Journal of Algebra 324 (2010) 2971–2989

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Classification of group gradings on simple Lie algebras of types $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ ☆

Yuri Bahturin *, Mikhail Kochetov

Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NL, A1C5S7, Canada

ARTICLE INFO

Article history:
Received 13 December 2009
Available online 17 March 2010
Communicated by Nicolás Andruskiewitsch and Robert Guralnick

MSC:
primary 17B70
secondary 17B60

Keywords:
Graded algebra
Simple Lie algebra
Grading
Involution

ABSTRACT

For a given abelian group G, we classify the isomorphism classes of G-gradings on the simple Lie algebras of types $\mathcal{A}_n$ $(n \geqslant 1)$, $\mathcal{B}_n$ $(n \geqslant 2)$, $\mathcal{C}_n$ $(n \geqslant 3)$ and $\mathcal{D}_n$ $(n > 4)$, in terms of numerical and group-theoretical invariants. The ground field is assumed to be algebraically closed of characteristic different from 2.

1. Introduction

Let U be an algebra (not necessarily associative) over a field $\mathbb{F}$ and let G be an abelian group, written multiplicatively.

Definition 1.1. A *G-grading* on U is a vector space decomposition

$$U = \bigoplus_{g \in G} U_g$$

☆ The first author acknowledges support by NSERC grant # 227060-04. The second author acknowledges support by NSERC Discovery Grant # 341792-07.

* Corresponding author.
E-mail addresses: bahturin@mun.ca (Y. Bahturin), mikhail@mun.ca (M. Kochetov).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2010.03.003

such that

$$U_g U_h \subset U_{gh} \quad \text{for all } g, h \in G.$$

U_g is called the *homogeneous component* of degree g. The *support* of the G-grading is the set

$$\{g \in G \mid U_g \neq 0\}.$$

Definition 1.2. We say that two G-gradings, $U = \bigoplus_{g\in G} U_g$ and $U = \bigoplus_{g\in G} U'_g$, are *isomorphic* if there exists an algebra automorphism $\psi : U \to U$ such that

$$\psi(U_g) = U'_g \quad \text{for all } g \in G,$$

i.e., $U = \bigoplus_{g\in G} U_g$ and $U = \bigoplus_{g\in G} U'_g$ are isomorphic as G-graded algebras.

The purpose of this paper is to classify, for a given abelian group G, the isomorphism classes of G-gradings on the classical simple Lie algebras of types $\mathcal{A}_n$ ($n \geqslant 1$), $\mathcal{B}_n$ ($n \geqslant 2$), $\mathcal{C}_n$ ($n \geqslant 3$) and $\mathcal{D}_n$ ($n > 4$), in terms of numerical and group-theoretical invariants. Descriptions of such gradings were obtained in [4,8,5,2,1], but the question of distinguishing non-isomorphic gradings was not addressed in those papers. Also, A. Elduque [13] has recently found a counterexample to [8, Proposition 6.4], which was used in the description of gradings on Lie algebras of type $\mathcal{A}$. The fine gradings (i.e., those that cannot be refined) on Lie algebras of types $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ (including $\mathcal{D}_4$) have been classified, up to equivalence, in [13] over algebraically closed fields of characteristic zero. For a discussion of the difference between classification up to equivalence and classification up to isomorphism see [16]. The two kinds of classification cannot be easily obtained from each other.

We will assume throughout this paper that the ground field $\mathbb{F}$ is algebraically closed. We will usually assume that $\operatorname{char}\mathbb{F} \neq 2$ and in one case also $\operatorname{char}\mathbb{F} \neq 3$. We obtain a description of gradings in type $\mathcal{A}$ without using [8, Proposition 6.4] and with methods simpler than those in [2,1]. We also obtain invariants that allow us to distinguish among non-isomorphic gradings in types $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$.

The paper is structured as follows. In Section 2 we recall the description of G-gradings on a matrix algebra $R = M_n(\mathbb{F})$ and determine when two such gradings are isomorphic (Theorem 2.6). We also obtain a canonical form for an anti-automorphism of R that preserves the grading and restricts to an involution on the identity component R_e (Theorem 2.10). In particular, this allows us to classify (up to isomorphism) the pairs (R, φ) where $R = M_n(\mathbb{F})$ is G-graded and φ is an involution that preserves the grading (Corollary 2.15). In Section 3 we use affine group schemes to show how one can reduce the classification of G-gradings on classical simple Lie algebras to the classification of G-gradings on $R = M_n(\mathbb{F})$ and of the pairs (R, φ) where φ is an involution or an anti-automorphism satisfying certain properties. In Section 4 we obtain a classification of G-gradings on simple Lie algebras of type $\mathcal{A}$—see Theorem 4.9. Finally, in Section 5 we state a classification of G-gradings on simple Lie algebras of types $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ (except $\mathcal{D}_4$)—see Theorem 5.2, which is an immediate consequence of Corollary 2.15.

2. Gradings on matrix algebras

Let $R = M_n(\mathbb{F})$ where $\mathbb{F}$ is an algebraically closed field of arbitrary characteristic. Let G be an abelian group. A description of G-gradings on R was obtained in [3,7,6]. In this section we restate that description in a slightly different form and obtain invariants that allow us to distinguish among non-isomorphic gradings. Criteria for isomorphism of the so-called “elementary” gradings (see below) on matrix algebras $M_n(\mathbb{F})$ and on the algebra of finitary matrices were obtained in [11] and [9], respectively.

We start with gradings $R=\bigoplus_{g\in G}R_g$ with the property $\dim R_g\leqslant 1$ for all $g\in G$. As shown in the proof of [3, Theorem 5], R is then a graded division algebra, i.e., any nonzero homogeneous element is invertible in R. Consequently, the support $T\subset G$ of the grading is a subgroup. Following [13], we will call such $R=\bigoplus_{g\in G}R_g$ a *division grading* (the terms used in [3,7,6] and in [14] are "fine gradings" and "Pauli gradings", respectively). Note that since $R\cong\mathbb{F}^\sigma T$ is semisimple, $\operatorname{char}\mathbb{F}$ does not divide $n^2=|T|$.

For each $t\in T$, let X_t be a nonzero element in the component R_t. Then

$$X_uX_v=\sigma(u,v)X_{uv}$$

for some nonzero scalar $\sigma(u,v)$. Clearly, the function $\sigma:T\times T\to\mathbb{F}^\times$ is a 2-cocycle, and the G-graded algebra R is isomorphic to the twisted group algebra $\mathbb{F}^\sigma T$ (with its natural T-grading regarded as a G-grading). Rescaling the elements X_t corresponds to replacing σ with a cohomologous cocycle. Let

$$\beta_\sigma(u,v):=\frac{\sigma(u,v)}{\sigma(v,u)}.$$

Then $\beta=\beta_\sigma$ depends only on the class of σ in $H^2(T,\mathbb{F}^\times)$ and $\beta:T\times T\to\mathbb{F}^\times$ is an *alternating bicharacter*, i.e., it is multiplicative in each variable and has the property $\beta(t,t)=1$ for all $t\in T$.

Clearly, $X_uX_v=\beta(u,v)X_vX_u$. Since the centre $Z(R)$ is spanned by the identity element, β is *nondegenerate* in the sense that $\beta(u,t)=1$ for all $u\in T$ implies $t=e$. Conversely, if σ is a 2-cocycle such that β_σ is nondegenerate, then $\mathbb{F}^\sigma T$ is a semisimple associative algebra whose centre is spanned by the identity element, so $\mathbb{F}^\sigma T$ is isomorphic to R. Therefore, the isomorphism classes of division G-gradings on $R=M_n(\mathbb{F})$ with support $T\subset G$ are in one-to-one correspondence with the classes $[\sigma]\in H^2(T,\mathbb{F}^\times)$ such that β_σ is nondegenerate.

The classes $[\sigma]$ and the corresponding gradings on R can be found explicitly as follows. As shown in the proof of [3, Theorem 5], there exists a decomposition of T into the direct product of cyclic subgroups:

$$T=H_1'\times H_1''\times\cdots\times H_r'\times H_r'' \tag{1}$$

such that $H_i'\times H_i''$ and $H_j'\times H_j''$ are β-orthogonal for $i\neq j$, and H_i' and H_i'' are in duality by β. Denote by ℓ_i the order of H_i' and H_i''. If we pick generators a_i and b_i for H_i' and H_i'', respectively, then $\varepsilon_i:=\beta(a_i,b_i)$ is a primitive ℓ_i-th root of unity, and all other values of β on the elements $a_1,b_1,\dots,a_r,b_r$ are 1. Pick elements $X_{a_i}\in R_{a_i}$ and $X_{b_i}\in R_{b_i}$ such that $X_{a_i}^{\ell_i}=X_{b_i}^{\ell_i}=1$. Then we obtain an isomorphism $\mathbb{F}^\sigma T\to M_{\ell_1}(\mathbb{F})\otimes\cdots\otimes M_{\ell_r}(\mathbb{F})$ defined by

$$X_{a_i}\mapsto I\otimes\cdots\otimes I\otimes X_i\otimes I\otimes\cdots\otimes I\quad\text{and}\quad X_{b_i}\mapsto I\otimes\cdots\otimes I\otimes Y_i\otimes I\otimes\cdots\otimes I, \tag{2}$$

where

$$X_i=\begin{bmatrix}\varepsilon_i^{n-1} & 0 & 0 & \dots & 0 & 0\\ 0 & \varepsilon_i^{n-2} & 0 & \dots & 0 & 0\\ \dots & & & & & \\ 0 & 0 & 0 & \dots & \varepsilon_i & 0\\ 0 & 0 & 0 & \dots & 0 & 1\end{bmatrix}\quad\text{and}\quad Y_i=\begin{bmatrix}0 & 1 & 0 & \dots & 0 & 0\\ 0 & 0 & 1 & \dots & 0 & 0\\ \dots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1\\ 1 & 0 & 0 & \dots & 0 & 0\end{bmatrix} \tag{3}$$

are in the i-th factor, $M_{\ell_i}(\mathbb{F})$.

It follows that the class $[\sigma]\in H^2(T,\mathbb{F}^\times)$, and hence the isomorphism class of the G-graded algebra $\mathbb{F}^\sigma T$, is uniquely determined by $\beta=\beta_\sigma$. Conversely, since the relation $X_uX_v=\beta(u,v)X_vX_u$ does not change when we rescale X_u and X_v, the values of β are determined by the G-grading. We summarize our discussion in the following

Proposition 2.1. *There exist division G-gradings on $R = M_n(\mathbb{F})$ with support $T \subset G$ if and only if* $\operatorname{char}\mathbb{F}$ *does not divide n and $T \cong \mathbb{Z}_{\ell_1}^2 \times \cdots \times \mathbb{Z}_{\ell_r}^2$ where $\ell_1 \cdots \ell_r = n$. The isomorphism classes of division G-gradings with support T are in one-to-one correspondence with nondegenerate alternating bicharacters $\beta : T \times T \to \mathbb{F}^\times$.*

We also note that taking

$$X_{(a_1^{i_1}, b_1^{j_1}, \ldots, a_r^{i_r}, b_r^{j_r})} = X_{a_1}^{i_1} X_{b_1}^{j_1} \cdots X_{a_r}^{i_r} X_{b_r}^{j_r},$$

we obtain a representative of the cohomology class $[\sigma]$ that is multiplicative in each variable, i.e., it is a bicharacter (not alternating unless T is the trivial subgroup). In what follows, we will always assume that σ is chosen in this way.

Definition 2.2. A concrete representative of the isomorphism class of division G-graded algebras with support T and bicharacter β can be obtained as follows. First decompose T as in (1) and pick generators $a_1, b_1, \ldots, a_r, b_r$. Then define a grading on $M_{\ell_i}(\mathbb{F})$ by declaring that X_i has degree a_i and Y_i has degree b_i, where X_i and Y_i are given by (3) and $\varepsilon_i = \beta(a_i, b_i)$. Then $M_{\ell_1}(\mathbb{F}) \otimes \cdots \otimes M_{\ell_r}(\mathbb{F})$ with tensor product grading is a representative of the desired class. We will call any representative obtained in this way a *standard realization*.

If R has a division grading, then its structure is quite rigid. Any automorphism of the graded algebra R must send X_t to a scalar multiple of itself, hence it is given by $X_t \mapsto \lambda(t) X_t$ where $\lambda : T \to \mathbb{F}^\times$ is a character of T. Since β is nondegenerate, it establishes an isomorphism between T and $\widehat{T}$. It follows that the automorphism of R corresponding to λ is given by $X \mapsto X_t^{-1} X X_t$ where $t \in T$ is determined by $\beta(u, t) = \lambda(u)$ for all $u \in T$.

It follows from [8, Lemma 6.1] that the graded algebra R admits anti-automorphisms only when T is an elementary 2-group (and hence $\operatorname{char}\mathbb{F} \neq 2$ or T is trivial). In this case, we can regard T as a vector space over the field of order 2 and think of $\sigma(u, v)$ as a bilinear form on T (recall that σ is chosen so that it is a bicharacter). Hence $\sigma(t, t)$ is a quadratic form, and $\beta(u, v)$ is the polar bilinear form for $\sigma(t, t)$. Note that $\sigma(t, t)$ depends on the choice of σ, so it is not an invariant of the graded algebra R. In fact, any quadratic form with polar form $\beta(u, v)$ can be achieved by changing generators a_i, b_i in the i-th copy of $\mathbb{Z}_2^2$. However, once we fix a standard realization of R, $\sigma(t, t)$ is uniquely determined. Following the usual convention regarding quadratic forms, we will denote $\sigma(t, t)$ by $\beta(t)$ so that $\beta(u, v) = \beta(uv)\beta(u)\beta(v)$. Note that

$$X^\beta = \beta(u) X \quad \text{for all } X \in R_u,\ u \in T, \tag{4}$$

is an involution of the graded algebra R. Hence any anti-automorphism of the graded algebra R is given by $X \mapsto X_t^{-1} X^\beta X_t$ for a suitable $t \in T$. In the standard realization of R as $M_2(\mathbb{F})^{\otimes r}$, the involution β is given by matrix transpose on each slot of the tensor power. We summarize the above discussion for future reference:

Proposition 2.3. *Suppose $R = M_n(\mathbb{F})$ has a division G-grading with support $T \subset G$ and bicharacter β. Then the mapping that sends $t \in T$ to the inner automorphism $X \mapsto X_t^{-1} X X_t$ is an isomorphism between T and the group of automorphisms $\operatorname{Aut}_G(R)$ of the graded algebra R. The graded algebra R admits anti-automorphisms if and only if T is an elementary 2-group. If this is the case, then, in any standard realization of R, the mapping $X \mapsto {}^tX$ is an involution of the graded algebra R. This involution can be written in the form (4), where $\beta : T \to \{\pm 1\}$ is a quadratic form. The bicharacter $\beta(u, v)$ is the polar bilinear form associated to β. The group $\overline{\operatorname{Aut}}_G(R)$ of automorphisms and anti-automorphisms of the graded algebra R is equal to $\operatorname{Aut}_G(R) \times \langle \beta \rangle$. In particular, any anti-automorphism of the graded algebra R is an involution, given by $X \mapsto X_t^{-1} X^\beta X_t$ for a uniquely determined $t \in T$.*

We now turn to general G-gradings on R. As shown in [3,7,6], there exist graded unital subalgebras C and D in R such that $D \cong M_\ell(\mathbb{F})$ has a division grading, $C \cong M_k(\mathbb{F})$ has an *elementary grading* given by a k-tuple $(g_1, \dots, g_k)$ of elements of G:

$$C_g = \operatorname{Span}\{E_{ij} \mid g_i^{-1} g_j = g\} \quad \text{for all } g \in G,$$

where E_{ij} is a basis of matrix units in C, and we have an isomorphism $C \otimes D \to R$ given by $c \otimes d \mapsto cd$. Moreover, the intersection of the support $\{g_i^{-1} g_j\}$ of the grading on C and the support T of the grading on D is equal to $\{e\}$.

Without loss of generality, we may assume that the k-tuple has the form

$$\left(g_1^{(k_1)}, \dots, g_s^{(k_s)}\right)$$

where the elements $g_1, \dots, g_s$ are pairwise distinct and we write $g^{(q)}$ for $\underbrace{g, \dots, g}_{q \text{ times}}$.

It is important to note that the subalgebras C and D are not uniquely determined. We are now going to obtain invariants of the graded algebra R. The partition $k = k_1 + \dots + k_s$ gives a block decomposition of C. Let e_i be the block-diagonal matrix $\operatorname{diag}(0, \dots, I_{k_i}, \dots, 0)$ where I_{k_i} is in the i-th position, $i = 1, \dots, s$. Consider the Peirce decomposition of C corresponding to the orthogonal idempotents $e_1, \dots, e_s$: $C_{ij} = e_i C e_j$. We will write C_i instead of C_{ii} for brevity. Then the identity component is

$$R_e = C_1 \otimes I \oplus \dots \oplus C_s \otimes I.$$

It follows that the idempotents $e_1, \dots, e_s$ and the (non-unital) subalgebras $C_1, \dots, C_s$ of R are uniquely determined (up to permutation). It is easy to verify that the centralizer of R_e in R is equal to $e_1 \otimes D \oplus \dots \oplus e_s \otimes D$. Hence the (non-unital) subalgebras $D_i := e_i \otimes D$ of R are uniquely determined (up to permutation). All D_i are isomorphic to D as G-graded algebras, so the isomorphism class of D is uniquely determined. This gives us invariants T and β according to Proposition 2.1. However, there is no canonical way to choose the isomorphisms of D with D_i. According to Proposition 2.3, the possible choices are parameterized by $t_i \in T$, $i = 1, \dots, s$. If we fix isomorphisms $\eta_i : D \to D_i$, then each Peirce component $R_{ij} = e_i R e_j$ becomes a D-bimodule by setting $d \cdot r = \eta_i(d) r$ and $r \cdot d = r \eta_j(d)$ for all $d \in D$ and $r \in R_{ij}$. Taking $\eta_i(d) = e_i \otimes d$ for all $d \in D$, we recover the subspaces C_{ij} for $i \neq j$ as the centres of these bimodules:

$$C_{ij} = \{r \in R_{ij} \mid d \cdot r = r \cdot d \text{ for all } d \in D\}.$$

Also, the subalgebra D of R can be identified:

$$D = \left\{\eta_1(d) + \dots + \eta_s(d) \,\middle|\, d \in D\right\}.$$

If we replace η_i by $\eta_i'(d) = \eta_i(X_{t_i}^{-1} d X_{t_i})$, then we get $C_{ij}' = \eta_i(X_{t_i}^{-1}) C_{ij} \eta_j(X_{t_j})$. Let $C' = C_1 \oplus \dots \oplus C_s \oplus \bigoplus_{i \neq j} C_{ij}'$ and $D' = \{\eta_1'(d) + \dots + \eta_s'(d) \mid d \in D\}$. Then C' and D' are graded unital subalgebras of R. Let $\Psi = e_1 \otimes X_{t_1} + \dots + e_s \otimes X_{t_s}$. Then Ψ is an invertible matrix and the mapping $\psi(X) = \Psi^{-1} X \Psi$ is an automorphism of the (ungraded) algebra R that sends C to C' and D to D'. The restriction of ψ to D preserves the grading, whereas the restriction of ψ to C sends homogeneous elements of degree $g_i^{-1} g_j$ to homogeneous elements of degree $t_i^{-1} g_i^{-1} g_j t_j$ (i.e., "shifts" the grading in the (i, j)-th Peirce components by $t_i^{-1} t_j$). We conclude that the G-grading of R associated to the k-tuple $(g_1^{(k_1)}, \dots, g_s^{(k_s)})$ is isomorphic to the G-grading associated to the k-tuple $((g_1 t_1)^{(k_1)}, \dots, (g_s t_s)^{(k_s)})$. Finally, we note that the cosets $g_i^{-1} g_j T$ are uniquely determined by the G-graded algebra R, because they are the supports of the grading on the Peirce components R_{ij} $(i \neq j)$. We have obtained an irredundant classification of G-gradings on R.

To state the result precisely, we introduce some notation. Let

$$\kappa = (k_1, \ldots, k_s) \quad \text{where } k_i \text{ are positive integers.}$$

We will write $|\kappa|$ for $k_1 + \cdots + k_s$ and e_i, $i = 1, \ldots, s$, for the orthogonal idempotents in $M_{|\kappa|}(\mathbb{F})$ associated to the block decomposition determined by κ. Let

$$\gamma = (g_1, \ldots, g_s) \quad \text{where } g_i \in G \text{ are such that } g_i^{-1} g_j \notin T \text{ for all } i \neq j.$$

Definition 2.4. We will write $(\kappa, \gamma) \sim (\widetilde{\kappa}, \widetilde{\gamma})$ if κ and $\widetilde{\kappa}$ have the same number of components s and there exist an element $g \in G$ and a permutation π of the symbols $\{1, \ldots, s\}$ such that $\widetilde{k}_i = k_{\pi(i)}$ and $\widetilde{g}_i \equiv g_{\pi(i)} g \pmod T$, for all $i = 1, \ldots, s$.

Definition 2.5. Let D be a standard realization of division G-graded algebra with support $T \subset G$ and bicharacter β. Let κ and γ be as above. Let $C = M_{|\kappa|}(\mathbb{F})$. We endow the algebra $M_{|\kappa|}(D) = C \otimes D$ with a G-grading by declaring the degree of $U \otimes d$ to be $g_i^{-1} t g_j$ for all $U \in e_i C e_j$ and $d \in D_t$. We will denote this G-graded algebra by $\mathcal{M}(G, T, D, \kappa, \gamma)$. By abuse of notation, we will also write $\mathcal{M}(G, T, \beta, \kappa, \gamma)$, since the isomorphism class of D is uniquely determined by β.

Theorem 2.6. *Let $\mathbb{F}$ be an algebraically closed field of arbitrary characteristic. Let G be an abelian group. Let $R = \bigoplus_{g \in G} R_g$ be a grading of the matrix algebra $R = M_n(\mathbb{F})$. Then the G-graded algebra R is isomorphic to some $\mathcal{M}(G, T, \beta, \kappa, \gamma)$ where $T \subset G$ is a subgroup, $\beta : T \times T \to \mathbb{F}^\times$ is a nondegenerate alternating bicharacter, κ and γ are as above with $|\kappa|\sqrt{|T|} = n$. Two G-graded algebras $\mathcal{M}(G, T_1, \beta_1, \kappa_1, \gamma_1)$ and $\mathcal{M}(G, T_2, \beta_2, \kappa_2, \gamma_2)$ are isomorphic if and only if $T_1 = T_2$, $\beta_1 = \beta_2$ and $(\kappa_1, \gamma_1) \sim (\kappa_2, \gamma_2)$.*

Remark 2.7. In fact, it follows from the above discussion that, for any permutation π as in Definition 2.4, there exists an isomorphism from $\mathcal{M}(G, T, \beta, \widetilde{\kappa}, \widetilde{\gamma})$ to $\mathcal{M}(G, T, \beta, \kappa, \gamma)$ that sends $\widetilde{e}_i$ to $e_{\pi(i)}$. We can construct such an isomorphism explicitly in the following way. Let $P = P_\pi$ be the block matrix with I_{k_i} in the $(i, \pi(i))$-th positions and 0 elsewhere (i.e., the block-permutation matrix corresponding to π). Pick $t_i \in T$ such that $\widetilde{g}_i = g_{\pi(i)} t_{\pi(i)} g$ and let B be the block-diagonal matrix $e_1 \otimes X_{t_1} + \cdots + e_s \otimes X_{t_s}$. Then the map $X \mapsto (BP)X(BP)^{-1}$ has the desired properties. We will refer to isomorphisms of this type as *monomial*.

Let $\mathrm{Sym}(s)$ be the group of permutations on $\{1, \ldots, s\}$. Let $\mathrm{Aut}(\kappa, \gamma)$ be the subgroup of $\mathrm{Sym}(s)$ that consists of all π such that, for some $g \in G$, we have $k_i = k_{\pi(i)}$ and $g_i \equiv g_{\pi(i)} g \pmod T$ for all $i = 1, \ldots, s$.

Proposition 2.8. *The group of automorphisms $\mathrm{Aut}_G(R)$ of the graded algebra $R = \mathcal{M}(G, T, \beta, \kappa, \gamma)$ is an extension of $\mathrm{Aut}(\kappa, \gamma)$ by $\mathrm{PGL}_\kappa(\mathbb{F}) \times \mathrm{Aut}_G(D)$ where*

$$\mathrm{PGL}_\kappa(\mathbb{F}) = \big(\mathrm{GL}_{\kappa_1}(\mathbb{F}) \times \cdots \times \mathrm{GL}_{\kappa_s}(\mathbb{F})\big)/\mathbb{F}^\times,$$

where $\mathbb{F}^\times$ is identified with nonzero scalar matrices.

Proof. Any $\psi \in \mathrm{Aut}_G(R)$ leaves the identity component R_e invariant and hence permutes the idempotents $e_1, \ldots, e_s$. This gives a homomorphism $f : \mathrm{Aut}_G(R) \to \mathrm{Sym}(s)$. Looking at the supports of the Peirce components, we see that $f(\psi) \in \mathrm{Aut}(\kappa, \gamma)$. Conversely, any element of $\mathrm{Aut}(\kappa, \gamma)$ is in $\mathrm{im}\, f$ by Remark 2.7, since it comes from a monomial automorphism of the graded algebra R. Finally, any $\psi \in \ker f$ leaves C_i and D_i invariant and hence is given by $\psi(X) = \Psi^{-1} X \Psi$ where

$\Psi = B_1 \otimes Q_1 \oplus \cdots \oplus B_s \otimes Q_s$ for some $B_i \in \mathrm{GL}_{\kappa_i}(\mathbb{F})$ and $Q_i \in D$. In view of Proposition 2.3, we may assume that $Q_i = X_{t_i}$ for some $t_i \in T$. It is easy to see that ψ preserves the grading if and only if $t_1 = \cdots = t_s$. The result follows. □

In order to classify gradings on Lie algebras of types $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, we will need to study involutions on G-graded matrix algebras. A description of such involutions was given in [5]. Here we will slightly simplify that description and obtain invariants that will allow us to distinguish among isomorphism classes. We start with a more general situation, which we will need for the classification of gradings in type $\mathcal{A}$.

Definition 2.9. Let G be an abelian group and let $U = \bigoplus_{g\in G} U_g$ be a G-graded algebra. We will say that an anti-automorphism φ of U is *compatible* with the grading if $\varphi(U_g) = U_g$ for all $g \in G$. If U_1 and U_2 are G-graded algebras and φ_1 and φ_2 are anti-automorphisms on U_1 and U_2, respectively, compatible with the grading, then we will say that (U, φ_1) and (U, φ_2) are *isomorphic* if there exists an isomorphism $\psi : U_1 \to U_2$ of G-graded algebras such that $\varphi_1 = \psi^{-1}\varphi_2\psi$.

Suppose $R = M_n(\mathbb{F})$ is G-graded and there exists an anti-automorphism φ compatible with the grading and such that $\varphi^2|_{R_e} = id$. Then φ leaves some of the components of R_e invariant and swaps the remaining components in pairs. Without loss of generality, we may assume that e_i are φ-invariant for $i = 1, \dots, m$ and not φ-invariant for $i > m$. It will be convenient to change the notation and write $e'_{m+1}, e''_{m+1}, \dots, e'_k, e''_k$ so that φ swaps e'_i and e''_i for $i > m$. (Thus the total number of orthogonal idempotents in question is $2k - m$.) It will also be convenient to distinguish φ-invariant idempotents of even and odd rank. Thus we assume that $e_1, \dots, e_\ell$ have odd rank and $e_{\ell+1}, \dots, e_m$ have even rank. We will change the notation for κ and γ accordingly:

$$\kappa = (q_1, \dots, q_\ell, 2q_{\ell+1}, \dots, 2q_m, q_{m+1}, q_{m+1}, \dots, q_k, q_k) \tag{5}$$

where q_i are positive integers with $q_1, \dots, q_\ell$ odd, and

$$\gamma = \left(g_1, \dots, g_\ell, g_{\ell+1}, \dots, g_m, g'_{m+1}, g''_{m+1}, \dots, g'_k, g''_k\right) \tag{6}$$

where $g_i \in G$ are such that $g_i^{-1} g_j \notin T$ for all $i \neq j$.

As shown in [8,5], the existence of the anti-automorphism φ places strong restrictions on the G-grading. First of all, note that the centralizer of R_e in R, which is equal to $D_1 \oplus \cdots \oplus D_m \oplus D'_{m+1} \oplus D''_{m+1} \oplus \cdots \oplus D'_k \oplus D''_k$, is φ-invariant. Since $e_1, \dots, e_m$ are φ-invariant and belong to $D_1, \dots, D_m$, respectively, we see that $D_1, \dots, D_m$ are also φ-invariant. By a similar argument, φ swaps D'_i and D''_i for $i > m$. Each of the D_i, D'_i and D''_i is an isomorphic copy of D, so we see that D admits an anti-automorphism. By Proposition 2.3, T must be an elementary 2-group and we have a standard realization $D \cong M_2(\mathbb{F})^{\otimes r}$.

Since φ preserves the G-grading and $\varphi(e_i R e_j) = e_j R e_i$ for $i, j \leqslant m$, the supports of these two Peirce components must be equal, which gives $g_i^{-1} g_j \equiv g_j^{-1} g_i \pmod{T}$ for $i, j \leqslant m$. Similarly, $\varphi(e'_i R e''_j) = e'_j R e''_i$ implies $(g'_i)^{-1} g''_j \equiv (g'_j)^{-1} g''_i \pmod{T}$ for $i, j > m$. Also, $\varphi(e_i R e'_j) = e''_j R e_i$ implies $g_i^{-1} g'_j \equiv (g''_j)^{-1} g_i \pmod{T}$ for $i \leqslant m$ and $j > m$. These conditions can be summarized as follows:

$$g_1^2 \equiv \cdots \equiv g_m^2 \equiv g'_{m+1} g''_{m+1} \equiv \cdots \equiv g'_k g''_k \pmod{T}. \tag{7}$$

If γ satisfies (7), then we have

$$g_1^2 t_1 = \cdots = g_m^2 t_m = g'_{m+1} g''_{m+1} t_{m+1} = \cdots = g'_k g''_k t_k$$

for some $t_1, \dots, t_k \in T$. We can replace the G-grading by an isomorphic one so that γ satisfies

$$g_1^2 t_1 = \cdots = g_m^2 t_m = g'_{m+1} g''_{m+1} = \cdots = g'_k g''_k. \tag{8}$$

Indeed, it suffices to replace g''_i by $g''_i t_i$, $i = m+1, \dots, k$ (which does not change the cosets mod T).

Theorem 2.10. *Let $\mathbb{F}$ be an algebraically closed field,* $\operatorname{char}\mathbb{F} \neq 2$. *Let G be an abelian group. Let $R = \mathcal{M}(G, T, \beta, \kappa, \gamma)$. Assume that R admits an anti-automorphism φ that is compatible with the grading and satisfies $\varphi^2|_{R_e} = id$. Write κ and γ in the form* (5) *and* (6)*, respectively. Then T is an elementary* 2*-group and γ satisfies* (7)*. Up to an isomorphism of the pair (R, φ), γ satisfies* (8) *for some $t_1, \dots, t_m \in T$ and φ is given by $\varphi(X) = \Phi^{-1}({}^t X)\Phi$ for all $X \in R$, where matrix Φ has the following block-diagonal form*:

$$\Phi = \sum_{i=1}^{\ell} I_{q_i} \otimes X_{t_i} \oplus \sum_{i=\ell+1}^{m} S_i \otimes X_{t_i} \oplus \sum_{i=m+1}^{k} \begin{pmatrix} 0 & I_{q_i} \\ \mu_i I_{q_i} & 0 \end{pmatrix} \otimes I \tag{9}$$

where, for $i = \ell+1, \dots, m$, each S_i is either I_{2q_i} or $\left(\begin{smallmatrix} 0 & I_{q_i} \\ -I_{q_i} & 0 \end{smallmatrix}\right)$, and $\mu_{m+1}, \dots, \mu_k$ are nonzero scalars.

Proof. There exists an invertible matrix Φ such that φ is given by $\varphi(X) = \Phi^{-1}({}^t X)\Phi$ for all $X \in R$. Recall that conjugating φ by the automorphism $\psi(X) = \Psi^{-1} X \Psi$ replaces matrix Φ by ${}^t\Psi\Phi\Psi$, i.e., Φ is transformed as the matrix of a bilinear form.

Recall that we fixed the idempotents

$$e_1, \dots, e_\ell, e_{\ell+1}, \dots, e_m, e'_{m+1}, e''_{m+1}, \dots, e'_k, e''_k. \tag{10}$$

It is also convenient to introduce $e_i = e'_i + e''_i$ for $i = m+1, \dots, k$.

Following the proof of [5, Lemma 6 and Proposition 1], we see that, up to an automorphism of the G-graded algebra R, Φ has the following block-diagonal form—in agreement with the idempotents given by (10):

$$\Phi = \sum_{i=1}^{\ell} S_i Y_i \otimes Q_i \oplus \sum_{i=\ell+1}^{m} S_i Y_i \otimes Q_i \oplus \sum_{i=m+1}^{k} S_i Y_i \otimes Q_i.$$

(This is formula (20) of just cited paper, rewritten according to our present notation.) For $i = 1, \dots, m$, the matrix Y_i is in the centralizer of the simple algebra C_i, i.e., has the form $Y_i = \xi_i I_{q_i}$. For $i = m+1, \dots, k$, the matrix Y_i is in the centralizer of the semisimple algebra $C'_i \oplus C''_i$, i.e., has the form $Y_i = \operatorname{diag}(\eta_i I_{q_i}, \xi_i I_{q_i})$. Each Q_i is in D_i, and the map $X \mapsto Q_i^{-1}({}^t X)Q_i$ is an anti-automorphism of D. Hence, by Proposition 2.3, each Q_i is, up to a scalar multiple, of the form X_{t_i}, for an appropriate choice of $t_i \in T$. The scalar can be absorbed in Y_i. Finally, the matrix S_i is I_{q_i} for $i = 1, \dots, \ell$, either I_{2q_i} or $\left(\begin{smallmatrix} 0 & I_{q_i} \\ -I_{q_i} & 0 \end{smallmatrix}\right)$ for $i = \ell+1, \dots, m$, and $\left(\begin{smallmatrix} 0 & I_{q_i} \\ I_{q_i} & 0 \end{smallmatrix}\right)$ for $i = m+1, \dots, k$. This allows us to rewrite the above formula as follows:

$$\Phi = \sum_{i=1}^{\ell} \xi_i I_{q_i} \otimes X_{t_i} \oplus \sum_{i=\ell+1}^{m} \xi_i S_i \otimes X_{t_i} \oplus \sum_{i=m+1}^{k} \begin{pmatrix} 0 & \xi_i I_{q_i} \\ \eta_i I_{q_i} & 0 \end{pmatrix} \otimes X_{t_i}.$$

Here ξ_i, η_i are some nonzero scalars. If we now apply the inner automorphism of the graded algebra R given by the matrix $P = \frac{1}{\sqrt{\xi_1}} e_1 \otimes I + \cdots + \frac{1}{\sqrt{\xi_k}} e_k \otimes I$, then φ is transformed to the anti-automorphism given by the following matrix (which we again denote by Φ):

$$\Phi = \sum_{i=1}^{\ell} I_{q_i} \otimes X_{t_i} \oplus \sum_{i=\ell+1}^{m} S_i \otimes X_{t_i} \oplus \sum_{i=m+1}^{k} \begin{pmatrix} 0 & I_{q_i} \\ \mu_i I_{q_i} & 0 \end{pmatrix} \otimes X_{t_i},$$

for an appropriate set of nonzero scalars $\mu_{m+1}, \dots, \mu_k$. It can be easily verified (and is shown in the proof of [5, Theorem 3]) that $t_1, \dots, t_k$ satisfy the following condition: $g_1^2 t_1 = \dots = g_m^2 t_m = g'_{m+1} g''_{m+1} t_{m+1} = \dots = g'_s g''_s t_s$.

Finally, the inner automorphism $\psi(X) = \Psi^{-1} X \Psi$ of R where

$$\Psi^{-1} = e_1 \otimes I + \dots + e_m \otimes I + e'_{m+1} \otimes I + e''_{m+1} \otimes X_{t_{m+1}} + \dots + e'_k \otimes I + e''_k \otimes X_{t_k}$$

sends the G-grading to the one given by

$$\left(g_1, \dots, g_m, g'_{m+1}, g''_{m+1} t_{m+1}, \dots, g'_k, g''_k t_k\right)$$

and transforms φ to the anti-automorphism given by a matrix of form (9). $\square$

If φ is an involution on R, then one can get rid of the parameters $\mu_{m+1}, \dots, \mu_s$, and the selection of $S_{\ell+1}, \dots, S_m$ is uniquely determined. Indeed, the matrix Φ is then either symmetric or skew-symmetric. In the first case, φ is called an *orthogonal* (or *transpose*) involution. In the second case, φ is called a *symplectic* involution. Set $\operatorname{sgn}(\varphi) = 1$ if φ is orthogonal and $\operatorname{sgn}(\varphi) = -1$ if φ is symplectic. Similarly, set $\operatorname{sgn}(S_i) = 1$ if ${}^t S_i = S_i$ and $\operatorname{sgn}(S_i) = -1$ if ${}^t S_i = -S_i$. We restate the main result of [5] in our notation (and setting $t_{m+1} = \dots = t_k = e$):

Theorem 2.11. *(See [5, Theorem 3].) Under the conditions of Theorem* 2.10*, assume that $\varphi^2 = id$. Then, up to an isomorphism of the pair (R, φ), γ satisfies* (8) *for some $t_1, \dots, t_m \in T$ and φ is given by $\varphi(X) = \Phi^{-1}({}^t X)\Phi$ for all $X \in R$, where matrix Φ has the following block-diagonal form*:

$$\Phi = \sum_{i=1}^{\ell} I_{q_i} \otimes X_{t_i} \oplus \sum_{i=\ell+1}^{m} S_i \otimes X_{t_i} \oplus \sum_{i=m+1}^{k} S_i \otimes I \tag{11}$$

where

- *for $i = \ell + 1, \dots, m$, each S_i is either I_{2q_i} or $\left(\begin{smallmatrix} 0 & I_{q_i} \\ -I_{q_i} & 0 \end{smallmatrix}\right)$, and*
- *for $i = m + 1, \dots, k$, all S_i are either $\left(\begin{smallmatrix} 0 & I_{q_i} \\ I_{q_i} & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & I_{q_i} \\ -I_{q_i} & 0 \end{smallmatrix}\right)$*

such that the following condition is satisfied:

$$\begin{aligned} \operatorname{sgn}(\varphi) &= \beta(t_1) = \dots = \beta(t_\ell) \\ &= \beta(t_{\ell+1}) \operatorname{sgn}(S_{\ell+1}) = \dots = \beta(t_m) \operatorname{sgn}(S_m) \\ &= \operatorname{sgn}(S_{m+1}) = \dots = \operatorname{sgn}(S_k). \end{aligned} \tag{12}$$

Conversely, if γ satisfies (8) *and condition* (12) *holds, then Φ defines an involution of the type indicated by* $\operatorname{sgn}(\varphi)$ *on the G-graded algebra R.*

It is convenient to introduce the following notation (for $m > 0$):

$$\tau = (t_1, \ldots, t_m). \tag{13}$$

Note that for the elements $t_1, \ldots, t_m$ in (8), the ratios $t_i^{-1} t_j$ are uniquely determined by the cosets of $g_1, \ldots, g_m$ mod T, so it is sufficient to specify only one t_i to find τ.

Definition 2.12. We will say that γ is *$*$-admissible* if it satisfies (7) and, for some $t_1, \ldots, t_\ell \in T$, we have $g_1^2 t_1 = \cdots = g_\ell^2 t_\ell$ and

$$\beta(t_1) = \cdots = \beta(t_\ell). \tag{14}$$

(If $\ell \leqslant 1$, then condition (14) is automatically satisfied.)

Definition 2.13. Let $T \subset G$ be an elementary 2-group (of even rank) with a nondegenerate alternating bicharacter β. Suppose γ is $*$-admissible, and γ and τ satisfy (8) and (14). If $\ell > 0$, let δ be the common value of $\beta(t_1), \ldots, \beta(t_\ell)$. If $\ell = 0$, select $\delta \in \{\pm 1\}$ arbitrarily. Consider $R = \mathcal{M}(G, T, \beta, \kappa, \gamma)$. Let Φ be the matrix given by (11) where the matrices S_i are selected so that Eq. (12) holds with $\operatorname{sgn}(\varphi) = \delta$. Then, by Theorem 2.11, $\varphi(X) = \Phi^{-1}({}^t X)\Phi$ is an involution on R that is compatible with the grading. We will denote (R, φ) defined in this way by $\mathcal{M}^*(G, T, \beta, \kappa, \gamma, \tau, \delta)$. (Here τ is empty if $m = 0$.)

Definition 2.14. Referring to Definition 2.13, we will write $(\kappa, \gamma, \tau) \approx (\widetilde{\kappa}, \widetilde{\gamma}, \widetilde{\tau})$ if κ and $\widetilde{\kappa}$ have the same number of components of each type, i.e., the same values of ℓ, m and k, and there exist an element $g \in G$ and a permutation π of the symbols $\{1, \ldots, k\}$ preserving the sets $\{1, \ldots, \ell\}$, $\{\ell + 1, \ldots, m\}$ and $\{m + 1, \ldots, k\}$ such that $\widetilde{q}_i = q_{\pi(i)}$ for all i, $\widetilde{g}_i \equiv g_{\pi(i)} g \pmod T$ for all $i = 1, \ldots, m$, $\{\widetilde{g}'_i, \widetilde{g}''_i\} \equiv \{g'_{\pi(i)} g, g''_{\pi(i)} g\} \pmod T$ for all $i = m + 1, \ldots, k$, and

- if $m > 0$, then $\widetilde{t}_i = t_{\pi(i)}$ for all $i = 1, \ldots, m$;
- if $m = 0$, then $\widetilde{g}'_i \widetilde{g}''_i = g'_{\pi(i)} g''_{\pi(i)} g^2$ for some (and hence all) $i = 1, \ldots, k$.

In the case $m = 0$, τ is empty, so we may write $(\kappa, \gamma) \approx (\widetilde{\kappa}, \widetilde{\gamma})$.

Corollary 2.15. *Let* $\operatorname{char} \mathbb{F} \neq 2$ *and* $R = \mathcal{M}(G, T, \beta, \kappa, \gamma)$. *Then the G-graded algebra R admits an involution if and only if T is an elementary 2-group and γ is $*$-admissible. If φ is an involution on R, then (R, φ) is isomorphic to some $\mathcal{M}^*(G, T, \beta, \kappa, \gamma, \tau, \delta)$ where $\delta = \operatorname{sgn}(\varphi)$. Two G-graded algebras with involution, $\mathcal{M}^*(G, T_1, \beta_1, \kappa_1, \gamma_1, \tau_1, \delta_1)$ and $\mathcal{M}^*(G, T_2, \beta_2, \kappa_2, \gamma_2, \tau_2, \delta_2)$, are isomorphic if and only if $T_1 = T_2$, $\beta_1 = \beta_2$, $(\kappa_1, \gamma_1, \tau_1) \approx (\kappa_2, \gamma_2, \tau_2)$ and $\delta_1 = \delta_2$.*

Proof. The first two statements are a combination of Theorems 2.10 and 2.11. It remains to prove the last statement.

Let $R_1 = \mathcal{M}(G, T_1, \beta_1, \kappa_1, \gamma_1)$, $R_2 = \mathcal{M}(G, T_2, \beta_2, \kappa_2, \gamma_2)$ and let φ_1 and φ_2 be the corresponding involutions. Suppose $T_1 = T_2$, $\beta_1 = \beta_2$, and $(\kappa_1, \gamma_1, \tau_1) \approx (\kappa_2, \gamma_2, \tau_2)$. Then, by Theorem 2.6, there exists an isomorphism of G-graded algebras $\psi : R_1 \to R_2$. By Remark 2.7, ψ can be chosen to be a monomial isomorphism associated to the permutation π in Definition 2.14. The matrix of the involution $\psi^{-1}\varphi_2\psi$ on R_1 is then obtained from the matrix of φ_2 by permuting the blocks on the diagonal so that they align with the corresponding blocks of φ_1 and possibly multiplying some of the blocks by -1 (the extra condition for the case $m = 0$ in Definition 2.14 guarantees that the second tensor factor in each block remains I). If $\delta_1 = \delta_2$, then $\psi^{-1}\varphi_2\psi$ can be transformed to φ_1 by an automorphism of the G-graded algebra R_1 (see the proof of Theorem 2.10).

Conversely, suppose there exists an isomorphism $\psi : (R_1, \varphi_1) \to (R_2, \varphi_2)$. First of all, δ_1 and δ_2 are determined by the type of involution (orthogonal or symplectic), so $\delta_1 = \delta_2$. By Theorem 2.6, we also have $T_1 = T_2$, $\beta_1 = \beta_2$, $(\kappa_1, \gamma_1) \sim (\kappa_2, \gamma_2)$. The partitions of κ_1 and κ_2 according to $\{1, \dots, \ell\}$, $\{\ell + 1, \dots, m\}$ and $\{m+1, \dots, k\}$ are determined by φ_1 and φ_2, hence they must correspond under ψ. At the same time, for some $g \in G$, the cosets of $\gamma_1 g T$ and $\gamma_2 T$ must correspond under ψ up to switching g_i' with g_i'' $(i > m)$. In the case $m > 0$, by Proposition 2.3, τ_1 and τ_2 are uniquely determined by the restrictions of φ_1 and φ_2 to $D_1, \dots, D_m$ and hence must match under the permutation determined by ψ. Therefore, in this case $(\kappa_1, \gamma_1, \tau_1) \approx (\kappa_2, \gamma_2, \tau_2)$. It remains to consider the case $m = 0$. Looking at the description of the automorphism group given by Proposition 2.8, we see that $\psi = \psi_0 \alpha$ where ψ_0 is a monomial isomorphism and α is in $\mathrm{PGL}_{\kappa_1}(\mathbb{F}) \times \mathrm{Aut}_G(D)$. The action of ψ_0 on φ_2 leads to the permutation of blocks and the replacement of the second tensor factor I by X_{t_0} for some $t_0 \in T$. Then α must transform $\psi_0^{-1} \varphi_2 \psi_0$ to φ_1. The effect of α on one block is the following (we omit subscripts to simplify notation):

$$\left(\begin{pmatrix} {}^tA & 0 \\ 0 & {}^tB \end{pmatrix} \otimes {}^tX_u\right)\left(\begin{pmatrix} 0 & I \\ \varepsilon I & 0 \end{pmatrix} \otimes X_{t_0}\right)\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \otimes X_u\right) = \pm\begin{pmatrix} 0 & {}^tAB \\ \varepsilon\, {}^tBA & 0 \end{pmatrix} \otimes X_{t_0}.$$

We see that α cannot change t_0. It follows that $t_0 = e$ and $(\kappa_1, \gamma_1) \approx (\kappa_2, \gamma_2)$. □

3. Correspondence between Lie gradings and associative gradings

Let U be an algebra and let G be a group. Then a G-grading on U is equivalent to a structure of an $\mathbb{F}G$-comodule algebra (see e.g. [17] for background). If we assume that U is finite-dimensional and G is abelian and finitely generated, then the comodule structure can be regarded as a morphism of (affine) algebraic group schemes $G^D \to \mathbf{Aut}(U)$ where G^D is the Cartier dual of G and $\mathbf{Aut}(G)$ is the automorphism group scheme of U (see e.g. [19] for background). Two G-gradings are isomorphic if and only if the corresponding morphisms $G^D \to \mathbf{Aut}(U)$ are conjugate by an automorphism of U. Note also that, if U is finite-dimensional, then we may always assume without loss of generality that G is finitely generated (just replace G by the subgroup generated by the support of the grading).

If $\operatorname{char}\mathbb{F} = 0$, then $G^D = \widehat{G}$, the algebraic group of characters on G, and $\mathbf{Aut}(G) = \mathrm{Aut}(G)$, the algebraic group of automorphisms. If $\operatorname{char}\mathbb{F} = p > 0$, then we can write $G = G_0 \times G_1$ where G_0 has no p-torsion and G_1 is a p-group. Hence $G^D = \widehat{G_0} \times G_1^D$, where $\widehat{G_0}$ is smooth and G_1^D is finite and connected. The algebraic group $\widehat{G_0}$ (which is equal to $\widehat{G}$) acts on U as follows:

$$\chi * X = \chi(g)X \quad \text{for all } X \in U_g \text{ and } g \in G.$$

The group scheme $\mathbf{Aut}(U)$ contains the group $\mathrm{Aut}(U)$ as the largest smooth subgroupscheme. The tangent Lie algebra of $\mathbf{Aut}(U)$ is $\mathrm{Der}(U)$, so $\mathbf{Aut}(U)$ is smooth if and only if $\mathrm{Der}(U)$ equals the tangent Lie algebra of the group $\mathrm{Aut}(U)$.

We will be interested in the following algebras: $M_n(\mathbb{F})$, $\mathfrak{psl}_n(\mathbb{F})$, $\mathfrak{so}_n(\mathbb{F})$ and $\mathfrak{sp}_n(\mathbb{F})$, where $\operatorname{char}\mathbb{F} \neq 2$. In all these cases the automorphism group scheme is smooth, i.e., coincides with the algebraic group of automorphisms (regarded as a group scheme). Indeed, for the associative algebra $R = M_n(\mathbb{F})$, it is well known that $\mathrm{Aut}(R) = \mathrm{PGL}_n(\mathbb{F})$ and $\mathrm{Der}(R) = \mathfrak{pgl}_n(\mathbb{F})$. For the Lie algebra $L = \mathfrak{so}_n(\mathbb{F})$ $(n \geqslant 5,\ n \neq 8)$ or $\mathfrak{sp}_n(\mathbb{F})$ $(n \geqslant 4)$, it is known that every automorphism of L is the conjugation by an element of $\mathrm{O}_n(\mathbb{F})$ or $\mathrm{Sp}_n(\mathbb{F})$, respectively—see [15] for the case $\operatorname{char}\mathbb{F} = 0$ and [18] for the case $\operatorname{char}\mathbb{F} = p$ $(p \neq 2)$. In particular, every automorphism of L is the restriction of an automorphism of R. Similarly, every derivation of L is the restriction of a derivation of R (see e.g. [10]).

Let φ be the involution of R such that $L = \mathcal{K}(R, \varphi)$, the space of skew-symmetric elements with respect to φ. Then the projectivizations of the groups $\mathrm{O}_n(\mathbb{F})$ and $\mathrm{Sp}_n(\mathbb{F})$ are equal to $\mathrm{Aut}(R, \varphi)$, and their tangent algebras are equal to $\mathrm{Der}(R, \varphi)$. Hence the restriction map $\theta : \mathrm{Aut}(R, \varphi) \to \mathrm{Aut}(L)$ is a surjective homomorphism of algebraic groups such that $d\theta : \mathrm{Der}(R, \varphi) \to \mathrm{Der}(L)$ is also surjective. It follows that $\mathbf{Aut}(L)$ is smooth. Since L generates R as an associative algebra, both θ and $d\theta$ are also injective. Hence $\theta : \mathrm{Aut}(R, \varphi) \to \mathrm{Aut}(L)$ is an isomorphism of algebraic groups. For G-gradings this means the following. Clearly, if $R = \bigoplus_{g \in G} R_g$ is a grading that is compatible with φ, then the restriction $L_g = R_g \cap L$ is a grading of L. Since $\theta : \mathrm{Aut}(R, \varphi) \to \mathrm{Aut}(L)$ is an isomorphism and the automorphism groups are equal to the automorphism group schemes, the restriction map gives a bijection between the isomorphism classes of G-gradings on L and the $\mathrm{Aut}(R,\varphi)$-orbits on the set of φ-compatible G-gradings on R. The orbits correspond to isomorphism classes of pairs (R,φ) where $R=M_n(\mathbb{F})$ is G-graded and φ is an involution on R that is compatible with the grading.

The case of $L=\mathfrak{psl}_n(\mathbb{F})$ is more complicated. We have a homomorphism of algebraic groups $\theta:\mathrm{Aut}(R)\to\mathrm{Aut}(L)$ given by restriction and passing to cosets modulo the centre. It is well known that this homomorphism is not surjective for $n\geqslant 3$, because the map $X\mapsto -{}^tX$ is not an automorphism of the associative algebra R, but it is an automorphism of the Lie algebra $R^{(-)}$ and hence induces an automorphism of L. Let $\overline{\mathrm{Aut}}(R)$ be the group of automorphisms and anti-automorphisms of R. Then we can extend θ to a homomorphism $\overline{\mathrm{Aut}}(R)\to\mathrm{Aut}(L)$ by sending an anti-automorphism φ of R to the map induced on L by $-\varphi$. This extended θ is surjective for any $n\geqslant 3$ if $\mathrm{char}\,\mathbb{F}\neq 2,3$ (see [18]) and for any $n>3$ if $\mathrm{char}\,\mathbb{F}=3$ (see [10]). It is easy to verify that θ and $d\theta$ are injective and hence θ is an isomorphism of algebraic groups (see e.g. [1, Lemma 5.3]). It is shown in [10] that, under the same assumptions on $\mathrm{char}\,\mathbb{F}$, every derivation of L is induced by a derivation of R. It follows that $\mathbf{Aut}(L)$ is smooth, i.e., $\mathbf{Aut}(L)=\mathrm{Aut}(L)$.

Now let $L=\bigoplus_{g\in G}L_g$ be a G-grading and let $\alpha:G^D\to\mathrm{Aut}(L)$ be the corresponding morphism. Then we have a morphism $\widetilde{\alpha}:=\theta^{-1}\alpha:G^D\to\overline{\mathrm{Aut}}(R)$, which gives a G-grading $R=\bigoplus_{g\in G}R_g$ on the Lie algebra $R^{(-)}$. The two gradings are related in the following way: $L_g=(R_g\cap[R,R]) \bmod Z(R)$.

Set $\Lambda=\widetilde{\alpha}^{-1}(\mathrm{Aut}(R))$. Then Λ is a subgroupscheme of G^D of index at most 2. Moreover, since G_1^D is connected, it is mapped by $\widetilde{\alpha}$ to $\mathrm{Aut}(R)$ and hence is contained in Λ. We have two possibilities: either $\Lambda=G^D$ or Λ has index 2. Following [8], we will say that the G-grading on L has *Type I* in the first case and has *Type II* in the second case. In Type I, the G-grading corresponding to $\widetilde{\alpha}$ is a grading of R as an associative algebra. In Type II, we consider $\Lambda^{\perp}$, which is a subgroup of order 2 in G. Let h be the generator of this subgroup. Note that, since $\mathrm{char}\,\mathbb{F}\neq 2$, the element h is in G_0.

Remark 3.1. For the readers more familiar with the language of Hopf algebras, there is an alternative way to define the element h. The Hopf algebra $\mathbb{F}[\overline{\mathrm{Aut}}(R)]$ of regular functions on the algebraic group $\overline{\mathrm{Aut}}(R)$ has a group-like element f defined by $f(\psi)=1$ if ψ is an automorphism and $f(\psi)=-1$ if ψ is an anti-automorphism. The morphism of group schemes $\widetilde{\alpha}:G^D\to\overline{\mathrm{Aut}}(R)$ corresponds to a homomorphism of Hopf algebras $\mathbb{F}[\overline{\mathrm{Aut}}(R)]\to\mathbb{F}G$. The element h is the image of f under this homomorphism.

Let $\overline{G}=G/\langle h\rangle$. Then the restriction $\widetilde{\alpha}:\Lambda\to\mathrm{Aut}(R)$ corresponds to the coarsening of the G-grading on R given by the quotient map $G\to\overline{G}$:

$$R=\bigoplus_{\bar{g}\in\overline{G}}R_{\bar{g}}\quad\text{where } R_{\bar{g}}=R_g\oplus R_{gh}.$$

This $\overline{G}$-grading is a grading of R as an associative algebra. The G-grading on $R^{(-)}$ can be recovered as follows. Fix $\chi\in\widehat{G_0}=\widehat{G}$ such that $\chi(h)=-1$. Then χ acts on R as $-\varphi$ where φ is an anti-automorphism preserving the $\overline{G}$-grading. Then we have

$$R_g=\bigl\{X\in R_{\bar{g}}\bigm| -\varphi(X)=\chi(g)X\bigr\}=\bigl\{-\varphi(X)+\chi(g)X\bigm| X\in R_{\bar{g}}\bigr\}.$$

Thus we obtain (1) a bijection between the isomorphism classes of G-gradings on L of Type I and the $\overline{\mathrm{Aut}}(R)$-orbits on the set of G-gradings on R and (2) a bijection between the isomorphism classes of G-gradings on L of Type II and $\overline{\mathrm{Aut}}(R)$-orbits on the set of pairs (R,φ) where $R=M_n(\mathbb{F})$ is $\overline{G}$-graded and φ is an anti-automorphism on R that is compatible with the $\overline{G}$-grading and has the property $\varphi^2(X)=\chi^2*X$ for all $X\in R$.

Remark 3.2. If $n=2$, then $\theta:\mathrm{Aut}(R)\to\mathrm{Aut}(L)$ is an isomorphism, so there are no gradings of Type II.

4. Gradings on Lie algebras of type $\mathcal{A}$

Let $L = \mathfrak{psl}_n(\mathbb{F})$ and $R = M_n(\mathbb{F})$, where $\operatorname{char}\mathbb{F} \neq 2$ and, for $n = 3$, also $\operatorname{char}\mathbb{F} \neq 3$. Let $L = \bigoplus_{g\in G} L_g$ be a grading of L by an abelian group G. As discussed in the previous section, this grading belongs to one of two types. Gradings of Type I are induced from G-gradings on the associative algebra R, which have been classified in Theorem 2.6.

Definition 4.1. Let $R = \mathcal{M}(G, T, \beta, \kappa, \gamma)$ and let $L_g = (R_g \cap [R, R]) \bmod Z(R)$. We will denote the G-graded algebra L obtained in this way as $\mathcal{A}^{(\mathrm{I})}(G, T, \beta, \kappa, \gamma)$.

Now assume that we have a grading of Type II. Then there is a distinguished element $h \in G$ of order 2. Let $\overline{G} = G/\langle h\rangle$. Then the G-grading on L is induced from a G-grading on the Lie algebra $R^{(-)}$ that is obtained by refining a $\overline{G}$-grading $R = \bigoplus_{\bar{g}\in\overline{G}} R_{\bar{g}}$ on the associative algebra R. Let $R = \mathcal{M}(\overline{G}, \overline{T}, \beta, \kappa, \gamma)$ as a $\overline{G}$-graded algebra. The refinement is obtained using the action of any character $\chi \in \widehat{G}$ with $\chi(h) = -1$, and the result does not depend on the choice of χ. So we fix $\chi \in \widehat{G}$ such that $\chi(h) = -1$.

Set $\varphi(X) = -\chi * X$ for all $X \in R$. Then φ is an anti-automorphism of the $\overline{G}$-graded algebra R. Moreover, $\varphi^2(X) = \chi^2 * X$. Since $\chi^2(h) = 1$, we can regard χ^2 as a character on $\overline{G}$ and hence its action on $X \in R_{\bar{g}}$ is given by $\chi^2 * X = \chi^2(\bar{g})X$. In particular, $\varphi^2|_{R_{\bar{e}}} = id$. By Theorem 2.10, $\overline{T}$ is an elementary 2-group, κ is given by (5) and γ is given by (6) with bars over the g's. We may also assume that γ satisfies

$$\bar{g}_1^2\bar{t}_1 = \cdots = \bar{g}_m^2\bar{t}_m = \bar{g}'_{m+1}\bar{g}''_{m+1} = \cdots = \bar{g}'_s\bar{g}''_s \tag{15}$$

for some $\bar{t}_1, \ldots, \bar{t}_m \in \overline{T}$, and φ is given by $\varphi(X) = \Phi^{-1}({}^tX)\Phi$ where

$$\Phi = \sum_{i=1}^{\ell} I_{q_i} \otimes X_{\bar{t}_i} \oplus \sum_{i=\ell+1}^{m} S_i \otimes X_{\bar{t}_i} \oplus \sum_{i=m+1}^{k} \begin{pmatrix} 0 & I_{q_i} \\ \mu_i I_{q_i} & 0 \end{pmatrix} \otimes I, \tag{16}$$

where μ_i are nonzero scalars. We will use the notation τ introduced in (13).

Our goal now is to determine the parameters $\mu_i \in \mathbb{F}^\times$ that appear in the above formula. On the one hand, the automorphism φ^2 is the conjugation by matrix ${}^t\Phi^{-1}\Phi$ given by

$${}^t\Phi^{-1}\Phi = \sum_{i=1}^{\ell} \beta(t_i) I_{q_i} \otimes I \oplus \sum_{i=\ell+1}^{m} \beta(t_i)\operatorname{sgn}(S_i) I_{2q_i} \otimes I \oplus \sum_{i=m+1}^{k} \begin{pmatrix} \mu_i I_{q_i} & 0 \\ 0 & \mu_i^{-1} I_{q_i} \end{pmatrix} \otimes I.$$

On the other hand, φ^2 acts as χ^2. We now derive the conditions that are necessary and sufficient for $\chi^2 * X = ({}^t\Phi^{-1}\Phi)^{-1} X ({}^t\Phi^{-1}\Phi)$ to hold for all $X \in R$.

Recall the idempotents $e_1, \ldots, e_k \in C$ defined earlier (where $e_i = e'_i + e''_i$ for $i > m$). We denote by U_{ij} any matrix in the Peirce component $e_i C e_j$. Then, for $1 \leqslant i, j \leqslant m$, we have, for all $\bar{t} \in \overline{T}$,

$$\chi^2 * (U_{ij} \otimes X_{\bar{t}}) = \chi^2\big(\bar{g}_i^{-1}\bar{g}_j\bar{t}\big) U_{ij} \otimes X_{\bar{t}}$$

while

$$\begin{aligned}\varphi^2(U_{ij} \otimes X_{\bar{t}}) &= \big({}^t\Phi^{-1}\Phi\big)^{-1}(U_{ij} \otimes X_{\bar{t}})\big({}^t\Phi^{-1}\Phi\big) \\ &= \beta(\bar{t}_i)\operatorname{sgn}(S_i)\beta(\bar{t}_j)\operatorname{sgn}(S_j)(U_{ij} \otimes X_{\bar{t}}).\end{aligned}$$

For $m+1 \leqslant i, j \leqslant k$, we write $U_{ij} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ according to the decompositions $e_i = e'_i + e''_i$ and $e_j = e'_j + e''_j$. Then, for all $\bar{t} \in \overline{T}$,

$$\chi^2 * (U_{ij} \otimes X_{\bar{t}}) = \begin{pmatrix} \chi^2((\bar{g}'_i)^{-1}\bar{g}'_j\bar{t})A & \chi^2((\bar{g}'_i)^{-1}\bar{g}''_j\bar{t})B \\ \chi^2((\bar{g}''_i)^{-1}\bar{g}'_j\bar{t})C & \chi^2((\bar{g}''_i)^{-1}\bar{g}''_j\bar{t})D \end{pmatrix},$$

while

$$\varphi^2(U_{ij} \otimes X_{\bar{t}}) = \begin{pmatrix} \mu_i^{-1}\mu_j A & \mu_i^{-1}\mu_j^{-1} B \\ \mu_i\mu_j C & \mu_i\mu_j^{-1} D \end{pmatrix}.$$

For $1 \leqslant i \leqslant m$ and $m+1 \leqslant j \leqslant k$, we write $U_{ij} = (\,A \quad B\,)$ according to the decomposition $e_j = e'_j + e''_j$. Then, for all $\bar{t} \in \overline{T}$,

$$\chi^2 * (U_{ij} \otimes X_{\bar{t}}) = \left(\, \chi^2(\bar{g}_i^{-1}\bar{g}'_j\bar{t})A \quad \chi^2(\bar{g}_i^{-1}\bar{g}''_j\bar{t})B \,\right),$$

while

$$\varphi^2(U_{ij} \otimes X_{\bar{t}}) = \beta(\bar{t}_i)\,\mathrm{sgn}(S_i)\left(\, \mu_j A \quad \mu_j^{-1} B \,\right).$$

For $m+1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant m$, we have a similar calculation.

By way of comparison, we derive $\chi^2(\bar{t}) = \text{const}$ for all $\bar{t} \in \overline{T}$, and so $\chi^2(\overline{T}) = 1$. Hence the natural epimorphism $\pi : G \to \overline{G}$ splits over $\overline{T}$, i.e., $\pi^{-1}(\overline{T}) = T \times \langle h \rangle$, where $T = \pi^{-1}(\overline{T}) \cap \ker \chi$. So we may identify T with $\overline{T}$ and write t_i for the representative of the coset $\bar{t}_i$ in T. Conversely, if $\pi : G \to \overline{G}$ splits over $\overline{T}$, then $\chi^2(\overline{T}) = 1$.

In the case $1 \leqslant i, j \leqslant m$, our relations are equivalent to $\beta(t_i)\,\mathrm{sgn}(S_i)\chi^2(\bar{g}_i) = \beta(t_j)\,\mathrm{sgn}(S_j)\chi^2(\bar{g}_j)$. Therefore, we have a fixed $\lambda \in \mathbb{F}^\times$ such that

$$\beta(t_i)\,\mathrm{sgn}(S_i)\chi^2(\bar{g}_i) = \lambda \quad \text{for all } i = 1, \dots, m. \tag{17}$$

In the case $m+1 \leqslant i, j \leqslant k$, our relations are equivalent to

$$\mu_i^{-1}\chi^2(\bar{g}'_i) = \mu_j^{-1}\chi^2(\bar{g}'_j)$$

and

$$\mu_i^{-1}\chi^2(\bar{g}'_i) = \mu_j\chi^2(\bar{g}''_j).$$

Therefore, we have a fixed $\mu \in \mathbb{F}^\times$ such that

$$\mu_i^{-1}\chi^2(\bar{g}'_i) = \mu_i\chi^2(\bar{g}''_i) = \mu \quad \text{for all } i = m+1, \dots, k. \tag{18}$$

In the case $1 \leqslant i \leqslant m$ and $m+1 \leqslant j \leqslant k$, our relations are equivalent to

$$\mu_j^{-1}\chi^2(\bar{g}'_j) = \beta(t_i)\,\mathrm{sgn}(S_i)\chi^2(\bar{g}_i) = \mu_j\chi^2(\bar{g}''_j). \tag{19}$$

If both (17) and (18) are present (i.e., $m \neq 0, k$), then (19) is equivalent to $\mu = \lambda$. We have proved that if the $\overline{G}$-grading on R is the coarsening a G-grading on $R^{(-)}$ induced by $\pi : G \to \overline{G}$, and χ acts on R as $-\varphi$, then $\pi^{-1}(\overline{T})$ splits and conditions (17) and (18) hold with $\lambda = \mu$. Conversely, if

$R=\mathcal{M}(\overline{G},\overline{T},\beta,\kappa,\gamma)$ is such that $\pi^{-1}(\overline{T})$ splits, and an anti-automorphism φ is given by matrix (16) such that (17) and (18) hold with $\lambda=\mu$, then φ^2 acts as χ^2 on R and hence $-\varphi$ defines a refinement of the $\overline{G}$-grading on R to a G-grading (as a vector space). The latter is automatically a grading of the Lie algebra $R^{(-)}$, since $-\varphi$ is an automorphism of $R^{(-)}$.

To summarize, we state the following

Proposition 4.2. *Let $h\in G$ be an element of order 2 and let $\pi: G\to\overline{G}=G/\langle h\rangle$ be the quotient map. Fix $\chi\in\widehat{G}$ with $\chi(h)=-1$. Let $R=\mathcal{M}(\overline{G},\overline{T},\beta,\kappa,\gamma)$ and let φ be the anti-automorphism of the $\overline{G}$-graded algebra R given by $\varphi(X)=\Phi^{-1}({}^tX)\Phi$ with Φ as in (16). Set $H=\pi^{-1}(\overline{T})$. Then*

$$R_g=\left\{X\in R_{\bar{g}}\;\middle|\;-\varphi(X)=\chi(g)X\right\}\quad \textit{for all } g\in G$$

defines a G-grading on $R^{(-)}$ if and only if H splits as $T\times\langle h\rangle$ with $T=H\cap\ker\chi$ and the following condition holds (identifying $\overline{T}$ with T):

$$\begin{aligned}\beta(t_1)\chi^2(\bar{g}_1)&=\cdots=\beta(t_\ell)\chi^2(\bar{g}_\ell)\\&=\beta(t_{\ell+1})\operatorname{sgn}(S_{\ell+1})\chi^2(\bar{g}_{\ell+1})=\cdots=\beta(t_m)\operatorname{sgn}(S_m)\chi^2(\bar{g}_m)\\&=\mu_{m+1}^{-1}\chi^2(\bar{g}'_{m+1})=\mu_{m+1}\chi^2(\bar{g}''_{m+1})=\cdots=\mu_k^{-1}\chi^2(\bar{g}'_k)=\mu_k\chi^2(\bar{g}''_k).\end{aligned}\tag{20}$$

It is convenient to distinguish the following three cases for a grading of Type II on L:

- The case with $\ell>0$ will be referred to as Type II_1;
- The case with $\ell=0$ but $m>0$, will be referred to as Type II_2;
- The case with $m=0$ will be referred to as Type II_3.

Definition 4.3. We will say that γ is *admissible* if it satisfies

$$\bar{g}_1^2\equiv\cdots\equiv\bar{g}_m^2\equiv\bar{g}'_{m+1}\bar{g}''_{m+1}\equiv\cdots\equiv\bar{g}'_k\bar{g}''_k\pmod{\overline{T}}\tag{21}$$

and, for some $\bar{t}_1,\dots,\bar{t}_\ell\in\overline{T}$, we have $\bar{g}_1^2\bar{t}_1=\cdots=\bar{g}_\ell^2\bar{t}_\ell$ and

$$\beta(\bar{t}_1)\chi^2(\bar{g}_1)=\cdots=\beta(\bar{t}_\ell)\chi^2(\bar{g}_\ell).\tag{22}$$

(If $\ell\leqslant 1$, then condition (22) is automatically satisfied.)

Note that the above definition does not depend on the choice of $\chi\in\widehat{G}$ with $\chi(h)=-1$. Indeed, if we replace χ by $\tilde{\chi}=\chi\psi$ where $\psi\in\widehat{G}$ satisfies $\psi(h)=1$, then ψ can be regarded as a character on $\overline{G}$ and we can compute:

$$\tilde{\chi}^2(\bar{g}_i^{-1}\bar{g}_j)=\chi^2(\bar{g}_i^{-1}\bar{g}_j)\psi^2(\bar{g}_i^{-1}\bar{g}_j)=\chi^2(\bar{g}_i^{-1}\bar{g}_j)\psi(\bar{g}_i^{-2}\bar{g}_j^2)=\chi^2(\bar{g}_i^{-1}\bar{g}_j)\psi(\bar{t}_i\bar{t}_j)$$

for all $1\leqslant i,j\leqslant\ell$. On the other hand, for $\bar{t}\in\overline{T}$, we have

$$\beta(\bar{t}\bar{t}_i)\beta(\bar{t}\bar{t}_j)=\beta(\bar{t})\beta(\bar{t}_i)\beta(\bar{t},\bar{t}_i)\beta(\bar{t})\beta(\bar{t}_j)\beta(\bar{t},\bar{t}_j)=\beta(\bar{t}_i)\beta(\bar{t}_j)\beta(\bar{t},\bar{t}_i\bar{t}_j).$$

Therefore, if condition (22) holds for χ and $\bar{t}_1,\dots,\bar{t}_\ell$, then it holds for $\tilde{\chi}$ and $\bar{t}\bar{t}_1,\dots,\bar{t}\bar{t}_\ell$ where $\bar{t}$ is the unique element of $\overline{T}$ such that $\beta(\bar{t},\bar{u})=\psi(\bar{u})$ for all $\bar{u}\in\overline{T}$.

As pointed out earlier, for γ satisfying (21), we can replace $\bar{g}''_i$, $i>m$, within their cosets mod $\overline{T}$ so that γ satisfies (15).

We now give our standard realizations for gradings of Type II. Let $H \subset G$ be an elementary 2-group of odd rank containing h. Let β be a nondegenerate alternating bicharacter on $\overline{T} = H/\langle h\rangle$. Fix κ. Choose γ formed from elements of $\overline{G} = G/\langle h\rangle$ and τ formed from elements of $\overline{T} = H/\langle h\rangle$ so that they satisfy (15). Let $R = \mathcal{M}(\overline{G}, \overline{T}, \beta, \kappa, \gamma)$. Fix $\chi \in \widehat{G}$ with $\chi(h) = -1$ and identify $\overline{T}$ with $T = H \cap \ker\chi$.

Definition 4.4. Suppose $\ell > 0$ and γ is admissible. Let Φ be the matrix given by (16) where the scalars μ_i and matrices S_i are determined by Eq. (20). Then, by Proposition 4.2, the anti-automorphism $\varphi(X) = \Phi^{-1}({}^tX)\Phi$ defines a refinement of the $\overline{G}$-grading on the associative algebra R to a G-grading $R = \bigoplus_{g\in G} R_g$ as a Lie algebra. Set $L_g = (R_g \cap [R,R]) \bmod Z(R)$. We will denote the G-graded algebra L obtained in this way as $\mathcal{A}^{(\mathrm{II}_1)}(G, H, h, \beta, \kappa, \gamma, \tau)$.

Definition 4.5. Suppose $\ell = 0$ and $m > 0$. Choose $\delta = (\delta_1, \ldots, \delta_m) \in \{\pm 1\}^m$ so that

$$\beta(t_1)\chi^2(\bar{g}_1)\delta_1 = \cdots = \beta(t_m)\chi^2(\bar{g}_m)\delta_m.$$

(Note that there are exactly two such choices.) Let Φ be the matrix given by (16) where the matrices S_i are selected by the rule $\operatorname{sgn}(S_i) = \delta_i$ and the scalars μ_i are determined by Eq. (20). Then, by Proposition 4.2, the anti-automorphism $\varphi(X) = \Phi^{-1}({}^tX)\Phi$ defines a refinement of the $\overline{G}$-grading on the associative algebra R to a G-grading $R = \bigoplus_{g\in G} R_g$ as a Lie algebra. Set $L_g = (R_g \cap [R,R]) \bmod Z(R)$. We will denote the G-graded algebra L obtained in this way as $\mathcal{A}^{(\mathrm{II}_2)}(G, H, h, \beta, \kappa, \gamma, \tau, \delta)$.

Definition 4.6. Suppose $m = 0$. Then we have

$$\chi^2(\bar{g}'_1\bar{g}''_1) = \cdots = \chi^2(\bar{g}'_k\bar{g}''_k).$$

Let μ be a scalar such that μ^2 is equal to the common value of $\chi^2(\bar{g}'_i\bar{g}''_i)$. (There are two choices.) Let Φ be the matrix given by (16) where the scalars μ_i are determined by equation

$$\mu_1^{-1}\chi^2(\bar{g}'_1) = \mu_1\chi^2(\bar{g}''_1) = \cdots = \mu_k^{-1}\chi^2(\bar{g}'_k) = \mu_k\chi^2(\bar{g}''_k) = \mu.$$

Then, by Proposition 4.2, the anti-automorphism $\varphi(X) = \Phi^{-1}({}^tX)\Phi$ defines a refinement of the $\overline{G}$-grading on the associative algebra R to a G-grading $R = \bigoplus_{g\in G} R_g$ as a Lie algebra. Set $L_g = (R_g \cap [R,R]) \bmod Z(R)$. We will denote the G-graded algebra L obtained in this way as $\mathcal{A}^{(\mathrm{II}_3)}(G, H, h, \beta, \kappa, \gamma, \mu)$.

Definition 4.7. Referring to Definition 4.5, we will write $(\kappa, \gamma, \tau, \delta) \approx (\widetilde{\kappa}, \widetilde{\gamma}, \widetilde{\tau}, \widetilde{\delta})$ if κ and $\widetilde{\kappa}$ have the same number of components of each type, i.e., the same values of m and k, and there exist an element $\bar{g} \in \overline{G}$ and a permutation π of the symbols $\{1, \ldots, k\}$ preserving the sets $\{1, \ldots, m\}$ and $\{m+1, \ldots, k\}$ such that $\widetilde{q}_i = q_{\pi(i)}$ for all i, $\widetilde{t}_i = t_{\pi(i)}$, $\widetilde{\bar{g}}_i \equiv \bar{g}_{\pi(i)}\bar{g} \pmod{\overline{T}}$ and $\widetilde{\delta}_i = \delta_{\pi(i)}$ for all $i = 1, \ldots, m$, and $\{\widetilde{\bar{g}}'_i, \widetilde{\bar{g}}''_i\} \equiv \{\bar{g}'_{\pi(i)}\bar{g}, \bar{g}''_{\pi(i)}\bar{g}\} \pmod{\overline{T}}$ for all $i = m+1, \ldots, k$.

Definition 4.8. Referring to Definition 4.6, we will write $(\kappa, \gamma, \mu) \approx (\widetilde{\kappa}, \widetilde{\gamma}, \widetilde{\mu})$ if κ and $\widetilde{\kappa}$ have the same number of components k and there exist an element $\bar{g} \in \overline{G}$ and a permutation π of the symbols $\{1, \ldots, k\}$ such that $\widetilde{q}_i = q_{\pi(i)}$, $\{\widetilde{\bar{g}}'_i, \widetilde{\bar{g}}''_i\} \equiv \{\bar{g}'_{\pi(i)}\bar{g}, \bar{g}''_{\pi(i)}\bar{g}\} \pmod{\overline{T}}$ and $\widetilde{\bar{g}}'_i\widetilde{\bar{g}}''_i = \bar{g}'_{\pi(i)}\bar{g}''_{\pi(i)}\bar{g}^2$ for all i, and, finally, $\widetilde{\mu} = \mu\chi^2(\bar{g})$.

Theorem 4.9. *Let $\mathbb{F}$ be an algebraically closed field,* $\operatorname{char}\mathbb{F} \neq 2$. *Let G be an abelian group. Let $L = \mathfrak{psl}_n(\mathbb{F})$ where $n \geqslant 3$. If $n = 3$, assume also that* $\operatorname{char}\mathbb{F} \neq 3$. *Let $L = \bigoplus_{g\in G} L_g$ be a G-grading. Then the graded algebra L is isomorphic to one of the following*:

- $\mathcal{A}^{(\mathrm{I})}(G, T, \beta, \kappa, \gamma)$,
- $\mathcal{A}^{(\mathrm{II}_1)}(G, H, h, \beta, \kappa, \gamma, \tau)$,
- $\mathcal{A}^{(\mathrm{II}_2)}(G, H, h, \beta, \kappa, \gamma, \tau, \delta)$,
- $\mathcal{A}^{(\mathrm{II}_3)}(G, H, h, \beta, \kappa, \gamma, \mu)$,

as in Definitions 4.1, 4.4, 4.5 *and* 4.6, *with* $|\kappa|\sqrt{|T|} = n$ *in Type I and* $|\kappa|\sqrt{|H|/2} = n$ *in Type II. Graded algebras belonging to different types listed above are not isomorphic. Within each type, we have the following*:

- $\mathcal{A}^{(\mathrm{I})}(G, T_1, \beta_1, \kappa_1, \gamma_1) \cong \mathcal{A}^{(\mathrm{I})}(G, T_2, \beta_2, \kappa_2, \gamma_2)$ *if and only if* $T_1 = T_2$, $\beta_1 = \beta_2$, *and* $(\kappa_1, \gamma_1) \sim (\kappa_2, \gamma_2)$ *or* $(\kappa_1, \gamma_1) \sim (\kappa_2, \gamma_2^{-1})$;
- $\mathcal{A}^{(\mathrm{II}_1)}(G, H_1, h_1, \beta_1, \kappa_1, \gamma_1, \tau_1) \cong \mathcal{A}^{(\mathrm{II}_1)}(G, H_2, h_2, \beta_2, \kappa_2, \gamma_2, \tau_2)$ *if and only if* $H_1 = H_2$, $h_1 = h_2$, $\beta_1 = \beta_2$, *and* $(\kappa_1, \gamma_1, \tau_1) \approx (\kappa_2, \gamma_2, \tau_2)$ *or* $(\kappa_1, \gamma_1, \tau_1) \approx (\kappa_2, \gamma_2^{-1}, \tau_2)$;
- $\mathcal{A}^{(\mathrm{II}_2)}(G, H_1, h_1, \beta_1, \kappa_1, \gamma_1, \tau_1, \delta_1) \cong \mathcal{A}^{(\mathrm{II}_2)}(G, H_2, h_2, \beta_2, \kappa_2, \gamma_2, \tau_2, \delta_2)$ *if and only if* $H_1 = H_2$, $h_1 = h_2$, $\beta_1 = \beta_2$, *and* $(\kappa_1, \gamma_1, \tau_1, \delta_1) \approx (\kappa_2, \gamma_2, \tau_2, \delta_2)$ *or* $(\kappa_1, \gamma_1, \tau_1, \delta_1) \approx (\kappa_2, \gamma_2^{-1}, \tau_2, \delta_2)$;
- $\mathcal{A}^{(\mathrm{II}_3)}(G, H_1, h_1, \beta_1, \kappa_1, \gamma_1, \mu_1) \cong \mathcal{A}^{(\mathrm{II}_3)}(G, H_2, h_2, \beta_2, \kappa_2, \gamma_2, \mu_2)$ *if and only if* $H_1 = H_2$, $h_1 = h_2$, $\beta_1 = \beta_2$, *and* $(\kappa_1, \gamma_1, \mu_1) \approx (\kappa_2, \gamma_2, \mu_2)$ *or* $(\kappa_1, \gamma_1, \mu_1) \approx (\kappa_2, \gamma_2^{-1}, \mu_2^{-1})$.

Proof. The first statement is a combination of Theorem 2.10 and Proposition 4.2. The non-isomorphism of graded algebras belonging to different types is clear.

For Type I, let $R_1 = \mathcal{M}(G, T_1, \beta_1, \kappa_1, \gamma_1)$ and $R_2 = (G, T_2, \beta_2, \kappa_2, \gamma_2)$. By Theorem 2.6, $R_1 \cong R_2$ if and only if $T_1 = T_2$, $\beta_1 = \beta_2$, and $(\kappa_1, \gamma_1) \sim (\kappa_2, \gamma_2)$. It remains to observe that the outer automorphism $X \mapsto -{}^tX$ transforms $\mathcal{M}(G, T, \beta, \kappa, \gamma)$ to $\mathcal{M}(G, T, \beta, \kappa, \gamma^{-1})$.

For Type II, the element h, the subgroup H, and the bicharacter β on $\overline{T} = H/\langle h\rangle$ are uniquely determined by the grading, so we may assume $H_1 = H_2$, $h_1 = h_2$, and $\beta_1 = \beta_2$. Let $R_1 = \mathcal{M}(\overline{G}, \overline{T}, \beta, \kappa_1, \gamma_1)$ and $R_2 = \mathcal{M}(\overline{G}, \overline{T}, \beta, \kappa_2, \gamma_2)$. Fix $\chi \in \widehat{G}$ with $\chi(h) = -1$. Let φ_1 and φ_2 be the corresponding anti-automorphisms. We have to check that $(R_1, \varphi_1) \cong (R_2, \varphi_2)$ if and only if

II$_1$) $(\kappa_1, \gamma_1, \tau_1) \approx (\kappa_2, \gamma_2, \tau_2)$,
II$_2$) $(\kappa_1, \gamma_1, \tau_1, \delta_1) \approx (\kappa_2, \gamma_2, \tau_2, \delta_2)$,
II$_3$) $(\kappa_1, \gamma_1, \mu_1) \approx (\kappa_2, \gamma_2, \mu_2)$.

For Type II$_1$, the "only if" part is clear, since (κ, γ, τ) is an invariant of (R, φ) (up to transformations indicated in the definition of the equivalence relation $\approx$). Indeed, (κ, γ) is an invariant of the $\overline{G}$-grading, and τ corresponds to the restrictions of φ to $D_1, \dots, D_m$ by Proposition 2.3. To prove the "if" part, assume $(\kappa_1, \gamma_1, \tau_1) \approx (\kappa_2, \gamma_2, \tau_2)$. Then, by Theorem 2.6, there exists an isomorphism of $\overline{G}$-graded algebras $\psi : R_1 \to R_2$. By Remark 2.7, we can take for ψ a monomial isomorphism associated to the permutation π in Definition 2.14. The matrix of the anti-automorphism $\psi^{-1}\varphi_2\psi$ on R_1 is then obtained from the matrix of φ_2 by permuting the blocks on the diagonal so that they align with the corresponding blocks of φ_1, and possibly multiplying some of the blocks by -1. Hence, by Theorem 2.10, $\psi^{-1}\varphi_2\psi$ can be transformed to φ_1 by an automorphism of the $\overline{G}$-graded algebra R_1.

For Type II$_2$, the proof is similar, since δ corresponds to the restrictions of φ to $C_1, \dots, C_m$ and thus is an invariant of (R, φ).

For Type II$_3$, we show in the same manner that if $(\kappa_1, \gamma_1, \mu_1) \approx (\kappa_2, \gamma_2, \mu_2)$, then $(R_1, \varphi_1) \cong (R_2, \varphi_2)$. Namely, we take a monomial isomorphism of $\overline{G}$-graded algebras $\psi : R_1 \to R_2$ associated to the permutation π in Definition 4.8. The effect of ψ on Φ_2 is just the permutation of blocks. The factor $\chi^2(\bar{g})$ in Definition 4.8 makes sure that the block with $\mu_i = \mu^{-1}\chi^2(\bar{g}'_i)$ in Φ_2 matches up with the block with $\mu_{\pi(i)} = \mu^{-1}\chi^2(\bar{g}'_{\pi(i)})$ in Φ_1. Conversely, suppose there exists an isomorphism $\psi : (R_1, \varphi_1) \to (R_2, \varphi_2)$. As in the proof of Corollary 2.15, we write $\psi = \psi_0\alpha$ where ψ_0 is a monomial isomorphism and α is in $\mathrm{PGL}_{\kappa_1}(\mathbb{F}) \times \mathrm{Aut}_G(D)$. The action of ψ_0 on φ_2 permutes the blocks and replaces the second tensor factor I by X_{t_0} for some $t_0 \in T$. The action of α on $\psi_0^{-1}\varphi_2\psi_0$ cannot change t_0 or the values of the scalars. We conclude that $t_0 = e$ and $(\kappa_1, \gamma_1, \mu_1) \approx (\kappa_2, \gamma_2, \mu_2)$. □

Remark 4.10. Let $\mathbb{F}$ and G be as in Theorem 4.9. Let $L = \mathfrak{sl}_2(\mathbb{F})$. If $L = \bigoplus_{g\in G} L_g$ is a G-grading, then the graded algebra L is isomorphic to $\mathcal{A}^{(\mathrm{I})}(G, T, \beta, \kappa, \gamma)$ where $|\kappa|\sqrt{|T|} = 2$. This, of course, gives two possibilities: either $T = \{e\}$ or $T \cong \mathbb{Z}_2^2$. In the first case the G-grading is induced from a Cartan decomposition by a homomorphism $\mathbb{Z} \to G$. The isomorphism classes of such gradings are in one-to-one correspondence with unordered pairs of the form $\{g, g^{-1}\}$, $g \in G$. In the second case the G-grading is given by Pauli matrices. The isomorphism classes of such gradings are in one-to-one correspondence with subgroups $T \subset G$ such that $T \cong \mathbb{Z}_2^2$.

Remark 4.11. The remaining case $L = \mathfrak{psl}_3(\mathbb{F})$ where $\operatorname{char}\mathbb{F} = 3$ can be handled using octonions. Let $\mathbb{O}$ be the algebra of octonions over an algebraically closed field $\mathbb{F}$. Then the subspace $\mathbb{O}'$ of zero trace octonions is a Malcev algebra with respect to the commutator $[x, y] = xy - yx$. If $\operatorname{char}\mathbb{F} = 3$, then $\mathbb{O}'$ is a Lie algebra isomorphic to L. Assuming $\operatorname{char}\mathbb{F} \neq 2$, we have $xy = \frac{1}{2}([x, y] - n(x, y)1)$ for all $x, y \in \mathbb{O}'$, where n is the norm of $\mathbb{O}$. We also have $(\operatorname{ad} x)^3 = -4n(x)(\operatorname{ad} x)$ for all $x \in \mathbb{O}'$. It follows that if ψ is an automorphism of $\mathbb{O}'$, then ψ preserves n and, setting $\psi(1) = 1$, we obtain an automorphism of $\mathbb{O}$. Hence the restriction map $\operatorname{Aut}(\mathbb{O}) \to \operatorname{Aut}(\mathbb{O}')$ is an isomorphism of algebraic groups. Similarly, one shows that the restriction map $\operatorname{Der}(\mathbb{O}) \to \operatorname{Der}(\mathbb{O}')$ is an isomorphism of Lie algebras.[1] It follows that $\mathbf{Aut}(\mathbb{O}')$ is smooth and can be identified with the algebraic group $\operatorname{Aut}(\mathbb{O})$. In particular, this means that the isomorphism classes of G-gradings on $\mathbb{O}$ are in one-to-one correspondence (via restriction) with the isomorphism classes of G-gradings on $\mathbb{O}'$ (cf. [12, Theorem 9]).

All gradings on $\mathbb{O}$ (in any characteristic) were described in [12]. For $\operatorname{char}\mathbb{F} \neq 2$, they are of two types:

- "elementary" gradings obtained by choosing $g_1, g_2, g_3 \in G$ with $g_1g_2g_3 = e$ and assigning degree e to e_1 and e_2, degree g_i to u_i and degree g_i^{-1} to v_i, $i = 1, 2, 3$, where $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ is a canonical basis for $\mathbb{O}$;
- "division" gradings by $\mathbb{Z}_2^3$ obtained by iterating the Cayley–Dickson doubling process three times.

It is easy to see when two G-gradings on $\mathbb{O}$ are isomorphic. The isomorphism classes of "elementary" gradings are in one-to-one correspondence with unordered pairs of the form $\{S, S^{-1}\}$ where S is an unordered triple $\{g_1, g_2, g_3\}$, $g_i \in G$ with $g_1g_2g_3 = e$. The isomorphism classes of "division" gradings are in one-to-one correspondence with subgroups $T \subset G$ such that $T \cong \mathbb{Z}_2^3$. An "elementary" grading is not isomorphic to a "division" grading.

If $\operatorname{char}\mathbb{F} = 3$, then the above is also the classification of G-gradings on $L = \mathfrak{psl}_3(\mathbb{F})$. As shown in [16], up to isomorphism, any grading on L is induced from the matrix algebra $M_3(\mathbb{F})$. Namely, any "elementary" grading on L can be obtained as a Type I grading, and any "division" grading on L is isomorphic to a Type II gradings. The only difference with the case of $\mathfrak{sl}_3(\mathbb{F})$ where $\operatorname{char}\mathbb{F} \neq 3$ is that there are fewer isomorphism classes of gradings in characteristic 3 (in particular, some "Type II" gradings are isomorphic to "Type I" gradings).

5. Gradings on Lie algebras of types $\mathcal{B}, \mathcal{C}, \mathcal{D}$

The classification of gradings for Lie algebras $\mathfrak{so}_n(\mathbb{F})$ and $\mathfrak{sp}_n(\mathbb{F})$ follows immediately from Corollary 2.15. We state the results here for completeness. Recall $\mathcal{M}^*(G, T, \beta, \kappa, \gamma, \tau, \delta)$ from Definition 2.13. Let $L = \mathcal{K}(R, \varphi) = \{X \in R \mid \varphi(X) = -X\}$. Then $L = \bigoplus_{g\in G} L_g$ where $L_g = R_g \cap L$.

Definition 5.1. Let $n = |\kappa|\sqrt{|T|}$.

- If $\delta = 1$ and n is odd, then necessarily $T = \{e\}$. We will denote the G-graded algebra L by $\mathcal{B}(G, \kappa, \gamma)$.

[1] This argument was communicated to us by A. Elduque.

- If $\delta = -1$ (hence n is even), then we will denote the G-graded algebra L by $\mathcal{C}(G, T, \beta, \kappa, \gamma, \tau)$.
- If $\delta = 1$ and n is even, then we will denote the G-graded algebra L by $\mathcal{D}(G, T, \beta, \kappa, \gamma, \tau)$.

Theorem 5.2. *Let $\mathbb{F}$ be an algebraically closed field,* $\operatorname{char}\mathbb{F} \neq 2$. *Let G be an abelian group.*

- *Let $L = \mathfrak{so}_n(\mathbb{F})$, with odd $n \geqslant 5$. Let $L = \bigoplus_{g\in G} L_g$ be a G-grading. Then the graded algebra L is isomorphic to $\mathcal{B}(G, \kappa, \gamma)$,*
- *Let $L = \mathfrak{sp}_n(\mathbb{F})$, with even $n \geqslant 6$. Let $L = \bigoplus_{g\in G} L_g$ be a G-grading. Then the graded algebra L is isomorphic to $\mathcal{C}(G, T, \beta, \kappa, \gamma, \tau)$,*
- *Let $L = \mathfrak{so}_n(\mathbb{F})$, with even $n \geqslant 10$. Let $L = \bigoplus_{g\in G} L_g$ be a G-grading. Then the graded algebra L is isomorphic to $\mathcal{D}(G, T, \beta, \kappa, \gamma, \tau)$,*

as in Definition 5.1. *Also, under the above restrictions on n, we have the following*:

- *$\mathcal{B}(G, \kappa_1, \gamma_1) \cong \mathcal{B}(G, \kappa_1, \gamma_1)$ if and only if $(\kappa_1, \gamma_1) \approx (\kappa_1, \gamma_1)$;*
- *$\mathcal{C}(G, T_1, \beta_1, \kappa_1, \gamma_1, \tau_1) \cong \mathcal{C}(G, T_2, \beta_2, \kappa_2, \gamma_2, \tau_2)$ if and only if $T_1 = T_2$, $\beta_1 = \beta_2$ and $(\kappa_1, \gamma_1, \tau_1) \approx (\kappa_2, \gamma_2, \tau_2)$;*
- *$\mathcal{D}(G, T_1, \beta_1, \kappa_1, \gamma_1, \tau_1) \cong \mathcal{D}(G, T_2, \beta_2, \kappa_2, \gamma_2, \tau_2)$ if and only if $T_1 = T_2$, $\beta_1 = \beta_2$ and $(\kappa_1, \gamma_1, \tau_1) \approx (\kappa_2, \gamma_2, \tau_2)$.*

References

[1] Y. Bahturin, M. Kochetov, Group gradings on the Lie algebra $\mathfrak{psl}_n$ in positive characteristic, J. Pure Appl. Algebra 213 (9) (2009) 1739–1749.
[2] Y. Bahturin, M. Kochetov, S. Montgomery, Group gradings on simple Lie algebras in positive characteristic, Proc. Amer. Math. Soc. 137 (4) (2009) 1245–1254.
[3] Y. Bahturin, S. Sehgal, M. Zaicev, Group gradings on associative algebras, J. Algebra 241 (2) (2001) 677–698.
[4] Y. Bahturin, I. Shestakov, M. Zaicev, Gradings on simple Jordan and Lie algebras, J. Algebra 283 (2) (2005) 849–868.
[5] Yuri Bahturin, M. Zaicev, Involutions on graded matrix algebras, J. Algebra 315 (2) (2007) 527–540.
[6] Y. Bahturin, M. Zaicev, Graded algebras and graded identities, in: Polynomial Identities and Combinatorial Methods, Pantelleria, 2001, in: Lect. Notes Pure Appl. Math., vol. 235, Dekker, New York, 2003, pp. 101–139.
[7] Y. Bahturin, M. Zaicev, Group gradings on matrix algebras, Canad. Math. Bull. 45 (4) (2002) 499–508.
[8] Y. Bahturin, M. Zaicev, Gradings on simple Lie algebras of type "A", J. Lie Theory 16 (4) (2006) 719–742.
[9] Y. Bahturin, M. Zaicev, Gradings on simple algebras of finitary matrices, J. Algebra 324 (6) (2010) 1279–1289, arXiv:0906.4595v1 [math.RA].
[10] K.I. Beidar, M. Brešar, M.A. Chebotar, W.S. Martindale 3rd, On Herstein's Lie map conjectures. III, J. Algebra 249 (1) (2002) 59–94.
[11] A.A. Chasov, Isomorphisms of graded matrix algebras, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 70 (5) (2004) 12–18 (in Russian, Russian summary); translation in Moscow Univ. Math. Bull. 59 (5) (2004) 12–18 (2005).
[12] A. Elduque, Gradings on octonions, J. Algebra 207 (1) (1998) 342–354.
[13] A. Elduque, Fine gradings on simple classical Lie algebras, arXiv:0906.0655v1 [math.RA].
[14] M. Havlíček, J. Patera, E. Pelantová, On Lie gradings. II, Linear Algebra Appl. 277 (1–3) (1998) 97–125.
[15] N. Jacobson, Lie Algebras, Dover Publications, Inc., New York, 1979.
[16] M. Kochetov, Gradings on finite-dimensional simple Lie algebras, Acta Appl. Math. 108 (2009) 101–127.
[17] S. Montgomery, Hopf Algebras and Their Actions on Rings, CBMS Reg. Conf. Ser. Math., vol. 82, American Mathematical Society, Providence, RI, 1993.
[18] G.B. Seligman, Modular Lie Algebras, Ergeb. Math. Grenzgeb., vol. 40, Springer-Verlag New York, Inc., New York, 1967.
[19] W.C. Waterhouse, Introduction to Affine Group Schemes, Grad. Texts in Math., vol. 66, Springer-Verlag, New York, Berlin, 1979.