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Particular solutions of singularly perturbed partial differential equations with constant coefficients in rectangular domains, Part I. Convergence analysis

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Abstract

The technique of separation of variables is used to derive explicit particular solutions for constant coefficient, singularly perturbed partial differential equations (PDEs) on a rectangular domain with Dirichlet boundary conditions. Particular solutions and exact solutions in closed form are obtained. An analysis of convergence for the series solutions is performed, which is useful in numerical solution of singularly perturbed differential equations for moderately small values of ε (e.g., $\varepsilon = 0.1$–10^{-4}). Two computational models are designed deliberately: Model I with waterfalls solutions and Model II with wedding-gauze solutions. Model II is valid for very small ε (e.g., $\varepsilon = 10^{-7}$), but Model I for a moderately small ε (=0.1–10^{-4}). The investigation contains two parts. The first part, reported in the present paper, focuses on the convergence analysis and some preliminary numerical experiments for both of the models, while the second part, to be reported in a forthcoming paper, will illustrate the solutions near the boundary layers.

Keywords: Singularly perturbed equation; Particular solutions; Separation of variables

1. Introduction

Recently, singularly perturbed differential equations [4] have drawn a great deal of attention from scientists, because of the presence of boundary and interior layers in the exact solutions which

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causes great difficulties in the numerical solution of this type of equations. Although a number of numerical methods have been developed, see, for example, [12,11,14–16], the results are still not very satisfactory, compared with those for the point singularities in elliptic boundary value problems. One reason is that the behaviors of the layers in solutions of such a problem are not very clear in physics and mathematics. This is in contrast to the point singularities of, say, a Laplace's equation, to which the exact solutions are well known for corners, interface intersections and infinities. Hence, a lot of efficient numerical methods have been developed, see [9], because typical models for these problems have been set up for us to test our numerical methods and to compare our results with others'. This paper is devoted to a fundamental issue: to seek particular solutions of the singularly perturbed differential equations, which may have an impact on the further research in this fields.

Using the technique of separation of variables [3,6,7], we are able to find the explicit analytic solutions to singularly perturbed, constant coefficient 2nd order PDEs on rectangles, sectors, or on some simple unbounded domains. Moreover, the exact solutions in closed form can also be found for three-dimensional singularly perturbed PDEs with constant coefficients and on some simple domains such as cubes, cylinders and spheres. Such simple cases do not prevent them from being testing models for numerical methods. Take the point singularities for example. Motz's problem [13] is a typical benchmark of singularity problems which is defined by the Laplace equation in a rectangle with the mixed Dirichlet and Neumann boundary conditions along the x axis. The singularity in the solution of Motz's problem behaves like $u = O(r^{1/2})$ as $0 < r \ll 1$. The exact solutions of Motz's problem provided in [9] help exploration on new and efficient numerical methods.

The motivation of this paper comes partially from the research experiments on the point singularity problems in [9], which are PDEs involving angular singularity, interface singularity or infinity singularity. We also refer simply the layer singularity problems to the singularly perturbed differential equations (cf. [11,14]). Since the derivatives are unbounded near the singularity when $\varepsilon \to 0$, the traditional finite element methods (FEMs) and finite difference methods (FDMs) provide numerical solutions with poor accuracies. Though the layer singularity problems are more challenging and more difficulties to solve than the point singularity problems, because of their complexities, we may still learn some from the point singularity problems, in order to develop new numerical methods for the layer singularity problems. In fact, most of numerical methods for point singularity problems may fall into two categories:

Case I: Local refinements of partitions in FEMs and FDMs, based on the knowledge of singularity and regularity of the solutions. Particular and exact solutions given in this paper may provide such knowledge even the solution domains discussed are rather simple.

Case II: Use of particular solutions in FEMs, partially or completely, see [9]. The particular solutions explored in this paper are a must for constructing a numerical method of this type. For instance, to resemble the penalty combination for the point singularity problems in [8], we may choose the particular solutions developed in this paper near a singular layer, use a traditional FEM in the rest domain, and employ the penalty coupling to combine both. New numerical methods can also be developed for the singularly perturbed differential equations, as reported in [10].

This paper is organized as follows. In the next section, we describe basic approaches to finding particular solutions, which consist of the methods for problems with and without the *zero corner conditions*. In Section 3, we discuss convergence properties of the series solutions. In Sections 4 and 5, we propose two computational models (called Models I and II) and present some numerical results for these models.

2. Basic approaches

Consider homogeneous, singularly perturbed PDEs with the Dirichlet boundary condition of the following form

$$\mathscr{L}u = \frac{\partial}{\partial x}\left(-\varepsilon\frac{\partial}{\partial x}u + \alpha u\right) + \frac{\partial}{\partial y}\left(-\varepsilon\frac{\partial}{\partial y}u + \beta u\right) + cu = 0 \quad \text{in } S, \tag{1}$$

$$u|_\Gamma = g \quad \text{on } \Gamma, \tag{2}$$

where S is the rectangle $S = \{(x, y), 0 < x < 2\pi, 0 < y < 2\pi\}$ and Γ is its boundary. The parameters $\varepsilon > 0, \alpha, \beta$ and $c(\geqslant 0)$ are all constants, but ε may be very small, i.e., $0 < \varepsilon \ll 1$. In this paper, we assume without loss of generality,

$$\alpha > 0, \quad \beta > 0. \tag{3}$$

In this case the solution to the problem contains two boundary layers along the boundary segments at $x = 2\pi$ and $y = 2\pi$ respectively. Since all $\varepsilon, \alpha, \beta$ and c are constants, it is possible to find explicit solutions to (1)–(2) by means of the separation of variables, as given in the next section.

2.1. Particular solutions

For (1) and (2), let the Dirichlet boundary conditions along four edges of S be given by (see Fig. 1)

$$u(x,0) = g(x,0) = g_1(x), \quad u(x,2\pi) = g(x,2\pi) = g_2(x),$$
$$u(0,y) = g(0,y) = g_3(y), \quad u(2\pi,y) = g(2\pi,y) = g_4(y), \tag{4}$$

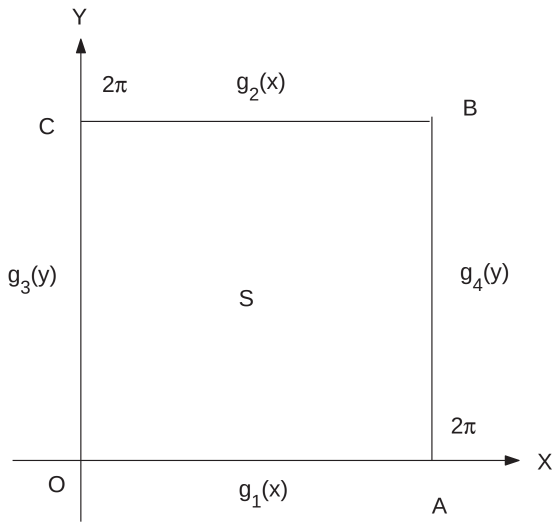

Fig. 1. A square solution domain.

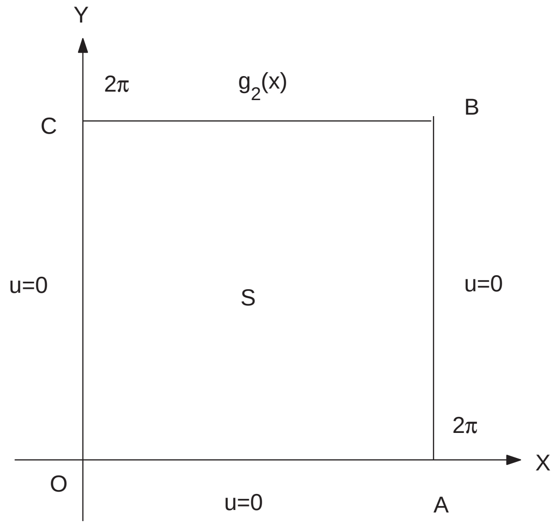

Fig. 2. A simple case.

where the function g is continuous at the four corners so that the corner continuity conditions hold:

$$g_1(0) = g_3(0), \quad g_1(2\pi) = g_4(0),$$

$$g_3(2\pi) = g_2(0), \quad g_2(2\pi) = g_4(2\pi).$$

Suppose that the boundary conditions at the four corners are all zero, i.e.,

$$g_i(0) = g_i(2\pi) = 0 \quad i = 1, 2, 3, 4. \tag{5}$$

In what follows we refer (5) to as the *corner zero conditions*. Now we split the solution to (1) and (2) into four parts, $u = \sum_{i=1}^{4} u^{(i)}$, such that $u^{(i)}$ satisfies

$$\mathscr{L}u^{(i)} = 0 \quad \text{in } S, \tag{6}$$

$$u^{(i)}|_{\Gamma_i} = g_i, \quad u^{(i)}|_{\partial S \setminus \Gamma_i} = 0, \tag{7}$$

for $i = 1, 2, 3, 4$, where $\partial S = \bigcup_{i=1}^{4} \Gamma_i$, $\Gamma_1 = \overline{OA}$, $\Gamma_2 = \overline{CB}$, $\Gamma_3 = \overline{OC}$ and $\Gamma_4 = \overline{AB}$. Fig. 2 illustrates the case for $u^{(2)}$.

Based on (5), we investigate explicit solutions to $u^{(i)}$ in this subsection using the technique of separation of variables, which was first proposed in Grunberg [6]. In what follows we use the case for $u^{(2)}$ to demonstrate this process.

Consider $u^{(2)}$ of the form $u^{(2)} = R(x)T(y)$. We have from $\mathscr{L}u^{(2)} = 0$,

$$-\varepsilon \frac{R''(x)}{R} + \alpha \frac{R'(x)}{R} = \varepsilon \frac{T''(y)}{T} - \beta \frac{T'(y)}{T} - c =: \mu.$$

From this we see that μ must be a constant. Assuming $\mu > 0$, we have from the above equation

$$-\varepsilon R''(x) + \alpha R'(x) = \mu R(x), \quad x \in [0, 2\pi],$$

$$R(0) = R(2\pi) = 0, \tag{8}$$

and

$$-\varepsilon T''(y)+\beta T'(y)+cT(y)=-\mu T(y),\quad y\in[0,2\pi]. \tag{9}$$

Under the transformation

$$R(x)=C\exp\left(\frac{\alpha}{2\varepsilon}x\right)v(x), \tag{10}$$

Eq. (8) can be rewritten as

$$-\varepsilon v''=\left(\mu-\frac{\alpha^2}{4\varepsilon}\right)v,\quad v(0)=v(2\pi)=0. \tag{11}$$

There are three cases for the values of $(\mu-\alpha^2/4\varepsilon)$.

Case I. $(\mu-\alpha^2/4\varepsilon)>0$. Let

$$\frac{1}{\varepsilon}\left(\mu-\frac{\alpha^2}{4\varepsilon}\right)=k^2, \tag{12}$$

we obtain from (11)

$$v''+k^2v=0,\quad v(0)=v(2\pi)=0.$$

The particular solutions to this equation are

$$v=C\sin kx,\quad k=1,2,\dots\ . \tag{13}$$

Obviously, for this case, μ can be obtained from (12)

$$\mu=\varepsilon k^2+\frac{\alpha^2}{4\varepsilon},\quad k=1,2,\dots\ . \tag{14}$$

Case II. $(\mu-\alpha^2/4\varepsilon)<0$. Let

$$\frac{1}{\varepsilon}\left(\frac{\alpha^2}{4\varepsilon}-\mu\right)=t^2,\quad k=1,2,\dots\ .$$

Then (11) becomes

$$v''-t^2v=0,\quad v(0)=v(2\pi)=0. \tag{15}$$

The particular solutions are given by

$$v=a\sinh(tx)+b\cosh(tx). \tag{16}$$

Applying the homogeneous boundary conditions in (15) to the above yields $a=b=0$. Therefore, (16) is a trivial solution, i.e., $v\equiv 0$.

Case III. $(\mu-\alpha^2/4\varepsilon)=0$. Then we have

$$v''=0,\quad v(0)=v(2\pi)=0.$$

This leads to the solution $v=ax+b$, which is also a trivial solution due to the homogeneous boundary conditions.

In summary, for the above three cases, the nontrivial solutions occur only in Case I. Now, consider (9), where μ is given in (14):

$$\mu_k=\varepsilon k^2+\frac{\alpha^2}{4\varepsilon},\quad k=1,2,\dots\ .$$

Using the transformation

$$T(y) = C \exp\left(\frac{\beta}{2\varepsilon} y\right) w(y), \tag{17}$$

we obtain from (9)

$$-\varepsilon w''(y) + \left(\mu_k + \frac{\beta^2}{4\varepsilon} + c\right) w(y) = 0. \tag{18}$$

Denote

$$t_k = \left\{\frac{1}{\varepsilon}\left(\mu_k + \frac{\beta^2}{4\varepsilon} + c\right)\right\}^{1/2} = \left\{k^2 + \frac{\alpha^2 + \beta^2 + 4\varepsilon c}{4\varepsilon^2}\right\}^{1/2}. \tag{19}$$

Particular solutions to (18) are given by

$$w = a_k \sinh(t_k y) + b_k \cosh(t_k y), \quad k = 1, 2, \ldots . \tag{20}$$

Hence, from (10), (13), (17) and (20), we obtain the following particular solutions to (6) as $i = 2$:

$$u^{(2)} = \exp\left(\frac{\alpha x + \beta y}{2\varepsilon}\right) \sum_{k=1}^{\infty} \{a_k \sinh(t_k y) + b_k \cosh(t_k y)\} \sin kx, \tag{21}$$

where t_k is given in (19).

Note that the coefficients a_k and b_k can be found from the boundary condition (7). In fact, since $u^{(2)}(x, 0) = 0$, we have from (21)

$$0 = u^{(2)}(x, 0) = \exp\left(\frac{\alpha x}{2\varepsilon}\right) \sum_{k=1}^{\infty} b_k \sin kx,$$

yielding

$$b_k = \frac{1}{\pi} \int_0^{2\pi} u^{(2)}(x, 0) \exp\left(-\frac{\alpha}{2\varepsilon} x\right) \sin kx \, dx = 0, \quad \forall k.$$

Also for $y = 2\pi$, we obtain from (21)

$$g_2(x) = u^{(2)}(x, 2\pi) = \exp\left(\frac{\alpha x + 2\pi\beta}{2\varepsilon}\right) \sum_{k=1}^{\infty} a_k \sinh(2\pi t_k) \sin kx,$$

which gives

$$a_k = \frac{1}{\pi} \frac{1}{\sinh(2\pi t_k)} \int_0^{2\pi} g_2(x) \exp\left(-\frac{\alpha x + 2\pi\beta}{2\varepsilon}\right) \sin kx \, dx.$$

Hence solution (21) reduces to

$$u^{(2)} = \exp\left(\frac{\alpha x + \beta y}{2\varepsilon}\right) \sum_{k=1}^{\infty} a_k \sinh(t_k y) \sin kx. \tag{22}$$

When $\alpha > 0$, the layer may occur at $x = 2\pi$. In this case we rewrite the solutions (22) as

$$u^{(2)} = \exp\left(\frac{-\alpha(2\pi - x) + \beta y}{2\varepsilon}\right) \sum_{k=1}^{\infty} a_k \sinh(t_k y) \sin kx, \tag{23}$$

where

$$a_k = \frac{1}{\pi}\frac{1}{\sinh(2\pi t_k)}\int_0^{2\pi} g_2(x)\exp\left(\frac{\alpha(2\pi - x) - 2\pi\beta}{2\varepsilon}\right)\sin kx\,\mathrm{d}x. \tag{24}$$

Similarly, it can be shown that the solution $u^{(1)}$ in (6) is given by

$$u^{(1)} = \exp\left(\frac{-\alpha(2\pi - x) + \beta y}{2\varepsilon}\right)\sum_{k=1}^{\infty} b_k \sinh(t_k(2\pi - y))\sin kx,$$

where

$$b_k = \frac{1}{\pi}\frac{1}{\sinh(2\pi t_k)}\int_0^{2\pi} g_1(x)\exp\left(\frac{\alpha(2\pi - x)}{2\varepsilon}\right)\sin kx\,\mathrm{d}x. \tag{25}$$

Furthermore, $u^{(3)}$ and $u^{(4)}$ have the following representations, respectively,

$$u^{(3)} = \exp\left(\frac{\alpha x - \beta(2\pi - y)}{2\varepsilon}\right)\sum_{k=1}^{\infty} d_k \sinh(t_k(2\pi - x))\sin ky, \tag{26}$$

$$u^{(4)} = \exp\left(\frac{\alpha x - \beta(2\pi - y)}{2\varepsilon}\right)\sum_{k=1}^{\infty} c_k \sinh(t_k x)\sin ky, \tag{27}$$

where

$$c_k = \frac{1}{\pi}\frac{1}{\sinh(2\pi t_k)}\int_0^{2\pi} g_4(y)\exp\left(\frac{-\alpha 2\pi + \beta(2\pi - y)}{2\varepsilon}\right)\sin ky\,\mathrm{d}y, \tag{28}$$

$$d_k = \frac{1}{\pi}\frac{1}{\sinh(2\pi t_k)}\int_0^{2\pi} g_3(y)\exp\left(\frac{\beta(2\pi - y)}{2\varepsilon}\right)\sin ky\,\mathrm{d}y. \tag{29}$$

In summary, we have the following solution to (6) and (7) satisfying the corner zero conditions (5):

$$\begin{aligned} u &= \sum_{i=1}^{4} u^{(i)} \\ &= \exp\left(\frac{-\alpha(2\pi - x) + \beta y}{2\varepsilon}\right)\sum_{k=1}^{\infty}\{a_k \sinh(t_k y) + b_k \sinh(t_k(2\pi - y))\}\sin kx \\ &\quad + \exp\left(\frac{\alpha x - \beta(2\pi - y)}{2\varepsilon}\right)\sum_{k=1}^{\infty}\{c_k \sinh(t_k x) + d_k \sinh(t_k(2\pi - x))\}\sin ky, \end{aligned} \tag{30}$$

where the coefficients a_k, b_k, c_k and d_k are given in (24), (25), (28) and (29), respectively.

2.2. The case that the corner zero conditions are not satisfied

Let us consider a transition function $\bar{u}$ satisfying

$$\mathscr{L}\bar{u} = 0 \quad \text{in } S, \tag{31}$$

$$\bar{u}(0,0)=g(0,0), \quad \bar{u}(2\pi,0)=g(2\pi,0),$$
$$\bar{u}(0,2\pi)=g(0,2\pi), \quad \bar{u}(2\pi,2\pi)=g(2\pi,2\pi). \tag{32}$$

The difference $(u-\bar{u})$ satisfies (1) and the corner zero conditions (5). In fact, the particular solutions in Cases II and III in the previous subsection may be chosen to satisfy the conditions in (32).

First, consider the solutions in Case III, i.e.,

$$\frac{1}{\varepsilon}\left(\mu_0-\frac{\alpha^2}{4\varepsilon}\right)=0,$$

from which we have

$$\mu_0=\frac{\alpha^2}{4\varepsilon} \quad \text{and} \quad t_0=\frac{1}{2\varepsilon}(\alpha^2+\beta^2+4\varepsilon c)^{1/2}.$$

We have the particular solution,

$$\bar{u}=\exp\left(\frac{-\alpha(2\pi-x)+\beta y}{2\varepsilon}\right)\{(a+\bar{a}x)\sinh(t_0y)+(b+\bar{b}x)\sinh(t_0(2\pi-y))\}$$
$$+\exp\left(\frac{\alpha x-\beta(2\pi-y)}{2\varepsilon}\right)\{(c+\bar{c}y)\sinh(t_0x)+(d+\bar{d}y)\sinh(t_0(2\pi-x))\}. \tag{33}$$

This expression contains eight unknown coefficients $a,b,c,d,\bar{a},\bar{b},\bar{c}$ and $\bar{d}$, while there are only four conditions in (32) to be satisfied by $\bar{u}$. Therefore, four of these eight coefficients are free. If set let $c=\bar{c}=d=\bar{d}=0$, then the other four coefficients can be determined by (32) as

$$a=\frac{u(2\pi,2\pi)}{2\pi\sinh(2\pi t_0)}\exp\left(-\frac{\beta\pi}{\varepsilon}\right), \quad \bar{a}=\frac{u(0,2\pi)}{2\pi\sinh(2\pi t_0)}\exp\left(\frac{(\alpha-\beta)\pi}{\varepsilon}\right),$$
$$b=\frac{u(2\pi,0)}{2\pi\sinh(2\pi t_0)}, \quad \bar{b}=\frac{u(0,0)}{2\pi\sinh(2\pi t_0)}\exp\left(\frac{\alpha\pi}{\varepsilon}\right).$$

The particular solution in (33), denoted as $\bar{u}_1$, then becomes

$$\bar{u}_1=\frac{1}{2\pi\sinh(2\pi t_0)}\left\{\left[u(2\pi,2\pi)x\exp\left(-\frac{\alpha(2\pi-x)+\beta(2\pi-y)}{2\varepsilon}\right)\right.\right.$$
$$\left.+u(0,2\pi)(2\pi-x)\exp\left(\frac{\alpha x-\beta(2\pi-y)}{2\varepsilon}\right)\right]\sinh(t_0y)$$
$$+\left[u(2\pi,0)x\exp\left(\frac{-\alpha(2\pi-x)+\beta y}{2\varepsilon}\right)+u(0,0)(2\pi-x)\exp\left(\frac{\alpha x+\beta y}{2\varepsilon}\right)\right]$$
$$\left.\times\sinh(t_0(2\pi-y))\right\}.$$

Similarly, if we choose $a = \bar{a} = b = \bar{b} = 0$ in (33) using (32), we have another particular solution

$$\bar{u}_2 = \frac{1}{2\pi \sinh(2\pi t_0)} \left\{ \left[u(2\pi, 2\pi) y \exp\left(-\frac{\alpha(2\pi - x) + \beta(2\pi - y)}{2\varepsilon} \right) \right.\right.$$
$$\left. + u(2\pi, 0)(2\pi - y) \exp\left(\frac{-\alpha(2\pi - x) + \beta y}{2\varepsilon} \right) \right] \sinh(t_0 x)$$
$$+ \left[u(0, 2\pi) y \exp\left(\frac{\alpha x - \beta(2\pi - y)}{2\varepsilon} \right) + u(0,0)(2\pi - y) \exp\left(\frac{\alpha x + \beta y}{2\varepsilon} \right) \right]$$
$$\left. \cdot \sinh(t_0(2\pi - x)) \right\}.$$

It is also possible to derive a symmetric solution of the form $\bar{u} = \frac{1}{2}(\bar{u}_1 + \bar{u}_2)$.

Since the particular solutions in Case II for $\mu < \alpha^2/4\varepsilon$ also satisfy equation $\mathscr{L}u = 0$, they can be chosen to satisfy the corner conditions (32) as well. Let $\mu = \alpha^2/8\varepsilon$, we have from Case II,

$$v = a \sinh(t_\mu x) + b \sinh(t_\mu(2\pi - x)),$$

where

$$t_\mu = \left\{ \frac{1}{\varepsilon} \left(\frac{\alpha^2}{4\varepsilon} - \mu \right) \right\}^{1/2} = \frac{1}{2\sqrt{2}} \frac{\alpha}{\varepsilon}.$$

Denote $p_\mu = (\beta^2 - \frac{1}{2}\alpha^2 + 4\varepsilon c)/4\varepsilon^2$. When $p_\mu > 0$, the particular solution is given by

$$\bar{u}_3 = \exp\left(\frac{-\alpha(2\pi - x) + \beta y}{2\varepsilon} \right) (a_\mu \sinh(t_\mu x) + b_\mu \sinh(t_\mu(2\pi - x)))$$
$$\times (c_\mu \sinh(\sqrt{p_\mu} y) + d_\mu \sinh(\sqrt{p_\mu}(2\pi - y))).$$

When $p_\mu < 0$, the particular solution is

$$\bar{u}_3 = \exp\left(\frac{-\alpha(2\pi - x) + \beta y}{2\varepsilon} \right) (a_\mu \sinh(t_\mu x) + b_\mu \sinh(t_\mu(2\pi - x)))$$
$$\times (c_\mu \sin(\sqrt{-p_\mu} y) + d_\mu \sin(\sqrt{-p_\mu}(2\pi - y))),$$

and, when $p_\mu = 0$,

$$\bar{u}_3 = \exp\left(\frac{-\alpha(2\pi - x) + \beta y}{2\varepsilon} \right) (a_\mu \sinh(t_\mu x) + b_\mu \sinh(t_\mu(2\pi - x)))$$
$$\times (c_\mu y + \bar{c}_\mu(2\pi - y)),$$

where $a_\mu, b_\mu, c_\mu, d_\mu$ and $\bar{c}_\mu$ are constants.

3. Convergence

For simplicity, we consider the case of $u = u^{(2)}$ in (22) with a slightly different form, i.e.,

$$u(x,y) = \exp\left(\frac{\alpha x + \beta y}{2\varepsilon}\right) \sum_{k=1}^{\infty} a_k \frac{\sinh(t_k y)}{\sinh(2\pi t_k)} \sin kx, \quad (x,y) \in [0,2\pi]^2. \tag{34}$$

When $y = 2\pi$, the coefficients in (34) are

$$a_k = \frac{1}{\pi} \int_0^{2\pi} g_2(t) \exp\left(-\frac{\alpha t + 2\beta\pi}{2\varepsilon}\right) \sin kt \, dt. \tag{35}$$

Using (35) we see that (34) can be rewritten as

$$u(x,y) = \sum_{k=1}^{\infty} A_k \frac{\sinh(t_k y)}{\sinh(2\pi t_k)} \sin kx, \quad (x,y) \in [0,2\pi]^2, \tag{36}$$

where

$$\begin{aligned} A_k &= \frac{1}{\pi} \int_0^{2\pi} g_2(t) \exp\left(\frac{\alpha(x-t) + \beta(y-2\pi)}{2\varepsilon}\right) \sin kt \, dt \\ &=: \int_0^{2\pi} f(t) \sin kt \, dt, \end{aligned} \tag{37}$$

and the function $f(t)$ is given by

$$f(t) = g_2(t) \frac{1}{\pi} \exp\left(\frac{\alpha(x-t) + \beta(y-2\pi)}{2\varepsilon}\right), \quad (x,y) \in [0,2\pi]^2. \tag{38}$$

Here $g_2(0) = g_2(2\pi) = 0$ and $g_2(t)$ is bounded on $[0,2\pi]$.

Before we derive convergence rates for (34), let us first make two assumptions on the function $f(t)$ in (38).

A1. The function $f(t)$ satisfies the *corner zero conditions*:

$$f(0) = f(2\pi) = 0.$$

A2. The nth derivatives of function $f(x)$ have the bounds,

$$|f^{(n)}(t)| \leqslant \frac{C}{\varepsilon^n} \quad \text{for } n = 0,1,2,\ldots, \tag{39}$$

where C is a generic positive constant independent of ε and n.

Define the partial sum of (36) by

$$u_N(x,y) = \sum_{k=1}^{N} A_k \frac{\sinh(t_k y)}{\sinh(2\pi t_k)} \sin kx, \quad (x,y) \in [0,2\pi]^2. \tag{40}$$

The upper bounds on the absolute errors in the solution and its derivatives are contained in the following theorem.

Theorem 3.1. *Let the function $f(t)$ satisfy the assumptions* A1 *and* A2, *and suppose that* $N \geqslant \frac{1}{\varepsilon}$. *Then, there exist the bounds,*

$$|u - u_N| \leqslant \frac{C}{\varepsilon^2 N^2} \exp\left(\frac{\alpha(x-2\pi) + \beta(y-2\pi)}{2\varepsilon}\right) \exp\left(\frac{\alpha\pi}{\varepsilon} - t_N(2\pi - y)\right), \tag{41}$$

$$|u_x - (u_N)_x| \leqslant \frac{C}{\varepsilon^2 N} \exp\left(\frac{\alpha(x-2\pi) + \beta(y-2\pi)}{2\varepsilon}\right) \exp\left(\frac{\alpha\pi}{\varepsilon} - t_N(2\pi - y)\right), \tag{42}$$

$$|u_y - (u_N)_y| \leqslant \frac{C}{\varepsilon^2 N} \exp\left(\frac{\alpha(x-2\pi) + \beta(y-2\pi)}{2\varepsilon}\right) \exp\left(\frac{\alpha\pi}{\varepsilon} - t_N(2\pi - y)\right), \tag{43}$$

where t_N is given in (19) *and C is a generic positive constant, independent of ε and N.*

Proof. Integrating by parts and using A1, we have from (37)

$$\begin{aligned}
A_k &= \int_0^{2\pi} f(t)\sin kt\,\mathrm{d}t = -\frac{1}{k}\int_0^{2\pi} f(t)\,\mathrm{d}\cos kt \\
&= -\frac{1}{k}[f(t)\cos kt]_0^{2\pi} + \frac{1}{k}\int_0^{2\pi} f'(t)\cos kt\,\mathrm{d}t \\
&= -\frac{1}{k}[f(2\pi) - f(0)] + \frac{1}{k}\int_0^{2\pi} f'(t)\cos kt\,\mathrm{d}t = \frac{1}{k}\int_0^{2\pi} f'(t)\cos kt\,\mathrm{d}t.
\end{aligned}$$

Integrating the last term in the above by parts twice gives

$$\begin{aligned}
A_k &= \frac{1}{k^2}\int_0^{2\pi} f'(t)\,\mathrm{d}\sin kt = \frac{1}{k^2}[f'(t)\sin kt]_0^{2\pi} - \frac{1}{k^2}\int_0^{2\pi} f''(t)\sin kt\,\mathrm{d}t \\
&= \frac{1}{k^3}\int_0^{2\pi} f''(t)\,\mathrm{d}\cos kt = \frac{1}{k^3}[f''(t)\cos kt]_0^{2\pi} - \frac{1}{k^3}\int_0^{2\pi}\cos kt\,\mathrm{d}f''(t).
\end{aligned} \tag{44}$$

From (44), A2 and the definition of f, we have

$$\begin{aligned}
|A_k| &\leqslant \frac{C}{k^3\varepsilon^2}\left|\exp\left(\frac{\alpha(x-2\pi)+\beta(y-2\pi)}{2\varepsilon}\right) + \exp\left(\frac{\alpha x + \beta(y-2\pi)}{2\varepsilon}\right)\right| \\
&\quad + \frac{C}{k^3}\int_0^{2\pi}\frac{1}{\varepsilon^3}\exp\left(\frac{\alpha(x-t)+\beta(y-2\pi)}{2\varepsilon}\right)\mathrm{d}t \\
&\leqslant \frac{C}{k^3\varepsilon^2}\exp\left(\frac{\alpha x+\beta(y-2\pi)}{2\varepsilon}\right) + \frac{C}{k^3}\int_0^{2\pi}\frac{1}{\varepsilon^2}\,\mathrm{d}\left\{\exp\left(\frac{\alpha(x-t)+\beta(y-2\pi)}{2\varepsilon}\right)\right\} \\
&\leqslant \frac{C}{k^3\varepsilon^2}\exp\left(\frac{\alpha(x-2\pi)+\beta(y-2\pi)}{2\varepsilon}\right)\exp\left(\frac{\alpha\pi}{\varepsilon}\right).
\end{aligned} \tag{45}$$

Moreover, when $k \geqslant N+1 > N$,

$$\left|\frac{\sinh(t_k y)}{\sinh(2t_k\pi)}\right| = \exp(-t_k(2\pi - y))\left(\frac{1-\exp(-2t_k y)}{1-\exp(-4t_k\pi)}\right)$$
$$\leqslant \exp(-t_k(2\pi - y)) \leqslant \exp(-t_N(2\pi - y)). \tag{46}$$

From (36), (40) and (46) we obtain

$$|u(x,y) - u_N(x,y)| \leqslant \sum_{k=N+1}^{\infty} |A_k| \left|\frac{\sinh(t_k y)}{\sinh(2t_k\pi)}\right| |\sin kx|$$
$$\leqslant \exp(-t_N(2\pi - y)) \sum_{k=N+1}^{\infty} |A_k|. \tag{47}$$

Combining (45) and (47) leads to the first result (41) by noting $\sum_{k=N+1}^{\infty}(1/k^3) \leqslant C/N^2$.

Next, let us consider the bounds on the errors in the derivatives. Noting $A_k = A_k(x,y)$ we have

$$u_x = \frac{\partial u}{\partial x} = \sum_{k=1}^{\infty} A_k \frac{\sinh(t_k y)}{\sinh(2t_k\pi)}\left(\frac{\alpha}{2\varepsilon}\sin kx + k\cos kx\right), \quad (x,y)\in[0,2\pi]^2,$$

$$(u_N)_x = \sum_{k=1}^{N} A_k \frac{\sinh(t_k y)}{\sinh(2t_k\pi)}\left(\frac{\alpha}{2\varepsilon}\sin kx + k\cos kx\right). \tag{48}$$

Following the arguments in (47), we obtain from the assumption $N \geqslant 1/\varepsilon$ and (45)

$$|u_x - (u_N)_x| \leqslant \exp(-t_N(2\pi - y)) \sum_{k=N+1}^{\infty}\left(k + \frac{1}{2\varepsilon}\right)|A_k|$$
$$\leqslant C\exp(-t_N(2\pi - y)) \sum_{k=N+1}^{\infty} k|A_k|$$
$$\leqslant \frac{C}{\varepsilon^2 N}\exp\left(\frac{\alpha(x-2\pi)+\beta(y-2\pi)}{2\varepsilon}\right)\exp\left(\frac{\alpha\pi}{\varepsilon} - t_N(2\pi - y)\right). \tag{49}$$

This is the second result (42).

Third, we have

$$u_y = \frac{\partial u}{\partial y} = \sum_{k=1}^{\infty} A_k \left\{\frac{\sinh(t_k y)}{\sinh(2t_k\pi)}\frac{\beta}{2\varepsilon} + \frac{\cosh(t_k y)}{\sinh(2t_k\pi)}t_k\right\}\sin kx, \tag{50}$$

$$(u_N)_y = \frac{\partial(u_N)}{\partial y} = \sum_{k=1}^{N} A_k \left\{\frac{\sinh(t_k y)}{\sinh(2t_k\pi)}\frac{\beta}{2\varepsilon} + \frac{\cosh(t_k y)}{\sinh(2t_k\pi)}t_k\right\}\sin kx. \tag{51}$$

Since for $k \geqslant N+1 > N$, there exist the bounds,

$$\frac{\cosh(t_k y)}{\sinh(2t_k \pi)} = \exp(-t_k(2\pi - y))\left(\frac{1+\exp(-2t_k y)}{1-\exp(-4t_k \pi)}\right) \leqslant 3\exp(-t_k(2\pi - y)) \leqslant 3\exp(-t_N(2\pi - y)), \tag{52}$$

and

$$t_k = \sqrt{k^2 + \frac{\alpha^2+\beta^2+4\varepsilon c}{4\varepsilon^2}} \leqslant k + \frac{\alpha+\beta+2\sqrt{\varepsilon c}}{2\varepsilon} \leqslant \left(k + \frac{C}{\varepsilon}\right). \tag{53}$$

Hence we obtain from $N \geqslant 1/\varepsilon$

$$\begin{aligned}|u_y - (u_N)_y| &\leqslant \sum_{k=N+1}^{\infty} |A_k| \left\{ \left|\frac{\sinh(t_k y)}{\sinh(2t_k \pi)}\right| \frac{\beta}{2\varepsilon} + \left|\frac{\cosh(t_k y)}{\sinh(2t_k \pi)}\right| \times t_k \right\} |\sin kx| \\ &\leqslant C\exp(-t_N(2\pi - y)) \sum_{k=N+1}^{\infty} \left(k + \frac{1}{\varepsilon}\right)|A_k| \\ &\leqslant C\exp(-t_N(2\pi - y)) \sum_{k=N+1}^{\infty} k|A_k| \\ &\leqslant \frac{C}{\varepsilon^2 N}\exp\left(\frac{\alpha(x-2\pi)+\beta(y-2\pi)}{2\varepsilon}\right)\exp\left(\frac{\alpha\pi}{\varepsilon} - t_N(2\pi - y)\right).\end{aligned} \tag{54}$$

This is the third result (43). This completes the proof of Theorem 3.1. □

When $\varepsilon > 0$ is given and fixed, Theorem 3.1 displays the convergence of the solutions and derivatives when $N \to \infty$. For real computation, suppose that the maximal errors of solutions and derivatives are less than δ over the entire $S = [0, 2\pi]^2$, then from Theorem 3.1 we may require

$$\frac{C}{\varepsilon^2 N^2}\exp\left(\frac{\alpha\pi}{\varepsilon} - t_N(2\pi - y)\right) \leqslant \delta, \quad \frac{C}{\varepsilon^2 N}\exp\left(\frac{\alpha\pi}{\varepsilon} - t_N(2\pi - y)\right) \leqslant \delta,$$

respectively. Solving these gives

$$N \geqslant \frac{\sqrt{C}}{\varepsilon\sqrt{\delta}}\exp\left(\frac{\alpha\pi}{2\varepsilon} - \frac{1}{2}t_N(2\pi - y)\right), \quad N \geqslant \frac{C}{\varepsilon^2\delta}\exp\left(\frac{\alpha\pi}{\varepsilon} - t_N(2\pi - y)\right). \tag{55}$$

When ε is small and $y = 2\pi$, the parameter N needed is large, which is prohibitive for computing the solutions numerically. Therefore, the series solution can be used only for problems with a moderately small value of ε. The above worst estimates for N happen only in the approximation of the derivatives at corner $(2\pi, 2\pi)$, while the solutions at $(2\pi, 2\pi)$ are known from the Dirichlet boundary condition.

To improve the convergence, we give another assumption.

A3. Let the periodic conditions satisfy

$$f^{(2\ell)}(0) = f^{(2\ell)}(2\pi), \quad \ell \geqslant 1,$$

where $f(t)$ is given in (38).

We have following theorem.

Theorem 3.2. *The function $f(t)$ satisfy assumptions* A1–A3 *with $\ell \geqslant 2$. Suppose $N \geqslant 1/\varepsilon$, there exist the bounds*,

$$|u - u_N| \leqslant \frac{C}{\varepsilon^{2\ell} N^{2\ell}} \exp\left(\frac{\alpha(x-2\pi)+\beta(y-2\pi)}{2\varepsilon}\right) \exp\left(\frac{\alpha\pi}{\varepsilon} - t_N(2\pi - y)\right), \tag{56}$$

$$|u_x - (u_N)_x| \leqslant \frac{C}{\varepsilon^{2\ell} N^{2\ell-1}} \exp\left(\frac{\alpha(x-2\pi)+\beta(y-2\pi)}{2\varepsilon}\right) \exp\left(\frac{\alpha\pi}{\varepsilon} - t_N(2\pi - y)\right), \tag{57}$$

$$|u_y - (u_N)_y| \leqslant \frac{C}{\varepsilon^{2\ell} N^{2\ell-1}} \exp\left(\frac{\alpha(x-2\pi)+\beta(y-2\pi)}{2\varepsilon}\right) \exp\left(\frac{\alpha\pi}{\varepsilon} - t_N(2\pi - y)\right), \tag{58}$$

where C is a generic positive constant, *independent of ε and N*.

Proof. We only show $\ell = 2$. Since the proof for $\ell \geqslant 2$ is similar. From A3 and (44), we have

$$\begin{aligned} A_k &= \frac{1}{k^3}[f''(2\pi) - f''(0)] - \frac{1}{k^3}\int_0^{2\pi} f'''(t)\cos kt \,\mathrm{d}t \\ &= -\frac{1}{k^4}\int_0^{2\pi} f'''(t)\,\mathrm{d}\sin kt \\ &= -\frac{1}{k^4}[f'''(t)\sin kt]_0^{2\pi} + \frac{1}{k^4}\int_0^{2\pi} f''''(t)\sin kt\,\mathrm{d}t \\ &= -\frac{1}{k^5}\int_0^{2\pi} f''''(t)\,\mathrm{d}\cos kt \\ &= -\frac{1}{k^5}[f''''(t)\cos kt]_0^{2\pi} + \frac{1}{k^5}\int_0^{2\pi} \sin kt\,\mathrm{d}f''''(t). \end{aligned} \tag{59}$$

Hence, we obtain from (45) and A2

$$|A_k| \leqslant \frac{C}{k^5 \varepsilon^4} \exp\left(\frac{\alpha(x-2\pi)+\beta(y-2\pi)}{2\varepsilon}\right) \exp\left(\frac{\alpha\pi}{\varepsilon}\right). \tag{60}$$

Following the arguments in (47) we have

$$\begin{aligned} |u(x,y) - u_N(x,y)| &\leqslant \exp(-t_N(2\pi - y)) \sum_{k=N+1}^{\infty} |A_k| \\ &\leqslant \frac{C}{N^4 \varepsilon^4} \exp\left(\frac{\alpha(x-2\pi)+\beta(y-2\pi)}{2\varepsilon}\right) \exp\left(\frac{\alpha\pi}{\varepsilon} - t_N(2\pi - y)\right). \end{aligned} \tag{61}$$

This is the result (56) for $\ell = 2$, the proofs for (57) and (58) are similar to the above. □

Suppose that the maximal errors of solutions and derivatives are less than δ, then for $\ell > 1$, we may require respectively,

$$N \geqslant \frac{1}{\varepsilon}\left(\frac{C}{\delta}\right)^{1/2\ell} \exp\left(\frac{\alpha\pi/\varepsilon - t_N(2\pi - y)}{2\ell}\right),$$

$$N \geqslant \left(\frac{C}{\varepsilon^{2\ell}\delta}\right)^{1/(2\ell-1)} \exp\left(\frac{\alpha\pi/\varepsilon - t_N(2\pi - y)}{2\ell - 1}\right). \qquad (62)$$

From (62), we see that the worst convergence rates happen at $x = y = 2\pi$,

$$N \geqslant C\frac{1}{\varepsilon}\left(\frac{C}{\delta}\right)^{1/2\ell} \exp\left(\frac{\alpha\pi}{2\ell\varepsilon}\right), \quad N \geqslant \left(\frac{C}{\varepsilon^{2\ell}\delta}\right)^{1/(2\ell-1)} \exp\left(\frac{\alpha\pi}{(2\ell-1)\varepsilon}\right). \qquad (63)$$

Although the lower bound of N in (63) is smaller than that in (55), ε is still limited to being not small over the entire S in practice.

For numerical computation, let us consider a subdomain $S^* \subset S$, where

$$S^* = \{(x, y), 0 \leqslant x \leqslant 2\pi - \mu\varepsilon^{1+q}, 0 \leqslant y \leqslant 2\pi - \mu\varepsilon^{1+q}\}. \qquad (64)$$

Here q and μ are positive integers. We have the following corollary of Theorem 3.1.

Corollary 3.1. *Let all the conditions in Theorem* 3.1 *hold. If it is required that the errors in the approximate derivatives are less than* $\delta = \exp(-M)$ *over the subdomain* S^*, *then* $N(\geqslant 1/\varepsilon^2)$ *should be chosen, to satisfy the following inequality,*

$$N \geqslant \frac{\alpha\pi/\varepsilon + \ln C + M}{\mu\varepsilon^{1+q}}, \qquad (65)$$

where C *is the constant in Theorem* 3.1.

Proof. From Theorem 3.1 we have

$$|u_x - (u_N)_x| \leqslant \frac{C}{\varepsilon^2 N} \exp\left(\frac{\alpha(x - 2\pi) + \beta(y - 2\pi)}{2\varepsilon}\right) \exp\left(\frac{\alpha\pi}{\varepsilon} - t_N(2\pi - y)\right)$$
$$\leqslant \delta = \exp(-M). \qquad (66)$$

The worst convergence happens at $(\bar{x}, \bar{y})$, where $\bar{x} = \bar{y} = 2\pi - \mu\varepsilon^{1+q}$. Then from $\varepsilon^2 N \geqslant 1$ and $t_N > N$, we may require

$$C \exp\left(\frac{\alpha\pi}{\varepsilon} - N\mu\varepsilon^{1+q}\right) \leqslant \exp(-M). \qquad (67)$$

This leads to

$$\ln C + \frac{\alpha\pi}{\varepsilon} - N\mu\varepsilon^{1+q} + M \leqslant 0, \qquad (68)$$

and the desired bound (65) is obtained. □

We comment that the above corollary gives specific lower bounds for N which guarantees that the absolute errors in the approximate derivatives are bounded above by δ. This result is of practical importance as it specifies the minimum number of terms in the series solution so that the errors in the approximate derivatives are less than a given (small) positive constant.

In the next two sections, we will design two different models by judicious choices of the boundary conditions. Numerical results on the two models will be presented.

4. Model I

Let us consider the numerical computation of the above series solutions. To do so, we design a model, called Model I, with a waterfalls profile of solutions on $S=[0,\pi]^2$. This model is chosen as follows.

We first require that the solution to this model satisfies the corner conditions $u(0,0)=0$ and $u(\pi,0)=u(0,\pi)=u(\pi,\pi)=1$. The boundary layers of this model occur at $x=\pi$ and $y=\pi$. The particular solution to $\mathscr{L}u=0$ is of the form

$$u(x,y)=\exp\left(\frac{\alpha x+\beta y}{2\varepsilon}\right)\times\{a_0\sinh(t_0 y)+b_0\sinh(t_0 x)+c_0(x\sinh(t_0 y)+y\sinh(t_0 x))\}\quad \text{in } [0,\pi]^2, \tag{69}$$

where $t_0=(1/2\varepsilon)(\alpha^2+\beta^2+4\varepsilon c)^{1/2}$. Clearly, u in (69) already satisfies $u(0,0)=0$. Thus, the coefficients a_0, b_0 and c_0 can be determined by the other three conditions, yielding

$$a_0=\frac{1}{\sinh(t_0\pi)}\exp\left(-\frac{\beta\pi}{2\varepsilon}\right),\quad b_0=\frac{1}{\sinh(t_0\pi)}\exp\left(-\frac{\alpha\pi}{2\varepsilon}\right),$$
$$c_0=\frac{1}{2\pi\sinh(t_0\pi)}\left\{\exp\left(-\frac{(\alpha+\beta)\pi}{2\varepsilon}\right)-\exp\left(\frac{-\beta\pi}{2\varepsilon}\right)-\exp\left(\frac{-\alpha\pi}{2\varepsilon}\right)\right\}. \tag{70}$$

The particular solution satisfying the corner conditions, denoted as $\bar{u}$, then becomes

$$\bar{u}(x,y)=\left(\frac{\sinh(t_0 y)}{\sinh(t_0\pi)}-\frac{x\sinh(t_0 y)+y\sinh(t_0 x)}{2\pi\sinh(t_0\pi)}\right)\exp\left(\frac{\alpha x-\beta(\pi-y)}{2\varepsilon}\right)$$
$$+\left(\frac{\sinh(t_0 x)}{\sinh(t_0\pi)}-\frac{x\sinh(t_0 y)+y\sinh(t_0 x)}{2\pi\sinh(t_0\pi)}\right)\exp\left(\frac{-\alpha(\pi-x)+\beta y}{2\varepsilon}\right)$$
$$+\frac{x\sinh(t_0 y)+y\sinh(t_0 x)}{2\pi\sinh(t_0\pi)}\exp\left(-\frac{\alpha(\pi-x)+\beta(\pi-y)}{2\varepsilon}\right). \tag{71}$$

Clearly, this particular solution is symmetric in x and y.

Now, we pose the following boundary conditions on $[0,\pi]^2$ for the model

$$u(x,\pi)=u(\pi,y)=1,$$
$$u(x,0)=\frac{\sinh(t_0 x)}{\sinh(t_0\pi)}\exp\left(-\frac{\alpha(\pi-x)}{2\varepsilon}\right)=\bar{u}(x,0)=g_1(x),$$
$$u(0,y)=\frac{\sinh(t_0 y)}{\sinh(t_0\pi)}\exp\left(-\frac{\beta(\pi-y)}{2\varepsilon}\right)=\bar{u}(0,y)=g_3(y). \tag{72}$$

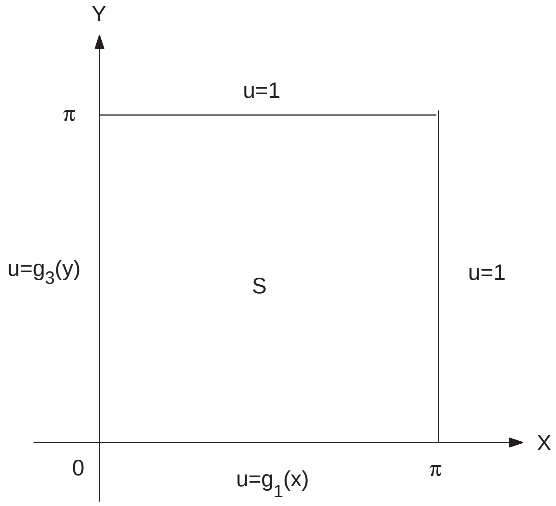

Fig. 3. The solution domain for Model I.

These conditions are illustrated in Fig. 3. Using the results in the previous sections, it is easy to see that the particular solution to $\mathscr{L}u = 0$ satisfying the above boundary conditions is given by

$$u(x,y) = \bar{u}(x,y) + \exp\left(\frac{-\alpha(\pi - x) + \beta y}{2\varepsilon}\right) \sum_{k=1}^{\infty} a_k \frac{\sinh(t_k y)}{\sinh(t_k \pi)} \sin kx$$

$$+ \exp\left(\frac{\alpha x - \beta(\pi - y)}{2\varepsilon}\right) \sum_{k=1}^{\infty} c_k \frac{\sinh(t_k x)}{\sinh(t_k \pi)} \sin ky, \tag{73}$$

where $\bar{u}(x,y)$ is given in (71) and

$$a_k = \frac{2}{\pi} \int_0^{\pi} (1 - \bar{u}(x,\pi)) \exp\left(\frac{\alpha(\pi - x) - \beta\pi}{2\varepsilon}\right) \sin kx \, dx, \tag{74}$$

$$c_k = \frac{2}{\pi} \int_0^{\pi} (1 - \bar{u}(\pi,y)) \exp\left(\frac{-\alpha\pi + \beta(\pi - y)}{2\varepsilon}\right) \sin ky \, dy. \tag{75}$$

The integrals on the right sides of (74) and (75) can be evaluated exactly using the results in [5], leading to

$$a_k = \frac{2}{\pi} \left\{ \left[\exp\left(\frac{(\alpha - \beta)\pi}{2\varepsilon}\right) - 1 \right] \left[\frac{(-1)^k}{2k} - \frac{k(-1)^k}{2t_k^2} \right] \right.$$

$$\left. + \exp\left(\frac{(\alpha - \beta)\pi}{2\varepsilon}\right) \left[\frac{k}{p_x^2 + k^2} - \frac{1}{k} \right] + \exp(p_y \pi) \left[\frac{(-1)^k}{2k} + \frac{k(-1)^k}{2t_k^2} - \frac{k(-1)^k}{p_x^2 + k^2} \right] \right\}, \tag{76}$$

$$c_k = \frac{2}{\pi}\left\{\left[\exp\left(\frac{(\beta-\alpha)\pi}{2\varepsilon}\right)-1\right]\left[\frac{(-1)^k}{2k}-\frac{k(-1)^k}{2t_k^2}\right]\right.$$
$$\left.+\exp\left(\frac{(\beta-\alpha)\pi}{2\varepsilon}\right)\left[\frac{k}{p_y^2+k^2}-\frac{1}{k}\right]+\exp(p_x\pi)\left[\frac{(-1)^k}{2k}+\frac{k(-1)^k}{2t_k^2}-\frac{k(-1)^k}{p_y^2+k^2}\right]\right\}, \quad (77)$$

where $p_x = -\alpha/2\varepsilon$ and $p_y = -\beta/2\varepsilon$. Truncating the series (73), we have

$$u(x,y) \approx \bar{u}(x,y) + \exp\left(\frac{-\alpha(\pi-x)+\beta y}{2\varepsilon}\right)\sum_{k=1}^{N} a_k \frac{\sinh(t_k y)}{\sinh(t_k\pi)}\sin kx$$
$$+\exp\left(\frac{\alpha x-\beta(\pi-y)}{2\varepsilon}\right)\sum_{k=1}^{N} c_k \frac{\sinh(t_k x)}{\sinh(t_k\pi)}\sin ky.$$
$$=: \bar{u}(x,y) + u_N^{(1)}(x,y) + u_N^{(2)}(x,y). \quad (78)$$

Note that since, in this case, $S=[0,\pi]^2$ rather than $[0,2\pi]^2$, the assumption A1 should be modified as $f(0)=f(\pi)=0$. Application of Theorem 3.1 to the second term, $u_N^{(1)}$, on the right-hand side in (78) with $\beta \leqslant \alpha$ gives

$$|u-u_N^{(1)}| \leqslant \frac{C}{\varepsilon^2 N^2}\exp\left(\frac{\alpha(x-\pi)+\beta(y-\pi)}{2\varepsilon}\right)\exp\left(\frac{\alpha\pi}{2\varepsilon}-t_N(\pi-y)\right), \quad (79)$$

$$|u_x-(u_N^{(1)})_x| \leqslant \frac{C}{\varepsilon^2 N}\exp\left(\frac{\alpha(x-\pi)+\beta(y-\pi)}{2\varepsilon}\right)\exp\left(\frac{\alpha\pi}{2\varepsilon}-t_N(\pi-y)\right), \quad (80)$$

$$|u_y-(u_N^{(1)})_y| \leqslant \frac{C}{\varepsilon^2 N}\exp\left(\frac{\alpha(x-\pi)+\beta(y-\pi)}{2\varepsilon}\right)\exp\left(\frac{\alpha\pi}{2\varepsilon}-t_N(\pi-y)\right), \quad (81)$$

where C is a generic positive constant independent of ε and N. When $\alpha=\beta$ and $x+y \leqslant \pi$, (79)–(81) become

$$|u-u_N^{(1)}| \leqslant \frac{C}{\varepsilon^2 N^2}\exp(-t_N(\pi-y)),$$

$$|u_x-(u_N^{(1)})_x| \leqslant \frac{C}{\varepsilon^2 N}\exp(-t_N(\pi-y)),$$

$$|u_y-(u_N^{(1)})_y| \leqslant \frac{C}{\varepsilon^2 N}\exp(-t_N(\pi-y)).$$

Similarly, we have the following bounds for the third term on the right-hand side in (78)

$$|u-u_N^{(2)}| \leqslant \frac{C}{\varepsilon^2 N^2}\exp(-t_N(\pi-x)),$$

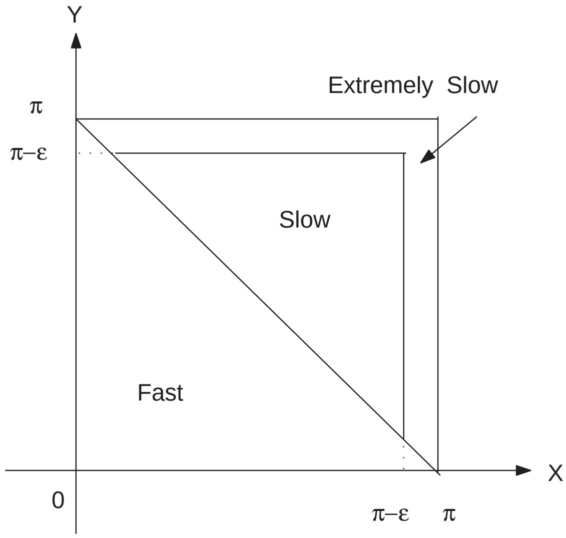

Fig. 4. Regions of different convergences for Model I.

$$|u_x - (u_N^{(2)})_x| \leqslant \frac{C}{\varepsilon^2 N}\exp(-t_N(\pi - x)),$$

$$|u_y - (u_N^{(2)})_y| \leqslant \frac{C}{\varepsilon^2 N}\exp(-t_N(\pi - x)).$$

From the above estimates on the errors in $u_N^{(1)}, u_N^{(2)}$ and their derivatives we see that the convergence of (78) on the region $S \cap (x + y \leqslant \pi)$ is fast. Moreover, (65) for the derivatives at $u_x(\pi - \mu\varepsilon^{1+q}, \pi - \mu\varepsilon^{1+q})$ should be modified as

$$N \geqslant \frac{\alpha\pi/2\varepsilon + \ln C + M}{\mu\varepsilon^{1+q}} = \mathrm{O}(\varepsilon^{-(2+q)}). \tag{82}$$

The convergence of the solutions and derivatives is still slow at points near the layer. In particular, the slowest convergence occurs in the approximation of the normal derivative u_n at the corner (π, π). The convergence rates of solutions and derivatives in Model I vary significantly at different locations of S. Fig. 4 illustrates different convergence rates in different sub-regions of S for $\alpha = \beta (> 0)$. However, from our extensive computational experience we notice that only the computation of the derivatives near the corner (π, π) suffers from poor convergence.

We now perform some numerical experiments for Model I using the expression in (78). To do so, we choose $\alpha = \beta = c = 1$, $\delta = 10^{-3}$, and $\varepsilon = 0.1$–0.002. The implementation details and results will be reported in a future paper, and here we only present some of the numerical results, as given in Table 1 and Fig. 5, to illustrate the solutions for $\varepsilon = 0.1$. Let us consider the solution behavior using the results. First, we examine the solutions along the horizontal line of $y = \pi/2$. The solutions

Table 1
The solution values near boundary along $y=\pi/2$ for Model I with $\varepsilon=0.1$

x	$1/2$	$\pi/2$	$\pi-5\varepsilon$	$\pi-4\varepsilon$	$\pi-3\varepsilon$
u	8.1903(−9)	5.6825(−8)	4.2388(−3)	1.2660(−2)	3.7771(−2)
u_x	1.7785(−8)	3.3622(−7)	4.6407(−2)	1.3844(−1)	4.1273(−1)
u_y	1.1053(−7)	3.3622(−7)	7.4644(−5)	1.2158(−4)	1.7770(−4)
x	$\pi-2\varepsilon$	$\pi-\varepsilon$	$\pi-\varepsilon^2$	$\pi-\varepsilon^3$	π
u	1.1261(−1)	3.3562(−1)	8.9657(−1)	9.8914(−1)	1.0000
u_x	1.2299	3.6645	9.7880	10.798	10.917
u_y	2.2044(−4)	1.9552(−4)	3.1418(−5)	3.2872(−6)	0.0000

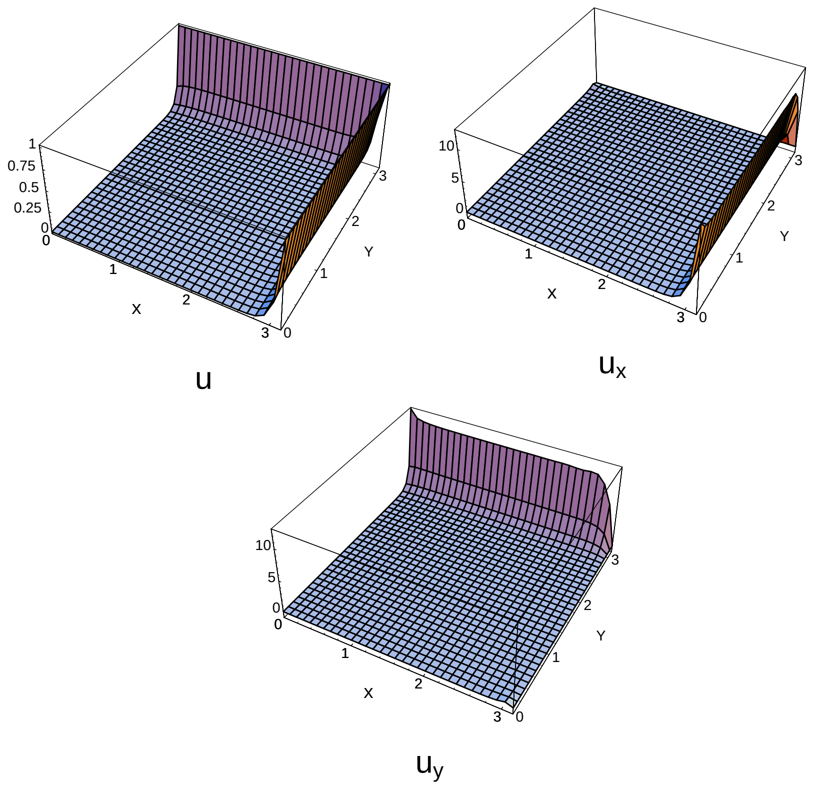

Fig. 5. The computed solution u and derivatives u_x and u_y to Model I with $\varepsilon=0.1$.

and the derivatives along this segment are computed and listed in Table 1. The value $u_y(\pi,\pi/2)=0$ is obtained directly from the boundary condition $u=0$ at $x=\pi$, and $u_x(\pi,\pi/2)$ is computed from (78). From the discrete values listed in Table 1 we see that the width of boundary layer at $x=\pi$ is about 2ε.[1] More observations can be seen from Table 1 as follows. The values of the derivative u_y parallel to the line $x=\pi$ are very small, due to facts that $\max|u_y| \leqslant 2.2\times 10^{-4}$, while the normal derivatives within the 2ε neighborhood of the boundary layer are large, due to $|u_n| = |u_x| \geqslant 1.23$ when $x \geqslant \pi - 2\varepsilon$. Moreover, the normal derivatives u_x increase monotonically when x increases, and the maximal normal derivative occurs at the boundary: $u_n(\pi,\pi/2) = u_x(\pi,\pi/2) = 10.917 \approx 1/\varepsilon$. This also illustrates that the normal derivatives are of order $O(1/\varepsilon)$ near the boundary layer. All the above observations agree with the results in [11,14].

5. Model II

While the convergence of the series solution to Model I is slow, we consider another special case to which the series solution converges fast. This is our second computational model, called Model II, to be defined below, in which only a few expansion terms of particular functions are needed. This is important to the combined methods explored in [10]. Let us consider the problem $\mathscr{L}u=0$ on $[0,\pi]^2$ with the following Dirichlet conditions,

$$u(x,\pi) = u(\pi,y) = 0, \tag{83}$$

$$u(x,0) = \gamma \exp(p\cos x)\sin(p\sin x)\exp\left(-\frac{\alpha(\pi-x)}{2\varepsilon}\right) = g_1(x), \tag{84}$$

$$u(0,y) = \gamma \exp(p\cos y)\sin(p\sin y)\exp\left(-\frac{\beta(\pi-y)}{2\varepsilon}\right) = g_3(y), \tag{85}$$

where $p^2 \leqslant 1$, and γ is a parameter used to adjust the solutions to be O(1). The solution is given by

$$u(x,y) = \sum_{n=1}^{\infty} (a_n\phi_n(x,y) + b_n\psi_n(x,y)), \tag{86}$$

where

$$\phi_n(x,y) = \exp\left(\frac{-\alpha(\pi-x)+\beta y}{2\varepsilon}\right)\frac{\sinh(t_n(\pi-y))}{\sinh(t_n\pi)}\sin nx,$$

$$\psi_n(x,y) = \exp\left(\frac{\alpha x-\beta(\pi-y)}{2\varepsilon}\right)\frac{\sinh(t_n(\pi-x))}{\sinh(t_n\pi)}\sin ny, \tag{87}$$

[1] Strictly speaking, the width of the boundary layer is a little larger than 2ε by noting $u(\pi-2\varepsilon,\pi/2)=0.11261$ and $u(\pi-3\varepsilon,\pi/2)=0.037771$ given in Table 1. Such a numerical observation coincides perfectly with the theoretical width of one dimensional problems in Miller et al. [11, p. 7],

$$\varepsilon \ln\frac{1}{\varepsilon} = \varepsilon \ln 10 = 2.3\varepsilon.$$

The reason for this remarkable coincidence is that the solution along the horizontal segment from $(\pi-5\varepsilon,\pi/2)$ to $(\pi,\pi/2)$ can be regarded approximately as that of a one dimensional problem, due to the fact that it is far from the boundary segments $y=0$ and $y=\pi$.

and

$$t_n = \left(n^2 + \frac{\alpha^2 + \beta^2 + 4\varepsilon c}{4\varepsilon^2}\right)^{1/2}.$$

Based on the formula in [5], i.e.,

$$\sum_{n=1}^{\infty} \frac{p^n \sin nx}{n!} = \exp(p\cos x)\sin(p\sin x), \quad p^2 \leqslant 1,$$

we derive from the orthogonality of trigonometric functions that

$$a_n = b_n = \gamma \frac{p^n}{n!}. \tag{88}$$

Let us find the value of γ in (88). First we conclude that the maximal solutions in (86) occurs on the boundary Γ. This can be justified by contradiction. Suppose that a maximal value of $u(x, y)$ occurs at an interior point $P \in S$. Then $u_x(P) = u_y(P) = 0$ and $\Delta u(P) < 0$. Since $u \geqslant 0$ and $c \geqslant 0$ we obtain from (1)

$$\begin{aligned}\mathscr{L}u &= \frac{\partial}{\partial x}\left(-\varepsilon \frac{\partial}{\partial x} u + \alpha u\right) + \frac{\partial}{\partial y}\left(-\varepsilon \frac{\partial}{\partial y} u + \beta u\right) + cu \\ &= -\varepsilon \Delta u + cu > 0\end{aligned}$$

at P. This contradicts the fact $\mathscr{L}u = 0$ in S.

Next, we seek the maximal value on the x axis. Consider the function $g_1(x)$ in (84). The maximal value occurs when $\partial g_1(\bar{x})/\partial x = 0$, where

$$\begin{aligned}\frac{\partial g_1(x)}{\partial x} &= \gamma \exp(p\cos x)\exp\left(-\frac{\alpha(\pi - x)}{2\varepsilon}\right) \\ &\quad \times \left\{\left(-p\sin x + \frac{\alpha}{2\varepsilon}\right)\sin(p\sin x) + \cos(p\sin x)p\cos x\right\} = 0.\end{aligned} \tag{89}$$

Let $\bar{x} = \pi - v\varepsilon$ with a positive constant v. Since $v\varepsilon \ll 1$, we have

$$\sin(\pi - v\varepsilon) \approx v\varepsilon \quad \text{and} \quad \cos(\pi - v\varepsilon) \approx -\left(1 - \frac{(v\varepsilon)^2}{2}\right),$$

and so from (89) we have

$$\left(-pv\varepsilon + \frac{\alpha}{2\varepsilon}\right)(pv\varepsilon) - p \approx 0.$$

This gives an approximation $v = 2/\alpha$, and then $\bar{x} = \pi - (2/\alpha)\varepsilon$. We choose $g_1(\bar{x}) \approx 1$, so that

$$g_1(\bar{x}) \approx 2\gamma p \frac{\varepsilon}{\alpha} \exp(-(\alpha + p)) = 1,$$

and then let

$$\gamma = \frac{\alpha}{2p\varepsilon}\exp(p + \alpha).$$

Similarly, for (85), we have $\gamma = \exp(p + \beta)\beta/2p\varepsilon$. Suppose that $\beta \leqslant \alpha$, without loss of generality. Then we may choose

$$\gamma = \frac{\alpha}{2p\varepsilon}\exp(p + \alpha).$$

Truncating (86), we have

$$u_N(x,y)=\sum_{n=1}^{N}(a_n\phi_n(x,y)+b_n\psi_n(x,y)), \tag{90}$$

where $\phi_n(x,y)$ and $\psi_n(x,y)$ are given in (87), and a_n and b_n in (88). The upper bound for the error in u_N is given in the following theorem.

Theorem 5.1. *For the solutions of Model* II *with* $p^2\leqslant 1$, *there exists the error bound,*

$$|u-u_N|\leqslant 2\gamma\,\frac{p^{N+1}}{N!N}. \tag{91}$$

Moreover, if $\beta\leqslant\alpha$, *and* $N\leqslant 1/\varepsilon$ *for* $\varepsilon\ll 1$, *then there exist the bounds,*

$$|u_x-(u_N)_x|\leqslant C_1\left(\frac{\gamma}{\varepsilon}\right)\frac{p^{N+1}}{N!N},\quad |u_y-(u_N)_y|\leqslant C_1\left(\frac{\gamma}{\varepsilon}\right)\frac{p^{N+1}}{N!N}, \tag{92}$$

where $C_1=4(2+\alpha)$.

Proof. Taking absolute value, we have

$$|\phi_k(x,y)|=\exp\left(\frac{-\alpha(\pi-x)+\beta y}{2\varepsilon}\right)\left|\frac{\sinh(t_k(\pi-y))}{\sinh(t_k\pi)}\right||\sin kx|. \tag{93}$$

For $0\leqslant y\leqslant\pi$,

$$\left|\frac{\sinh(t_k(\pi-y))}{\sinh(t_k\pi)}\right|=\frac{\exp(t_k(\pi-y))}{\exp(t_k\pi)}\times\frac{1-\exp(-2t_k(\pi-y))}{1-\exp(-2t_k\pi)}\leqslant\exp(-t_k y).$$

Using this, we obtain from (93)

$$\begin{aligned}|\phi_k(x,y)|&\leqslant\exp\left(\frac{-\alpha(\pi-x)+\beta y}{2\varepsilon}\right)\exp(-t_k y)\\&\leqslant\exp\left(\frac{-\alpha(\pi-x)}{2\varepsilon}\right)\exp\left(-\left(t_k-\frac{\beta}{2\varepsilon}\right)y\right)\leqslant 1,\end{aligned} \tag{94}$$

by noting $t_k>\beta/2\varepsilon$ in (19). Similarly,

$$|\psi_k(x,y)|\leqslant 1. \tag{95}$$

Using (86), (90), (94), (95) and (88), we obtain

$$\begin{aligned}|u-u_N|&\leqslant\sum_{k=N+1}^{\infty}\{a_k|\phi_k(x,y)|+b_k|\psi_k(x,y)|\}\\&\leqslant 2\sum_{k=N+1}^{\infty}a_k=2\gamma\sum_{k=N+1}^{\infty}\frac{p^k}{k!}.\end{aligned} \tag{96}$$

Moreover, we have from $p^2 \leqslant 1$

$$\begin{aligned}\sum_{k=N+1}^{\infty} \frac{p^k}{k!} &= \frac{p^N}{N!}\left\{\frac{p}{N+1} + \frac{p^2}{(N+1)(N+2)} + \cdots\right\}\\ &\leqslant \frac{p^N}{N!}\sum_{n=1}^{\infty}\left\{\frac{p}{N+1}\right\}^n = \frac{p^N}{(N!)}\frac{p/(N+1)}{1-p/(N+1)}\\ &= \frac{p^N}{(N!)}\frac{p}{N+(1-p)} \leqslant \frac{p^{N+1}}{N!N}. \end{aligned} \tag{97}$$

Eqs. (96) and (97) lead to the first desired result in (91).

Next, we have from (86) and (90),

$$\begin{aligned} u_x - (u_N)_x = &\sum_{k=N+1}^{\infty}\left\{\left(\frac{\alpha}{2\varepsilon}\right)a_k\phi_k(x,y) + \left(\frac{\alpha}{2\varepsilon}\right)b_k\psi_k(x,y)\right\}\\ &+\sum_{k=N+1}^{\infty}\left\{a_k k\exp\left(\frac{-\alpha(\pi-x)+\beta y}{2\varepsilon}\right)\frac{\sinh(t_k(\pi-y))}{\sinh(t_k\pi)}\cos kx\right.\\ &\left.-t_k b_k\exp\left(\frac{\alpha x-\beta(\pi-y)}{2\varepsilon}\right)\frac{\cos(t_k(\pi-x))}{\sinh(t_k\pi)}\sin ky\right\}. \end{aligned} \tag{98}$$

Moreover, for $k \geqslant N$,

$$\begin{aligned}\left|\frac{\cosh(t_k(\pi-y))}{\sinh(t_k\pi)}\right| &= \frac{\exp(t_k(\pi-y))}{\exp(t_k\pi)} \times \frac{1+\exp(-2t_k(\pi-y))}{1-\exp(-2t_k\pi)}\\ &\leqslant 3\exp(-t_k y). \end{aligned} \tag{99}$$

We obtain from $t_k \geqslant \alpha/2\varepsilon$ and $t_k \geqslant \beta/2\varepsilon$,

$$\begin{aligned} |u_x - (u_N)_x| \leqslant &\sum_{k=N+1}^{\infty}\left(\frac{\alpha}{2\varepsilon}\right)\{a_k|\phi_k(x,y)| + b_k|\psi_k(x,y)|\}\\ &+\sum_{k=N+1}^{\infty}\left\{a_k k\exp\left(\frac{-\alpha(\pi-x)}{2\varepsilon}\right)\exp\left(-\left(t_k-\frac{\beta}{2\varepsilon}\right)y\right)|\cos kx|\right.\\ &\left.+3t_k b_k\exp\left(\frac{-\beta(\pi-y)}{2\varepsilon}\right)\exp\left(-\left(t_k-\frac{\alpha}{2\varepsilon}\right)x\right)|\sin kx|\right\}\\ \leqslant &\sum_{k=N+1}^{\infty}\left\{\left(\frac{\alpha}{2\varepsilon}\right)(a_k+b_k) + ka_k + 3t_k b_k\right\}. \end{aligned} \tag{100}$$

From $\beta \leqslant \alpha$ and $\sqrt{x^2+y^2} \leqslant |x|+|y|$, we obtain

$$t_k = \sqrt{k^2 + \frac{\alpha^2+\beta^2+4\varepsilon c}{4\varepsilon^2}} = \sqrt{k^2 + \frac{2\alpha^2+4\varepsilon c}{4\varepsilon^2}}$$
$$\leqslant k + \frac{\sqrt{2}\alpha + 2\sqrt{\varepsilon c}}{2\varepsilon} \leqslant k + \frac{\alpha}{\varepsilon}. \tag{101}$$

Hence we obtain from (100), (101) and (88)

$$|u_x - (u_N)_x| \leqslant \sum_{k=N+1}^{\infty} \left\{ \left(\frac{\alpha}{2\varepsilon}\right)(a_k+b_k) + k a_k + 3\left(k+\frac{\alpha}{\varepsilon}\right) b_k \right\}$$
$$= 4\frac{\alpha}{\varepsilon}\sum_{k=N+1}^{\infty} a_k + 4\sum_{k=N+1}^{\infty} k a_k$$
$$= 4\gamma\frac{\alpha}{\varepsilon}\sum_{k=N+1}^{\infty}\frac{p^k}{k!} + 4\gamma\sum_{k=N+1}^{\infty}\frac{p^k}{(k-1)!}. \tag{102}$$

Now, using $p^2 \leqslant 1$ and $N \geqslant 1$, we have

$$\sum_{k=N+1}^{\infty}\frac{p^k}{(k-1)!} = \frac{p^{N+1}}{N!}\left\{1 + \frac{p}{N+1} + \frac{p^2}{(N+1)(N+2)} + \cdots\right\}$$
$$\leqslant \frac{p^{N+1}}{N!}\sum_{n=0}^{\infty}\left\{\frac{p}{N+1}\right\}^n = \frac{p^{N+1}}{(N!)}\,\frac{1}{1-p/N+1}$$
$$= \frac{p^{N+1}}{(N!)}\,\frac{N+1}{N+(1-p)} \leqslant \frac{p^{N+1}}{(N!)}\,\frac{N+1}{N}$$
$$\leqslant 2\,\frac{p^{N+1}}{N!}. \tag{103}$$

Noting that $N\varepsilon \leqslant 1$ for $\varepsilon \ll 1$, we obtain from (102), (103) and (97)

$$|u_x - (u_N)_x| \leqslant 4\gamma\left(2+\frac{\alpha}{N\varepsilon}\right)\frac{p^{N+1}}{N!}$$
$$\leqslant 4(2+\alpha)\frac{\gamma}{\varepsilon}\frac{p^{N+1}}{N!N} = C_1\frac{\gamma}{\varepsilon}\frac{p^{N+1}}{N!N}. \tag{104}$$

This is the left-hand result in (92), and the proof for the result on the right-hand side in (92) is similar. □

Let us consider the minimum number of terms, N, which guarantees that the error in the approximation $(u_N)_x \leqslant \delta$ for a given (small) positive number δ. Clearly, it is required that

$$|u_x - (u_N)_x| \leqslant C_1\left(\frac{\gamma}{\varepsilon}\right)\frac{p^{N+1}}{N!N} \leqslant \delta.$$

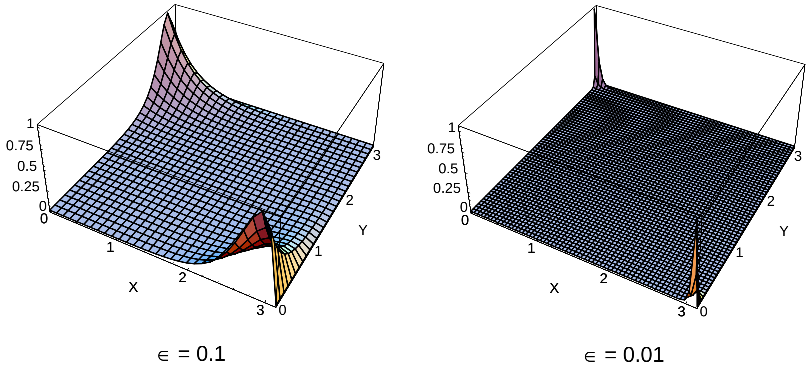

Fig. 6. The computed solutions to Model II with different values of ε.

From the Stirling formula in [1, p. 257], we have

$$N! = \left(\frac{N}{e}\right)^N \sqrt{2\pi N}\left(1 + \frac{1}{12N} + \mathrm{O}\left(\frac{1}{N^2}\right)\right). \tag{105}$$

Using this, the above inequality becomes

$$\frac{C_1}{\sqrt{2\pi N}}\left(\frac{\gamma}{e}\right)\left(\frac{ep}{N}\right)^{N+1} \leqslant \varepsilon\delta.$$

From this we see that we may choose $N \ll 1/\varepsilon$. When $\delta = 10^{-3}$ and $\varepsilon \ll 1$, N needed is very small. In fact, $N = 15$ is good enough for $\varepsilon = 10^{-7}$.

Let us now consider the computation of the series solution (90). We choose $p=1$ and $\alpha=\beta=c=1$. We also choose $\delta = 10^{-3}$, and N is chosen to be in the range $(10, 15)$ for $\varepsilon = 0.1, 0.01, 10^{-4}$ and 10^{-7}. The profiles of the numerical solutions are plotted in Fig. 6. From the figure it is seen that the solutions contain two towers at $(\pi - 2\varepsilon, 0)$ and $(0, \pi - 2\varepsilon)$ with the height close to one. The areas of the cross-sections of two towers are $\mathrm{O}(\varepsilon^2)$. Hence, when ε is very small, the towers become two spikes. From Fig. 6 it is also seen that the solution profiles are like wedding gauze on a bride.

6. Concluding remarks

To conclude this paper, let us give a few remarks.

(1) The motivation of this paper is to discover particular solutions of singularly perturbed differential equations in simple cases, i.e., those with constant coefficients in rectangular domains. These explicit particular solutions are important for exploring the solution behavior, see Babuska and Zhang [2]. In Section 2, a number of particular solutions of (1) have been derived. From the given particular solutions, one may design deliberately some useful models which may be used to test other numerical methods.

(2) A Convergence analysis is given in Sections 3 and 5 for Model II. Considering rather arbitrary values given in the Dirichlet condition on Γ, when choose the solutions with the first leading N terms of particular solutions, Theorems 3.1 and 3.2 confirm the convergence of the first N

leading terms of particular solutions as $N \to \infty$. However, in real numerical simulations, only a moderately small ε can be used due to slow convergence. In our numerical experiments, if we require that the absolute errors of solutions and derivatives are less than $\delta = 10^{-3}$, we may choose $N = O(10^4)$ for $\varepsilon = 0.01$, to obtain the solutions over the sub-region $S^* = \{(x, y),\ 0 \leqslant x \leqslant \pi - \varepsilon,\ 0 \leqslant y \leqslant \pi - \varepsilon\}$. Of course, it is possible to increase N up to, say, $N = O(10^8)$, so that ε may be chosen as small as $\varepsilon = 10^{-4}$. We will report these results in detail in a future paper.

(3) We have designed two models: Model I with waterfalls profiles of the solutions, and Model II with wedding-gauze profiles of the solutions. The convergence of the series solutions to Model I is slow and valid for $\varepsilon = 0.1$–10^{-4}, but the convergence for Model II is fast and valid for $\varepsilon = 0.1$–10^{-7}. In practice, the study of problems with $\varepsilon = 0.1$–0.01 is often useful, and easy to illustrate the changes of solutions with respect to those of the parameters α, β, and c. Some numerical simulations of solutions in Models I and II have been given in this paper, which provide intrinsic characteristics of solutions to the test problems.

(4) Since the numerical values of $\exp((\alpha x + \beta y)/2\varepsilon)$ change in a huge range when $\varepsilon \leqslant 0.1$, Mathematica with unlimited working digits needs to be used for Model I. In this case, the rounding errors may be ignored. More detailed numerical techniques and computational results will be reported in a forthcoming paper. Nevertheless, the computation of Model I with $\varepsilon \geqslant 0.1$ and Model II with $\varepsilon \geqslant 10^{-7}$ may be carried out in double precision.

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