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On deletions of largest bonds in graphs

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Abstract

A well-known conjecture of Scott Smith is that any two distinct longest cycles of a k-connected graph must meet in at least k vertices when $k \geq 2$. We provide a dual version of this conjecture for two distinct largest bonds in a graph. This dual conjecture is established for $k \leqslant 6$.

Keywords: Bond; Cocircuit; Cycle intersection

1. Introduction

Scott Smith in 1979 gave the following fundamental assertion about cycle intersections in k-connected graphs.

Conjecture 1.1. If C and D are longest cycles of a k-connected graph G for $k \geqslant 2$, then C and D meet in at least k vertices.

The generalization of Smith's Conjecture for matroids [3] can be stated as follows. The rank of a set S in a matroid is denoted by $r(S)$ (see [4] for matroid terminology).

Conjecture 1.2. If C and D are largest circuits of a k-connected matroid M with at least $2(k-1)$ elements, then $r(C \cup D) \leqslant r(C) + r(D) - k + 1$.

This leads to a dual version of Smith's Conjecture for graphs. When two distinct bonds are removed from a graph, the remaining graph might have only three components. If the two bonds are large and have few edges in common, we may expect their removal to leave many components. The number of components of a graph G is denoted by $\omega(G)$, and all graphs considered here are simple.

Conjecture 1.3. If C and D are the edge sets of distinct largest bonds of a k-connected graph G, then $\omega(G-(C \cup D)) \geqslant k + 2 - |C \cap D|$.

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The bound of Conjecture 1.3 is tight when $k = 2$. To see this, consider the example of a cycle on n vertices. Every largest bond consists of exactly two edges. If C and D are largest bonds and $|C \cap D| = 1$, then $\omega(G - (C \cup D)) = 3$. If $|C \cap D| = 0$, then $\omega(G - (C \cup D)) = 4$. Checking the tightness of this bound is more complex when $k > 2$.

Conjecture 1.1 has been verified in the literature for $k \leqslant 6$ by Grötschel [1] and Grötschel and Nemhauser [2]. The main result of the paper strengthens Conjecture 1.3 for $k \leqslant 6$. Thus we provide a dual result to those of Grötschel and Nemhauser.

Theorem 1.4. *If C and D are the edge sets of distinct bonds of a k-connected graph G with C a largest bond and $|D| \geqslant |C| - 1$ for $k \leqslant 6$, then $\omega(G - (C \cup D)) \geqslant k + 2 - |C \cap D|$.*

2. Proof of the Theorem

In this section some notation, technical lemmas, and the proof of Theorem 1.4 are given. Let X and Y be disjoint non-empty sets of vertices in a graph G. Then $[X, Y]$ denotes the set of all edges with an end-vertex in each of X and Y. The subgraph of G induced by X is denoted by $G[X]$. We say that $[X, Y]$ is a *bond* when (X, Y) partitions $V(G)$ and both $G[X]$ and $G[Y]$ are connected. A path with an end-vertex in each of X and Y and no internal vertices in $X \cup Y$ is called a path *from X to Y*. If all internal vertices of the path are contained in some vertex set V, we say it is a path *through V*.

Lemma 2.1. *Let $G = (V, E)$ be a connected graph with V partitioned into sets V_1, V_2 and V_3 so that V_3 induces a connected subgraph. Suppose that there are two internally disjoint paths from V_1 to V_2 through V_3. Then V_3 can be partitioned into sets T_1 and T_2 that both have neighbors in V_1 and V_2, induce connected subgraphs of G, and are joined by at least one edge.*

Proof. Let $P_1 = P(x_1, x_2)$ and $P_2 = P(y_1, y_2)$ be the parts of these two paths in V_3 so that vertices x_1 and y_1 have a neighbor in V_1 while x_2 and y_2 have a neighbor in V_2 (see Fig. 1). The connected graph $G[V_3]$ contains a spanning tree T that contains P_1 and P_2 as subgraphs. Delete an edge e on the path from P_1 to P_2 in T. LetT_1 and T_2 be the vertex sets of the two resulting subtrees of T with $V(P_i) \subseteq T_i$ for $i = 1, 2$. Then T_1 and T_2 satisfy the conditions of the lemma. $\square$

Proof of Theorem 1.4. Assume $\hat{\omega} := \omega(G - (C \cup D)) \leqslant k + 1 - |C \cap D|$. Note that $\hat{\omega} \geqslant 3$ since C and D are distinct bonds. Let C_1 and C_2 denote the vertex sets of the two components of $G - C$ and D_1 and D_2 denote the vertex sets of the two components of $G - D$. Fig. 2 illustrates this with $X = C_1 \cap D_1$, $Y = C_2 \cap D_1$, $W = C_1 \cap D_2$ and $U = C_2 \cap D_2$. Here the edges of C are those from the left vertex sets X and W to the right vertex sets Y and U. The edges of D are those in this figure from the upper vertex sets X and Y to the lower vertex sets W and U. The edges of $C \cap D$ are the diagonal edges in this figure. $\square$

Claim 1. *For each $i, j \in \{1, 2\}$, the set $C_i \cap D_j$ is nonempty.*

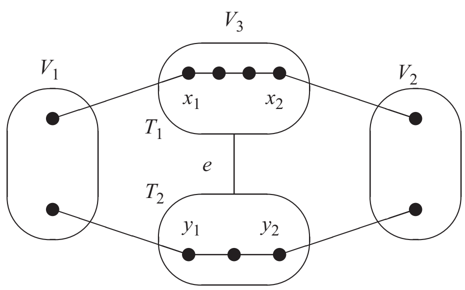

Fig. 1. Split paths in Lemma 2.1.

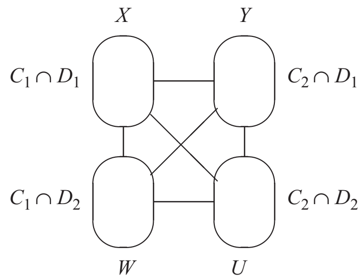

Fig. 2. The bonds C and D.

Proof of Claim 1. Suppose otherwise and assume without loss of generality that $C_2 \cap D_1 = \emptyset$. Then $X = D_1$ and $U = C_2$ so that $G[X]$ and $G[U]$ are connected. Note that $C \cap D = [X, U]$. Since $\hat{\omega} \geqslant 3$, $W = C_1 \cap D_2 \neq \emptyset$. Let W_1, $W_2, \ldots, W_s$ denote the vertex sets of the components of $G[W]$.

It follows from $s + 2 = \hat{\omega} \leqslant k + 1 - |C \cap D|$ that $s < k$. Since G is k-connected, there exist two internally disjoint paths from X to U through some component of W, say W_1. It follows by Lemma 2.1 that there exists a partition of W_1 into sets T_1 and T_2 so that both sets have neighbors in X and U, induce connected subgraphs, and are joined by at least one edge.

Let $S_1 = [X, U] \cup [X, (W - T_2)] \cup [T_2, U] \cup [T_1, T_2]$ and $S_2 = [U, X] \cup [U, (W - T_2)] \cup [T_2, X] \cup [T_1, T_2]$. Then $|S_1| + |S_2| \geqslant |[X, W]| + |[U, W]| + 2|[X, U]| + 2 = |C| + |D| + 2$. So one of S_1 or S_2 is a bond of G that is larger than C, a contradiction. □

Claim 2. *Let* $i, j \in \{1, 2\}$. *If* $\omega(G[C_i \cap D_j] \cup G[C_{3-i} \cap D_{3-j}]) = 2$, *then* $|[(C_i \cap D_{3-j}), (C_{3-i} \cap D_j)]| \geqslant 2$.

Proof of Claim 2. Let $W = C_i \cap D_j$, $Y = C_{3-i} \cap D_{3-j}$, $X = C_i \cap D_{3-j}$, and $U = C_{3-i} \cap D_j$. (Fig. 2 corresponds to $i = 1$, $j = 2$.) Then $\omega(G[W] \cup G[Y]) = 2$, so each of $G[W]$ and $G[Y]$ is connected. Since $G[X]$ and $G[U]$ together contain at most $k - 1 - |C \cap D|$ components and $G - (C \cap D)$ is $(k - |C \cap D|)$-connected, there exist two internally disjoint paths from W to Y through some component of either $G[X]$ or $G[U]$. Assume this component is induced by vertex set $X_1 \subseteq X$. It follows from Lemma 2.1 that there exists a partition of X_1 into sets T_1 and T_2 so that both sets have neighbors in W and Y, induce connected subgraphs, and are joined by at least one edge.

Note that each of $G[W \cup X] = C_1$, $G[Y \cup U] = C_2$, $G[X \cup Y] = D_1$, and $G[W \cup U] = D_2$ is connected, and that there are edges from each of T_1 and T_2 to each of W and Y. When $l \in \{1, 2\}$, each of $G[(X - T_l) \cup W]$, $G[(X - T_l) \cup Y]$, $G[T_l \cup W \cup U]$ and $G[T_l \cup Y \cup U]$ is connected. So $S_1 := [(X - T_l) \cup W, T_l \cup Y \cup U]$ and $S_2 := [(X - T_l) \cup Y, T_l \cup W \cup U]$ are bonds.

If $[T_l, U] = \emptyset$, then

$$\begin{aligned} |S_1| + |S_2| &= (|C| - |[T_l, Y]| + |[T_l, W]| + |[T_1, T_2]|) \\ &\quad + (|D| - |[T_l, W]| + |[T_l, Y]| + |[T_1, T_2]|) \\ &= |C| + |D| + 2|[T_1, T_2]| \\ &\geqslant |C| + |D| + 2 \\ &\geqslant 2|C| + 1. \end{aligned}$$

This contradicts that C is a largest bond of G. Hence $[T_l, U] \neq \emptyset$ for each $l \in \{1, 2\}$, and $|[X, U]| \geqslant 2$. □

Our next claim is a generalized form of the following idea. Suppose $G[W] \cup G[Y]$ has exactly three components, and there are two internally disjoint paths from W to Y through a single component of $G[X] \cup G[U]$. Then $|[X, U]| \geqslant 1$.

Claim 3. *Let* $i, j \in \{1, 2\}$. *Suppose* $\omega(G[C_i \cap D_j] \cup G[C_{3-i} \cap D_{3-j}]) = 3$, *and sets* V_1, V_2, *and* V_3 *induce components of* $G[C_i \cap D_j]$, $G[C_{3-i} \cap D_{3-j}]$, *and* $G[(C_i \cap D_{3-j})] \cup G[(C_{3-i} \cap D_j)]$, *respectively. If there are two internally disjoint paths from* V_1 *to* V_2 *through* V_3, *then* $|[V_3, (C_i \cap D_{3-j}) \cup (C_{3-i} \cap D_j) - V_3]| \geqslant 1$.

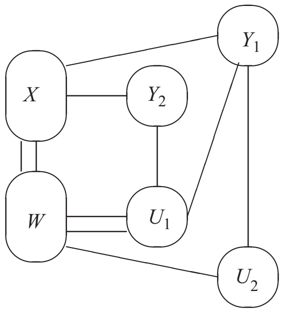

Fig. 3. Paths from W to Y when $\hat{\omega} \geqslant 6$.

Proof of Claim 3. Again let $W = C_i \cap D_j$, $Y = C_{3-i} \cap D_{3-j}$, $X = C_i \cap D_{3-j}$, and $U = C_{3-i} \cap D_j$. (Fig. 2 corresponds to $i = 1$, $j = 2$.) Assume by symmetry that $G[W]$ is connected, and that $G[Y]$ has exactly two components induced by vertex sets Y_1 and Y_2. Assume further that $V_1 = W$, $V_2 = Y_2$, and $V_3 \subseteq X$, so there are two internally disjoint paths from W to Y_2 through some component of $G[X]$. Then we have X_1, T_1 and T_2 as in the proof of Claim 2 above. Again each of $G[(X - T_l) \cup W]$, $G[T_l \cup W \cup U]$ and $G[T_l \cup Y \cup U]$ is connected for $l \in \{1, 2\}$. But $G[(X - T_l) \cup Y]$ may not be connected, as there may be no edge from $X - T_l$ to Y_1. However, at least one of $G[(X - T_1) \cup Y]$ or $G[(X - T_2) \cup Y]$ is connected. So for at least one $l \in \{1, 2\}$, the sets S_1, S_2 as defined above are bonds. Hence $[T_l, U] \neq \emptyset$, and $|[V_3, (C_{3-i} \cap D_j) - V_3]| \geqslant 1$. □

It follows from Claim 1 and $k \leqslant 6$ that $4 \leqslant \hat{\omega} \leqslant k + 1 - |C \cap D| \leqslant 7 - |C \cap D|$. We consider the cases $4 \leqslant \hat{\omega} \leqslant 7$ below. The sets X, Y, U, W are again defined as in Fig. 2.

If $\hat{\omega} = 4$, then each of X, Y, U, and W induces a connected subgraph. By Claim 2, $|[X, U]| \geqslant 2$ and $|[W, Y]| \geqslant 2$, so $|C \cap D| \geqslant 4$. Hence $4 = \hat{\omega} \leqslant k - 3 \leqslant 3$, a contradiction. Thus $\hat{\omega} > 4$.

Suppose that $\hat{\omega} = 5$. We may assume by symmetry that each of X, U, W induces a connected subgraph, and $G[Y]$ has exactly two components. It follows from Claim 2 that $|[W, Y]| \geqslant 2$. Hence $5 = \hat{\omega} \leqslant k - 1$, so $k = 6$, $|C \cap D| = 2$, and $[X, U] = \emptyset$. Since $G - (C \cap D)$ is 4-connected, there are at least four internally disjoint paths from W to Y_i for each $i \in \{1, 2\}$. If two of these paths are both through one of X or U then by Claim 3 we find $|[X, U]| \geqslant 1$, a contradiction. So we must have one path from W to each of Y_1 and Y_2 through X, one path from W to each of Y_1 and Y_2 through U, and two paths from Y_1 to Y_2 each through one of X or U. Suppose by symmetry that there are paths P_1 from W to Y_1, and P_2 from W to Y_2, and P_3 from Y_1 to Y_2, each through X. Since $G[X]$ is connected, there is some path Q from $V(P_3)$ to $V(P_1) \cup V(P_2)$ in $G[X]$. If Q has an end-vertex in P_1, then $P_1 \cup Q \cup P_3$ contains a path from W to Y_2 which is internally disjoint from P_2. Likewise, if Q has an end in P_2, we have internally disjoint paths from W to Y_1. Each of these produces a contradiction.

Finally, suppose $\hat{\omega} \geqslant 6$. The following argument will encompass three distinct subcases: $\hat{\omega} = 6$, $k = 5$, and $C \cap D = \emptyset$; or $\hat{\omega} = 6$, $k = 6$, and $|C \cap D| \leqslant 1$; or $\hat{\omega} = 7$, $k = 6$, and $C \cap D = \emptyset$.

In any of these cases, we may assume by symmetry that $\omega(G[W] \cup G[Y]) \leqslant 3$. Assume further that $G[W]$ is connected. If $G[Y]$ is connected, then by Claim 2, $|[X, U]| \geqslant 2$, contradicting $|C \cap D| \leqslant 1$. We have a similar contradiction if $G[X]$ and $G[U]$ are both connected. So we assume by symmetry that $G[Y]$ has exactly two components, induced by vertex sets Y_1 and Y_2, and $G[U]$ has at least two components, with two being induced by vertex sets U_1 and U_2 respectively (Fig. 3). We may further assume $[X, U] = \emptyset$; otherwise, $|C \cap D| = 1$, so we are in the case $k = 6$ and $\hat{\omega} = 6$, and we may symmetrically exchange the set pair X, U for the set pair W, Y.

By Claim 3, we may assume that for $i \in \{1, 2\}$, there is no pair of internally disjoint paths from W to Y_i through some component of $G[X] \cup G[U]$. Otherwise, we contradict the assumption that $[X, U] = \emptyset$.

The graph $G[X] \cup G[U]$ has exactly $\hat{\omega} - 3$ components. $G - (C \cup D))$ is $k - |C \cap D|$-connected, and $k - |C \cap D| \geqslant (\hat{\omega} - 3) + 2$. So we may assume that there are two components of $G[X] \cup G[U]$ which each contain the internal vertices of two internally disjoint paths from W to Y. Suppose without loss of generality that there are internally disjoint paths from W to Y_1 and from W to Y_2 through U_1. By Lemma 2.1 there exists a partition of U_1 into sets T_1 and T_2 so that both sets have neighbors in W and Y, induce connected subgraphs, and are joined by at least one edge. Set T_1 has some neighbor in Y_1 and T_2 has some neighbor in Y_2.

We claim that one of $[Y_1, U - U_1]$ and $[Y_2, U - U_1]$ is empty. To see this, suppose both Y_1 and Y_2 have neighbors in $U - U_1$. Let $l \in \{1, 2\}$. Since $G[W \cup U]$ is connected and W has a neighbor in T_{3-k}, $G[W \cup (U - T_l)]$ is connected. Since $G[X \cup W]$ is connected and W has a neighbor in T_l, $G[X \cup W \cup T_l]$ is connected. Since $G[Y \cup U]$ is connected and both Y_1 and Y_2 have neighbors in U_2, $G[Y \cup (U - T_l)]$ is connected. Finally, since $G[X \cup Y]$ is connected and T_l has a neighbor in Y, $G[X \cup Y \cup T_l]$ is connected. So $S_1 := [X \cup W \cup T_l, Y \cup (U - T_l)]$ and $S_2 := [X \cup Y \cup T_l, W \cup (U - T_l)]$ are each bonds.

If $[T_l, X] = \emptyset$, then

$$\begin{aligned}|S_1| + |S_2| &= (|C| - |[T_l, W]| + |[T_l, Y]| + |[T_1, T_2]|)\\ &\quad + (|D| - |[T_l, Y]| + |[T_l, W]| + |[T_1, T_2]|)\\ &= |C| + |D| + 2|[T_1, T_2]|\\ &\geqslant 2|C| + 1.\end{aligned}$$

This is a contradiction as C is the largest bond of G. Hence $[T_l, X] \neq \emptyset$ for each $l \in \{1, 2\}$, and $|[X, U]| \geqslant 2$, contradicting $|C \cap D| \leqslant 1$. So either Y_1 or Y_2 has no neighbors in $U - U_1$.

Assume by symmetry that $[Y_1, U - U_1] = \emptyset$. There is some vertex $u \in U_1$ which separates Y_1 from W in $G[W \cup U_1 \cup Y] - (C \cap D)$; otherwise there would be two internally disjoint paths from W to Y_1 through U_1. So u separates Y_1 from W in $G[W \cup U \cup Y] - (C \cap D)$. Similarly, for each $X_i \subset X$ which induces a component of $G[X]$, there is a vertex set $\{x_i, v_i\} \subset X_i$ separating W from Y in $G[W \cup X_i \cup Y] - (C \cap D)$. If $\hat{\omega} = 6$, then $k - |C \cap D| \geqslant \hat{\omega} - 1 \geqslant 5$ so $G - (C \cap D)$ is 5-connected. But $G[X]$ has only one component, so we have a cutset $\{x_1, v_1, u\}$ in $G - (C \cap D)$, a contradiction. If $\hat{\omega} = 7$, then $G - (C \cap D)$ is 6-connected. But $G[X]$ has at most two components, so we have a cutset $\{x_1, v_1, x_2, v_2, u\}$ in $G - (C \cap D)$, a contradiction.

This completes the proof of Theorem 1.4. □

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