ln[(1+g_{jk})y_j(\tau_k)]|=V(\tau_k).$$

The above analysis shows that, for all $t>\tau_p>T_0$,

$$D^+V(t)<-\phi_\varepsilon V(t). \tag{3.14}$$

Applying the differential inequality theorem and the variation of constants formula of solutions of first-order linear differential equation, we have

$$V(t)\le V(\tau_p)\exp(-\phi_\varepsilon(t-\tau_p)). \tag{3.15}$$

It is obvious that $V(t)\to 0$ as $t\to+\infty$, that is

$$\sum_{i=1}^{n}\rho_i|\ln u_i(t)-\ln x_i(t)|+\sum_{j=1}^{m}\mu_j|\ln v_j(t)-\ln y_j(t)|\to 0\quad \text{as } t\to+\infty.$$

From (3.12), it is not hard to prove that

$$\sum_{i=1}^{n}\frac{\rho_i}{\theta_i^1+\varepsilon}|u_i(t)-x_i(t)|+\sum_{j=1}^{m}\frac{\mu_j}{\theta_j^2+\varepsilon}|v_j(t)-y_j(t)|\to 0\quad \text{as } t\to+\infty.$$

It is easy to obtain that

$$\lim_{t\to+\infty}|x_i(t)-u_i(t)|=0,\quad i=1,2,\ldots,n;$$
$$\lim_{t\to+\infty}|y_j(t)-v_j(t)|=0,\quad j=1,2,\ldots,m.$$

The proof is completed. □

**Remark 3.1.** Assume that $h_{ik}\equiv 0\ (i=1,2,\ldots,n)$ and $g_{jk}\equiv 0\ (j=1,2,\ldots,m)$ in system (1.2), then Theorems (3.1)–(3.3) generalize the main results in Zhao and Chen [7].

From Theorem 3.1 to Theorem 3.3, we consider the permanence and stability of system (1.2) with all $\alpha_i>0,\ \beta_j(n)>0\ (i=1,2,\ldots,n;\ j=1,2,\ldots,m)$. But with the impulsive perturbations in system (1.2), the property of system (1.2) will be changed with some $\alpha_i<0$ and $\beta_i(n_1)<0\ (n_1\le n)$, then we have the following results.

(IV) Consider the periodic logistic equation with impulses

$$\begin{cases} y_j'(t)=y_j(t)\left(-r_j(t)+\sum_{l=1}^{n_1}d_{jl}(t)X_l^*(t)-e_{jj}(t)y_j(t)\right), & t\ne\tau_k,\\ y_j(\tau_k^+)=(1+g_{jk})y_j(\tau_k), \quad t=\tau_k,\ k=1,2,\ldots, \end{cases} \tag{3.16}$$

where $j=1,2,\ldots,m_1$.

If

$$\sum_{k=1}^{q}\ln(1+g_{jk})+Tm\left[-r_j(t)+\sum_{l=1}^{n_1}d_{jl}(t)X_l^*(t)\right]>0$$

holds, it follows from Lemma 2.3 that system (3.16) has a unique $T$-periodic solution $Y_j^{1*}(t)$, which is globally asymptotically stable.

(VI) Consider the periodic logistic equation with impulses

$$\begin{cases} x_i'(t)=x_i(t)\left(b_i(t)-\sum_{l=1}^{m_1}c_{il}(t)Y_l^{1*}(t)-\sum_{l=1,l\ne i}^{n_1}a_{il}(t)X_l^*(t)-a_{ii}(t)x_i(t)\right), & t\ne\tau_k,\\ x_i(\tau_k^+)=(1+h_{ik})x_i(\tau_k), \quad t=\tau_k,\ k=1,2,\ldots, \end{cases} \tag{3.17}$$

where $i=1,2,\ldots,n_1$.

If

$$\sum_{k=1}^{q}\ln(1+h_{ik})+Tm\left[b_i(t)-\sum_{l=1}^{m}c_{il}(t)Y_l^{1*}(t)-\sum_{l=1,l\ne i}^{n}a_{il}(t)X_l^*(t)\right]>0$$

holds, it follows from Lemma 2.3 that system (3.17) has a unique $T$-periodic solution $X_{*i}^1(t)$, which is globally asymptotically stable.

**Theorem 3.4.** *Assume* ($H_2$) *holds. Assume further that*

$$\sum_{k=1}^{q}\ln(1+h_{ik})+Tm\left[b_i(t)-\sum_{l=1}^{m_1}c_{il}(t)Y_l^{1*}(t)-\sum_{l=1,l\neq i}^{n_1}a_{il}(t)X_l^*(t)\right]>0,\quad i=1,2,\ldots,n_1; \tag{B$_1$}$$

$$\sum_{k=1}^{q}\ln(1+g_{jk})+Tm\left[-r_j(t)+\sum_{l=1}^{n_1}d_{jl}(t)X_{*l}^1(t)-\sum_{l=1,l\neq j}^{m_1}e_{jl}(t)Y_l^{1*}(t)\right]>0,\quad j=1,2,\ldots,m_1, \tag{B$_2$}$$

*hold, where* $X_i^*(t)$, $Y_j^{1*}(t)$ *and* $X_{*i}^1(t)$ *are the unique T-positive solutions of systems* (2.6), (3.16) *and* (3.17)*, respectively. Let* $F(t)=(X(t),Y(t))=(x_1(t),x_2(t),\ldots,x_n(t),y_1(t),y_2(t),\ldots,y_m(t))^T$ *be any solution of system* (1.2) *with* $x_i(0^+)>0$ *and* $y_j(0^+)>0$ $(i=1,2,\ldots,n;j=1,2,\ldots,m)$*, then there exist positive constants* $\theta_i^1$, $\theta_j^{21}$, $\lambda_i^{11}$, $\lambda_j^{21}$ $(i=1,2,\ldots,n;j=1,2,\ldots,m)$ *such that*

$$\lambda_i^{11}\le\liminf_{t\to+\infty}x_i(t)\le\limsup_{t\to+\infty}x_i(t)\le\theta_i^1,\quad i=1,2,\ldots,n_1;$$
$$\lambda_j^{21}\le\liminf_{t\to+\infty}y_j(t)\le\limsup_{t\to+\infty}y_j(t)\le\theta_j^{21},\quad j=1,2,\ldots,m_1;$$
$$\lim_{t\to+\infty}x_i(t)=0,\quad i=n_1+1,\ldots,n;$$
$$\lim_{t\to+\infty}y_j(t)=0,\quad j=m_1+1,\ldots,m.$$

**Proof.** Let $F(t)=(X(t),Y(t))=(x_1(t),x_2(t),\ldots,x_n(t),y_1(t),y_2(t),\ldots,y_m(t))^T$ be any positive solution of system (1.2). From the $i$th equation of system (1.2), we have

$$x_i'(t)\le x_i(t)\left[b_i(t)-a_{ii}(t)x_i(t)\right],\quad t\neq\tau_k,\ k=1,2,\ldots.$$

Consider the following system

$$\begin{cases}z_i'(t)=z_i(t)\left[b_i(t)-a_{ii}(t)z_i(t)\right], & t\neq\tau_k,\ i=1,2,\ldots,n\\ z_i(\tau_k^+)=(1+h_{ik})z_i(\tau_k), & t=\tau_k,\ k=1,2,\ldots.\end{cases} \tag{3.18}$$

By Lemma 2.1, we obtain that $x_i(t)\le z_i(t)$ $(i=1,2,\ldots,n)$, where $z_i(t)$ is the solution of (3.18) with initial value $x_i(0^+)=z_i(0^+)$. Since ($H_2$) holds, it follows from Lemma 2.3 and the proof of Theorem 3.1 that

$$\limsup_{t\to+\infty}x_i(t)\le\theta_i^1,\quad i=1,2,\ldots,n_1;$$
$$\lim_{t\to+\infty}x_i(t)=0,\quad i=n_1+1,\ldots,n,$$

where $\theta_i^1$ $(i=1,2,\ldots,n_1)$ are defined in Theorem 3.1.

Since ($H_2$) and ($B_1$) hold, it is easy to get that for any small enough positive constant $\varepsilon_1>0$,

$$\sum_{k=1}^{q}\ln(1+g_{jk})+Tm\left[-r_j(t)+\sum_{l=1}^{n_1}d_{jl}(t)(X_l^*(t)+\varepsilon_1)+\sum_{l=n_1+1}^{n}d_{jl}(t)\varepsilon_1\right]>0, \tag{3.19}$$

where $j=1,2,\ldots,m_1$;

$$\sum_{k=1}^{q}\ln(1+g_{jk})+Tm\left[-r_j(t)+\sum_{l=1}^{n_1}d_{jl}(t)(X_l^*(t)+\varepsilon_1)+\sum_{l=n_1+1}^{n}d_{jl}(t)\varepsilon_1\right]<0, \tag{3.20}$$

where $j=m_1+1,m_1+2,\ldots,m$, and

$$\sum_{k=1}^{q}\ln(1+h_{ik})+Tm\left[b_i(t)-\sum_{l=1}^{m_1}c_{il}(t)Y_l^{1*}(t)-\sum_{l=1,l\neq i}^{n_1}a_{il}(t)(X_l^*(t)+\varepsilon_1)-\sum_{l=n_1}^{n}a_{il}(t)\varepsilon_1\right]>0, \tag{3.21}$$

where $i=1,2,\ldots,n_1$.

Also there exists a $T_{i1}'>0$ $(i=1,2,\ldots,n)$ such that for $t>T_{i1}'$

$$\begin{aligned}&x_i(t)<X_i^*(t)+\varepsilon_1,\quad i=1,2,\ldots,n_1;\\&x_i(t)<\varepsilon_1,\quad i=n_1+1,\ldots,n.\end{aligned} \tag{3.22}$$

Let $T_1' = \max\{T_{i1}' | i = 1, 2, \ldots, n\}$. From the $(n+j)$th equation of systems (1.2) and (3.22), for $t > T_1'$ we have

$$y_j'(t) \le y_j(t)\left[-r_j(t) + \sum_{l=1}^{n_1} d_{jl}(t)(X_l^*(t)+\varepsilon_1) + \sum_{l=n_1+1}^{n} d_{jl}(t)\varepsilon_1 - e_{jj}(t)y_j(t)\right], \quad t \ne \tau_k,$$

where $j = 1, 2, \ldots, m$.

Consider the following system

$$\begin{cases} z_j'(t) = z_j(t)\left[-r_j(t) + \sum_{l=1}^{n_1} d_{jl}(t)(X_l^*(t)+\varepsilon_1) + \sum_{l=n_1+1}^{n} d_{jl}(t)\varepsilon_1 - e_{jj}(t)z_j(t)\right], \quad t \ne \tau_k, \\ z_j(\tau_k^+) = (1+g_{jk})z_j(\tau_k), \quad t = \tau_k,\ k = 1, 2, \ldots . j = 1, 2, \ldots, m. \end{cases} \tag{3.23}$$

By Lemma 2.1, for all $t > T_1'$, we have $y_j(t) \le z_j(t)$ $(j = 1, 2, \ldots, m)$, where $z_j(t)$ is the solution of (3.23) with initial value $y_j(T_1'^+) = z_j(T_1'^+)$. Since (3.19) holds, it follows from Lemma 2.3 that system (3.23) admits a unique $T$-periodic solution $Y_{j\varepsilon_1}^{1*}(t)$ $(j = 1, 2, \ldots, m_1)$, which is globally asymptotically stable. Setting $\varepsilon_1 \to 0$, from Lemma 2.4 we have

$$Y_{j\varepsilon_1}^{1*}(t) \to Y_j^{1*}(t).$$

Since (3.21) and (B$_2$) hold, it follows that for a positive constant $\varepsilon_2$ small enough,

$$\sum_{k=1}^{q} \ln(1+h_{ik}) + Tm\left[b_i(t) - \sum_{l=1}^{m_1} c_{il}(t)(Y_l^{1*}(t)+\varepsilon_2) - \sum_{l=m_1+1}^{m} c_{il}(t)\varepsilon_2 \right.$$
$$\left. - \sum_{l=1,l\ne i}^{n_1} a_{il}(t)(X_l^*(t)+\varepsilon_1) - \sum_{l=n_1}^{n} a_{il}(t)\varepsilon_1\right] > 0, \quad i = 1, 2, \ldots, n_1 \tag{3.24}$$

and

$$\sum_{k=1}^{q} \ln(1+g_{jk}) + Tm\left[-r_j(t) + \sum_{l=1}^{n_1} d_{jl}(t)X_{*l}^1(t) - \sum_{l=1,l\ne j}^{m_1} e_{jl}(t)(Y_l^{1*}(t)+\varepsilon_2) - \sum_{l=m_1+1}^{m} e_{jl}(t)\varepsilon_2\right] > 0,$$
$$j = 1, 2, \ldots, m_1. \tag{3.25}$$

Let $\theta_j^{21} = \sup\left\{Y_j^{1*}(t)\big| t \in [0, T]\right\}$, $j = 1, 2, \ldots, m_1$. From Lemma 2.3, (3.19) and (3.20), there exists a $T_{j2}' > T_1'$ $(j = 1, 2, \ldots, m)$ such that for $t > T_{j2}'$

$$\begin{aligned} &y_j(t) < Y_j^{1*}(t) + \varepsilon_2 < \theta_j^{21} + \varepsilon_2, \quad j = 1, 2, \ldots, m_1; \\ &y_j(t) < \varepsilon_2, \quad j = m_1+1, \ldots, m. \end{aligned} \tag{3.26}$$

Setting $\varepsilon_2 \to 0$, we obtain

$$\begin{aligned} &\limsup_{t\to+\infty} y_j(t) \le \theta_j^{21}, \quad j = 1, 2, \ldots, m_1; \\ &\lim_{t\to+\infty} y_j(t) = 0, \quad j = m_1+1, \ldots, m. \end{aligned}$$

Let $T_2' = \max\{T_{j2}' | j = 1, 2, \ldots, m\}$. From the $i$th equation of systems (1.2), (3.22) and (3.26), for $t > T_2'$, and $t \ne \tau_k$ we have

$$x_i'(t) \ge x_i(t)\left(b_i(t) - \sum_{l=1}^{m_1} c_{il}(t)(Y_l^{1*}(t)+\varepsilon_2) - \sum_{l=m_1+1}^{m} c_{il}(t)\varepsilon_2 \right.$$
$$\left. - \sum_{l=1,l\ne i}^{n_1} a_{il}(t)(X_l^*(t)+\varepsilon_1) - \sum_{l=n_1}^{n} a_{il}(t)\varepsilon_1 - a_{ii}(t)x_i(t)\right).$$

Consider the following system

$$\begin{cases} z_i'(t) = z_i(t)\left(b_i(t) - \sum_{l=1}^{m_1} c_{il}(t)(Y_l^{1*}(t)+\varepsilon_2) - \sum_{l=m_1+1}^{m} c_{il}(t)\varepsilon_2 \right. \\ \qquad \left. - \sum_{l=1,l\ne i}^{n_1} a_{il}(t)(X_l^*(t)+\varepsilon_1) - \sum_{l=n_1}^{n} a_{il}(t)\varepsilon_1 - a_{ii}(t)z_i(t)\right), \quad t \ne \tau_k \\ z_i(\tau_k^+) = (1+h_{ik})z_i(\tau_k), \quad t = \tau_k,\ k = 1, 2, \ldots,\ i = 1, 2, \ldots, n_1. \end{cases} \tag{3.27}$$

By Lemma 2.1, for all $t > T_2'$, we obtain that $x_i(t) \geq z_i(t)$ $(i = 1, 2, \ldots, n_1)$, where $z_i(t)$ is the solution of (3.27) with $x_i(T_2'^{+}) = z_i(T_2'^{+})$. Since (3.24) holds, it follows from Lemma 2.3 that system (3.27) admits a unique $T$-periodic solution $X^1_{*i(\varepsilon_1,\varepsilon_2)}(t)$ $(i = 1, 2, \ldots, n_1)$, which is globally asymptotically stable. Setting $\varepsilon_1, \varepsilon_2 \to 0$, from Lemma 2.4 we have

$$X^1_{*i(\varepsilon_1,\varepsilon_2)}(t) \to X^1_{*i}(t).$$

Let $\lambda_i^{11} = \inf\left\{X^1_{*i}(t) | t \in [0, T]\right\}$, $i = 1, 2, \ldots, n_1$. By (3.25), for a positive constant $\varepsilon_3 < \frac{1}{2}\min\left\{\lambda_i^{11} | i = 1, 2, \ldots, n_1\right\}$ small enough, we have

$$\sum_{k=1}^{q}\ln(1+g_{jk}) + Tm\left[-r_j(t) + \sum_{l=1}^{n_1} d_{jl}(t)(X^1_{*l}(t) - \varepsilon_3) - \sum_{l=1,l\neq j}^{m_1} e_{jl}(t)(Y_l^{1*}(t) + \varepsilon_2) - \sum_{l=m_1+1}^{m} e_{jl}(t)\varepsilon_2\right] > 0,$$
$$j = 1, 2, \ldots, m_1. \tag{3.28}$$

There exists a $T'_{i3} > T'_2$ $(i = 1, 2, \ldots, n_1)$ such that for $t > T'_{i3}$

$$x_i(t) > X^1_{*i}(t) - \varepsilon_3 \geq \lambda_i^{11} - \varepsilon_3 > 0, \quad i = 1, 2, \ldots, n_1. \tag{3.29}$$

Setting $\varepsilon_3 \to 0$, we obtain

$$\liminf_{t\to+\infty} x_i(t) \geq \lambda_i^{11}, \quad i = 1, 2, \ldots, n_1.$$

Let $T'_3 = \max\{T'_{i3} | i = 1, 2, \ldots, n_1\}$. From the $(n + j)$th equation of systems (1.2), (3.26) and (3.29), for $t > T'_3$ and $t \neq \tau_k$, we have

$$y_j'(t) \geq y_j(t)\left(-r_j(t) + \sum_{l=1}^{n_1} d_{jl}(t)(X^1_{*l}(t) - \varepsilon_3) - \sum_{l=1,l\neq j}^{m_1}(e_{jl}(t)(Y_l^{1*}(t) + \varepsilon_2)) - \sum_{l=m_1+1}^{m} e_{jl}(t)\varepsilon_2 - e_{jj}(t)y_j(t)\right).$$

Consider the following system

$$\begin{cases} z_j'(t) = z_j(t)\left(-r_j(t) + \sum_{l=1}^{n_1} d_{jl}(t)(X^1_{*l}(t) - \varepsilon_3) - \sum_{l=1,l\neq j}^{m_1}(e_{jl}(t)(Y_l^{1*}(t) + \varepsilon_2)) \right. \\ \qquad \left. - \sum_{l=m_1+1}^{m} e_{jl}(t)\varepsilon_2 - e_{jj}(t)z_j(t)\right), \quad t \neq \tau_k, \\ z_j(\tau_k^+) = (1 + g_{jk})z_j(\tau_k), \quad t = \tau_k,\ k = 1, 2, \ldots,\ j = 1, 2, \ldots, m_1. \end{cases} \tag{3.30}$$

By Lemma 2.1, for all $t > T'_3$, we obtain that $y_j(t) \geq z_j(t)$ $(j = 1, 2, \ldots, m_1)$, where $z_j(t)$ is the solution of (3.30) with $y_j(T_3'^{+}) = z_j(T_3'^{+})$. Since (3.28) holds, it follows from Lemma 2.3 that system (3.30) admits a unique $T$-periodic solution $Y^1_{*j(\varepsilon_2,\varepsilon_3)}(t)$ $(j = 1, 2, \ldots, m_1)$, which is globally asymptotically stable. Setting $\varepsilon_2, \varepsilon_3 \to 0$, from Lemma 2.4 we have

$$Y^1_{*j(\varepsilon_2,\varepsilon_3)}(t) \to Y^1_{*j}(t).$$

Let $\lambda_j^{21} = \inf\left\{Y^1_{*j}(t) | t \in [0, T]\right\}$, $j = 1, 2, \ldots, m_1$. For any positive constant $\varepsilon_4 < \frac{1}{2}\min\left\{\lambda_j^{21} | j = 1, 2, \ldots, m_1\right\}$, there exists a $T'_{j4} > T'_3$ $(j = 1, 2, \ldots, m_1)$ such that for $t > T'_{j4}$

$$y_j(t) > Y^1_{*j}(t) - \varepsilon_4 > \lambda_j^{21} - \varepsilon_4 > 0, \quad j = 1, 2, \ldots, m_1.$$

Setting $\varepsilon_4 \to 0$, we obtain

$$\liminf_{t\to+\infty} y_j(t) \geq \lambda_j^{21}, \quad j = 1, 2, \ldots, m_1.$$

The proof is completed. □

For each $n_1 \leq n$ and $m_1 \leq m$, let $H^{n_1+m_1}$ denote the $n_1 + m_1$-dimensional coordinate subspace on which $x_{n_1+1}, \ldots, x_n$, $y_{m_1+1}, \ldots, y_m$ vanish. We use the variable $u$, $v$ to denote the restriction of system (1.2) to $H^{n_1+m_1}$,

$$\begin{cases} u_i'(t) = u_i(t)\left[b_i(t) - \sum_{l=1}^{n_1} a_{il}(t)u_l(t) - \sum_{l=1}^{m_1} c_{il}(t)v_l(t)\right], \\ v_j'(t) = v_j(t)\left[-r_j(t) + \sum_{l=1}^{n_1} d_{jl}(t)u_l(t) - \sum_{l=1}^{m_1} e_{jl}(t)v_l(t)\right], \quad t \neq \tau_k, \\ u_i(\tau_k^+) = (1 + h_{ik})u_i(\tau_k), \\ v_j(\tau_k^+) = (1 + g_{jk})v_j(\tau_k), \quad t = \tau_k,\ k = 1, 2, \ldots, \end{cases} \tag{3.31}$$

where $i = 1, 2, \ldots, n_1$; $j = 1, 2, \ldots, m_1$.

Now let us consider the global attractivity of partial species $x_i$ $(i = 1, 2, \ldots, n_1)$, $y_j$ $(j = 1, 2, \ldots, m_1)$ of system (1.2), and we obtain the following result.

**Theorem 3.5.** *Assume that* $(H_2)$, $(B_1)$ *and* $(B_2)$ *hold. Assume further that there exist positive constants* $\rho_i^*$ $(i = 1, 2, \ldots, n_1)$ *and* $\mu_i^*$ $(j = 1, 2, \ldots, m_1)$ *such that*

$$\rho_i^* a_{ii}(t) > \sum_{l=1, l\neq i}^{n_1} \rho_l^* a_{li}(t) + \sum_{l=1}^{m_1} \mu_l^* d_{li}(t); \tag{B$_3$}$$

$$\mu_j^* e_{jj}(t) > \sum_{l=1, l\neq j}^{m_1} \mu_l^* e_{lj}(t) + \sum_{l=1}^{n_1} \rho_l^* c_{lj}(t), \tag{B$_4$}$$

*then the species* $x_i, y_j$ $(i = 1, 2, \ldots, n_1,\ j = 1, 2, \ldots, m_1)$ *are globally attractive, that is, for any positive solution* $F(t) = (X(t), Y(t)) = (x_1(t), x_2(t), \ldots, x_n(t), y_1(t), y_2(t), \ldots, y_m(t))^T$ *of system* (1.2) *and any positive solution* $W(t) = (U(t), V(t)) = (u_1(t), u_2(t), \ldots, u_{n_1}(t), v_1(t), v_2(t), \ldots, v_{m_1}(t))^T$ *of subsystem* (3.31)*, one has*

$$\lim_{t\to+\infty} |x_i(t) - u_i(t)| = 0, \quad i = 1, 2, \ldots, n_1;$$
$$\lim_{t\to+\infty} |y_j(t) - v_j(t)| = 0, \quad j = 1, 2, \ldots, m_1.$$

**Proof.** Let $F(t) = (X(t), Y(t)) = (x_1(t), x_2(t), \ldots, x_n(t), y_1(t), y_2(t), \ldots, y_m(t))^T$ be any positive solution of system (1.2) and $W(t) = (U(t), V(t)) = (u_1(t), u_2(t), \ldots, u_{n_1}(t), v_1(t), v_2(t), \ldots, v_{m_1}(t))^T$ be any positive solution of subsystem (3.31). For any positive constant $\varepsilon < \min\{\lambda_i^{11}, \lambda_j^{21} \mid i = 1, \ldots, n_1;\ j = 1, \ldots, m_1\}$, from Theorem 3.2, it immediately follows that there exists a $T_0' > 0$ large enough such that for all $t > T_0'$,

$$\begin{aligned}
&x_i(t) < \varepsilon, \quad i = n_1 + 1, \ldots, n;\\
&y_j(t) < \varepsilon, \quad j = m_1 + 1, \ldots, m;\\
&\lambda_i^{11} - \varepsilon \le x_i(t), u_i(t) \le \theta_i^1 + \varepsilon, \quad i = 1, 2, \ldots, n_1;\\
&\lambda_j^{21} - \varepsilon \le y_j(t), v_j(t) \le \theta_j^{21} + \varepsilon, \quad j = 1, 2, \ldots, m_1.
\end{aligned} \tag{3.32}$$

Set

$$V(t) = V_1(t) + V_2(t),$$

where

$$V_1(t) = \sum_{i=1}^{n_1} \rho_i^* |\ln u_i(t) - \ln x_i(t)|;$$

$$V_2(t) = \sum_{i=1}^{m_1} \mu_j^* |\ln v_j(t) - \ln y_j(t)|.$$

For $t \ge 0$, and $t \ne \tau_k$, $k = 1, 2, \ldots$, calculating the upper right derivatives of $V_1(t)$ and $V_2(t)$, respectively, we have

$$\begin{aligned}
D^+V_1(t) &= \sum_{i=1}^{n_1} \rho_i^* \left(\frac{u_i'(t)}{u_i(t)} - \frac{x_i'(t)}{x_i(t)}\right) \operatorname{sgn}(u_i(t) - x_i(t))\\
&\le \sum_{i=1}^{n_1} \left[-\rho_i^* a_{ii}(t) + \sum_{l=1, l\neq i}^{n_1} \rho_l^* a_{li}(t)\right] |x_i(t) - u_i(t)| + \sum_{i=1}^{n_1}\sum_{j=1}^{m_1} \rho_i^* c_{ij}(t) |y_j(t) - v_j(t)| + g_1(t);\\
D^+V_2(t) &= \sum_{j=1}^{m_1} \mu_j^* \left(\frac{v_j'(t)}{v_j(t)} - \frac{y_j'(t)}{y_j(t)}\right) \operatorname{sgn}(v_j(t) - y_j(t))\\
&\le \sum_{j=1}^{m_1} \left[-\mu_j^* e_{jj}(t) + \sum_{l=1, l\neq j}^{m_1} \mu_l^* e_{lj}(t)\right] |v_j(t) - y_j(t)| + \sum_{i=1}^{n_1}\sum_{l=1}^{m_1} \mu_l^* d_{li}(t) |u_i(t) - x_i(t)| + g_2(t),
\end{aligned}$$

where

$$g_1(t) = \sum_{i=1}^{n_1}\sum_{l=n_1+1}^{n} \rho_i^* a_{il}(t) x_l(t) + \sum_{i=1}^{n_1}\sum_{l=m_1+1}^{m} \rho_i^* c_{il}(t) y_l(t);$$

$$g_2(t) = \sum_{j=1}^{m_1}\sum_{l=n_1+1}^{n} \mu_j^* d_{jl}(t) x_l(t) + \sum_{j=1}^{m_1}\sum_{l=m_1+1}^{m} \mu_j^* e_{jl}(t) y_l(t).$$

It follows from conditions (B$_3$) and (B$_4$) that there exists a constant $\delta_0 > 0$ such that

$$\rho_i^* a_{ii}(t) > \sum_{l=1,l\neq i}^{n_1} \rho_l^* a_{li}(t) + \sum_{l=1}^{m_1} \mu_l^* d_{li}(t) + \delta_0;$$
$$\mu_j^* e_{jj}(t) > \sum_{l=1,l\neq j}^{m_1} \mu_l^* e_{lj}(t) + \sum_{l=1}^{n_1} \rho_l^* c_{lj}(t) + \delta_0.$$

Similarly to the analysis of the proof of Theorem 3.3, for $t > T_0'$ and $t \neq \tau_k,\ k = 1, 2, \ldots,$ it follows that

$$\begin{aligned} D^+V(t) &\le \sum_{i=1}^{n_1}\left[-\rho_i^* a_{ii}(t) + \sum_{l=1,l\neq i}^{n_1} \rho_l a_{li}(t) + \sum_{l=1}^{m_1} \mu_l^* d_{li}(t)\right] \\ &\quad + \sum_{j=1}^{m_1}\left[-\mu_j^* e_{jj}(t) + \sum_{l=1,l\neq j}^{m_1} \mu_l^* e_{lj}(t) + \sum_{l=1}^{n-1} \rho_l^* c_{lj}(t)\right] + g_1(t) + g_2(t) \\ &\le -\delta_0\left[\sum_{i=1}^{n_1} |x_i(t) - u_i(t)| + \sum_{j=1}^{m_1} |v_j(t) - y_j(t)|\right] + g_1(t) + g_2(t) \\ &\le -\delta_0\left[\sum_{i=1}^{n_1} \frac{\lambda_i^{11} - \varepsilon}{\rho_i^*}\rho_i^* |\ln x_i(t) - \ln u_i(t)| + \sum_{j=1}^{m_1} \frac{\lambda_j^{21} - \varepsilon}{\mu_j^*}\mu_j^* |\ln v_j(t) - \ln y_j(t)|\right] + g_1(t) + g_2(t) \\ &\le -\phi_\varepsilon V(t) + g_1(t) + g_2(t), \end{aligned}$$

where

$$\phi_\varepsilon = \delta_0 \min\left\{ \frac{\lambda_j^{21} - \varepsilon}{\mu_j^*},\ \frac{\lambda_i^{11} - \varepsilon}{\rho_i^*} \,\middle|\, i = 1, \ldots, n_1;\ j = 1, \ldots, m_1 \right\}.$$

It is obvious that for $t = \tau_k,\ k = 1, 2, \ldots, V(\tau_k^+) = V(\tau_k)$.

The above analysis shows that, for all $t > T_0'$,

$$D^+V(t) < -\phi_\varepsilon V(t) + g_1(t) + g_2(t). \tag{3.33}$$

Applying the differential inequality theorem and the variation of constants formula of solutions of first-order linear differential equation, we have

$$V(t) \le \exp(-\phi_\varepsilon(t - T_0'))\left(\int_{T_0'}^{t} (g_1(s) + g_2(s)) \exp(\phi_\varepsilon(s - T_0'))ds + V(T_0')\right). \tag{3.34}$$

Since $g_1(t) + g_2(t) \to 0$ as $t \to +\infty$, it is not hard to prove $V(t) \to 0$ as $t \to +\infty$. This implies that

$$\lim_{t\to+\infty} |x_i(t) - u_i(t)| = 0, \quad i = 1, 2, \ldots, n_1;$$
$$\lim_{t\to+\infty} |y_j(t) - v_j(t)| = 0, \quad j = 1, 2, \ldots, m_1,$$

that is, species $x_i\ (i \le n_1),\ y_j\ (j \le m_1)$ are globally attractive. The proof is completed. □

## 4. Examples

The following examples show the feasibility of our results.

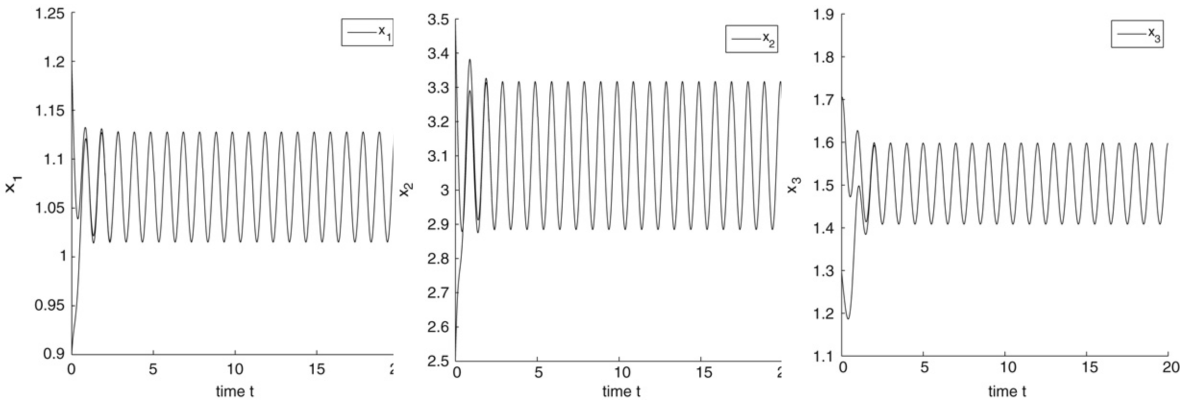
**Example 1.** Consider the following continuous system

$$\begin{cases} x_1'(t) = x_1(t)\left(3 - (2.35 + 0.15\cos(2\pi t))x_1(t) - 0.01x_2(t) - (0.3 + 0.2\sin(2\pi t))y_1(t)\right), \\ x_2'(t) = x_2(t)\left(5 - (0.8 + 0.2\sin(2\pi t))x_1(t) - (1.1 + 0.1\sin(2\pi t))x_2(t) - 0.5y_1(t)\right), \\ y_1'(t) = y_1(t)\left(-1.5 + (1.35 + 0.05\sin(2\pi t))x_1(t) + 0.6x_2(t) - (1.2 + 0.2\sin(2\pi t))y_1(t)\right). \end{cases} \tag{4.1}$$

From the main result in [7], it follows that

$$\lim_{t\to+\infty} |x_i(t) - u_i(t)| = 0, \qquad \lim_{t\to+\infty} |y_1(t) - v_1(t)| = 0, \quad i = 1, 2,$$

where $(x_1(t), x_2(t), y_1(t))$ and $(u_1(t), u_2(t), v_1(t))$ are any positive solutions of system (4.1). Fig. 1 shows the dynamic behaviors of system (4.1).




(a) species $x_1$. (b) species $x_2$. (c) species $y_1$.

**Fig. 1.** Numeric simulations of the solution $(x_1(t), x_2(t), y_1(t))$ to system (4.1) with the initial conditions (1.2, 3.5, 1.7) and (0.9, 2.5, 1.3), respectively.

**Example 2.** Consider the following impulsive system

$$\begin{cases} x_1'(t) = x_1(t)\,(3 - (2.35 + 0.15\cos(2\pi t))x_1(t) - 0.01x_2(t) - (0.3 + 0.2\sin(2\pi t))y_1(t))\,, \\ x_2'(t) = x_2(t)\,(5 - (0.8 + 0.2\sin(2\pi t))x_1(t) - (1.1 + 0.1\sin(2\pi t))x_2(t) - 0.5y_1(t))\,, \\ y_1'(t) = y_1(t)\,(-1.5 + (1.35 + 0.05\sin(2\pi t))x_1(t) + 0.6x_2(t) - (1.2 + 0.2\sin(2\pi t))y_1(t))\,, \quad t \neq \tau_k \\ x_1(\tau_k^+) = \exp(2)x_1(\tau_k), \\ x_2(\tau_k^+) = \exp(1.5)x_2(\tau_k), \\ y_1(\tau_k^+) = \exp(2)y_1(\tau_k), \quad t = \tau_k = kT,\ k = 1, 2, \ldots, \end{cases} \tag{4.2}$$

where $T = 1,\ h_{1k} \equiv h_1 = \exp(2) - 1, h_{2k} \equiv h_2 = \exp(1.5) - 1, g_{1k} \equiv g_1 = \exp(2) - 1$.

Conditions $(A_1)$ and $(A_2)$ are respectively equivalent to inequalities

$$\begin{cases} \ln(1 + h_1) + Tm[b_1(t) - c_{11}(t)Y_1^*(t) - a_{12}(t)X_2^*(t)] \geq 2.7394 > 0; \\ \ln(1 + h_2) + Tm[b_2(t) - c_{21}(t)Y_1^*(t) - a_{21}(t)X_1^*(t)] \geq 2.8734 > 0, \end{cases}$$

and

$$\ln(1 + g_1) + Tm[-r_1(t) + d_{11}(t)X_{*1}(t) + d_{12}(t)X_{*2}(t)] \geq 1.3734 > 0.$$

Taking $\rho_1 = 8$, $\rho_2 = 5$, $\mu_1 = 8$, one could easily verify

$$\begin{cases} \rho_1 a_{11}(t) - \rho_2 a_{21}(t) - \mu_1 d_{11}(t) \geq 1.4 > 0; \\ \rho_2 a_{22}(t) - \rho_1 a_{12}(t) - \mu_1 d_{12}(t) \geq 0.12 > 0; \\ \mu_1 e_{11}(t) - \rho_1 c_{11}(t) - \rho_2 c_{21}(t) \geq 1.5 > 0. \end{cases}$$

The above three inequalities show that conditions $(A_3)$ and $(A_4)$ in Theorem 3.3 hold.

From Theorems 3.1–3.3, we have

$$\lim_{t\to+\infty} |x_i(t) - u_i(t)| = 0, \qquad \lim_{t\to+\infty} |y_1(t) - v_1(t)| = 0, \quad i = 1, 2,$$

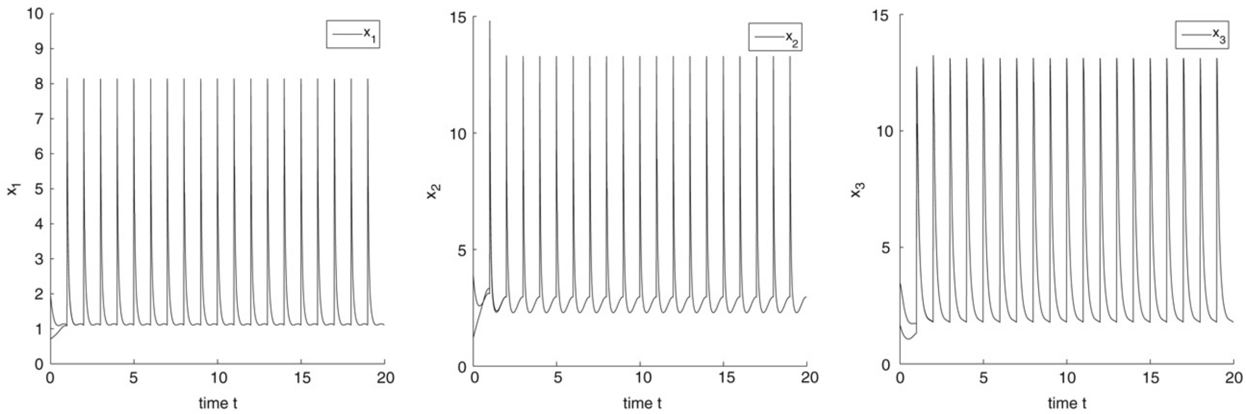
where $(x_1(t), x_2(t), y_1(t))$ and $(u_1(t), u_2(t), v_1(t))$ are any positive solutions of system (4.2). Fig. 2 shows the dynamic behaviors of system (4.2).

The above two examples show that if all the species are beneficial, we can control those densities to the levels acceptable to the public, that is impulsivity has no influence on the long time dynamic behaviors such as survival and stability of the species of the system.

**Example 3.** Consider the following impulsive system

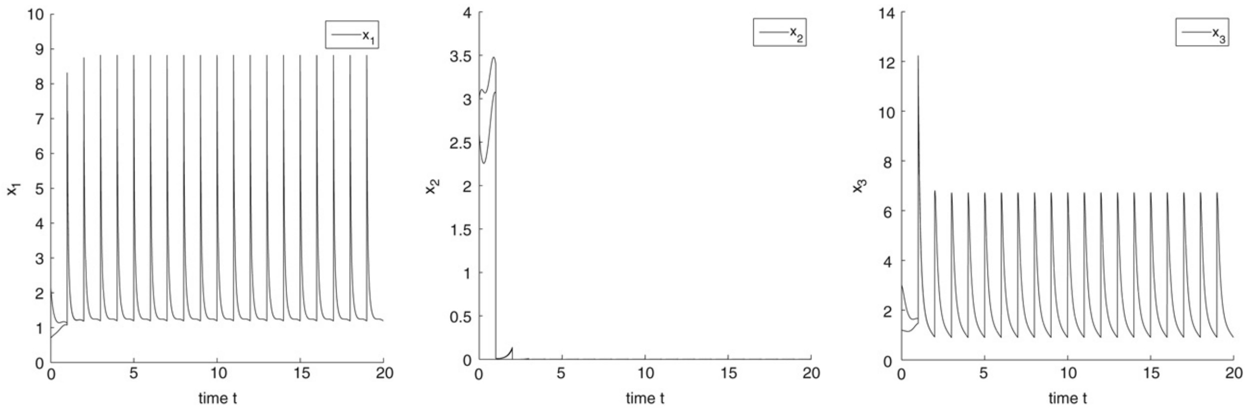
$$\begin{cases} x_1'(t) = x_1(t)\,(3 - (2.35 + 0.15\cos(2\pi t))x_1(t) - 0.01x_2(t) - (0.3 + 0.2\sin(2\pi t))y_1(t))\,, \\ x_2'(t) = x_2(t)\,(5 - (0.8 + 0.2\sin(2\pi t))x_1(t) - (1.1 + 0.1\sin(2\pi t))x_2(t) - 0.5y_1(t))\,, \\ y_1'(t) = y_1(t)\,(-1.5 + (1.35 + 0.05\sin(2\pi t))x_1(t) + 0.6x_2(t) - (1.2 + 0.2\sin(2\pi t))y_1(t))\,, \quad t \neq \tau_k \\ x_1(\tau_k^+) = (\exp(2))x_1(\tau_k), \\ x_2(\tau_k^+) = (\exp(-5.5))x_2(\tau_k), \\ y_1(\tau_k^+) = (\exp(2))y_1(\tau_k), \quad t = \tau_k = kT,\ k = 1, 2, \ldots, \end{cases} \tag{4.3}$$

where $T = 1, h_{1k} \equiv h_1 = \exp(2) - 1, h_{2k} \equiv h_2 = \exp(-5.5) - 1, g_{1k} \equiv g_1 = \exp(2) - 1$.




(a) species $x_1$. (b) species $x_2$. (c) species $y_1$.

**Fig. 2.** Numeric simulations of the solution $(x_1(t), x_2(t), y_1(t))$ to system (4.2) with the initial conditions (0.7, 1.2, 1.7) and (2, 4, 3.5), respectively.




(a) species $x_1$. (b) species $x_2$. (c) species $y_1$.

**Fig. 3.** Numeric simulations of the solution $(x_1(t), x_2(t), y_1(t))$ to system (4.3) with the initial conditions (0.7, 3, 1.2) and (2.2, 2.6, 3), respectively.

Conditions $(B_1)$ and $(B_2)$ are respectively equivalent to inequalities

$$\ln(1 + h_1) + Tm[b_1(t) - c_{11}(t)Y_1^*(t)] \geq 4.6343 > 0$$

and

$$\ln(1 + g_1) + Tm[-r_1(t) + d_{11}(t)X_{*1}(t)] \geq 1.9234 > 0.$$

Also

$$\ln(1 + h_2) + Tm[b_2] \leq -0.5 < 0.$$

The above three inequalities show that condition $(H_2)$ holds.

Taking $\rho_1^* = 8$, $\mu_1^* = 8$, one could easily verify

$$\begin{cases} \rho_1^* a_{11}(t) - \mu_1^* d_{11}(t) \geq 6.4; \\ \mu_1^* e_{11}(t) - \rho_1^* c_{11}(t) \geq 4. \end{cases}$$

The above two inequalities show that conditions $(B_3)$ and $(B_4)$ in Theorem 3.5 hold.

From Theorems 3.4 and 3.5, we have

$$\lim_{t\to+\infty} |x_1(t) - u_1(t)| = 0, \qquad \lim_{t\to+\infty} x_2(t) = \lim_{t\to+\infty} u_2(t) = 0, \qquad \lim_{t\to+\infty} |y_1(t) - v_1(t)| = 0,$$

where $(x_1(t), x_2(t), y_1(t))$ and $(u_1(t), u_2(t), v_1(t))$ are any positive solutions of system (4.3). Fig. 3 shows the dynamic behaviors of system (4.3).

Examples 1 and 3 show that if only one prey species $x_2$ is the target pest, we can choose the value of $h_{2k}$ $(k = 1, 2, \ldots)$ to eradicate the pest population and make the remaining two species permanent. To sum up, just changing the value of $h_{ik}$ and $g_{jk}$ leads to a great change in the dynamic behaviors of the predator–prey system.

## Acknowledgement

This work was supported by the National Natural Science Foundation of China (10501007).

## References

[1] S. Ahmad, F. Montes de Oca, Extinction in nonautonomous *T*-periodic competitive Lotka–Volterra system, Appl. Math. Comput. 90 (2–3) (1998) 155–166.
[2] T. Ding, H. Huang, F. Zanolin, A priori bounds and periodic solution for a class of planar systems with applications to Lotka–Volterra equations, Discrete Contin. Dyn. Syst. 1 (1995) 103–117.
[3] B. Lisena, Global attractive periodic models of predator–prey type, Nonlinear Anal. RWA 6 (2005) 133–144.
[4] J. Lopez-Gomez, R. Ortega, A. Tineo, The periodic predator–prey Lotka–Volterra model, Adv. Differential Equations. 1 (1996) 403–432.
[5] Z. Teng, Uniform persistence of the periodic predator–prey Lotka–Volterra systems, Appl. Anal. 72 (1998) 339–352.
[6] P. Yang, R. Xu, Global attractivity of the periodic Lotka–Volterra system, J. Math. Anal. Appl. 233 (1) (1999) 221–232.
[7] J.D. Zhao, W.C. Chen, Global asymptotic stability of a periodic ecological model, Appl. Math. Comput. 147 (3) (2004) 881–892.
[8] Y.H. Xia, F.D. Chen, A.P. Chen, J.D. Cao, Existence and global attractivity of an almost periodic ecological model, Appl. Math. Comput. 157 (2) (2004) 449–475.
[9] J.D. Zhao, J.F. Jiang, Permanence in nonautonomous Lotka–Volterra system with predator–prey, Appl. Math. Comput. 152 (2004) 99–109.
[10] J.D. Zhao, J.F. Jiang, A.C. Lazer, The permanence and global attractivity in a nonautonomous Lotka–Volterra system, Nonlinear Anal. RWA 5 (4) (2004) 265–276.
[11] F.D. Chen, Permanence in nonautonomous multi-species predator–prey system with feedback controls, Appl. Math. Comput. 173 (2) (2006) 694–709.
[12] F.D. Chen, Permanence and global stability of nonautonomous Lotka–Volterra system with predator–prey and deviating arguments, Appl. Math. Comput. 173 (2) (2006) 1082–1100.
[13] F.D. Chen, On a periodic multi-species ecological model, Appl. Math. Comput. 171 (1) (2005) 492–510.
[14] F.D. Chen, On a nonlinear nonautonomous predator–prey model with diffusion and distributed delay, J. Comput. Appl. Math. 180 (1) (2005) 33–49.
[15] F.D. Chen, Permanence and global attractivity of a discrete multispecies Lotka–Volterra competition predator–prey systems, Appl. Math. Comput. 182 (1) (2006) 3–12.
[16] F.D. Chen, C.L. Shi, Global attractivity in an almost periodic multi-species nonlinear ecological model, Appl. Math. Comput. 180 (1) (2006) 376–392.
[17] F.D. Chen, X.D. Xie, J.L. Shi, Existence, uniqueness and stability of positive periodic solution for a nonlinear prey-competition model with delays, J. Comput. Appl. Math. 194 (2) (2006) 368–387.
[18] F.D. Chen, X.D. Xie, Periodicity and stability of a nonlinear periodic integro-differential prey-competition model with infinite delays, Commun. Nonlinear Sci. Numer. Simul. 12 (6) (2007) 876–885.
[19] J.C. Panetta, A mathematical model of periodically pulsed chemotherapy: Tumor recurrence and metastasis in a competition environment, Bull. Math. Biol. 58 (1996) 425–447.
[20] D.D. Bainov, P.S. Simeonov, Impulsive differential equations: Periodic solutions and applications, Longman Scientific and Technical, 1993.
[21] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of impulsive differential equations, World Scientific, 1989.
[22] S. Ahmad, I.M. Stamova, Asymptotic stability of an *N*-dimensional impulsive competitive systems, Nonlinear Anal. RWA 8 (2) (2007) 654–663.
[23] G. Ballinger, X. Liu, Permanence of population growth models with impulsive effects, Math. Comput. Modelling 26 (1997) 59–72.
[24] Z. Jin, M.A. Han, G.H. Li, The persistence in a Lotka–Volterra competition systems with impulsive, Chaos Solitons Fractals 24 (2005) 1105–1117.
[25] Z. Jin, Z.E. Ma, M.A. Han, The existence of periodic solutions of the *n*-species Lotka–Volterra competition systems with impulsive, Chaos Solitons Fractals 22 (2004) 181–188.
[26] B. Liu, Z.D. Teng, W.B. Liu, Dynamic behaviors of the periodic Lotka–Volterra competing system with impulsive perturbations, Chaos Solitons Fractals 31 (2007) 356–370.
[27] X.N. Liu, L.S. Chen, Global dynamics of the periodic logistic system with periodic impulsive perturbations, J. Math. Anal. Appl. 289 (2004) 279–291.
[28] X.N. Liu, L.S. Chen, Complex dynamics of Holling type II Lotka–Volterra predator–prey system with impulsive perturbations on the predator, Chaos Solitons Fractals 16 (2003) 311–320.
[29] S.Y. Tang, L.S. Chen, The periodic predator–prey Lotka–Volterra model with impulsive effect, J. Mach. Med. Biol. 2 (2002) 267–296.
[30] S.W. Zhang, L.Z. Dong, L.S. Chen, The study of predator–prey system with defensive ability of prey and impulsive perturbations on the predator, Chaos Solitons Fractals 23 (2005) 631–643.
[31] S.W. Zhang, D.J. Tan, L.S. Chen, The periodic *n*-species Gilpin–Ayala competition system with impulsive effect, Chaos Solitons Fractals 26 (2005) 507–517.
[32] A. Lakmeche, O. Arino, Bifurcation of non-trivial periodic solutions of impulsive differential equations arising chemotherapeutic treatment, Dyn. Contin. Discrete Impuls. Syst. 7 (2000) 265–287.