Note on “\((\alpha, \beta)\)-fuzzy ideals of hemirings”

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1. Introduction

Dudek et al. [1] introduced the concept of \((\varepsilon, \varepsilon \lor q)\)-fuzzy h-ideal (k-ideal) of a hemiring, and gave its characterizations by using subsets \(U(\lambda; t), Q(\lambda; t)\) and \([\lambda]_t\), where \(\lambda\) is a fuzzy subset of a hemiring \(R\) and \(t \in A \subseteq (0, 1]\). They said \(U(\lambda; t)\) and \([\lambda]_t\) are ideals for every \(t \in (0, 0.5]\), and stated that \(Q(\lambda; t)\) is not an ideal in general by providing an example (see [1, Example 4.19]). We note that the given fuzzy subset \(\lambda\) in [1, Example 4.19] is not an \((\varepsilon, \varepsilon \lor q)\)-fuzzy ideal of \(R\), and \(U(\lambda; t)\) and \([\lambda]_t\) are not ideals in [1, Example 4.19].

The first aim of this paper is to show that [1, Theorem 4.17] and [1, Corollary 4.18] coincide with [1, Theorem 4.11] and [1, Corollary 4.12] respectively. The second purpose of this article is to verify that the given fuzzy subset \(\lambda\) in [1, Example 4.19] is not an \((\varepsilon, \varepsilon \lor q)\)-fuzzy ideal of \(R\), and then we prove that if \(\lambda\) is an \((\varepsilon, \varepsilon \lor q)\)-fuzzy h-ideal of \(R\), then the set \(Q(\lambda; t)\) is an h-ideal of \(R\) when it is nonempty for all \(t \in (0.5, 1]\) (see Theorem 3.7). Finally, we discuss more updated results than [1, Theorem 4.17] and [1, Corollary 4.18].

2. Preliminaries

A semiring is an algebraic system \((R, +, \cdot)\) consisting of a non-empty set \(R\) together with two binary operations called addition (+) and multiplication (\(\cdot\)), here \(x, y\) will be denoted by juxtaposition for all \(x, y \in R\), such that \((R, +)\) and \((R, \cdot)\) are semigroups connected by the following distributive laws: \((b + c)a = ba + ca\) and \((b + c)a = ba + ca\) for all \(a, b, c \in R\). An element \(0 \in R\) is called a zero of \(R\) if \(a + 0 = 0 + a = a\) for all \(a \in R\). A semiring with zero and a commutative addition is called a hemiring. A non-empty subset \(I\) of a semiring \(R\) is said to be a left (resp. right) ideal of \(R\) if it is closed under the addition and \(RI \subseteq I\) (resp. \(IR \subseteq I\)). A left ideal which is also a right ideal is called an ideal. A left (resp. right) ideal \(I\) of a hemiring \(R\) is called a left (resp. right) \(k\)-ideal of \(R\) if for any \(a, b \in I\) and \(x \in R\) whenever \(x + a = b\) then \(x \in I\). A left (resp. right) ideal \(I\) of a hemiring \(R\) is called a left (resp. right) \(h\)-ideal of \(R\) if for any \(a, b \in I\) and all \(x, y \in R\) whenever \(x + a + y = b + y\) then \(x \in I\). Every left (resp. right) \(h\)-ideal is a left (resp. right) \(k\)-ideal but the converse is not true in general. For a set \(R\), let

\[ \mathcal{F}(R) := \{ \lambda : \lambda : R \to [0, 1] \text{ is a mapping} \} \]

Elements of \(\mathcal{F}(R)\) are called fuzzy subsets of \(R\). For any \(\lambda \in \mathcal{F}(R)\) and any \(t \in [0, 1]\), the set

\[ U(\lambda; t) = \{ x \in R \mid \lambda(x) \geq t \} \]
is called a level subset of \( \lambda \). Given a point \( x \in R \), consider a mapping

\[
\lambda : R \rightarrow [0, 1], \quad y \mapsto \begin{cases} t \in (0, 1) & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}
\]

Then \( \lambda \in \mathcal{F}(R) \), and it is said to be a fuzzy point with support \( x \) and value \( t \) and is denoted by \( x_t \).

For a fuzzy point \( x_t \) and a fuzzy subset \( \lambda \) of a set \( R \), Pu and Liu [2] introduced the symbol \( x_t \alpha \lambda \), where \( \alpha \in \{ e, q, \epsilon \vee q, \epsilon \wedge q \} \). To say that \( x_t \in \lambda \) (resp. \( x_t q \lambda \)), we mean \( \lambda(x) \geq t \) (resp. \( \lambda(x) + t > 1 \)), and in this case, \( x_t \) is said to belong to (resp. be quasi-coincident with) a fuzzy subset \( \lambda \). To say that \( x_t \in \epsilon \vee q \lambda \) (resp. \( x_t \in \epsilon \wedge q \lambda \)), we mean \( x_t \in \lambda \) or \( x_t q \lambda \) (resp. \( x_t \in \lambda \) and \( x_t q \lambda \)).

A fuzzy subset \( \lambda \) of a hemiring \( R \) is called an \((e, \epsilon \vee q)\)-fuzzy left (resp. right) ideal of \( R \) (see [1]) if

\[
x_{t_1} \in \lambda, \quad y_{t_2} \in \lambda \implies (x + y)_{\min(t_1, t_2)} \in \epsilon \vee q \lambda, \quad (2.1)
x_t \in \lambda \implies (yx)_t \in \epsilon \vee q \lambda \quad \text{(resp. } (xy)_t \in \epsilon \vee q \lambda) \quad (2.2)
\]

for all \( x, y \in R \) and \( t_1, t_2 \in (0, 1] \). A fuzzy subset which is an \((e, \epsilon \vee q)\)-fuzzy left and right ideal is called an \((e, \epsilon \vee q)\)-fuzzy ideal.

An \((e, \epsilon \vee q)\)-fuzzy ideal \( \lambda \) of a hemiring \( R \) satisfying the following condition:

\[
x + a = b, \quad a_{t_1} \in \lambda, \quad b_{t_2} \in \lambda \implies x_{\min(t_1, t_2)} \in \epsilon \vee q \lambda \quad (2.3)
\]

for all \( a, b, x \in R \) and \( t_1, t_2 \in (0, 1] \) is called an \((e, \epsilon \vee q)\)-fuzzy \( k \)-ideal of \( R \) (see [1]).

An \((e, \epsilon \vee q)\)-fuzzy ideal \( \lambda \) of a hemiring \( R \) satisfying the following condition:

\[
x + a + y = b + y, \quad a_{t_1} \in \lambda, \quad b_{t_2} \in \lambda \implies x_{\min(t_1, t_2)} \in \epsilon \vee q \lambda \quad (2.4)
\]

for all \( a, b, x \in R \) and \( t_1, t_2 \in (0, 1] \) is called an \((e, \epsilon \vee q)\)-fuzzy \( h \)-ideal of \( R \) (see [1]).

**Lemma 2.1** ([1]). A fuzzy subset \( \lambda \) of a hemiring \( R \) is an \((e, \epsilon \vee q)\)-fuzzy \( h \)-ideal of \( R \) if and only if it satisfies:

1. \( (\forall x, y \in R) (\lambda(x + y) \geq \min(\lambda(x), \lambda(y), 0.5)) \).
2. \( (\forall x, y \in R) (\lambda(xy) \geq \min(\lambda(x), 0.5)) \).
3. \( (\forall x, y \in R) (\lambda(xy) \geq \min(\lambda(x), 0.5)) \).
4. \( (\forall a, b, x, y \in R) (x + a + y = b + y \Rightarrow \lambda(x) \geq \min(\lambda(a), \lambda(b), 0.5)) \).

**Corollary 2.2** ([1]). A fuzzy subset \( \lambda \) of a hemiring \( R \) is an \((e, \epsilon \vee q)\)-fuzzy \( k \)-ideal of \( R \) if and only if it satisfies:

1. \( (\forall x, y \in R) (\lambda(x + y) \geq \min(\lambda(x), \lambda(y), 0.5)) \).
2. \( (\forall x, y \in R) (\lambda(xy) \geq \min(\lambda(x), 0.5)) \).
3. \( (\forall x, y \in R) (\lambda(xy) \geq \min(\lambda(x), 0.5)) \).
4. \( (\forall a, b, x \in R) (x + a = b \Rightarrow \lambda(x) \geq \min(\lambda(a), \lambda(b), 0.5)) \).

**3. Note on \((\alpha, \beta)\)-fuzzy ideals of hemirings**

In what follows, let \( R \) denote a hemiring. For any fuzzy subset \( \lambda \) of \( R \) and any \( t \in (0, 1] \), we consider two subsets:

\[
Q(\lambda; t) := \{ x \in R \mid x_t \in \epsilon \vee q \lambda \} \quad \text{and} \quad [\lambda]_t := \{ x \in R \mid x_t \in \epsilon \vee q \lambda \}.
\]

It is clear that \( [\lambda]_t = U(\lambda; t) \cup Q(\lambda; t) \) (see [1]). Dudek et al. [1] discussed the following results:

**Theorem 3.1** ([1, Theorem 4.11]). A fuzzy subset \( \lambda \) of \( R \) is an \((e, \epsilon \vee q)\)-fuzzy (left, right) ideal of \( R \) if and only if \( U(\lambda; t) \neq \emptyset \) is a (left, right) ideal of \( R \) for all \( 0 < t \leq 0.5 \).

**Corollary 2.2** ([1, Corollary 4.12]). A fuzzy subset \( \lambda \) of \( R \) is an \((e, \epsilon \vee q)\)-fuzzy (left, right) \( h \)-ideal (k-ideal) of \( R \) if and only if \( U(\lambda; t) \neq \emptyset \) is a (left, right) \( h \)-ideal (k-ideal) of \( R \) for all \( 0 < t \leq 0.5 \).

**Theorem 3.3** ([1, Theorem 4.17]). A fuzzy subset \( \lambda \) of \( R \) is an \((e, \epsilon \vee q)\)-fuzzy (left, right) ideal of \( R \) if and only if \([\lambda]_t \) is a (left, right) ideal of \( R \) for all \( t \in (0, 0.5] \).

**Corollary 3.4** ([1, Corollary 4.18]). A fuzzy subset \( \lambda \) of \( R \) is an \((e, \epsilon \vee q)\)-fuzzy (left, right) \( h \)-ideal (k-ideal) of \( R \) if and only if \([\lambda]_t \) is a (left, right) \( h \)-ideal (k-ideal) of \( R \) for all \( t \in (0, 0.5] \).

We know that the set \( Q(\lambda; t) \) has no any role in Theorem 3.3 and Corollary 3.4. We first consider the following lemma.
Lemma 3.5. Every fuzzy subset $\lambda$ of $R$ satisfies the following assertion:

\[ t \in (0, 0.5) \implies [\lambda]_t = U(\lambda; t). \]  

(3.1)

Proof. Let $t \in (0, 0.5)$. It is clear that $U(\lambda; t) \subseteq [\lambda]_t$. Let $x \in [\lambda]_t$. If $x \notin U(\lambda; t)$, then $\lambda(x) < t$ and so $\lambda(x) + t \leq 2t < 1$. This shows that $x \notin Q(\lambda; t)$, i.e., $x \notin Q(\lambda; t)$ and thus $x \notin U(\lambda; t) \cup Q(\lambda; t) = [\lambda]_t$. This is a contradiction, and thus $x \in U(\lambda; t)$. Therefore $[\lambda]_t \subseteq U(\lambda; t)$. \qed

According to Lemma 3.5, we know that the set $Q(\lambda; t)$ has no any role in Theorem 3.3 and Corollary 3.4. Hence Theorem 3.3 and Corollary 3.4 coincide with Theorem 3.1 and Corollary 3.2, respectively.

In [1], the authors said that “$U(\lambda; t)$ and $[\lambda]_t$ are ideals for every $t \in (0, 0.5)$, but $Q(\lambda; t)$ is not an ideal in general”. They provided an example to show that $Q(\lambda; t)$ is not an ideal, as follows:

Example 3.6 ([1, Example 4.19]). Consider a hemiring $R = \{0, 1, a, b, c\}$ with two binary operations defined by Table 1. Let $\lambda$ be a fuzzy subset of $R$ defined by

$\lambda(x) := \begin{cases} 1 & \text{if } x \in [0, a] \\ 0.4 & \text{if } x \in [b, c], \\ 0 & \text{if } x = 1. \end{cases}$

Then $Q(\lambda; t) = \{0, a\}$ for $0 < t \leq 0.6$, and $Q(\lambda; t) = \{0, a, b, c\}$ for $0.6 < t < 1$. It is not difficult to see that $Q(\lambda; t)$ is not an ideal of a hemiring $R$ for any $t \in (0, 1)$.

Of course, in this example, $Q(\lambda; t)$ is not an ideal of $R$. But we know that $U(\lambda; t)$ and $[\lambda]_t$ are not ideals for some $t \in (0, 0.5)$. In fact,

$U(\lambda; t) = \begin{cases} [0, a, b, c] & \text{if } 0 < t \leq 0.4, \\ [0, a] & \text{if } 0.4 < t \leq 0.5. \end{cases}$

Note that $\{0, a, b, c\}$ is not an ideal of $R$. Hence if $t \in (0, 0.4)$, then $U(\lambda; t)$ is not an ideal of $R$. If $t \in (0, 0.5)$, then $[\lambda]_t = U(\lambda; t)$ by Lemma 3.5. Thus $[\lambda]_t$ is also not an ideal of $R$ for any $t \in (0, 0.4)$. We also note that the given fuzzy subset $\lambda$ in Example 3.6 is not an $(\varepsilon, \in, \vee, q)$-fuzzy ideal of $R$ since $b_{0.2} \in \lambda$ and $b_{0.3} \in \lambda$, but $(b + b)_{\min(0.2, 0.3)} = 1_{0.2} \in \vee q \lambda$.

Now, we would like to give a role to the set $Q(\lambda; t)$.

Theorem 3.7. If $\lambda$ is an $(\varepsilon, \in, \vee, q)$-fuzzy h-ideal of $R$, then the set $Q(\lambda; t)$ is an h-ideal of $R$ when it is nonempty for all $t \in (0, 0.5)$.

Proof. Assume that $\lambda$ is an $(\varepsilon, \in, \vee, q)$-fuzzy h-ideal of $R$ and let $t \in (0.5, 1)$ be such that $Q(\lambda; t) \neq \emptyset$. Let $x, y \in Q(\lambda; t)$. Then $\lambda(x) + t > 1$ and $\lambda(y) + t > 1$. Using Lemma 2.1(1), we have

$\lambda(x + y) \geq \min(\lambda(x), \lambda(y), 0.5).$  

(3.2)

If $\min(\lambda(x), \lambda(y)) \geq 0.5$, then $\lambda(x + y) \geq 0.5 > 1 - t$ by (3.2). If $\min(\lambda(x), \lambda(y)) < 0.5$, then $\lambda(x + y) \geq \min(\lambda(x), \lambda(y)) > 1 - t$ by (3.2). Hence $x + y \in Q(\lambda; t)$. Let $x \in Q(\lambda; t)$ and $y \in R$. Then $\lambda(x) + t > 1$. If $\lambda(x) \geq 0.5$, then

$\lambda(y) \geq \min(\lambda(x), 0.5) = 0.5 > 1 - t,$

and if $\lambda(x) < 0.5$, then $\lambda(y) \geq \min(\lambda(x), 0.5) = \lambda(x) > 1 - t$ by Lemma 2.1(2). Thus $xy \in Q(\lambda; t)$. Similarly, $yx \in Q(\lambda; t)$. Hence $Q(\lambda; t)$ is an ideal of $R$. Now, let $a, b \in Q(\lambda; t)$ and $x, y \in R$ be such that $x + a + y = b + y$. Then $\lambda(a) + t > 1$ and $\lambda(b) + t > 1$. Using Lemma 2.1(4), we get

$\lambda(x) \geq \min(\lambda(a), \lambda(b), 0.5).$  

(3.3)

If $\min(\lambda(a), \lambda(b)) \geq 0.5$, then $\lambda(x) \geq 0.5 > 1 - t$, and if $\min(\lambda(a), \lambda(b)) < 0.5$, then $\lambda(x) \geq \min(\lambda(a), \lambda(b)) > 1 - t$ by (3.3). Therefore $x \in Q(\lambda; t)$, and consequently $Q(\lambda; t)$ is an h-ideal of $R$. \qed

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Taking \( y = 0 \) in Theorem 3.7 and using Corollary 2.2, we have the following corollary.

**Corollary 3.8.** If \( \lambda \) is an \( (\varepsilon, \in \vee q) \)-fuzzy \( k \)-ideal of \( R \), then the set \( Q(\lambda; t) \) is a \( k \)-ideal of \( R \) when it is nonempty for all \( t \in (0.5, 1] \).

The following theorem is a more updated result than [1, Theorem 4.17].

**Theorem 3.9.** For any fuzzy subset \( \lambda \) of \( R \), the following are equivalent:

1. \( \lambda \) is an \( (\varepsilon, \in \vee q) \)-fuzzy \( h \)-ideal of \( R \).
2. \( \forall t \in (0, 1) \) \( \{ \lambda \} \neq \emptyset \implies \{ \lambda \} \) is an \( h \)-ideal of \( R \).

**Proof.** Assume that \( \lambda \) is an \( (\varepsilon, \in \vee q) \)-fuzzy \( h \)-ideal of \( R \) and let \( t \in (0, 1] \) be such that \( \{ \lambda \} \neq \emptyset \). Let \( x, y \in \{ \lambda \} \). Then
\[
\lambda(x) \geq t \text{ or } \lambda(y) + t > 1, \quad \text{and } \lambda(y) \geq t \text{ or } \lambda(y) + t > 1.
\]
We can consider four cases:

1. \( \lambda(x) \geq t \) and \( \lambda(y) \geq t \),
2. \( \lambda(x) \geq t \) and \( \lambda(y) + t > 1 \),
3. \( \lambda(x) + t > 1 \) and \( \lambda(y) \geq t \),
4. \( \lambda(x) + t > 1 \) and \( \lambda(y) + t > 1 \).

For the first case, Lemma 2.1(1) implies that
\[
\lambda(x + y) \geq \min\{\lambda(x), \lambda(y), 0.5\} \geq \min\{t, 0.5\} = \begin{cases} 0.5 & \text{if } t > 0.5, \\ t & \text{if } t \leq 0.5, \end{cases}
\]
and so \( \lambda(x + y) + t > 0.5 + 0.5 = 1 \), i.e., \( x + y \in U(\lambda; t) \). Therefore \( x + y \in U(\lambda; t) \cup Q(\lambda; t) = \{ \lambda \} \). For the case (ii), assume that \( t > 0.5 \). Then \( 1 - t < 0.5 \). If \( \min\{\lambda(y), 0.5\} \leq \lambda(y) \), then
\[
\lambda(x + y) \geq \min\{\lambda(y), 0.5\} > 1 - t,
\]
and if \( \min\{\lambda(y), 0.5\} > \lambda(y) \), then \( \lambda(x + y) \geq \lambda(y) \). Hence \( x + y \in Q(\lambda; t) \cup U(\lambda; t) = \{ \lambda \} \) for \( t > 0.5 \). Suppose that \( t \leq 0.5 \). Then \( 1 - t \geq 0.5 \). If \( \min\{\lambda(x), 0.5\} \leq \lambda(x) \), then
\[
\lambda(x + y) \geq \min\{\lambda(x), 0.5\} \geq t,
\]
and if \( \min\{\lambda(x), 0.5\} > \lambda(x) \), then \( \lambda(x + y) \geq \lambda(y) \). \( x + y \in U(\lambda; t) \cup Q(\lambda; t) = \{ \lambda \} \) for \( t \leq 0.5 \). We have similar result for the case (iii). For the final case, if \( t > 0.5 \) then \( 1 - t < 0.5 \). Hence
\[
\lambda(x + y) \geq \min\{\lambda(x), \lambda(y), 0.5\} \geq \begin{cases} 0.5 & \text{if } \min\{\lambda(x), \lambda(y)\} \geq 0.5, \\ 1 - t & \text{if } \min\{\lambda(x), \lambda(y)\} < 0.5, \end{cases}
\]
and so \( x + y \in Q(\lambda; t) \subseteq \{ \lambda \} \) for \( t \leq 0.5 \). Thus
\[
\lambda(x + y) \geq \min\{\lambda(x), \lambda(y), 0.5\} \geq \begin{cases} 0.5 & \text{if } \min\{\lambda(x), \lambda(y)\} \geq 0.5, \\ 1 - t & \text{if } \min\{\lambda(x), \lambda(y)\} < 0.5, \end{cases}
\]
which implies that \( x + y \in U(\lambda; t) \cup Q(\lambda; t) = \{ \lambda \} \). Let \( x \in \{ \lambda \} \) and \( y \in R \). Then \( \lambda(x) \geq t \) or \( \lambda(x) + t > 1 \). Assume that \( \lambda(x) \geq t \). Lemma 2.1(2) implies that
\[
\lambda(xy) \geq \min\{\lambda(x), 0.5\} \geq \min\{t, 0.5\} \geq \begin{cases} t & \text{if } t \leq 0.5, \\ 0.5 & \text{if } t > 0.5, \end{cases}
\]
so that \( xy \in U(\lambda; t) \cup Q(\lambda; t) = \{ \lambda \} \). Suppose that \( \lambda(x) + t > 1 \). If \( t > 0.5 \), then
\[
\lambda(xy) \geq \min\{\lambda(x), 0.5\} = \begin{cases} 0.5 & \text{if } \lambda(x) \geq 0.5, \\ \lambda(x) + 1 - t & \text{if } \lambda(x) < 0.5, \end{cases}
\]
and thus \( xy \in Q(\lambda; t) \subseteq \{ \lambda \} \). If \( t \leq 0.5 \) then
\[
\lambda(xy) \geq \min\{\lambda(x), 0.5\} = 0.5 \geq t
\]
and so \( xy \in U(\lambda; t) \subseteq \{ \lambda \} \). Similarly, \( xy \in \{ \lambda \} \). Now, let \( a, b \in \{ \lambda \} \) and \( x, y \in R \) be such that \( x + a + y = b + y \). Then we have the following four cases:
\[
\begin{align*}
\lambda(a) & \geq t \quad \text{and} \quad \lambda(b) \geq t, \\
\lambda(a) & \geq t \quad \text{and} \quad \lambda(b) + t > 1, \\
\lambda(a) + t & > 1 \quad \text{and} \quad \lambda(b) \geq t, \\
\lambda(a) + t & > 1 \quad \text{and} \quad \lambda(b) + t > 1.
\end{align*}
\]
and we have
\[ \lambda(x) \geq \min(\lambda(a), \lambda(b), 0.5) \]  
(3.8) 
by Lemma 2.1(4). Using (3.4) and (3.8), we get \( \lambda(x) \geq \min(t, 0.5) \). If \( t \leq 0.5 \), then \( \lambda(x) \geq t \), i.e., \( x \in U(\lambda; t) \subseteq [\lambda]_t \). If \( t > 0.5 \), then \( \lambda(x) \geq 0.5 \), and so \( \lambda(x) + t > 0.5 + 0.5 = 1 \). Hence \( x \in Q(\lambda; t) \subseteq [\lambda]_t \). For the case (3.5), if \( t > 0.5 \) then \( 1 - t < 0.5 \) and \( \lambda(a) \geq t > 0.5 \) and so \( \lambda(x) \geq \min(\lambda(a), \lambda(b), 0.5) \) \[ \begin{cases} 0.5 > 1 - t & \text{if } \lambda(b) \geq 0.5, \\ \lambda(b) > 1 - t & \text{if } \lambda(b) < 0.5. \end{cases} \]
Hence \( x \in Q(\lambda; t) \subseteq [\lambda]_t \). If \( t \leq 0.5 \) then \( \lambda(x) \geq \min(\lambda(a), \lambda(b), 0.5) \) \[ \begin{cases} \min(\lambda(a), 0.5) \geq t & \text{if } \min(\lambda(a), 0.5) < \lambda(b), \\ \lambda(b) > 1 - t & \text{if } \min(\lambda(a), 0.5) \geq \lambda(b) \end{cases} \]
which implies that \( x \in U(\lambda; t) \cup Q(\lambda; t) \subseteq [\lambda]_t \). Similarly, we have \( x \in [\lambda]_t \) from (3.6) and (3.8). For the case (3.7), assume first that \( t > 0.5 \). Then \( 1 - t < 0.5 \). If \( \min(\lambda(a), \lambda(b)) \geq 0.5 \), then \( \lambda(x) \geq \min(\lambda(a), \lambda(b), 0.5) = 0.5 > 1 - t \), and if \( \min(\lambda(a), \lambda(b)) < 0.5 \) then \( \lambda(x) \geq \min(\lambda(a), \lambda(b), 0.5) = \min(\lambda(a), \lambda(b)) > 1 - t \).
Therefore \( x \in Q(\lambda; t) \subseteq [\lambda]_t \). Now suppose that \( t \leq 0.5 \). Then \( 1 - t \geq 0.5 \). If \( \min(\lambda(a), \lambda(b)) \geq 0.5 \), then \( \lambda(x) \geq \min(\lambda(a), \lambda(b), 0.5) = 0.5 \geq t \).
If \( \min(\lambda(a), \lambda(b)) < 0.5 \) then \( \lambda(x) \geq \min(\lambda(a), \lambda(b), 0.5) = \min(\lambda(a), \lambda(b)) > 1 - t \).
Thus \( x \in U(\lambda; t) \cup Q(\lambda; t) = [\lambda]_t \). Consequently, \( [\lambda]_t \) is an \( h \)-ideal of \( R \).
Conversely, suppose that (2) is valid. If there exist \( x_0, y_0 \in R \) such that \( \lambda(x_0 + y_0) < \min(\lambda(x_0), \lambda(y_0), 0.5) \), then \( \lambda(x_0 + y_0) < t_0 \leq \min(\lambda(x_0), \lambda(y_0), 0.5) \) for some \( t_0 \in (0, 0.5] \). It follows that \( x_1, y_1 \in U(\lambda; t_0) \subseteq [\lambda]_{t_0} \), but so \( x_1 + y_1 \in [\lambda]_{t_0} \) and \( \lambda(x_1 + y_1) > t_0 \), a contradiction. Therefore \( \lambda(x + y) \geq \min(\lambda(x), \lambda(y), 0.5) \) for all \( x, y \in R \). Assume that \( \lambda(ba) < \min(\lambda(a), 0.5) \) for some \( a, b \in R \). Then there exists \( t_b \in (0, 1] \) such that \( \lambda(ba) < t_b \leq \min(\lambda(a), 0.5) \). Then \( t_b \in (0, 0.5] \) and \( a \in U(\lambda; t_b) \subseteq [\lambda]_{t_b} \), but \( ba \in U(\lambda; t_b) \cup Q(\lambda; t_b) \subseteq [\lambda]_{t_b} \), a contradiction. Hence \( \lambda(xy) \geq \min(\lambda(x), 0.5) \) for all \( x, y \in R \). Similarly, \( \lambda(xy) \geq \min(\lambda(x), 0.5) \) for all \( x, y \in R \). Finally, let \( a, b, x, y \in R \) be such that \( x + a + y = b + y \). Suppose that \( \lambda(x) < \min(\lambda(a), \lambda(b), 0.5) \). Then there exists \( t_x \in (0, 1] \) such that \( \lambda(x) < t_x \). Then \( x \in U(\lambda; t_x) \subseteq [\lambda]_{t_x} \), but \( x \notin U(\lambda; t_x) \subseteq [\lambda]_{t_x} \), a contradiction. Hence \( \lambda(xy) \geq \min(\lambda(x), 0.5) \) for all \( x, y \in R \). Similarly, \( \lambda(xy) \geq \min(\lambda(x), 0.5) \) for all \( x, y \in R \). Consequently, \( x \notin [\lambda]_{t_x} \), a contradiction. Therefore \( \lambda(x) \geq \min(\lambda(a), \lambda(b), 0.5) \).

The following corollary is a more updated result than [1, Corollary 4.18].

**Corollary 3.10.** For any fuzzy subset \( \lambda \) of \( R \), the following are equivalent:

1. \( \lambda \) is an \((\epsilon, \epsilon \vee q)\)-fuzzy \( k \)-ideal of \( R \).
2. \( \forall t \in (0, 1)] ([\lambda]_t \neq \emptyset \implies [\lambda]_t \) is a \( k \)-ideal of \( R \).

**References**