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Convergence of subdivision schemes and smoothness of limit functions ☆

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Abstract

Starting with vector $\lambda = (\lambda(k))_{k\in\mathbb{Z}} \in \ell_p(\mathbb{Z})$, the subdivision scheme generates a sequence $\{S_a^n\lambda\}_{n=1}^\infty$ of vectors by the subdivision operator

$$S_a\lambda(k) = \sum_{j\in\mathbb{Z}} \lambda(j)a(k-2j), \quad k \in \mathbb{Z}.$$

Subdivision schemes play an important role in computer graphics and wavelet analysis. It is very interesting to understand under what conditions the sequence $\{S_a^n\lambda\}_{n=1}^\infty$ converges to a L_p-function in an appropriate sense. This problem has been studied extensively.

In this paper, we consider the convergence of subdivision scheme in Sobolev spaces with the tool of joint spectral radius. Firstly, the conditions under which the sequence $\{S_a^n\lambda\}_{n=1}^\infty$ converges to a W_p^k-function in an appropriate sense are given. Then, we show that the subdivision scheme converges for any initial vector in $W_p^k(\mathbb{R})$ provided that it does for one nonzero vector in that space. Moreover, if the shifts of the refinable function are stable, the smoothness of the limit function corresponding to the vector λ is also independent of λ, where the smoothness of a given function is measured by the generalized Lipschitz space.

Keywords: Subdivision scheme; Joint spectral radius; Sobolev space; Generalized Lipschitz space

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1. Introduction

Subdivision schemes play an important role in computer graphics (see [10]) and wavelet analysis (see [2]). There has been an intensive study on convergence of subdivision schemes. The purpose of this paper is to investigate the convergence of subdivision schemes in Sobolev space and, when the scheme converges, the smoothness of the corresponding limit functions.

Let $a = (a(m))_{m\in\mathbb{Z}}$. We assume throughout this paper that

$$\sum_{m\in\mathbb{Z}} a(m) = 2$$

and that there is a positive integer N such that, for $m \notin \{0, 1, \dots, N\}$, $a(m) = 0$. The *subdivision operator* S_a, associated with mask a, on $\ell_p(\mathbb{Z})$ is defined by

$$S_a\lambda(k) = \sum_{j\in\mathbb{Z}} \lambda(j)a(k-2j), \quad k \in \mathbb{Z}. \tag{1.1}$$

Starting with an initial vector $\lambda = (\lambda(k)) \in \ell_p(\mathbb{Z})$, the *subdivision scheme* with mask a generates control points $\lambda^n = (\lambda(j))_{j\in\mathbb{Z}}$ at dyadic points $j/2^n$, $j \in \mathbb{Z}$, recursively by $\lambda^0 = \lambda$, and

$$\lambda^n = S_a\lambda^{n-1} = \cdots = S_a^n\lambda, \quad n = 1, 2, \dots. \tag{1.2}$$

Let $W_p^k(\mathbb{R})$ be the kth Sobolev space, that is

$$W_p^k(\mathbb{R}) = \left\{ f^{(j)} \in L_p(\mathbb{R}) \colon 0 \leqslant j \leqslant k \right\}. \tag{1.3}$$

The norm on $W_p^k(\mathbb{R})$ is defined by

$$\|f\|_{W_p^k} := \sum_{j=0}^{k} \left\| f^{(j)} \right\|_p, \quad 1 \leqslant p \leqslant \infty.$$

Let

$$B_1 = \chi_{[0,1]}(x), \qquad B_{k+1} = \underbrace{B_1 * \cdots * B_1}_{k+1},$$

i.e., the k times of convolution of B_1. They are the B splines. It is easily seen that $B_{k+1} \in W_p^k(\mathbb{R})$ for $1 \leqslant p < \infty$.

With B_{k+1}, we give the notions concerning the convergence of subdivision schemes in Sobolev spaces as follows.

Definition 1.1. Let $1 \leqslant p < \infty$ and $\lambda \in \ell_p(\mathbb{Z})$. The sequence $\{S_a^n\lambda\}_{n=1}^{\infty}$ is said to be convergent in $W_p^k(\mathbb{R})$ if there is a function $f_\lambda \in W_p^k(\mathbb{R})$ such that

$$\lim_{n\to\infty} \left\| \sum_{j\in\mathbb{Z}} S_a^n\lambda(j) B_{k+1}\left(2^n \cdot -j\right) - f_\lambda \right\|_{W_p^k} = 0.$$

Definition 1.2. We say that subdivision scheme $\{S_a^n\}$, associated with mask a, converges in $W_p^k(\mathbb{R})$ if for each $\lambda \in \ell_p(\mathbb{Z})$ the sequence $\{S_a^n\lambda\}_{n=1}^\infty$ is convergent and $f_\lambda \neq 0$ for some $\lambda \in \ell_p(\mathbb{Z})$.

The convergence of subdivision schemes is essentially close to that of cascade algorithms. The *cascade algorithm*, staring with an appropriately initial function ϕ_0, generates a sequence $\{Q_a^n\phi_0\}_{n=1}^\infty$ by the *cascade operator* Q_a defined by

$$Q_a f = \sum_{j\in\mathbb{Z}} a(j) f(2\cdot - j). \tag{1.4}$$

There is a lot of papers considering the convergence of cascade algorithms and subdivision schemes. We mention here some works with no attempt of completeness at all. Dyn et al. [9] and Cavaretta et al. [2] already found necessary and sufficient conditions ensuring that subdivision scheme converges uniformly to a continuous limit function. The L_p-convergence of vector cascade algorithms was characterized by Jia et al. [12,13] and Han et al. [11], in terms of the p-norm joint spectral radius. By factorization of mask, Micchelli et al. [14] discussed the convergence of subdivision schemes in L_p. For the characterization of vector cascade algorithms in W_p^k ($1 \leqslant p \leqslant \infty$), in terms of the p-norm joint spectral radius, we refer to [4].

In [5], an interesting problem of how the convergence of the subdivision scheme depends on the initial vector λ was proposed. We established an independence of the initial vector $\lambda \in \ell_p(\mathbb{Z})$ in the convergence of subdivision scheme. More precisely, it was proved that the subdivision scheme $\{S_a^n\}_{n=1}^\infty$ converges in $L_p(\mathbb{R})$ provided that, for *one* nonzero vector $\lambda \in \ell_p(\mathbb{Z})$, the sequence $\{S_a^n\lambda\}_{n=1}^\infty$ converges in $L_p(\mathbb{R})$. One of our purposes is to establish such an independence of initial vectors $\lambda \in \ell_p(\mathbb{Z})$ in the convergence of subdivision schemes in Sobolev space.

Once the subdivision scheme converges for an initial vector $\lambda \in \ell_p(\mathbb{Z})$, it is desired to obtain the smoothness of the limit function f_λ. Our second purpose is to discuss the independence of λ in the critical exponent of f_λ.

The paper is organized as follows. In Section 2, we first recall the definition of the p-norm joint spectral radius. Then we establish a formula for the p-norm joint spectral radius, which is independent of the initial vector of subdivision operator. In Section 3, we prove that the subdivision scheme $\{S_a^n\}_{n=1}^\infty$ converges in $W_p^k(\mathbb{R})$ provided that, for *one* nonzero vector $\lambda \in \ell_p(\mathbb{Z})$, the sequence $\{S_a^n\lambda\}_{n=1}^\infty$ converges in $W_p^k(\mathbb{R})$. In Section 4, we will investigate the smoothness of the limit functions. Under a condition for the refinable function, it is shown that all the limit functions have the same critical exponent.

2. A formula for joint spectral radii

The p-norm joint spectral radius was introduced by Jia [12]. Let us recall from [12] the definition of the p-norm joint spectral radius. Let V be a finite-dimensional vector space equipped with a vector norm $\|\cdot\|$. For a linear operator A on V, define

$$\|A\| := \max_{\|v\|=1} \|Av\|.$$

Let $\mathcal{A}$ be a finite multiset of linear operators on V. For a positive integer n we denote by $\mathcal{A}^n$ the Cartesian power of $\mathcal{A}$:

$$\mathcal{A}^n = \{(A_1, \ldots, A_n):\ A_1, \ldots, A_n \in \mathcal{A}\}.$$

For $1 \leqslant p < \infty$, let

$$\|\mathcal{A}^n\|_p := \left(\sum_{(A_1,\ldots,A_n)\in\mathcal{A}^n} \|A_1 \cdots A_n\|^p \right)^{1/p},$$

and for $p = \infty$, define

$$\|\mathcal{A}^n\|_\infty := \max\{\|A_1 \cdots A_n\|:\ (A_1, \ldots, A_n) \in \mathcal{A}^n\}.$$

For $1 \leqslant p \leqslant \infty$, the *p-norm joint spectral radius* of $\mathcal{A}$ is defined to be

$$\rho_p(\mathcal{A}) := \lim_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n}. \tag{2.1}$$

It is easily seen that this limit indeed exists, and

$$\lim_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n} = \inf_{n\geqslant 1} \|\mathcal{A}^n\|_p^{1/n}. \tag{2.2}$$

Clearly, $\rho_p(\mathcal{A})$ is independent of the choice of the vector norm on V.

Furthermore, for $X = \{x_i\}_{i=1}^s \subseteq V$, let $U(X)$ be the minimal common invariant subspace of $A \in \mathcal{A}$ containing X. Then $\mathcal{A}|_{U(X)} = \{A|_{U(X)}:\ A \in \mathcal{A}\}$ is a set of operators on subspace $U(X)$. We define for $1 \leqslant p < \infty$,

$$\|\mathcal{A}^n X\|_p := \left(\sum_{i=1}^s \sum_{(A_1,\ldots,A_n)\in\mathcal{A}^n} \|A_1 \cdots A_n x_i\|^p \right)^{1/p},$$

and for $p = \infty$, define

$$\|\mathcal{A}^n X\|_\infty := \max\{\|A_1 \cdots A_n x_i\|:\ (A_1, \ldots, A_n) \in \mathcal{A}^n,\ i = 1, \ldots, s\},$$

where the norms in the right-hand sides are any fixed norms on V. Then [12] there is a positive constant κ, independent of the norm on V, such that

$$\kappa^{-1} \|\mathcal{A}|_{U(X)}^n\|_p \leqslant \|\mathcal{A}^n X\|_p \leqslant \kappa \|\mathcal{A}|_{U(X)}^n\|_p, \quad n = 1, 2, \ldots. \tag{2.3}$$

Consequently we obtain by (2.1)–(2.3),

$$\rho_p\{\mathcal{A}|_{U(X)}\} \leqslant \kappa^{1/n} \|\mathcal{A}^n X\|_p^{1/n}, \quad n = 1, 2, \ldots. \tag{2.4}$$

Using (2.1) and (2.3) we get that

$$\lim_{n\to\infty} \|\mathcal{A}^n X\|_p^{1/n} = \lim_{n\to\infty} \|\mathcal{A}|_{U(X)}^n\|_p^{1/n} = \rho_p(\mathcal{A}|_{U(X)}). \tag{2.5}$$

The difference operator ∇ on $\ell_p(\mathbb{Z})$ is given by

$$\nabla\lambda(j) := \lambda(j) - \lambda(j-1), \quad \lambda \in \ell_p(\mathbb{Z}).$$

For any integer $k \geqslant 2$, let $\nabla^k = \nabla\nabla^{k-1}$.

Now, we quote a result in [3].

Lemma 2.1. *Let* $1 \leqslant p < \infty$ *and* $\lambda \in \ell_p(\mathbb{Z})\backslash\{0\}$*, then*

$$\|x\| = \left(\sum_{\gamma\in\mathbb{Z}} \left|{}^{\gamma}\lambda^T x\right|^p\right)^{1/p}, \quad x \in \mathbb{C}^N,$$

defines a norm on $\mathbb{C}^N$*, where, for any* $\gamma \in \mathbb{Z}$*, the vector* ${}^{\gamma}\lambda \in \mathbb{C}^N$ *is defined by*

$$ {}^{\gamma}\lambda = \big(\lambda(\gamma-1), \lambda(\gamma-2), \ldots, \lambda(\gamma-N)\big)^T \in \mathbb{C}^N. \tag{2.6}$$

Associated to the mask a, there are two matrices A_0, A_1 as follows:

$$A_0 := \big(a(2i-j-1)\big)_{1\leqslant i,j\leqslant N}, \qquad A_1 := \big(a(2i-j)\big)_{1\leqslant i,j\leqslant N}.$$

With the help of Lemma 2.1, we can prove the following result.

Theorem 2.2. *Let* $\|\cdot\|$ *be the norm of* $\mathbb{C}^N$ *given in Lemma* 2.1*. For* $\varepsilon = 0, 1$*, let* A_ε *be the matrix on* $R^{N\times N}$ *given as above, where* $N \geqslant k+2$*. Let*

$$X = \left\{\sum_{i=0}^{k+1} (-1)^i C_{k+1}^i e_{m+i}\right\}_{m=1}^{N-(k+1)},$$

$\mathcal{A} = \{A_0, A_1\}$ *and* $U(X)$ *be the minimal common invariant subspace of* A_0 *and* A_1 *containing* X*. For* $1 \leqslant p < \infty$*, and* $\lambda \in \ell_p(\mathbb{Z})\backslash\{0\}$*, we have a positive constant* C *such that*

$$\rho_p\big(\{\mathcal{A}|_{U(X)}\}\big) \leqslant C^{1/n} \big\|\nabla^{k+1} S_a^n \lambda\big\|_p^{1/n}, \quad n = 1, 2, \ldots. \tag{2.7}$$

Moreover,

$$\lim_{n\to\infty} \big\|\nabla^{k+1} S_a^n \lambda\big\|_p^{1/n} = \rho_p\big(\{\mathcal{A}|_{U(X)}\}\big). \tag{2.8}$$

Proof. Let $j \in \mathbb{Z}$. For any nonnegative integer n, there are uniquely $\varepsilon_i \in \{0, 1\}$, $1 \leqslant i \leqslant n$, and $\gamma \in \mathbb{Z}$ such that $j = 2^n\gamma + 2^{n-1}\varepsilon_n + \cdots + \varepsilon_1$. For any $\lambda \in \ell_p(\mathbb{Z})$, it is known (see [12]) that

$$S_a^n(j+1-m) = A_{\varepsilon_1}^T \cdots A_{\varepsilon_n}^{T\,\gamma}\lambda(m), \quad 1 \leqslant m \leqslant N,\ j \in \mathbb{Z}.$$

Therefore, for $m = 1, 2, \ldots, N-1$,

$$\begin{aligned}
\nabla S_a^n(j+1-m) &= S_a^n\lambda(j+1-m) - S_a^n(j-m) \\
&= A_{\varepsilon_1}^T \cdots A_{\varepsilon_n}^{T\,\gamma}\lambda(m) - A_{\varepsilon_1}^T \cdots A_{\varepsilon_n}^{T\,\gamma}\lambda(m+1) \\
&= \big({}^{\gamma}\lambda\big)^T A_{\varepsilon_n} \cdots A_{\varepsilon_1}(e_m - e_{m+1}).
\end{aligned}$$

For $m = 1, 2, \ldots, N-2$,

$$\begin{aligned}
\nabla^2 S_a^n(j+1-m) &= \nabla S_a^n\lambda(j+1-m) - \nabla S_a^n(j-m) \\
&= \big({}^{\gamma}\lambda\big)^T A_{\varepsilon_n} \cdots A_{\varepsilon_1}(e_m - e_{m+1} - e_{m+1} + e_{m+2}) \\
&= \big({}^{\gamma}\lambda\big)^T A_{\varepsilon_n} \cdots A_{\varepsilon_1}(e_m - 2e_{m+1} + e_{m+2}).
\end{aligned}$$

By the induction argument, for $m = 1, 2, \ldots, N-(k+1)$,

$$\nabla^{k+1} S_a^n (j+1-m) = \left({}^{\gamma}\lambda\right)^T A_{\varepsilon_n} \cdots A_{\varepsilon_1} \left(\sum_{i=0}^{k+1} (-1)^i C_{k+1}^i e_{m+i} \right).$$

It follows that for $m = 1, 2, \ldots, N-(k+1)$,

$$\sum_{j\in\mathbb{Z}} \left| \nabla^{k+1} S_a^n \lambda(j+1-m) \right|^p$$
$$= \sum_{\varepsilon_1,\ldots,\varepsilon_n\in\{0,1\}} \sum_{\gamma\in\mathbb{Z}} \left| \left({}^{\gamma}\lambda\right)^T A_{\varepsilon_n} \cdots A_{\varepsilon_1} \left(\sum_{i=0}^{k+1} (-1)^i C_{k+1}^i e_{m+i} \right) \right|^p.$$

Summing over $m = 1, 2, \ldots, N-(k+1)$ gives

$$(N-k-1) \left\| \nabla^{k+1} S_a^n \lambda \right\|_p^p$$
$$= \sum_{\varepsilon_1,\ldots,\varepsilon_n\in\{0,1\}} \sum_{\gamma\in\mathbb{Z}} \sum_{m=1}^{N-k-1} \left| \left({}^{\gamma}\lambda\right)^T A_{\varepsilon_n} \cdots A_{\varepsilon_1} \left(\sum_{i=0}^{k+1} (-1)^i C_{k+1}^i e_{m+i} \right) \right|^p.$$

Choosing the norm in $\mathbb{C}^N$ given as in Lemma 2.1 yields that

$$(N-k-1) \left\| \nabla^{k+1} S_a^n \lambda \right\|_p^p = \left\| \mathcal{A}^n X \right\|_p^p.$$

The proof is completed by (2.4) and (2.5). □

It is interesting that the limit in above theorem is independent of $\lambda \in \ell_p(\mathbb{Z}) \setminus \{0\}$.

3. Convergence of subdivision scheme in W_p^k

In this section, we discuss the convergence of the subdivision scheme in W_p^k. First, we gave a characterization for the convergence of $\{S_a^n \lambda\}_{n=1}^{\infty}$ in W_p^k.

Let $y \in \mathbb{R}$ and f be a function defined on $\mathbb{R}$. The difference operator ∇_y is defined by

$$\nabla_y f = f - f(\cdot - y).$$

The modulus of continuity of a function f is given by

$$\omega(f, h)_p := \sup_{|y|\leqslant h} \|\nabla_y f\|_p, \quad h \geqslant 0.$$

Lemma 3.1. *Assume that, for some $\lambda \in \ell_p(\mathbb{Z}) \setminus \{0\}$, $\{S_a^n \lambda\}_{n=1}^{\infty}$ converges in $W_p^k(\mathbb{R})$. Then we have*

$$\lim_{n\to\infty} 2^{-n/p} 2^{nk} \left\| \nabla^{k+1} S_a^n \lambda \right\|_p = 0. \tag{3.1}$$

Proof. We first recall that $\phi_0 = B_{k+1}$ satisfies a stability condition. This means that there exist two positive constants C_1 and C_2 such that, for any $b \in \ell_p(\mathbb{Z})$,

$$C_1\|b\|_p \leqslant \left\|\sum_{j\in\mathbb{Z}} b(j)\phi_0(\cdot - j)\right\|_p \leqslant C_2\|b\|_p.$$

Suppose $f \in W_p^k(\mathbb{R})$, by the equation

$$\nabla_{2^{-n}}^{k+1} f = \int_0^{2^{-n}} \cdots \int_0^{2^{-n}} \nabla_{2^{-n}} f^{(k)}\big(x + 2^{-n} + t_1 + \cdots + t_k\big)\, dt_1 \cdots dt_k$$

and Minkowski's inequality, it is obtained that

$$\big\|\nabla_{2^{-n}}^{k+1} f\big\|_p \leqslant 2^{-nk}\omega\big(f^{(k)}, 2^{-n}\big)_p.$$

Therefore, as $n \to \infty$, we have

$$2^{nk}\big\|\nabla_{2^{-n}}^{k+1} f\big\|_p \leqslant \omega\big(f^{(k)}, 2^{-n}\big)_p \to 0. \tag{3.2}$$

We now let

$$g_n = \sum_{j\in\mathbb{Z}} S_a^n\lambda(j)\phi_0\big(2^n \cdot - j\big). \tag{3.3}$$

By Eq. (3.2), as $n \to \infty$, we have

$$2^{nk}\big\|\nabla_{2^{-n}}^{k+1}(g_n - f_\lambda)\big\|_p \leqslant \omega\big((g_n - f_\lambda)^{(k)}, 2^{-n}\big)_p \to 0.$$

So,

$$2^{nk}\big\|\nabla_{2^{-n}}^{k+1} g_n\big\|_p \leqslant 2^{nk}\big\|\nabla_{2^{-n}}^{k+1}(g_n - f_\lambda)\big\|_p + 2^{nk}\big\|\nabla_{2^{-n}}^{k+1} f_\lambda\big\|_p \to 0. \tag{3.4}$$

On the other hand, applying the difference operator $\nabla_{2^{-n}}$ to both sides of (3.3), we obtain

$$\begin{aligned}\nabla_{2^{-n}} g_n &= \sum_{j\in\mathbb{Z}} S_a^n\lambda(j)\big[\phi_0\big(2^n \cdot - j\big) - \phi_0\big(2^n \cdot - j - 1\big)\big] \\ &= \sum_{j\in\mathbb{Z}} \nabla S_a^n\lambda(j)\phi_0\big(2^n \cdot - j\big).\end{aligned}$$

An induction argument tells us that

$$\nabla_{2^{-n}}^{k+1} g_n = \sum_{j\in\mathbb{Z}} \nabla^{k+1} S_a^n\lambda(j)\phi_0\big(2^n \cdot - j\big). \tag{3.5}$$

Consequently,

$$\big\|\nabla_{2^{-n}}^{k+1} g_n\big\|_p^p = 2^{-n}\left\|\sum_{j\in\mathbb{Z}} \nabla^{k+1} S_a^n\lambda(j)\phi_0(\cdot - j)\right\|_p^p.$$

Therefore, (3.1) follows from the stability condition on ϕ_0, and (3.4). The proof is complete. □

Corollary 3.2. *Assume that* $1 \leqslant p < \infty$. *Under the condition of Lemma* 3.1, *we have*

$$\rho_p\{\mathcal{A}|_{U(X)}\} < 2^{-k+1/p},$$

where $\mathcal{A}$ *and* X *defined as in Theorem* 2.2.

Proof. Write $\rho = \rho_p\{\mathcal{A}|_{U(X)}\}$. If $\rho \geqslant 2^{-k+1/p}$, by (2.7), we have

$$\left\|\nabla^{k+1} S_a^n \lambda\right\|_p \geqslant C^{-1}\rho^n \geqslant C^{-1} 2^{-nk+n/p}, \quad n = 1, 2, \ldots,$$

where C is a positive constant independent of n as in Theorem 2.2. So

$$2^{nk-n/p}\left\|\nabla^{k+1} S_a^n \lambda\right\|_p \geqslant C^{-1} > 0, \quad n = 1, 2, \ldots.$$

It contradicts with (3.1). □

Corollary 3.3. *Assume that* $1 \leqslant p < \infty$ *and that the condition of Lemma* 3.1 *holds, then for any eigenvalue* σ *of* S_a *on* $\ell_p(\mathbb{Z})$, *we have*

$$|\sigma| < 2^{-k+1/p}.$$

Proof. By assumption, there is an $\eta \in \ell_p(\mathbb{Z}) \setminus \{0\}$ such that $S_a^n \eta = \sigma\eta$. Thus

$$\nabla^{k+1} S_a^n \eta = \sigma^n \nabla^{k+1} \eta, \quad \forall n \in \mathbb{N}.$$

By Theorem 2.2, $|\sigma| = \rho_p\{\mathcal{A}|_{U(X)}\}$. The proof is complete by Corollary 3.2. □

We are in a position to present the main result of this section.

Theorem 3.4. *Assume that* $1 \leqslant p < \infty$. *The following statements are equivalent*:

(i) *Subdivision scheme* $\{S_a^n\}$ *converges in* $W_p^k(\mathbb{R})$ *for one* $\lambda \in \ell_p(\mathbb{Z})\backslash\{0\}$.
(ii) $\rho_p\{\mathcal{A}|_{U(X)}\} < 2^{-k+1/p}$, *where* $\mathcal{A}$ *and* X *defined as in Theorem* 2.2.
(iii) *Subdivision scheme* $\{S_a^n \lambda\}$ *converges in* $W_p^k(\mathbb{R})$ *for any* $\lambda \in \ell_p(\mathbb{Z})$.

Proof. Since (i) ⇒ (ii) is just Corollary 3.2, we only need to establish (ii) ⇒ (iii). Suppose now that (ii) is true. Let $\phi_n = Q_a^n \phi_0$, where Q_a is the cascade operator defined as in (1.3) and $\phi_0 = B_{k+1}$. Since ϕ_0 is a compactly supported function, there is a compact set $E \subset \mathbb{R}$ such that ϕ_n is supported on E for any n.

Moreover, it follows from $\rho_p\{\mathcal{A}|_{U(X)}\} < 2^{-k+1/p}$ and [4, Theorem 4.1] that ϕ_n converges to ϕ_a in W_p^k, i.e.,

$$\lim_{n\to\infty} \|\phi_n - \phi_a\|_{W_p^k} = 0. \tag{3.6}$$

Consequently, for any $\lambda \in \ell_p(\mathbb{Z})$, the function $f_\lambda := \sum_{j\in\mathbb{Z}} \lambda(j)\phi_a(\cdot - j) \in W_p^k(\mathbb{R})$. Furthermore, it is not difficult by an induction argument to obtain

$$\sum_{j\in\mathbb{Z}} S_a^n \lambda(j)\phi_0\big(2^n \cdot - j\big) = \sum_{j\in\mathbb{Z}} \lambda(j)\phi_n(\cdot - j). \tag{3.7}$$

Therefore, for any $\lambda \in \ell_p(\mathbb{Z})$,

$$\lim_{n\to\infty}\Big\|\sum_{j\in\mathbb{Z}} S_a^n\lambda(j)\phi_0\big(2^n x - j\big) - f_\lambda\Big\|_{W_p^k} = \lim_{n\to\infty}\Big\|\sum_{j\in\mathbb{Z}}\lambda(j)\big(\phi_n(\cdot - j) - \phi_a(\cdot - j)\big)\Big\|_{W_p^k} = 0.$$

The last equality holds by (3.6), $\lambda \in \ell_p(\mathbb{Z})$ and the fact that $\operatorname{supp}\phi_n, \operatorname{supp}\phi_a \subset E$.

Let $\delta = (\delta(\alpha))_\alpha \in \ell_p(\mathbb{Z})$ is given by $\delta(0) = 1$ and $\delta(\alpha) = 0$ for any $\alpha \neq 0$. To conclude $\phi_a \neq 0$, let us recall that $\sum_{j\in\mathbb{Z}} a(j) = 2$. It is true by induction on n that

$$\sum_{j\in\mathbb{Z}} S_a^n\delta(j) = 2^n, \quad n = 1, 2, \ldots. \tag{3.8}$$

It yields by the stability of ϕ_0 that

$$C \leqslant C2^{-n}\big\|S_a^n\delta\big\|_1 \leqslant \|\phi_n\|_1, \tag{3.9}$$

where C is a positive constant.

Again, since ϕ_n is supported on E for any n, there exists a constant M, independent of n, satisfying

$$\|\phi_n\|_1 \leqslant M\|\phi_n\|_p, \quad \forall n = 1, 2, \ldots. \tag{3.10}$$

It follows from (3.9) and (3.10) that $\phi_a \neq 0$. The proof is complete. □

4. Smoothness of limit functions

In this section, we consider the smoothness of the limit functions of a subdivision scheme. We prove that, under a stability condition, all the limit functions have the same smoothness.

Let us recall from [7] the definition of the generalized Lipschitz space. For $y \in \mathbb{R}$, recall that the difference operator ∇_y is defined in Section 3. Moreover, for any integer $k \geqslant 2$, let $\nabla_y^k = \nabla_y^{k-1}\nabla_y$. The kth *modulus of smoothness* of $f \in L_p(\mathbb{R})$ is defined by

$$\omega_k(f, h)_p := \sup_{|y|\leqslant h}\big\|\nabla_y^k f\big\|_p, \quad h \geqslant 0.$$

For $\nu > 0$, let k be an integer greater than ν. The *generalized Lipschitz space* $\mathrm{Lip}^*(\nu, L_p(\mathbb{R}))$ consists of those functions $f \in L_p(\mathbb{R})$ for which

$$\omega_k(f, h)_p \leqslant Ch^\nu, \quad \forall h > 0,$$

where C is a positive constant independent of h.

For a constant $\nu > 0$ which is not an integer, we have a positive integer k such that $\nu \in [k-1, k)$. Then $f \in \mathrm{Lip}^*(\nu, L_p(\mathbb{R}))$ if and only if there exists a function $g \in W_p^{k-1}(\mathbb{R})$ such that $g = f$ a.e. and $g^{(k-1)} \in \mathrm{Lip}^*(k-\nu, L_p(\mathbb{R}))$. See [7] for the details.

The optimal smoothness of a function $f \in L_p(\mathbb{R})$ is described by its *critical exponent* $\nu_p(f)$ defined by

$$\nu_p(f) := \sup\bigl\{\nu\colon\ f \in \mathrm{Lip}^*\bigl(\nu, L_p(\mathbb{R})\bigr)\bigr\}.$$

Suppose that the subdivision scheme $\{S_a^n\}_{n=1}^\infty$ converges in $W_p^k(\mathbb{R})$. We denote by V_p the set of all limit functions f_λ, $\lambda \in \ell_p(\mathbb{Z})$. As is known from the proof of Theorem 3.4, f_λ has a representation as follows:

$$f_\lambda = \sum_{j\in\mathbb{Z}} \lambda(j)\phi(\cdot - j), \tag{4.1}$$

where ϕ is the limit function corresponding to $\lambda = \delta$. It is referred to as the *refinement function* associated with mask a.

Applying the difference operator $\nabla_{2^{-n}}^{k+1}$ to both sides of (4.1), we obtain as (3.5),

$$\nabla_{2^{-n}}^{k+1} f_\lambda = \sum_{j\in\mathbb{Z}} \nabla^{k+1} S_a^n \lambda(j)\phi\bigl(2^n \cdot - j\bigr). \tag{4.2}$$

It follows that

$$2^{n/p}\bigl\|\nabla_{2^{-n}}^{k+1} f_\lambda\bigr\|_p \leqslant C_1\bigl\|\nabla^{k+1} S_a^n \lambda\bigr\|_p, \quad \forall n \in \mathbb{N},$$

where C_1 is a constant independent of n. Let

$$\nu_p = \frac{1}{p} - \log_2 \rho_p\bigl(\{\mathcal{A}|_{U(X)}\}\bigr).$$

Therefore, for any $\nu < \nu_p$, there exists a constant C such that

$$\bigl\|\nabla_{2^{-n}}^{k+1} f_\lambda\bigr\|_p \leqslant C 2^{-n\nu}, \quad \forall n \in \mathbb{N}.$$

This implies $f_\lambda \in \mathrm{Lip}^*(\nu, L_p(\mathbb{R}))$ (see, e.g., [1,8]). Thus we have established the following result.

Lemma 4.1. *Let $1 \leqslant p < \infty$. Suppose that the subdivision scheme $\{S_a^n\}_{n=1}^\infty$ converges in $W_p^k(\mathbb{R})$. Then for any $\lambda \in \ell_p(\mathbb{Z})$ and $\nu < \nu_p$, $f_\lambda \in \mathrm{Lip}^*(\nu, L_p)$. Consequently, $\nu_p \leqslant \nu_p(f)$ for any $f \in V_p$.*

We now can prove the main result of this section.

Theorem 4.2. *Assume that $1 \leqslant p < \infty$. Let ν_p given as above satisfy $\nu_p < k+1$. Suppose that the shifts of the refinement function ϕ are stable, i.e., there exist two positive constants C_1 and C_2 such that, for any $\lambda \in \ell_p(\mathbb{Z})$,*

$$C_1\|\lambda\|_p \leqslant \Bigl\|\sum_{j\in\mathbb{Z}} \lambda(j)\phi(\cdot - j)\Bigr\|_p \leqslant C_2\|\lambda\|_p.$$

Then the subdivision scheme $\{S_a^n\}_{n=1}^\infty$ converges in $W_p^k(\mathbb{R})$. Moreover, for any $f_\lambda \in V_p$, its critical exponent $\nu_p(f_\lambda)$ satisfies $\nu_p(f_\lambda) = \nu_p$.

Proof. It is known [6] that the stability implies convergence of the subdivision scheme $\{S_a^n\}_{n=1}^{\infty}$ in $W_p^k(\mathbb{R})$. Therefore, Lemma 4.1 applies. For any $f_\lambda \in V_p(\phi) \setminus \{0\}$, by Lemma 4.1, we only need to prove

$$\nu_p(f_\lambda) \leqslant \nu_p. \tag{4.3}$$

If this is not true, then there exists μ such that $1/p - \log_2 \rho < \mu < k+1$ and $f_\lambda \in \mathrm{Lip}^*(\mu, L_p(\mathbb{R}))$. Therefore, there exists a constant C such that

$$\left\| \nabla_{2^{-n}}^{k+1} f_\lambda \right\|_p \leqslant C 2^{-n\mu}, \quad \forall n \in \mathbb{N}.$$

On the other hand, the stability condition of ϕ yields that

$$2^{n/p} \left\| \nabla_{2^{-n}}^{k+1} f_\lambda \right\|_p \geqslant C_2 \left\| \nabla^{k+1} S_a^n \lambda \right\|_p, \quad \forall n \in \mathbb{N}.$$

Consequently,

$$\left\| \nabla^{k+1} S_a^n \lambda \right\|_p \leqslant C C_2 2^{-n(\mu - 1/p)}, \quad \forall n \in \mathbb{N}.$$

By Theorem 2.2, we get

$$\rho_p\big(\{\mathcal{A}|_{U(X)}\}\big) = \lim_{n\to\infty} \left\| \nabla^{k+1} S_a^n \lambda \right\|_p^{1/n} \leqslant 2^{-\mu + 1/p}.$$

It follows that

$$\mu \leqslant \frac{1}{p} - \log_2 \rho_p\big(\{\mathcal{A}|_{U(X)}\}\big),$$

which contradicts with the assumption $\mu > 1/p - \log_2 \rho_p(\{\mathcal{A}|_{U(X)}\})$. The contradiction gives (4.3). The proof is complete. □

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