Adomian's method of decomposition: critical review and examples of failure

Paul NELSON *
Department of Applied Mathematics, California Institute of Technology, Pasadena, CA 91125, U.S.A.

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Abstract: Adomian's method of decomposition is considered in application to initial-boundary value problems for the one space-dimensional spatially homogeneous heat conduction equation. It is shown that the fundamental equation of the method is well-defined only for certain restricted types of boundary conditions. Within the class of such boundary conditions, examples are given such that the fundamental equation fails to have a unique solution, and such that the sequence produced by iteration of this equation is divergent. The latter is a counterexample to a published assertion of convergence.

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In a remarkable series of recent publications [1-26], G. Adomian and co-workers have described potential use of a technique that he terms the "method of decomposition" for a wide variety of applications. Many of these applications involve stochastic or nonlinear considerations, but in [14] and Sections 9.5 and 9.6 of [23] the application considered is simple deterministic linear heat conduction. We similarly limit the considerations of the present note, and even further to the one space-dimensional spatially homogeneous (Fourier) heat conduction equation. We now describe, within this context, our understanding of the method of decomposition.

We follow [14] in writing the underlying differential equation as

\[ L_t u - L_x u + g, \]

where \( L_t = \partial / \partial t \), \( L_x = \partial^2 / \partial x^2 \), and \(-g\) is the source term. We suppose that \( u(0, t)\) is specified, and that further auxiliary conditions (e.g. boundary conditions) are imposed on \( u \), under which
(1) has a unique solution. Let us define operators \( \hat{L}_t \) and \( \hat{L}_x \) so that \( \hat{L}_t f(x, t) \) is the definite integral of \( f \), relative to its second argument, from 0 to \( t \), and \( \hat{L}_x f(x, t) \) is the indefinite two-fold integral \(^1\) of \( f \), relative to its first argument. If we apply \( \hat{L}_x \) to (1), the result can be written as

\[
\begin{align*}
    u &= \hat{L}_x u + \hat{L}_x g + \gamma_1(t) + \gamma_2(t)x,
\end{align*}
\]

where \( \gamma_1 \) and \( \gamma_2 \) are to be determined from the auxiliary conditions. Similarly, we obtain

\[
\begin{align*}
    u &= \hat{L}_t L_x u - \hat{L}_x g + u(\cdot, 0).
\end{align*}
\]

Upon adding (2) and (3), and solving the result for \( u \), we obtain

\[
\begin{align*}
    u = K u + u_0,
\end{align*}
\]

where

\[
\begin{align*}
    u_0 := \frac{1}{2} \left\{ \left( \hat{L}_x - \hat{L}_t \right) g + \gamma_1(t) + \gamma_2(t)x + u(\cdot, 0) \right\}
\end{align*}
\]

and the operator \( K \) is given by

\[
\begin{align*}
    K := \frac{1}{2} \left[ \hat{L}_x L_t + \hat{L}_t L_x \right].
\end{align*}
\]

We shall term (4a) as the "fundamental equation of Adomian's method of decomposition." By iteration of (4a), we find that

\[
\begin{align*}
    \phi_n := \sum_{i=0}^{n-1} u_i
\end{align*}
\]

is an approximation to \( u \), where

\[
\begin{align*}
    u_{n+1} := Ku_n.
\end{align*}
\]

In [14] it is asserted that convergence of the \( \phi_n \) thus defined is established in [10,23]. In the following we give a counterexample to this assertion. However, before discussing such details, let us present a brief critical overview of the procedure described in the preceding paragraph, with emphasis on how the contents of the remainder of this note relate to this procedure.

Recall that we have assumed (1) has a unique solution satisfying the specified auxiliary conditions. It follows that if \( \hat{L}_x \) can be specified so that \( \gamma_1 \) and \( \gamma_2 \) can be determined in terms of the data of these auxiliary conditions, if (4) has a unique solution, if the sequence \( \{ \phi_n \} \) defined by (5) converges, and if the sense of convergence permits passage to the limit in

\[
\begin{align*}
    \phi_{n+1} := K \phi_n + u_0,
\end{align*}
\]

then indeed \( u = \lim_{n \to \infty} \phi_n \), where \( u \) is the specified solution of (1). An affirmative answer to the four issues highlighted by the italicized if’s in the preceding sentence would seem to be necessary (and sufficient) for applicability of the method of decomposition to any particular problem. The basic purpose of this note is to present the results of an initial study of these issues.

Specifically, an outline of the remainder of this note is as follows. First we show that the requirement that \( \gamma_1 \) and \( \gamma_2 \) be determinable from the auxiliary conditions requires that the given

\(^1\) More precisely, we take it as some definite two-fold integral of \( f \), where the lower limits of these integrals are to be selected in order to be able to compute \( u_0 \), as given by (4b), from the boundary conditions. The precise definition of \( \hat{L}_x \) (there written \( L_{x-1} \)) intended in [14] is not made clear, but on p. 136 of [23] one finds \( L_{x-1} = \int_0^x \int_0^t dx' dx \), which presumably restricts the lower limits to the left end of the spatial interval under consideration. Our approach is more general, but nonetheless we find the method is quite restricted as regards the types of boundary condition for which it is applicable.
initial value be supplemented by data of the rather restricted form of specified values for \( u(b_1, t) \) and \( (\partial u/\partial x)(b_2, t) \), for some values of \( b_1 \) and \( b_2 \). Second, we present an example of a problem corresponding to boundary conditions of this type, such that the fundamental equation (4a) does not have a unique solution. Third, we present a (related) example of an initial-boundary value problem, for (1) on \((x, t) \in [0, \pi/2] \times [0, \infty)\), such that the corresponding \( \phi_n(x, t) \) diverge, in any of a wide variety of senses. We do not consider questions relating to passage to the limit in (6), inasmuch as the third result just outlined suggests this question is irrelevant until (and unless) convergence in some sense is established for a reasonably general class of problem.

From the above definition, \( \hat{L}_x \) is of the form

\[
\hat{L}_x f(x, t) = \int_{b_1}^{x} \int_{b_2}^{x_1} f(x_2, t) \, dx_2 \, dx_1,
\]

where the points \( b_1, b_2 \) can be selected as is convenient for the problem at hand. In the present context the overriding consideration is that of being able to compute \( \gamma_1 \) and \( \gamma_2 \) from the auxiliary data, so that (4b) indeed expresses \( u_0 \) in terms of known functions. The manipulations leading to (2) give

\[
\gamma_1(t) = u(b_1, t) - \frac{\partial u}{\partial x}(b_2, t) b_1, \quad \gamma_2(t) = \frac{\partial u}{\partial x}(b_2, t).
\]

(7a, b)

It clearly follows that, for unknown \( u, \gamma_1 \) and \( \gamma_2 \) are determined by the boundary conditions only if these are of the form

\[
\frac{\partial u}{\partial x}(b_2, t) = g_2(t), \quad u(b_1, t) = g_1(t),
\]

(8a, b)

for some known functions \( g_1 \) and \( g_2 \). Of course if \( u \) is known a priori, then \( \gamma_1 \) and \( \gamma_2 \) can be determined from (7), but this approach hardly seems to qualify as a 'method'.

For the reason just described, we henceforth assume the given initial value for \( u \) is supplemented by data of the form (8). In this case (4a) constitutes a well-defined problem for the unknown \( u \), with \( u_0 \) and \( K \) known respectively from (4b) and (4c). The appropriate issue now becomes whether or not (4a) is well-posed; that is, we must examine the questions of existence, uniqueness and continuous dependence upon the data for (4a). The existence question obviously has an affirmative answer, as we have assumed that (1) has a unique solution, subject to (8) and the specified initial value, and the manipulations leading to (4) show that this also is a solution of (4a). Uniqueness is equivalent to the assertion that \( \lambda = 1 \) is not an eigenvalue of the operator \( K \).

In fact this assertion cannot generally be true, as straightforward calculations show that

\[
\omega(t, x) := (1 - t) e^{-t} \sin x
\]

(9)

is an eigenfunction of \( K \), with associated eigenvalue unity, in the case \( b_1 = 0, b_2 = \pi/2 \). The latter correspond to boundary conditions of the form

\[
u(0, t) = g_1(t), \quad \frac{\partial u}{\partial x}(\pi/2, t) = g_2(t).
\]

Of course this leaves open the possibility that the fundamental equation (4a) may have a unique solution in other cases, but it does show that the unicity issue cannot simply be assumed away.

We note that a negative answer to the uniqueness question need not be fatal to the utility of the method of decomposition. It is (just) conceivable that the sequence \( \{ \phi_n \} \), as defined by (5),
will somehow converge to the solution of (4a) that is of interest, even in the presence of other solutions. Specifically, we give a detailed presentation of the counterexample to this converge that was promised earlier.

Consider the one-dimensional heat equation

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + (-t^2 + 4t - 2) e^{-t} \sin x
\]

on \(x \in [0, \pi/2], \ t > 0\), subject to the initial condition

\[u(x, 0) = 0,\]

and the boundary conditions

\[u(0, t) = u(\pi/2, t) = 0.\]

A solution of this problem is

\[u(x, t) = (\frac{3}{2} t^3 - 2t^2 + 2t) e^{-t} \sin x,\]

and it is readily proved that it is only solution. The corresponding value of \(u_0\) can be computed, from (4b) and (7), as \(u_0 = w\), where \(w\) is given by (9). It follows from the properties of \(w\) established above that \(u_n = w\) for all \(n\), and hence

\[\phi_n(x, t) = nw(x, t) = n(1 - t) e^{-t} \sin x.\]

Thus \(\phi_n(x, t)\) diverges (pointwise) for \((x, t) \in (0, \pi/2) \times ([0, \infty) - \{1\})\), and also in any other useful sense of which the author is aware.

There arises then the question of where lies gap in the convergence result for the \(\{\phi_n\}\) that asserted in [14] to have been established in [10,23]. In [23] (cf. p. 223 and §11.4) one again is referred to [10] for such proof. But the iteration operator \((K)\) used in [10] is very different from that of [14]. Specifically, in [10] we find \(K = \hat{L}_tN\), where \(N\) is a nonlinear operator, and the plausibility argument for convergence that is given in [10] (for a particular \(N\)) seems to rely heavily, but tacitly, on the same considerations that are well-known to imply Volterra integral operators have zero spectral radius. These same considerations do not apply for the \(K\) of [14], as the above results show.

I would like to conclude this work by expressing the opinion that Adomian's method of decomposition has not received the appropriate amounts of either attention from mathematicians or space in the mathematical literature. I hope the present work will constitute an initial step toward remedying that situation.

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References


