Extension of vector-lattice homomorphisms revisited

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ABSTRACT

A new method of extending vector-lattice homomorphisms is developed. Its advantage, when compared with the previous ones due to Hayes, the author, Luxemburg and Schep, and Aron, Hager and Madden, is that it does not resort to the axiom of choice in the situation where the larger vector lattice is generated by its vector sublattice and an additional element.

The problem of extending lattice-group and vector-lattice homomorphisms was first solved in the special case where the range space is the reals by Hayes [6]. The general case was more recently examined in [2], [5], [7]–[9] and [11] and its positive solution found an application in [1], p. 270. The methods used so far are very noneffective in the sense that, even in the simplest situation where the larger vector lattice is generated (as a vector lattice) by its vector sublattice and a single additional element, an uncountable form of the axiom of choice is involved. In [11], p. 146, the authors write about “unsurmountable problems” arising already in this situation.

Nevertheless, the method used below, which is based on a version of a simple extension theorem of [10] (Proposition below), yields an effective first-step extension (Theorem). Moreover, it can also be applied in case the domain space is merely an Abelian lattice-ordered group (see the remark at the end of the paper). On the other hand, it is worth-while to point out that this method has two disadvantages. Firstly, it is somewhat more complicated than that of [8]. Secondly, it does not seem to yield some more general results concerning extension of positive operators ([8], Theorem, and [9], Section 2).
For the rest of the paper (excluding the closing remark) we shall fix the following notation and assumptions. Let \( X \) and \( Y \) be vector lattices over the reals \( \mathbb{R} \) with \( Y \) order complete\(^1\) and let \( M \) be a majorizing (i.e., cofinal) vector sublattice of \( X \). Given a positive (linear) operator \( T: M \to Y \), we put

\[
T_e(x) = \inf \{ T(z) : x \leq z \in M \}
\]

for all \( x \in X \). As easily seen, \( T_e: X \to Y \) is subadditive and positively homogeneous and \( T_e(z) = T(z) \) for \( z \in M \). The following two properties of \( T_e \) are also simple to check.

(i) \( T_e(x + z) = T_e(x) + T(z) \) for \( x \in X \) and \( z \in M \).

(ii) \( T_e(x_1 \vee x_2) = T_e(x_1) \vee T_e(x_2) \) for \( x_1, x_2 \in X \) provided \( T \) is a vector-lattice homomorphism.

Furthermore, we shall need the following well-known formula (cf. [12], Theorem 11.5(v)):

(iii) \((x_1 - x_2) \vee (x'_1 - x'_2) = ((x_1 + x'_1) \vee (x_1 + x_2)) - (x_2 + x'_2) \) for \( x_1, x'_1, x_2, x'_2 \in X \).

Our first result is partially known. Namely, the first half of part (b) follows from [10], Theorem (cf. also [3], Theorem 3.3). However, for the reader’s convenience we reprove it below.

**Proposition.** Let \( V \subset X \) be a cone. We have

(a) \( V - V \) is a vector subspace of \( X \). If \( V \) is, moreover, closed under finite suprema, then \( V - V \) is a sublattice of \( X \).

(b) Suppose \( M \subset V \). A positive operator \( T: M \to Y \) extends (uniquely) to a positive operator \( S: V - V \to Y \) such that \( S(v) = T_e(v) \) for all \( v \in V \) if and only if \( T_e \) is additive on \( V \). If moreover, \( V \) is closed under finite suprema and \( T \) is a vector-lattice homomorphism, then so is \( S \).

**Proof.** The first half of (a) is clear while the second one follows by (iii).

To prove the first half of (b) in the nontrivial direction, observe that, given \( v_1, w_1, v_2, w_2 \in V \),

\[
v_1 - v_2 = w_1 - w_2 \text{ implies } T_e(v_1) - T_e(v_2) = T_e(w_1) - T_e(w_2).
\]

Indeed, we have \( v_1 + w_2 = w_1 + v_2 \), whence \( T_e(v_1) + T_e(w_2) = T_e(w_1) + T_e(v_2) \). It follows that we can uniquely define \( S \) by

\[
S(x) = T_e(v_1) - T_e(v_2), \text{ where } x = v_1 - v_2 \text{ and } v_1, v_2 \in V.
\]

Evidently, \( S: V - V \to Y \) extends \( T_e \) \( 1V \) and is linear. Also, \( S \) is positive, since \( v_1 \geq v_2 \) implies \( T_e(v_1) \geq T_e(v_2) \).

Finally, to establish the second half of (b), note that, in virtue of (ii),

\[
S(v_1 \vee v_2) = S(v_1) \vee S(v_2)
\]

\(^1\) In an alternative terminology [12], \( X \) and \( Y \) are Riesz spaces with \( Y \) Dedekind complete.
for \( v_1, v_2 \in V \). Hence, using (iii), we obtain
\[
S((v_1 - v_2) \lor (w_1 - w_2)) = (v_1 + w_2) \lor (w_1 + v_2) - S(v_2 + w_2) =
\]
\[
(S(v_1 + w_2) \lor S(w_1 + v_2)) - S(v_2 + w_2) = S(v_1 - v_2) \lor S(w_1 - w_2)
\]
for all \( v_1, w_1, v_2, w_2 \in V \). Thus \( S \) is a vector-lattice homomorphism.

**THEOREM.** Let \( x_0 \in X \) and let \( T : M \to Y \) be a vector-lattice homomorphism. Then \( T \) extends (uniquely) to a positive operator \( S : N \to Y \), where \( N \) is the vector sublattice of \( X \) generated by \( M \cup \{x_0\} \), such that \( S(v) = T_e(v) \) for every \( v \) of the form
\[
(*) \quad v = \bigvee_{1 \leq i \leq n} (z_i + t_i x_0) \quad \text{with } z_i \in M, \ t_i \in \mathbb{R}_+ \quad \text{and} \quad n \in \mathbb{N}.
\]
Moreover, \( S \) is a vector-lattice homomorphism.

**PROOF.** Denote by \( V \) the set of all elements \( v \) of the form (\(*\)). Clearly, \( V \) is closed under finite suprema and multiplication by positive scalars. We shall check that \( V \) is also closed under addition and that \( T_e \) is additive on \( V \). Let
\[
v_k = \bigvee_{1 \leq i \leq n_k} (z_{ik} + t_{ik} x_0) \quad \text{with } z_{ik} \in M, \ t_{ik} \in \mathbb{R}_+ \quad \text{for } k = 1, 2.
\]
We have
\[
v_1 + v_2 = \bigvee_{1 \leq i \leq n_1, 1 \leq j \leq n_2} (z_{i1} + z_{j2} + (t_{i1} + t_{j2}) x_0),
\]
whence \( v_1 + v_2 \in V \). Moreover, in view of (i) and (ii),
\[
T_e(v_1 + v_2) = \bigvee_{i, j} T_e(z_{i1} + z_{j2} + (t_{i1} + t_{j2}) x_0) =
\]
\[
\bigvee_{i, j} (T_e(z_{i1} + t_{i1} x_0) + T_e(z_{j2} + t_{j2} x_0)) =
\]
\[
\bigvee_i T_e(z_{i1} + t_{i1} x_0) + \bigvee_j T_e(z_{j2} + t_{j2} x_0) = T_e(v_1) + T_e(v_2).
\]
Thus \( V \) is a cone and it follows from the proposition that \( N = V - V \) and that \( T \) can be extended as desired.

We note that, in the situation of the theorem above, a vector-lattice homomorphism \( S : N \to Y \) extending \( T \) is uniquely determined by the condition that \( S(x_0) = T_e(x_0) \). Accordingly, although the present method differs from that of [8], the resulting extensions coincide (see [8], proof of Theorem).

By a standard application of the Kuratowski-Zorn lemma, we get from the theorem

**COROLLARY** ([7], Corollary 2, [8], Corollary, [11], Theorem 3.1). Every vector-lattice homomorphism \( T : M \to Y \) extends to a vector-lattice homomorphism \( S : X \to Y \).
REMARK. With suitable modifications, the results above remain valid in case $X$ is merely an Abelian lattice-ordered group, $M$ is its lattice subgroup and $T: M \to Y$ is a positive group homomorphism or a lattice-group homomorphism, respectively. (The corresponding version of the corollary appears, for $Y = \mathbb{R}$, in [6], Theorem 7, and, in the general situation, in [2], Theorem 1.1; see also [5], Theorem 1.1) The major modifications consist in redefining $T_e$ as follows

$$T_e(x) = \inf \left\{ \frac{1}{n} T(z) : nx \leq z \in M \text{ and } n \in \mathbb{N} \right\}$$

for all $x \in X$ and replacing "$\mathbb{R}^+$" with "$\mathbb{N} \cup \{0\}$" in (*). Then our proofs still work in the group setting, since the redefined $T_e$ is subadditive,

$$T_e(nx) = nT_e(x)$$

for all $x \in X$ and $n \in \mathbb{N}$,

and $T_e$ has properties (i) and (ii). We shall confine ourselves to checking the subadditivity of $T_e$ and (ii). Let, for $k = 1, 2$,

$$x_k \in X, \ z_k \in M, \ n_k \in \mathbb{N} \text{ and } n_kx_k \leq z_k.$$ 

Then

$$n_1n_2(x_1 + x_2) \leq n_2z_1 + n_1z_2,$$

so that

$$T_e(x_1 + x_2) \leq \frac{1}{n_1n_2} T(n_2z_1 + n_1z_2) = \frac{1}{n_1} T(z_1) + \frac{1}{n_2} T(z_2).$$

Moreover, in view of a well-known formula (see, e.g., [4], 1.6.1), we have

$$n_1n_2(x_1 \vee x_2) = (n_1n_2x_1) \vee (n_1n_2x_2) \leq (n_2z_1) \vee (n_1z_2),$$

so that

$$T_e(x_1 \vee x_2) \leq \frac{1}{n_1n_2} T((n_2z_1) \vee (n_1z_2)) = \left( \frac{1}{n_1} T(z_1) \right) \vee \left( \frac{1}{n_2} T(z_2) \right).$$

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REFERENCES