On products of discretely generated spaces✩

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Abstract

Assuming a measurable cardinal exists, we construct a pair of discretely generated spaces whose product fails to be weakly discretely generated. Under the Continuum Hypothesis, a similar result is obtained for a pair of countable Fréchet spaces as well as for two compact discretely generated spaces whose product is not discretely generated. A somewhat weaker example is presented assuming Martin’s Axiom for countable posets. Further, the class of strongly discretely generated compacta is shown to preserve discrete generability in products.

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The study of discretely generated and related spaces, i.e. topological spaces in which the closure operator can be defined by adding cluster points of discrete subsets, has been initiated recently in [5]. In the paper several questions about productivity of certain classes of these spaces were asked. We shall give a partial answer to [5],

Problem 4.1. Let $X$ be a discretely generated compact space. Is it true that $X \times X$ is discretely generated?

Problem 4.4. Is $X \times X$ weakly discretely generated assuming that $X$ is such?

By the notion of a discrete subset of a given topological space we shall understand a subset which, endowed with the subspace topology, is discrete.

Definition. A topological space $X$ is called discretely generated if whenever $A \subseteq X$ and $x \in \overline{A}$ then exists a discrete subset $D \subseteq A$ such that $x \in \overline{D}$.

We say that $X$ is weakly discretely generated if for every non-closed $A \subseteq X$ there is a point $x \in \overline{A} \setminus A$ and a discrete $D \subseteq A$ such that $x \in \overline{D}$.

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A set $D \subseteq X$ is strongly discrete if there exists a disjoint family \( \{ U_x : x \in D \} \) consisting of open sets such that \( x \in U_x \) for all \( x \in D \). A space $X$ is called strongly discretely generated if for every $A \subseteq X$ and every $x \in \overline{A}$ there is a strongly discrete set $D \subseteq A$ such that $x \in \overline{D}$.

In the sequel, an ordinal number is considered as the set of all its predecessors as well as an element of a larger set. For a cardinal $\kappa$, \([\kappa]^{<\kappa} = \{ A \subseteq \kappa : |A| < \kappa \}$.

The symbol $c$ is used for cardinality of the continuum, $c = |2^\omega|$. We shall denote by $\pi_X$ the natural projection of a product to the space $X$ entering this product.

The “simplest” nontrivial case among products of a pair of spaces is when one of the factors has only one nonisolated point. In this context, a measurable filter provides a counterexample.

**Theorem 1.** Suppose a measurable cardinal exists. Then there are strongly discretely generated spaces $X$ and $Y$ such that $X \times Y$ is not weakly discretely generated.

**Proof.** Fix a measurable cardinal $\kappa$ and a free $\kappa$-complete ultrafilter $\mathcal{U}$ on $\kappa$. Then $\kappa$ is regular, $\kappa > c$ and, in particular, $\mathcal{U}$ is $c^+$-complete.

Further, consider a remote point $P \in \beta I$, where $I$ stands for the half-open interval $[0, 1]$ with Euclidean topology. In other words, $P$ is a maximal filter consisting of open subsets of $I$ such that $P \not\subseteq R^\beta I$ for any nowhere dense $R \subseteq I$ (such an object exists in ZFC, see [3, Theorem 4.3]). Note that for every $\varepsilon > 0$, $G \in P$ implies $G \cap (0, \varepsilon) \in P$.

The space $X$ will be defined as a modification of the metric hedgehog of spineness $\kappa$. Let $X = (\kappa \times I) \cup \{ \infty \}$, where every $I_\alpha = \{ \alpha \} \times I$ ($\alpha < \kappa$) is an open subspace of $X$. Let an open base of $\infty \in X$ consist of all the sets

$$O_{S,G} = \{ \infty \} \cup \left( (\kappa \setminus S) \times G \right),$$

where $S \in [\kappa]^{<\kappa}$ and $G \in P$. As $I \cup \{ P \}$ embeds to $\beta I$, it is easy to verify that $X$ is a regular space.

To prove that $X$ is strongly discretely generated, take $A \subseteq \kappa \times I$ such that $\infty \in \overline{A}$. Let $\{G_\alpha : \alpha < \kappa\}$ be an enumeration of $P$ with every element occurring $\kappa$ many times. Then a one-to-one sequence $\{\beta_\alpha : \alpha < \kappa\}$ and points $x_\alpha \in G_\alpha$ can be chosen inductively such that $\langle \beta_\alpha, x_\alpha \rangle \in A$ for each $\alpha < \kappa$. Indeed at step $\alpha$, $|S| < \kappa$ for $S = \{ \beta_\gamma : \gamma < \alpha \}$, consequently using $\infty \in \overline{A}$ we can choose $\langle \beta_\alpha, x_\alpha \rangle \in A \cap O_{S,G_\alpha}$ with $\beta_\alpha \not= \beta_\gamma$ for each $\gamma < \alpha$.

Put

$$D = \{ \langle \beta_\alpha, x_\alpha \rangle : \alpha < \kappa \}. $$

If $S \in [\kappa]^{<\kappa}$ and $G \in P$ then $G = G_\alpha$ for some $\alpha$ such that $\beta_\alpha \in \kappa \setminus S$, hence $\langle \beta_\alpha, x_\alpha \rangle \in D \cap O_{S,G}$. I.e. $\infty \in \overline{D}$. Moreover, $D$ is a strongly discrete subset of $A$.

The space $Y$ is defined as $\kappa + 1$ with $\kappa \subseteq Y$ discrete and with an open base in $\kappa \subseteq Y$ induced by the ultrafilter $U$, i.e. basic open neighbourhoods of $\kappa$ are all the sets of the form $\{ \kappa \} \cup U$, $U \in \mathcal{U}$. Then $Y$ is obviously strongly discretely generated.

To prove that $X \times Y$ is not weakly discretely generated, consider the set

$$A = \bigcup_{\alpha < \kappa} I_\alpha \times \{ \alpha \} = \{ \langle \alpha, x, \alpha \rangle : \alpha < \kappa \text{ & } x \in I \}. $$

It is easy to see that $\overline{\Delta} \setminus A = \{ (\infty, \kappa) \}$. Suppose $D \subseteq A$ is discrete. Then, in particular, each $D_\alpha = \{ x \in I : \langle \alpha, x, \alpha \rangle \in D \}$ ($\alpha < \kappa$) is a discrete subset of $I$. As there are only $c$ many discrete subsets of the real line and the ultrafilter $\mathcal{U}$ is $c^+$-complete, there is a discrete $R \subseteq I$ such that $U = \{ \alpha < \kappa : D_\alpha = R \} \in \mathcal{U}$. But then, for the open neighbourhood $O = O_{\emptyset, I \setminus R}$ of $x$, $(O \times U) \cap D = \emptyset$, hence $\langle \infty, \kappa \rangle \not\in \overline{D}$. □

In the following two constructions, another clear type of a discrete subset of a product occurs: having two spaces with the same underlying set, whose topologies differ in just one point $\infty$, the respective part of the diagonal is homeomorphic to a subset of the spaces while the neighbourhood systems in $\infty$ can interfere so that $\langle \infty, \infty \rangle$ is not in the closure of any discrete subset of the diagonal.

The proofs of Theorems 2 and 3 stem from corresponding ones in [2] (Theorems 7 and 8 and Lemma 2; history of this method is outlined in the quoted paper). We adopt the notation used there too.
Put $\mathbb{M} = \omega \times C$, where $C$ stands for the Cantor set $\{0, 1\}^\omega$, and for every $n \in \omega$, $C_n = \{n\} \times C$. Let us call a set $A \subset \mathbb{M}$ nice if $(\exists n_0) \ (\forall n \geq n_0) \ A \cap C_n \neq \emptyset$. For $A$, $B \subset \mathbb{M}$ we say that $A$ is essentially embedded in $B$ (and write $A \subset_{\text{ess}} B$) if $(\exists n_0) \ (\forall n \geq n_0) \ (A \setminus B) \cap C_n = \emptyset$. Moreover, for a natural number $k$, $A \subset_k B$ if $(\forall n \geq k) \ (A \setminus B) \cap C_n = \emptyset$. We say that $A$ and $B$ are compatible and write $A \parallel B$ if $\{n \in \omega : A \cap B \cap C_n \neq \emptyset\}$ is infinite. Finally, a set $A$ is admissible if $A \parallel \mathbb{M}$.

**Theorem 2.** Assume CH. There are discretely generated compact spaces $X$ and $Y$, whose product $X \times Y$ is not discretely generated.

**Theorem 3.** Assume CH. There are spaces $X_1$, $Y_1$, both countable and Fréchet (hence strongly discretely generated) such that $X_1 \times Y_1$ is not weakly discretely generated.

Let us remark that since every compact space is weakly discretely generated [5, Proposition 3.1], the conclusion of Theorem 2 cannot be strengthened.

**Proof of Theorems 2 and 3.** In Lemma 1 below we shall show that under CH there are sets $U_\alpha$, $V_\alpha$ ($\alpha < \omega_1$) such that

1. $U_\alpha$, $V_\alpha$ are clopen subsets of $\mathbb{M}$;
2. $\beta < \alpha < \omega_1 \implies U_\alpha \subset_{\text{ess}} U_\beta$, $V_\alpha \subset_{\text{ess}} V_\beta$;
3. $\beta < \alpha < \omega_1 \implies U_\beta \setminus U_\alpha$ as well as $V_\beta \setminus V_\alpha$ is nice;
4. $U_\alpha \cap V_\alpha$ is nice;
5. if $S \subset \mathbb{M}$ is a countable set such that $(\forall \alpha < \omega_1) \ S \parallel U_\alpha$, then there is an infinite discrete set $D \subset S$ such that $(\forall \alpha < \omega_1) \ D \setminus U_\alpha$ is finite, analogously for $V_\alpha$;
6. if $D \subset \mathbb{M}$ is discrete then there is $\beta < \omega_1$ such that $D \parallel U_\beta \cap V_\beta$.

For Theorem 2, let the underlying set of both spaces $X$ and $Y$ be $\mathbb{M} \cup \omega_1 \cup \{\infty\}$, $\mathbb{M} \cong \bigoplus_{n \in \omega} C_n \cong \bigoplus_{n \in \omega} C$ an open subspace of $X$ and $Y$. Let

$$\{(\beta, \alpha) \cup (U_\beta \setminus U_\alpha) \setminus \bigcup_{k \in \omega} C_k : \beta < \alpha \land k \in \omega\}$$

be an open base at $\alpha \in \omega_1 \subset X$ and

$$\{p \cup (\alpha, \omega_1) \cup U_\alpha \setminus \bigcup_{k \in \omega} C_k : \alpha < \omega_1 \land k \in \omega\}$$

an open base at $\infty \in X$. The topology of $Y$ is defined similarly using the sets $V_\alpha$. From (1) it follows that the spaces $X$ and $Y$ are zero-dimensional. The fact that $\omega_1 \cup \{\infty\}$ as a subset of $X$ or $Y$ is homeomorphic to $\omega_1 + 1$ with the order topology together with (5) gives that the spaces are discretely generated (in fact, they are radial). Indeed, for $A \subset \mathbb{M}$, $\infty \in \bar{A}$ in $X$ if and only if $(\forall \alpha < \omega_1) \ A \parallel U_\alpha$. In such a case (5) applies to a countable set $S$ dense in $A$. For proving that the spaces are compact see [2, Proof of Theorem 7].

Let us show the product $X \times Y$ is not discretely generated. Consider

$$\Delta = \{(x, x) : x \in \mathbb{M}\}$$

and $\nabla = \Delta \cup \{(\infty, \infty)\} \subset X \times Y$. Identify $\nabla$ naturally with $\mathbb{M} \cup \{\infty\}$ via the map $(z, z) \mapsto z$. In this setting $\Delta$ is homeomorphic to $\mathbb{M}$ while the sets $(U_\alpha \cap V_\alpha) \setminus \bigcup_{k \in \omega} C_k$ are traces of basic open neighbourhoods of $\infty$ on $\Delta$. From (4) it follows that $\infty \in \bar{\Delta}$. Finally, (6) yields that for any discrete $D \subset \Delta$, $\infty \notin \bar{D}$.

To prove Theorem 3, fix a countable dense subset $Q$ of $C$. Consider the subspaces

$$X_1 = (\omega \times Q) \cup \{\infty\} \subset X,$$

$$Y_1 = (\omega \times Q) \cup \{\infty\} \subset Y.$$
It is easy to see that $X_1$ and $Y_1$ are Fréchet. Now, $\Delta_1 = \Delta \cap (X_1 \times Y_1)$ satisfies $\Delta_1 \setminus \Delta_1 = \{ (\infty, \infty) \}$, but, again, $\langle \infty, \infty \rangle$ is not in the closure of any discrete subset of $\Delta_1$. Hence $X_1 \times Y_1$ has a closed subspace $V_1 = \Delta \cap (X_1 \times Y_1)$ which is not weakly discretely generated, therefore it cannot be weakly discretely generated either. \hfill \Box

Now it suffices to prove

**Lemma 1.** Assuming CH list all discrete subsets of $\mathbb{M}$ as $\{D_\alpha; \ 0 < \alpha < \omega_1 \}$ and all countable subsets of $\mathbb{M}$ as $\{S_\alpha; \ 0 < \alpha < \omega_1 \}$. There exist $U_\alpha, V_\alpha \subset \mathbb{M} (\alpha < \omega_1)$ satisfying (1)–(4) and

$$(5') \text{ if } S_\alpha \parallel U_\alpha (S_\alpha \parallel V_\alpha, \text{ respectively}) \text{ then there is a partial function } \Phi_\alpha \subset S_\alpha (\Psi_\alpha \subset S_\alpha, \text{ respectively}) \text{ from } \omega \text{ to } C \text{ such that (for all such functions defined)}$$

(a) $$(\forall \beta < \omega_1) \Phi_\alpha \subset \text{ess } U_\beta, \Psi_\alpha \subset \text{ess } V_\beta;$$

(b) $\Phi_\alpha \parallel V_\alpha$ and $\Psi_\alpha \parallel U_\alpha$.

$$(6') \text{ } D_\alpha \parallel U_\alpha \cap V_\alpha.$$ 

**Proof.** Put $U_0 = V_0 = \mathbb{M}$.

At step $\alpha \geq 1$ suppose that (1)–(6') are satisfied for all indices smaller than $\alpha$. We shall approximate $U_\alpha$ and $V_\alpha$ by $U_\alpha^0 \supset U_\alpha^1 \supset U_\alpha (V_\alpha^0 \supset V_\alpha \supset V_\alpha^1)$ to fulfill all the conditions.

Enumerate $\alpha = \{ \beta_n; \ n \in \omega \}$ and define $G_n = \bigcap_{k \leq n} U_\beta_k$, $H_n = \bigcap_{k \leq n} V_\beta_k$. Let $\Phi^k (\Psi^k, \text{ respectively}) (k \in \omega)$ be all the sets resulting from (5') for $S_\beta, 0 < \beta < \alpha$.

Note that $(\forall k, n \in \omega) \Phi^k \subset \text{ess } G_n$ and $\Psi^k \subset \text{ess } H_n$. Moreover, as follows from (4), there is an increasing sequence $k_0 < k_1 < \cdots$ of natural numbers such that $\Phi^0, \ldots, \Phi^m \subset \text{ess } G_m, \Psi^0, \ldots, \Psi^m \subset \text{ess } H_m$ and $(\forall n \geq m) G_m \cap H_m \cap C_n \neq \emptyset$. Further, fix for every $m$ and every $n \in [k_m, k_{m+1}]$ a nonempty clopen set $O_n \subset G_m \cap C_n$ not intersecting $\Phi^0 \cup \cdots \cup \Phi^m$ and a nonempty clopen $P_n \subset H_m \cap C_n$ not intersecting $\Psi^0 \cup \cdots \cup \Psi^m$ such that $G_m \cap H_m \cap C_n \setminus (O_n \cup P_n) \neq \emptyset$. Put

$$U_\alpha^0 = \bigcup \{ G_m \cap C_n \setminus O_n; \ m \in \omega \land n \in [k_m, k_{m+1}] \},$$

$$V_\alpha^0 = \bigcup \{ H_m \cap C_n \setminus P_n; \ m \in \omega \land n \in [k_m, k_{m+1}] \}.$$ 

In this setting $U_\alpha^0, V_\alpha^0$ satisfy (1)–(4), and (5'(a)) for $\Phi^k, \Psi^l$.

If

$$S_\alpha \parallel U_\alpha^0 \cap V_\alpha^0,$$  \hspace{1cm} (1)

then choose functions $\Phi_\alpha, \Psi_\alpha \subset S_\alpha \cap U_\alpha^0 \cap V_\alpha^0$ with disjoint infinite domains. Now, suppose that $S_\alpha \parallel U_\alpha^0 \cap V_\alpha^0$. In the case

$$S_\alpha \parallel U_\alpha^0 \setminus V_\alpha^0$$  \hspace{1cm} (2)

choose an infinite $\Phi_\alpha \subset S_\alpha \cap U_\alpha^0 \setminus V_\alpha^0$. Since $(\forall \beta < \alpha) U_\alpha^0 \subset \text{ess } U_\beta$, it also follows that $S_\alpha \parallel U_\beta$ and $\Phi_\alpha \subset \text{ess } U_\beta$.

Define symmetrically $\Psi_\alpha \subset V_\alpha^0 \setminus U_\alpha^0$ if $S_\alpha \parallel V_\alpha^0 \setminus U_\alpha^0$.

It remains to eliminate $\Phi_\alpha$ from $U_\alpha^0, \Psi_\alpha$ from $V_\alpha^0$ and $D_\alpha$ from $U_\alpha^0 \cap V_\alpha^0$.

If $\Phi_\alpha \parallel U_\alpha^0$ it means that $\Phi_\alpha$ has been defined according to case (1). Notice that

$$\langle \forall k \in \omega \rangle \Phi_\alpha \cap \Psi^k \text{ is finite},$$

because $\Phi_\alpha \subset U_\alpha^0$ while $\Psi^k \parallel U_\alpha^0$, as follows from (5'(b)). Also, $\Phi_\alpha \cap \Psi_\alpha = \emptyset$.

Applying 0-dimensionality of $C_k \simeq C$ we can find a clopen neighbourhood $O_k \subset C_k$ of $\Phi_\alpha \cap C_k$ such that

$$(O_k \cap (\Psi_\alpha \cup \Psi^0 \cup \cdots \cup \Psi^k)) = \Phi_\alpha \cap C_k \cap (\Psi_\alpha \cup \Psi^0 \cup \cdots \cup \Psi^k)$$

and if $U_\alpha^0 \cap V_\alpha^0 \cap C_k \neq \emptyset$ then $U_\alpha^0 \cap V_\alpha^0 \cap C_k \not\subset O_k$. Define

$$V_\alpha^1 = V_\alpha^0 \setminus \bigcup_{k \in \omega} O_k.$$
Put $V_α^1 = V_0^α$ if $Φ_α ∥ V_0^α$. Using symmetric arguments define $U_α^1$ such that $Ψ_α ∥ U_α^1$ and the respective properties are preserved.

Assume $D_α ∥ U_α^1 \cap V_α^1$. Find clopen sets $P_k \subset U_α^1 \cap V_α^1 \cap C_k$ such that $P_k \cap \overline{D_α} = \emptyset$ and if $U_α^1 \cap V_α^1 \cap C_k \neq \emptyset$ then $P_k \neq \emptyset$. Finally, define

$$U_α = (U_α^1 \setminus V_α^1) \cup \bigcup_{k \in \omega} P_k,$$

$$V_α = (V_α^1 \setminus U_α^1) \cup \bigcup_{k \in \omega} P_k,$$

or put $U_α = U_α^1, V_α = V_α^1$ in case $D_α ∥ U_α^1 \cap V_α^1$. Since $Φ_α \subsetess U_α^1 \setminus V_α^1$, condition (5\(’b\)) is satisfied; the others are routine. □

The weakest assumption we are able to derive a counterexample from is Martin’s Axiom for countable posets (MA\(_{ctble}\); in what follows, the same arguments are valid if $C$ is replaced by $Q$ as in Theorem 3). In this case list all countable parts of $\mathbb{M}$ as \{ $S_α: 0 < α < c$ \} and all its discrete parts as \{ $D_α: 0 < α < c$ \}. Put $U_0 = V_0 = \mathbb{M}$. At a step $α$ with $0 < α < c$ denote by $U$ (by $V$, respectively) the filter base generated by \{ $U_β: β < α$ \} (by \{ $V_β: β < α$ \}, respectively). Obviously, $|U| + |V| < c$.

We shall find clopen subsets $U_α, V_α$ of $\mathbb{M}$ as well as a partial function $Φ_α \subset S_α$ ($Ψ_α \subset S_α$, respectively) from $ω$ to $C$ defined in the case

$$∀U ∈ U) S_α ∥ U \quad (3)$$

(in the case

$$∀V ∈ V) S_α ∥ V, \quad (4)$$

respectively) such that

(i) $(∀U ∈ U) (∀V ∈ V) U \cap V \cap U_α \cap V_α$ is admissible;
(ii) $(∀β ≤ α)$ if $Φ_β$ is defined then $(∀U ∈ U) Φ_β ∥ U \cap U_α$, analogously for $Ψ_α$ and elements of $V$;
(iii) $D_α \cap U_α \cap V_α = \emptyset$.

Indeed, suppose (3) and (4) hold. Let $C$ be a countable clopen base of $C$. Consider the poset

$$\mathcal{P} = \{ (u, v, φ, ψ): u, v, φ, ψ \text{ are finite functions}, \quad \text{dom} u \subset ω, \quad \text{dom} v \subset ω, \quad \text{rng} u \subset C, \quad \text{rng} v \subset C, \quad φ \subset S, \quad ψ \subset S, \quad D_α \cap u \cap v = \emptyset \}$$

with $(u_1, v_1, φ_1, ψ_1) \leq (u_2, v_2, φ_2, ψ_2)$ iff $u_1 \supset u_2 \& v_1 \supset v_2 \& φ_1 \supset φ_2 \& ψ_1 \supset ψ_2$ (or an obvious variant if none or just one of conditions (3), (4) holds).

Then for a generic filter $G$ given by MA\(_{ctble}\) we can define

$$U_α = \bigcup \{ \text{rng} u: (∃v, φ, ψ) (u, v, φ, ψ) \in G \},$$

$$V_α = \bigcup \{ \text{rng} v: (∃u, φ, ψ) (u, v, φ, ψ) \in G \},$$

$$Φ_α = \bigcup \{ φ: (∃u, v, φ) (u, v, φ, ψ) \in G \},$$

$$Ψ_α = \bigcup \{ ψ: (∃u, v, φ) (u, v, φ, ψ) \in G \}.$$

It is easy to verify that $U_α, V_α$ are clopen sets and (iii) holds. Since for every $U ∈ U, V ∈ V, n ∈ ω$ and every $β < α$ such that $Φ_β$ is defined, the sets

$$\{ (u, v, φ, ψ) ∈ \mathcal{P}: (∃m ≥ n) U \cap V \cap u \cap v \cap C_m \neq \emptyset \}$$

as well as
\[ \{ (u, v, \phi, \psi) \in \mathcal{P} : (\exists m \geq n) \bigcap_{u \cap \phi \cap C_m \neq \emptyset} \} \]

and the respective sets defined for the space \( Y \) are dense in \((\mathcal{P}, \leq)\), conditions (i) and (ii) are satisfied too.

Defining an open base at \( \infty \in X = \mathcal{M} \cup \{ \infty \} \) to be the system
\[
\bigcup_{i<n} C_i: k, n \in \omega, \alpha_0, \ldots, \alpha_k < \epsilon
\]

and analogously on the space \( Y \), we obtain

**Theorem 4.** Assume \( \text{MA}_{\text{cld}} \) holds. There is a pair of (countable) discretely generated spaces whose product is not weakly discretely generated.

A routine proof shows that the product of a weakly discretely generated space and a compact space is weakly discretely generated. To show that discrete generability is preserved in certain classes of compacta the following lemma applies.

**Lemma 2.** Suppose \( X \) is a discretely generated space, \( Y \) is a compact space. Let \( A \subset X \times Y, \langle x, y \rangle \in \overline{A} \setminus A \). Define
\[
M = \{ z \in Y : \langle x, z \rangle \in D \text{ for some discrete set } D \subset A \}.
\]

Then \( y \in \overline{M} \).

**Proof.** Assume \( y \notin \overline{M} \). There is a closed neighbourhood of \( y \) not intersecting \( M \), hence, without loss of generality, we can assume that
\[
(\forall z \in Y) (\forall D \subset A) \, D \text{ discrete } \Rightarrow \langle x, z \rangle \notin \overline{D}.
\]

In particular, we suppose that \( \{ z \in Y : \langle x, z \rangle \in A \} \) is empty. Since \( x \in \pi_X[A] = \overline{\pi_X[A]} \), and \( X \) is discretely generated, there is a discrete \( D_0 \subset \pi_X[A] \) such that \( x \in \overline{D_0} \). For every \( x_0 \in D_0 \) pick a \( y_{x_0} \) such that \( \langle x_0, y_{x_0} \rangle \in A \). Obviously,
\[
D = \{ \langle x_0, y_{x_0} \rangle : x_0 \in D_0 \}
\]
is discrete. Now, \( x \in \overline{D_0} = \pi_X[D] = \pi_X[\overline{D}] \), hence \( \overline{D} \cap \pi_X^{-1}(x) \neq \emptyset \)—a contradiction. \( \square \)

The spaces constructed in the proof of Theorem 2 are not strongly discretely generated, since they have countable cellularity and uncountable tightness. The next theorem shows that the lack of this property is essential.

**Theorem 5.** The product of any discretely generated space and a compact strongly discretely generated space is discretely generated.

**Proof.** Suppose \( X \) is discretely generated and \( Y \) is compact strongly discretely generated. Let \( A \subset X \times Y, \langle x, y \rangle \in \overline{A} \setminus A \). As \( Y \) is discretely generated, we may assume \( x \notin \pi_X[A] \). According to Lemma 2, \( y \in \overline{M} \), where
\[
M = \{ z \in Y : \langle x, z \rangle \in \overline{D} \text{ for some discrete set } D \subset A \}.
\]

Fix a strongly discrete \( D_0 \subset M \) such that \( y \in D_0 \) and choose pairwise disjoint neighbourhoods \( O_z \) of each of \( z \in D_0 \). By the assumption, there are discrete \( D_z \subset A \cap (X \times O_z) \) with \( \langle x, z \rangle \in \overline{D_z} \). Then
\[
D = \bigcup_{z \in D_0} D_z
\]
is discrete, \( \langle x, y \rangle \in \overline{D} \). \( \square \)

A minor modification of the proof shows that the product of finitely many compact strongly discretely generated spaces is strongly discretely generated. From this we can easily derive

**Corollary.** The product of countably many strongly discretely generated compacta is strongly discretely generated.
Proof. Let $A \subset X = \prod_{n \in \omega} X_n$, where each $X_n$ is compact strongly discretely generated, and $x = (x_n)_{n \in \omega} \in \bar{A} \setminus A$. For $n \in \omega$, denote by $\pi_n : X \to \prod_{k < n} X_k$ the natural projection. Since $x' = (x_k)_{k < n} \in \pi_n[A]$, there is a strongly discrete set $E_n \subset \pi_n[A]$ such that $x' \in \overline{E_n}$. Fix a strongly discrete set $D_n \subset A$ with $\pi_n[D_n] = E_n$.

If $(3n \in \omega) \ x \in \overline{D_n}$, we are done. Otherwise we can construct, inductively, a sequence $(U_n)_{n \in \omega}$ of canonical neighbourhoods of $x$ satisfying $(\forall n \in \omega) \ U_{n+1} \subset U_n$ and an increasing sequence $(k_n)_{n \in \omega}$ of natural numbers such that $(\forall n \in \omega) \ (\forall k \geq k_n) \ \pi_X[X_n] = X_k$ and

$$D_0 \cup \cdots \cup D_{k_n} \cap U_{n+1} = \emptyset.$$ 

Now, $U_n \setminus U_{n+1}$ are pairwise disjoint neighbourhoods of the strongly discrete sets $D_{k_n} \cap U_n$, hence $D = \bigcup_{n \in \omega} (D_{k_n} \cap U_n)$ is strongly discrete. Moreover, $x \in \overline{D}$. Indeed, for an open neighbourhood $O$ of $x$, fix $m \in \omega$ such that $(\forall k \geq k_m) \ \pi_X[X_n] = X_k$. Since $O \cap U_m \cap D_{k_m} \neq \emptyset$, we conclude that $O \cap D = \emptyset$. □

The above corollary cannot be extended to larger products: under CH or somewhat weaker assumption the space $[0, 1]^\omega$ fails to be discretely generated ([5, Example 3.3], [2, Proposition 2]).

Let us close by observations on countable spaces. We shall show that such spaces with enough small weight have to be strongly discretely generated. Recall that

$$d = \min \{ |D| : D \subset \omega^\omega \ \& \ (\forall f \in \omega^\omega) (\exists d \in D) (\forall n \in \omega) d(n) > f(n) \}$$

and

$$\text{cof} \mathbb{K} = \min \{ |K| : K \text{ consists of meager subsets of } \mathbb{R} \ \& \ (\forall M \subset \mathbb{R}) \ (M \text{ meager } \implies (\exists K \in K) \ M \subset K) \}.$$ 

It is well known that $\omega_1 \leq d \leq \text{cof} \mathbb{K} \leq \mathfrak{c}$, and these cardinals can consistently differ (see, e.g., [1]).

**Proposition.** If $X$ is a countable Hausdorff space, $w(X) < d$, then $X$ is strongly discretely generated.

**Proof.** Let $A \subset X$, $x \in \bar{A} \setminus A$ and let $B$ be a local base of $X$ at $x$, $|B| < d$. We can assume that $A$ has no isolated points. Fix a maximal disjoint family $(O_n)_{n \in \omega}$ of open subsets of $X$ such that $(\forall n \in \omega) x \notin \overline{O_n}$. It is easy to see that $x \in \bigcap_{n \in \omega} O_n$.

Enumerate each $O_n$ as $\{x_{n,k} : k \in \omega\}$. As each $B \in B$ intersects infinitely many $O_n$’s, a partial function $f_B : \text{dom}(f_B) \to \omega$ with infinite domain $\text{dom}(f_B) \subset \omega$ can be chosen so that $x_{n,f_B(n)} \in O_n \cap B$ for every $n \in \text{dom}(f_B)$. Since $|B| < d$ there is a function $g : \omega \to \omega$ such that

$$(\forall B \in B)(\exists n \in \text{dom}(f_B)) \ g(n) > f_B(n)$$

(see [4, Theorem 3.6]).

Now, $D = \{x_{n,k} : k \leq g(n)\}$ is a strongly discrete subset of $A$, $x \in \overline{D}$. □

Every countable weakly discretely generated regular space is discretely generated, as follows from [5, Theorem 2.8].

**Example.** There exists a countable Hausdorff space $X$, $w(X) = \text{cof} \mathbb{K}$, which is not discretely generated. Consider the set of rationals, $X = \mathbb{Q}$, and let the open base of $X$ consist of all the sets of the form $(p, q) \setminus D$, where $p, q \in \mathbb{Q}$, $p < q$, and $D$ is a nowhere dense subset of $\mathbb{Q}$. As the ideal of nowhere dense subsets of $\mathbb{Q}$ has cofinality $\text{cof} \mathbb{K}$ [6], $w(X) = \text{cof} \mathbb{K}$. Moreover, $X$ is not discretely generated, because every discrete $D \subset X$ is nowhere dense in $\mathbb{Q}$ with the Euclidean topology and therefore is closed in $X$.

**References**