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On the Computation of the Values of Zeta Functions of Totally Real Cubic Fields

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Based on earlier papers of the first author we give a concise formula for the values of class zeta functions of totally real cubic fields at even positive integers which is the exact analogue of the Barn–Siegel formula for real quadratic fields. For this purpose we use a rather complicated series representation for the aforementioned values depending on a parameter x which is analyzed for $x \to 0$. The final formula is well suited for actual computations; two tables of values of class zeta functions are given at the end of the paper.

This paper is concerned with the computation of the values of zeta functions of ideal classes of totally real cubic fields K at integral arguments. Given a basis of the integers in K and a system of fundamental units we develop an algorithm which is well suited for the calculation of the aforementioned values. Our starting point is a revised version of the results in [5, 6] (cited as I and II in the sequel). We check our values by comparing them with the results in [9] if K is not normal; if K is normal and $h_K = 1$, we have the well-known factorization of the zeta function into L-functions

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which again enables us to control our results. (Of course, for $h_K = 1$ the Euler product of the zeta function can be used for numerical purposes in both cases.) In principle Shintani's method [10] can be used constructively in either case; however, to our knowledge that has not yet been worked out.

Our method uses a complicated series representation for the zeta values considered. It does not yet work in the case of totally real fields of degree $n \geqslant 4$, because the series representation of the zeta values developed in I and II becomes even more complicated, and there remains also much work to be done on the limiting process $x \to 0$ (compare the proof of Theorem 1).

For brevity's sake we shall use throughout this paper the definitions and proofs of I and II. All symbols not defined here will be found there. The main result (Theorem 1), however, can be understood without recourse to I or II.

THEOREM 1. *Let K be a totally real cubic field with discriminant Δ. The conjugates of K will be denoted by K' and K''; for $\alpha \in K$ the conjugates are denoted by α' and α'', respectively. Furthermore, for $\alpha \in K$ let* $\mathrm{Tr}(\alpha) := \alpha + \alpha' + \alpha''$, $N(\alpha) := \alpha \cdot \alpha' \cdot \alpha''$. *Let* $\hat{K} := K(\sqrt{\Delta})$, $k \in \mathbb{N}$, $k \geqslant 2$, *and* $\varepsilon_1, \varepsilon_2$ *be independent units of K with* $N(\varepsilon_1)^k = N(\varepsilon_2)^k = 1$. *Define L by* $L := \ln |\varepsilon_1/\varepsilon_1''| \ln |\varepsilon_2'/\varepsilon_2''| - \ln |\varepsilon_1'/\varepsilon_1''| \ln |\varepsilon_2/\varepsilon_2''|$. *Let* $\omega_j \in K$, $j = 1, 2, 3$, *be algebraic integers, and* $W := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \oplus \mathbb{Z}\omega_3$ *a module of rank 3 of K with the property* $\varepsilon_1 W = \varepsilon_2 W = W$. *$W$ splits into equivalence classes with respect to* $M = \{\varepsilon_1^m \varepsilon_2^n : (m, n) \in \mathbb{Z}^2\}$ *(note that* $-1 \notin M$*); let W_0 be a system of representatives of* $(W \backslash \{0\})/M : W \backslash \{0\} = \bigcup_{(m,n) \in \mathbb{Z}^2} \varepsilon_1^m \varepsilon_2^n W_0$. *We choose* $\rho \in K$, $\rho \neq 0$, *with the property* $\mathrm{Tr}(\rho\omega_j) \in \mathbb{Z}$, $j = 1, 2, 3$. *Set* $\lambda := \gcd(\mathrm{Tr}(\rho\omega_1), \mathrm{Tr}(\rho\omega_2), \mathrm{Tr}(\rho\omega_3))$ *and*

$$E_j = \begin{pmatrix} 1 & 1 & 1 \\ \varepsilon_j & \varepsilon_j' & \varepsilon_j'' \\ \varepsilon_1\varepsilon_2 & \varepsilon_1'\varepsilon_2' & \varepsilon_1''\varepsilon_2'' \end{pmatrix}, \qquad j = 1, 2, \qquad B_\rho = \begin{pmatrix} \rho\omega_1 & \rho\omega_2 & \rho\omega_3 \\ \rho'\omega_1' & \rho'\omega_2' & \rho'\omega_3' \\ \rho''\omega_1'' & \rho''\omega_2'' & \rho''\omega_3'' \end{pmatrix}.$$

Defining[1]

$$\zeta(k, W, \varepsilon_1, \varepsilon_2) := \sum_{\omega \in W_0} N(\omega)^{-k}$$

and setting for $\tau_1, \tau_2 \in K$, $v = 1, 2$, $M(k, v, \tau_1, \tau_2) := 0$ *if* $\det E_v = 0$, *otherwise*

[1] For $\varepsilon_1, \varepsilon_2$ a system of fundamental units we simply write $\zeta(k, W)$.

$$M(k, \nu, \tau_1, \tau_2)$$
$$:= \operatorname{sign}(L)(-1)^\nu [\hat{K} : \mathbb{Q}]^{-1} \frac{(2\pi i)^{3k}}{(3k)!} N(\rho)^k$$
$$\cdot \sum_{m_1=0}^{3k} \sum_{m_2=0}^{3k} \binom{3k}{m_1, m_2}$$
$$\cdot \left\{ \frac{\det E_\nu}{|\det(E_\nu B_\rho)|^3} B(3, m_1, m_2, 3k-(m_1+m_2), (E_\nu B_\rho)^*, \mathbf{0}) \right.$$
$$\cdot \sum_{\kappa_1=0}^{k-1} \sum_{\kappa_2=0}^{k-1} \sum_{\mu_1=0}^{k-1} \sum_{\mu_2=0}^{k-1} \binom{m_1-1}{k-1-(\kappa_1+\kappa_2), k-1-(\mu_1+\mu_2)}$$
$$\cdot \binom{m_2-1}{\kappa_1, \mu_1} \binom{3k-1-(m_1+m_2)}{\kappa_2, \mu_2}$$
$$\cdot \operatorname{Tr}_{\hat{K}/\mathbb{Q}} (\tau_1^{\kappa_1+\kappa_2} \tau_1'^{\mu_1+\mu_2} \tau_1''^{3k-2-(m_1+\kappa_1+\kappa_2+\mu_1+\mu_2)}$$
$$\left. \cdot \tau_2^{\kappa_2} \tau_2'^{\mu_2} \tau_2''^{3k-1-(m_1+m_2+\kappa_2+\mu_2)}) \right\}$$

for $\det E_\nu \neq 0$,

$$C(k, \nu, \tau_1, \tau_2)$$
$$:= \operatorname{sign}(L)(-1)^{\nu+1} \frac{(2\pi i)^{3k}}{12 \cdot (3k-2)(k-1)!^3}$$
$$\cdot N(\rho)^k \lambda^{-3k+3} \tilde{B}_{3k-2}(0) \, |\det B_\rho|^{-1} \operatorname{sign}(\det E_\nu)$$
$$\cdot \{\operatorname{sign}((\tau_1\tau_2 - \tau_1'\tau_2')(\tau_1 - \tau_1')) + \operatorname{sign}((\tau_1'\tau_2' - \tau_1''\tau_2'')(\tau_1' - \tau_1''))$$
$$+ \operatorname{sign}((\tau_1''\tau_2'' - \tau_1\tau_2)(\tau_1'' - \tau_1)) + \operatorname{sign}(\tau_1''(\tau_1 - \tau_1')(\tau_2' - \tau_2))$$
$$+ \operatorname{sign}(\tau_1(\tau_1' - \tau_1'')(\tau_2'' - \tau_2')) + \operatorname{sign}(\tau_1'(\tau_1'' - \tau_1)(\tau_2 - \tau_2''))$$
$$+ N(\tau_2)[\operatorname{sign}(\tau_1''(\tau_2 - \tau_2')(\tau_1\tau_2 - \tau_1'\tau_2'))$$
$$+ \operatorname{sign}(\tau_1(\tau_2' - \tau_2'')(\tau_1'\tau_2' - \tau_1''\tau_2''))$$
$$+ \operatorname{sign}(\tau_1'(\tau_2'' - \tau_2)(\tau_1''\tau_2'' - \tau_1\tau_2))]\},$$

the following equation holds:

$$\zeta(k, W, \varepsilon_1, \varepsilon_2) = M(k, 1, \varepsilon_1, \varepsilon_2) + M(k, 2, \varepsilon_2, \varepsilon_1)$$
$$+ C(k, 1, \varepsilon_1, \varepsilon_2) + C(k, 2, \varepsilon_2, \varepsilon_1). \qquad (*)$$

Remark 2. (a) For the convenience of the reader we restate the definitions necessary for understanding the assertion of Theorem 1:

(α) For $k, l, m \in \mathbb{Z}$, $\binom{k}{l, m} := k!/l!\, m!\, (k-(l+m))!$, if $k, l, m, k-(l+m) \in \mathbb{N} \cup \{0\}$, $\binom{-1}{l, m} := (-1)^{l+m}\binom{l+m}{l}$, if $l, m \in \mathbb{N} \cup \{0\}$, $\binom{k}{l, m} = 0$ otherwise.

(β) Let $A=(a_{ij})_{n,n}$ a regular (n,n)-matrix with integral coefficients, $\det A\cdot A^{-1}=:(A_{ij})_{n,n}$. Let

$$\tilde{B}_r(x):=\begin{cases}B_r(x-[x]) & r=0 \text{ or } r\geqslant 2 \text{ or } r=1\wedge x\notin\mathbb{Z}\\ 0 & r=1\wedge x\in\mathbb{Z},\end{cases}$$

where $B_r(y)$ is defined as usual by $ze^{yz}(e^z-1)^{-1}=\sum_{r=0}^{\infty}B_r(y)\,z^r/r!$. Then, for $\mathbf{r}=(r_1,...,r_n)\in(\mathbb{N}\cup\{0\})^n$,

$$B(n,\mathbf{r},A,\mathbf{0})=\sum_{\kappa_1=0}^{|\det A|-1}\cdots\sum_{\kappa_n=0}^{|\det A|-1}\prod_{i=1}^{n}\tilde{B}_{r_i}\left(\frac{1}{\det A}\sum_{j=1}^{n}A_{ij}\kappa_j\right).$$

(γ) For a matrix A the transposed matrix is denoted by A^*.

(b) For the zeta function $\zeta(s,K_0)$ of an absolute ideal class K_0 of K we get with $c\in K_0^{-1}$, c integral, $\varepsilon_1,\varepsilon_2$ a system of fundamental units of K, and $k\in\mathbb{N}$,

$$\zeta(2k,K_0)=\tfrac{1}{2}\operatorname{Norm}(c)^{2k}\cdot\zeta(2k,c,\varepsilon_1,\varepsilon_2).$$

(c) In Theorem 1, $C(k,1,\varepsilon_1,\varepsilon_2)+C(k,2,\varepsilon_2,\varepsilon_1)=0$ for ε_j totally positive, $j=1,2$, and $\operatorname{sign}(\det(E_1E_2))=1$.

Contrary to Siegel's formula [11 (22)], Theorem 1 is well suited for actual computations. The required input data of the field K under consideration are

- the coefficients $a,b,c\in\mathbb{Z}$ of a generating equation $f(x)=x^3+ax^2+bx+c=0$, $K=\mathbb{Q}(\alpha)$ for a zero α of f;
- an integral basis $\tilde{\omega}_1,\tilde{\omega}_2,\tilde{\omega}_3$ of K (respectively of W) which for K we assume in the form $\tilde{\omega}_1=1$, $\tilde{\omega}_2=\alpha$, $\tilde{\omega}_3=(d+e\alpha+\alpha^2)/f$ $(d,e,f\in\mathbb{Z})$;
- the coefficients e_1,e_2,e_3, $f_1,f_2,f_3\in\mathbb{Z}$ of two fundamental units $\varepsilon_1=e_1\tilde{\omega}_1+e_2\tilde{\omega}_2+e_3\tilde{\omega}_3$, $\varepsilon_2=f_1\tilde{\omega}_1+f_2\tilde{\omega}_2+f_3\tilde{\omega}_3$ of K.

Also the argument k and sufficiently many Bernoulli numbers (up to the index $3k$) are input. We note that k is always even since the result for odd k is zero.

In the following we only compute $\frac{1}{2}N(W)^k\sqrt{\Delta}^{2k-1}\pi^{-3k}\zeta(k,W)$ for integral ideals W (compare Remark 2(b)). This is not very difficult by the formula of Theorem 1. Essentially all it takes is arithmetic with rational numbers. Of course the numerators and denominators can become quite large and we need to do computations with multiprecision integers. We add a few comments on the calculation of the generalized 3-fold Dedekind sums $B(3,\mathbf{m},A,\mathbf{0})$ and of the traces of power products of conjugates of the fundamental units (computed in the normal extension $K(\sqrt{\Delta})$ of K).

For the computation of the 3-fold Dedekind sums $B(3,\mathbf{m},A,\mathbf{0})$ ($\mathbf{m}\in\mathbb{Z}^3$, $A\in GL(3,\mathbb{Z})$) we make use of Bemerkung 15 and Hilfssatz 16 of II. We note that the matrix A is of the form $(E_j B_\rho)^*$ $(j=1,2)$ in our case (see Theorem 1). We have the freedom to choose $\rho \in K^\times$ quite arbitrarily. We set $\rho = \hat{\omega}_3$ for a dual basis $\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3$ of K subject to

$$\mathrm{Tr}(\tilde{\omega}_i \hat{\omega}_j) = \delta_{ij} \qquad (1 \leqslant i, j \leqslant 3).$$

Hence, the first column of A becomes 0, 0, 1. The remaining entries of A are also easily determined. We compute $\tilde{\varepsilon}_j \tilde{\omega}_i$ in terms of the basis $\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3$ and the corresponding entry a_{ij} of A just becomes the coefficient of $\tilde{\omega}_3$ $(j=2, 3; i=1, 2, 3;$ $\tilde{\varepsilon}_2 \in \{\varepsilon_1, \varepsilon_2\}, \tilde{\varepsilon}_3 = \varepsilon_1 \varepsilon_2)$. We transform A by elementary row operations into an upper triangular matrix of the form

$$\begin{pmatrix} 1 & * & * \\ 0 & \gcd(a_{22}, a_{13}) & * \\ 0 & 0 & \mathrm{lcm}(a_{22}, a_{13}) \end{pmatrix}$$

which we denote by $\tilde{A}$. The Dedekind sum $B(3, \mathbf{m}, \tilde{A}, \mathbf{0}) = B(3, \mathbf{m}, A, \mathbf{0})$ can then be evaluated with formula (b) of II, Hilfssatz 16.

The computation of traces depends on whether the normal closure $\hat{K} = K(\sqrt{\Delta})$ of K coincides with K (K is cyclic) or is an extension of K of degree 2. We simplify the notation by replacing $\mathrm{Tr}_{\hat{K}/\mathbb{Q}}$ by

$$\widehat{\mathrm{Tr}} = \begin{cases} 2\,\mathrm{Tr}_{K/\mathbb{Q}} & \text{for } K \text{ cyclic} \\ \mathrm{Tr}_{\hat{K}/\mathbb{Q}} & \text{for } K \neq \hat{K}. \end{cases}$$

Clearly, $\tau'' = N(\tau)/(\tau\tau')$ for $\tau \in K^\times$ can be expressed in terms of $\tilde{\omega}_i, \tilde{\omega}_i'$ $(1 \leqslant i \leqslant 3)$, too. Hence it suffices to know the values

$$\widehat{\mathrm{Tr}}(\alpha) = 2\,\mathrm{Tr}(\alpha) = -2a = \widehat{\mathrm{Tr}}(\alpha'),$$

$$\widehat{\mathrm{Tr}}(\alpha^2) = 2\,\mathrm{Tr}(\alpha^2) = 2(a^2 - 2b) = \widehat{\mathrm{Tr}}(\alpha'^2),$$

$$\widehat{\mathrm{Tr}}(\alpha\alpha') = 2b,$$

$$\widehat{\mathrm{Tr}}(\alpha^2\alpha'^2) = 2(b^2 - 2ac),$$

$$\widehat{\mathrm{Tr}}(\alpha\alpha'^2) = \begin{cases} 3c - ab + \delta & \text{for } K \text{ cyclic} \\ 3c - ab & \text{for } K \neq \hat{K}, \end{cases}$$

$$\widehat{\mathrm{Tr}}(\alpha^2\alpha') = \begin{cases} 3c - ab - \delta & \text{for } K \text{ cyclic} \\ 3c - ab & \text{for } K \neq \hat{K}, \end{cases}$$

where $\delta = (\alpha - \alpha')(\alpha' - \alpha'')(\alpha'' - \alpha) \in \mathbb{Z}$ for K cyclic, which are easily calculated from the coefficients of the generating equation of K. The remaining computations are straightforward. We remark that the trace computations can also be done by approximating $\alpha, \alpha', \alpha''$ and thus $\varepsilon_j, \varepsilon_j', \varepsilon_j''$ $(j = 1, 2)$ sufficiently well by floating-point numbers since we know that the result will be a rational integer. However, the required precision strongly varies with the size of the coefficients of the fundamental units and it seems to be easier (and more adequate to the problem) to do all calculations with (multiple precision) integers.

Regarding the complexity of our method we have the following

PROPOSITION 3. *The number of arithmetic operations required for the computation of the zeta values according to Theorem* 1 *is* $O(k^8 E^6 \cdot \Delta^{-1})$, *and the integers on which these operations are performed have a binary length* $O(k(\log k + \log E))$. *Here E is an upper bound for the size of the input data.*

Proof. We obtain this result by counting the arithmetic operations in evaluating the formula for $M(k, \nu, \tau_1, \tau_2)$ in Theorem 1. In addition we note that the special choice of ρ yields

$$|\det(E_\nu B_\rho)| \leqslant E^3 \Delta^{-1}. \quad \blacksquare$$

In two tables at the end of the paper we list some (ideal class) zeta function values of totally real cubic fields of moderate discriminant.

PROOF OF THEOREM 1

Our first task will be to simplify the assumptions of II, Satz 28, as the regularity conditions on the units are too complicated for practical purposes. This is possible by simplifying the assumptions of I, Satz 6, and by performing the limiting process $x \to 0$ in I (26) at a later stage of the proof. Thus we can substitute the absolute convergence of the double series in I (65) by the absolute convergence of the double series (46) in this paper. However, the limiting process $x \to 0$ must now be dealt with in a different manner; this is done in Theorem 7 which recalls the determination of boundary values of elliptic functions. Indeed there are natural connections between the two problems which will become more obvious when the Kronecker limit formula for the cubic case [8] is established.

First, we have to generalize I, Satz 6:

THEOREM 4. *Let* $k \in \mathbb{N}$, $x \in \mathbb{R}$, $x \neq 0$, $\mathbf{a} := (a, b, c)$ *with* $a = a_1 + ixb_1$, $b = a_2 + ixb_2$, $c = a_3 + ixb_3$, $a_1 \cdot a_2 \cdot a_3 \neq 0$, $(a_j, b_j) \in \mathbb{R}^2$, $j = 1, 2, 3$, *pairwise linearly independent. Let* $\boldsymbol{\beta} := (\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4$ *with* $\operatorname{sign}(1 - |\alpha|) = \operatorname{sign}(1 - |\gamma|) \neq 0$, $\operatorname{sign}(1 - |\beta|) = \operatorname{sign}(1 - |\delta|) \neq 0$ *and* $R := \ln|\alpha| \ln|\delta| - \ln|\beta| \ln|\gamma| \neq 0$. *Let* $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ *have the property* (*see* I, *Definition* 1): *for all*

$\mathbf{m} \in \mathbb{Z}^2$ with $a\alpha^m\beta^n + b\gamma^m\delta^n + c = 0$ it follows that $\varphi(\mathbf{m}) = 0$. Then the following equation holds:

$$(-1)^{k-1}(k-1)!^3 \operatorname{sign}(R) \cdot (abc)^{-k}$$

$$= \sum_{\mathbf{m} \in \mathbb{Z}^2} \boldsymbol{\beta}^{\mathbf{m}}(h_{21}^{(k)}(\mathbf{a}, \mathbf{m}) + h_{22}^{(k)}(\mathbf{a}, \mathbf{m}))$$

$$+ \sum_{\mathbf{m} \in \mathbb{Z}^2} \alpha^m \beta^n (1 - \varphi(\mathbf{m}))\, H^{(k)}(\mathbf{a}, \mathbf{m}).$$

Proof. 1. *Case.* Let $(|\alpha|, |\beta|, |\gamma|, |\delta|) \in (1, \infty)^4$ and $R > 0$. The proof then is similar to the proof of I, Satz 6; the only change occurs in establishing the equations I, (2)–(5). As in the proof of I (2) it is easy to see that for demonstrating the analog of I (2) it suffices to show

$$\lim_{K \to \infty} \sum_{n=-\infty}^{\infty} |\alpha^{-K}\beta^n h_{12}(\mathbf{a}, -K, n)| = 0. \tag{1}$$

Using the notations of the proof of I, Satz 6, we have again

$$\tau - \rho\sigma > 0. \tag{2}$$

Now let $\mu > 0$ with $-1 + (\sigma\tau^{-1} + \mu)\rho < 0$, $K_1 := [K(\sigma\tau^{-1} + \mu)]$. As in I (10) we deduce

$$\sum_{n=-\infty}^{\infty} |\alpha^{-K}\beta^n h_{12}(\mathbf{a}, -K, n)|$$

$$\leqslant |b|^{-1} D^{-2}\{|\alpha|^{-K+K_1\rho}(1 - |\alpha|^{-\rho})^{-1}$$

$$+ |\alpha|^{K\sigma - (K_1+1)\tau}(1 - |\alpha|^{-\tau})^{-1}\} \to 0 \qquad (K \to \infty). \tag{3}$$

In the same way we prove I (3)–(5). The rest of the proof of I, Satz 6, can be transfered literally.

2. *Case.* Let $(|\alpha|, |\beta|, |\gamma|, |\delta|) \in (1, \infty)^4$ and $R < 0$. By interchanging α and β, γ and δ in the first case we can reduce the second case to the first case.

All other cases can be dealt with by interchanging $K \leftrightarrow M$ resp. $L \leftrightarrow N$ resp. $K \leftrightarrow M$ and $L \leftrightarrow N$ in I (13). Thus the proof is concluded. ∎

In the following lemma we consider the convergence of certain series whose summands consist essentially of the reciprocals of certain linear forms over $\mathbb{Z}[i]$.

LEMMA 5. (a) *Let $\mathbf{a}_j \in \mathbb{Z}^3$, $\mathbf{b}_j \in \mathbb{Z}^3$, $j = 1, 2, 3$, $\mathbf{a}_j, \mathbf{a}_\mu, \mathbf{b}_\nu$, $j, \mu, \nu \in \{1, 2, 3\}$, $j \neq \mu$, linearly independent. Let $\mathbf{u} := (u, v, w) \in \mathbb{R}^3$. Let $\hat{A}$ be the matrix with the rows $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. Let $T(x)$ be the (3, 3)-matrix with the columns $\mathbf{a}_j^* + ix\mathbf{b}_j^*$, $j = 1, 2, 3$ (compare Remark 2(a) (γ)). Then we have that*

$$\sum_{\mathbf{m} \in \mathbb{Z}^3} f(\det \hat{A}) \prod_{j=1}^{3} f(\mathbf{a}_j \mathbf{m}^*) \cdot e^{2\pi i \mathbf{u} \mathbf{m}^*} g_{\mathbf{r}}(\mathbf{m} T(x))$$

is absolutely convergent for $x \in \mathbb{R}$, $x \neq 0$, *and* $\mathbf{r} \in \mathbb{N}^3$, $r_1 + r_2 + r_3 \geqslant 6$.

(b) *Let* $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{Z}^3$, $\mathbf{a}_1, \mathbf{a}_2$ *linearly independent*, $\mathbf{b}_j, \mathbf{c}_j \in \mathbb{Z}^3$, $j = 1, 2, 3$. *Let* $r_j \in \mathbb{N}$, $j = 1, 2, 3$, $x \in \mathbb{R}$. *Then*

$$\sum_{\mathbf{m} \in \mathbb{Z}^3} \prod_{j=1}^{2} (1 - f(\mathbf{a}_j \mathbf{m}^*)) \prod_{j=1}^{3} f(\mathbf{b}_j \mathbf{m}^*)((\mathbf{b}_j + ix\mathbf{c}_j)\mathbf{m}^*)^{-r_j}$$

converges absolutely and is a continuous function of x *on* $\mathbb{R}$.

Proof. (a) Let $\det \hat{A} \neq 0$. The case $r_j \geqslant 2$, $j = 1, 2, 3$, is clear. Thus it suffices to consider the case $r_j = 3$, $r_2 = r_3 = 1$, from which all other cases follow. Let A be the matrix with the rows $\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_2$. Setting

$$\mathbf{a}_3 := \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{b}_1 + \lambda_3 \mathbf{a}_2 \tag{4}$$

we have $\lambda_3 \neq 0$. Now II, Lemma 19, yields

$$\begin{aligned} &\sum_{\mathbf{m} \in \mathbb{Z}^3} |g_{\mathbf{r}}(\mathbf{m} T(x))| \\ &\quad \leqslant \sum_{\mathbf{m} \in \mathbb{Z}^3} f(m_1 m_3 (\lambda_1 m_1 + \lambda_2 m_2 + \lambda_3 m_3))(m_1^2 + x^2 m_2^2)^{-3/2} \\ &\qquad \cdot |m_3(\lambda_1 m_1 + \lambda_2 m_2 + \lambda_3 m_3)|^{-1} \\ &\quad =: X \end{aligned} \tag{5}$$

if we can show $X < \infty$. Application of the Hölder inequality for series yields

$$\begin{aligned} X \leqslant &\sum_{(m_1, m_3) \in \mathbb{Z}^2} f(m_1 m_3 (\lambda_1 m_1 + \lambda_3 m_3)) \, |m_1|^{-3} \, |m_3(\lambda_1 m_1 + \lambda_3 m_3)|^{-1} \\ &+ |x|^{-3/2} \left\{ \left(\sum_{\mathbf{m} \in \mathbb{Z}^3} f(m_1 m_2 m_3) \, |m_1|^{-3/2} \, |m_2|^{-3/2} \, |m_3|^{-2} \right)^{1/2} \right. \\ &\left. \cdot \left(\sum_{\mathbf{m} \in \mathbb{Z}^3} \frac{f(m_1) f(m_2) f(\lambda_1 m_1 + \lambda_2 m_2 + \lambda_3 m_3)}{|m_1|^{3/2} \, |m_2|^{3/2} \, |\lambda_1 m_1 + \lambda_2 m_2 + \lambda_3 m_3|^2} \right)^{1/2} \right\} \\ < \infty & \end{aligned} \tag{6}$$

(compare [3, Hilfssatz 4a]). Thus (a) is demonstrated.

(b) As $\mathbf{a}_1, \mathbf{a}_2$ are linearly independent there exists $\mathbf{d} \in \mathbb{Z}^3$ such that $\mathbf{a}_1, \mathbf{a}_2, \mathbf{d}$ form a basis of $\mathbb{R}^3$. Setting A the matrix with rows $\mathbf{a}_1, \mathbf{a}_2, \mathbf{d}$ in II, Lemma 19, and proceeding as in (a), we get the desired result. ∎

We need the following lemma in the proof of Theorem 7:

LEMMA 6. *Let* $A = (a_{ij})_{n,n}$ *be an integral regular* (n, n)*-matrix*, $n \in \mathbb{N}$.

(a) *Then there exist exactly* $|\det A|^{n-1}$ *elements* $\mathbf{v} \in \{0, ..., |\det A|-1\}^n$ *with* $A^{-1}\mathbf{v}^* \in \mathbb{Z}^n$.

(b) *Let* $\boldsymbol{\tau} := (\tau_1, ..., \tau_n) \in (\mathbb{Z}\backslash\{0\})^n$, $A(\boldsymbol{\tau}) = (\tau_i a_{ij})_{n,n}$. *Then it follows for* $\mathbf{r} \in \mathbb{N}^n$, $\mathbf{u} \in \mathbb{R}^n$ *that*

$$B(n, \mathbf{r}, A(\boldsymbol{\tau})^*, \mathbf{u}) = \prod_{i=1}^{n} (|\tau_i|^n \tau_i^{-r_i})\, B(n, \mathbf{r}, A^*, \tau_1 u_1, ..., \tau_n u_n).$$

Proof. (a) The proof of Theorem 2.2 in [7] can be adopted almost literally.

(b) In the case $r_j \geq 2$, $j = 1, ..., n$, we easily establish a series representation for n-fold Dedekind sums (see II, Satz 20(c)) from which the result follows; in all other cases the result follows from the case $r_j \geq 2$, $j = 1, ..., n$, by differentiation with respect to u_j, $j = 1, ..., n$. If $u_j \in \mathbb{Z}$, $j \in \{1, ..., n\}$, we get the result by a limiting process (substitute u_i by $u_i \pm \varepsilon$, differentiate, add the results and let $\varepsilon \to 0$). ∎

We can now prove the following theorem which together with Theorem 4 will be the main tool in the proof of Theorem 1:

THEOREM 7. *Let* $A = (a_{ij})_{3,3}$ *and* $B = (b_{ij})_{3,3}$ *be integral matrices,* $\prod_{i,j=1}^{3} b_{ij} \neq 0$, $BA =: (c_{ij})_{3,3}$. *Let* $E = (\delta_{ij})_{3,3}$, $\delta_{ij} = 1$ *for* $i = j$, $\delta_{ij} = 0$ *for* $i \neq j$. *Set* $\lambda_i = \gcd(a_{i1}, a_{i2}, a_{i3})$, $i = 1, 2, 3$. *Let* $\mathbf{r} = (r_1, r_2, r_3) \in \mathbb{N}^3$, $R := r_1 + r_2 + r_3 \geq 6$, $x \in \mathbb{R}$, $x \neq 0$, *and* $\mathbf{u} := (u_1, u_2, u_3) \in \mathbb{R}^3$. *Let*

$$F(\mathbf{r}, A, \mathbf{u}) := -f(\det A) \frac{1}{2} \frac{(2\pi i)^R}{r_1!\, r_2!\, r_3!} |\det A|^{-1} \sum_{j=1}^{3} \tilde{B}_{r_j}(u_j)$$

$$\cdot \lambda_j^{-r_j+1} \prod_{\substack{v=1 \\ v \neq j}}^{3} \left[(1 - f(1 - r_v)) \operatorname{sign}(b_{jv}) \cdot \sum_{\mu \in \mathbb{Z}} (1 - f(u_v - \mu)) \right],$$

$$S(A, B, \mathbf{u}, x) := \sum_{\mathbf{m} \in \mathbb{Z}^3} f(\det A)\, e^{2\pi i \mathbf{u}\mathbf{m}^*} \prod_{v=1}^{3} \frac{f(\sum_{j=1}^{3} a_{vj} m_j)}{(\sum_{j=1}^{3} (a_{vj} + ixc_{vj})\, m_j)^{r_v}}.$$

Then the series defining $S(A, B, \mathbf{u}, x)$ *converges absolutely for* $x \neq 0$, *and we have*

(a)
$$\lim_{x \to 0} (S(E, B, \mathbf{u}, x) + S(E, B, \mathbf{u}, -x))$$
$$= -2 \prod_{j=1}^{3} \frac{(2\pi i)^{r_j}}{r_j!} \tilde{B}_{r_j}(u_j) + F(\mathbf{r}, E, \mathbf{u}),$$

(b)
$$\lim_{x \to 0} (S(A, B, \mathbf{0}, x) + S(A, B, \mathbf{0}, -x))$$
$$= -2f(\det A) \frac{(2\pi i)^R}{r_1!\, r_2!\, r_3!} |\det A|^{-3} B(3, \mathbf{r}, A^*, \mathbf{0}) + F(\mathbf{r}, A, \mathbf{0}).$$

Proof. Let $\det A \neq 0$ (the case $\det A = 0$ is trivial). The absolute convergence of the series follows from Lemma 5.

(a) For $r_j \geqslant 2$, $j = 1, 2, 3$, the assertion is evident. Now let $r_j = 1$ for some $j \in \{1, 2, 3\}$. We may assume $r_1 \geqslant 3$ and will only consider the most difficult case $r_2 = r_3 = 1$.

If $u_2 \notin \mathbb{Z}$ and $u_3 \notin \mathbb{Z}$ the assertion follows by twofold partial summation (see [1, p. 97]); if $u_2 \notin \mathbb{Z}$ and $u_3 \in \mathbb{Z}$ it is obtained by partial summation with respect to m_2 and application of the Euler summation formula and partial fraction decomposition (compare (8), (11) below), which will also be used in the following case, $u_2 \in \mathbb{Z}$ and $u_3 \in \mathbb{Z}$.

Denoting by B_0 the matrix resulting from B by setting $b_{21} = b_{31} = 0$ we easily deduce

$$\lim_{x \to 0} (S(E, B, \mathbf{u}, x) - S(E, B_0, \mathbf{u}, x)) = 0. \tag{7}$$

Thus it is sufficient to determine $\lim_{x \to 0}(S(E, B_0, \mathbf{u}, x) + S(E, B_0, \mathbf{u}, -x))$. From the equation

$$\begin{aligned}
&(m_1 + ix(b_{11}m_1 + b_{12}m_2 + b_{13}m_3))^{-r_1}(m_2 + ix(b_{22}m_2 + b_{23}m_3))^{-1} \\
&\quad \cdot (m_3 + ix(b_{32}m_2 + b_{33}m_3))^{-1} \\
&= ((1 + ixb_{22})(1 + ixb_{33}) + x^2 b_{23} b_{32})^{-1} \\
&\quad \cdot (m_1 + ix(b_{11}m_1 + b_{12}m_2 + b_{13}m_3))^{-r_1} \\
&\quad \cdot \{-ixb_{23}(m_2(m_2 + ix(b_{22}m_2 + b_{23}m_3)))^{-1} \\
&\quad - ixb_{32}(m_3(m_3 + ix(b_{32}m_2 + b_{33}m_3)))^{-1} + (m_2 m_3)^{-1}\}
\end{aligned} \tag{8}$$

we conclude that we have only to determine $\lim_{x \to 0} S_j(x)$, $j = 1, 2, 3$, where

$$S_1(x) := \sum_{\mathbf{m} \in \mathbb{Z}^3} \frac{e^{2\pi i u_1 m_1} f(m_1 m_2 m_3)}{(m_1 + ix(b_{11}m_1 + b_{12}m_2 + b_{13}m_3))^{r_1} m_2 m_3},$$

$$S_2(x) := \sum_{\mathbf{m} \in \mathbb{Z}^3} \frac{-ixb_{23}\, e^{2\pi i u_1 m_1} f(m_1 m_2 m_3)}{\begin{pmatrix} (m_1 + ix(b_{11}m_1 + b_{12}m_2 + b_{13}m_3))^{r_1} \\ \times m_2(m_2 + ix(b_{22}m_2 + b_{23}m_3)) \end{pmatrix}},$$

$$S_3(x) := \sum_{\mathbf{m} \in \mathbb{Z}^3} \frac{-ixb_{32}\, e^{2\pi i u_1 m_1} f(m_1 m_2 m_3)}{\begin{pmatrix} (m_1 + ix(b_{11}m_1 + b_{12}m_2 + b_{13}m_3))^{r_1} \\ \times m_3(m_3 + ix(b_{32}m_2 + b_{33}m_3)) \end{pmatrix}};$$

the series defining $S_j(x)$, $j = 1, 2, 3$, are absolutely convergent for $x \neq 0$. By means of the Euler summation formula we show by some calculations

$$\lim_{x \to 0} S_3(x) = -ib_{32} \sum_{m_1 \in \mathbb{Z}} \sum_{m_3 \in \mathbb{Z}} \left(\frac{f(m_1 m_3)}{m_3} \int_{-\infty}^{\infty} \frac{e^{2\pi i u_1 m_1}\, dt}{(m_1 + ib_{12}t)^{r_1}(m_3 + ib_{32}t)} \right), \tag{9}$$

$$\lim_{x \to 0} S_2(x) = -ib_{23} \sum_{m_1 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \left(\frac{f(m_1 m_2)}{m_2} \int_{-\infty}^{\infty} \frac{e^{2\pi i u_1 m_1}\, dt}{(m_1 + ib_{13} t)^{r_1} (m_2 + ib_{23} t)} \right). \tag{10}$$

The integrals can be evaluated elementarily; this however is without importance for the following. Determining the limiting value of $S_1(x)$ proves to be the most difficult part of our task. Applying (38) from [4, p. 36], three times (see also II (87)) we get

$$\begin{aligned}
&(m_1 + ix(b_{11}m_1 + b_{12}m_2 + b_{13}m_3))^{-r_1}(m_2 m_3)^{-1} \\
&\quad = (m_2 m_3)^{-1}(m_1 + ix(b_{11}m_1 + b_{13}m_3))^{-r_1} \\
&\qquad - ixb_{12} \sum_{r=0}^{r_1 - 1} \Big\{ ((1 + ixb_{11})m_1)^{-(r+1)} \\
&\qquad \cdot (m_1 + ix(b_{11}m_1 + b_{12}m_2))^{-(r_1 - r)} m_3^{-1} \\
&\qquad - ixb_{13} \sum_{t=0}^{r_1 - 1 - r} ((1 + ixb_{11})m_1)^{-(r+1)} (m_1 + ix(b_{11}m_1 + b_{12}m_2))^{-(t+1)} \\
&\qquad \cdot (m_1 + ix(b_{11}m_1 + b_{12}m_2 + b_{13}m_3))^{-(r_1 - r - t)} \\
&\qquad - ixb_{13} \sum_{s=0}^{r} ((1 + ixb_{11})\, m_1)^{-(s+1)} (m_1 + ix(b_{11}m_1 + b_{13}m_3))^{-(r+1-s)} \\
&\qquad \cdot (m_1 + ix(b_{11}m_1 + b_{12}m_2 + b_{13}m_3))^{-(r_1 - r)} \Big\}.
\end{aligned} \tag{11}$$

In the equation defining $S_1(x)$ we now substitute $\sum_{\mathbf{m} \in \mathbb{Z}^3}$ by $\sum^*_{\mathbf{m} \in \mathbb{Z}^3}$ (see II, Definition 13). This is possible because of the absolute convergence of the series in question. Taking into consideration (11) we get by using the Euler summation formula with respect to the summation over m_3 after performing several lengthy estimates

$$\begin{aligned}
&\lim_{x \to 0} S_1(x) \\
&\quad = \lim_{x \to 0} \sum^*_{\mathbf{m} \in \mathbb{Z}^3} \Bigg\{ \sum_{r=0}^{r_1 - 1} \Bigg[\sum_{t=0}^{r_1 - r - 1} (-x^2)\, b_{12} b_{13}\, e^{2\pi i u_1 m_1} f(m_1 m_2 m_3) \\
&\qquad \cdot ((1 + ixb_{11})m_1)^{-(r+1)} (m_1 + ix(b_{11}m_1 + b_{12}m_2))^{-(t+1)} \\
&\qquad \cdot (m_1 + ix(b_{11}m_1 + b_{12}m_2 + b_{13}m_3))^{-(r_1 - r - t)} \\
&\qquad + \sum_{s=0}^{r} (-x^2)\, b_{12} b_{13}\, e^{2\pi i u_1 m_1} f(m_1 m_2 m_3)((1 + ixb_{11})m_1)^{-(s+1)} \\
&\qquad \cdot (m_1 + ix(b_{11}m_1 + b_{13}m_3))^{-(r+1-s)} \\
&\qquad \cdot (m_1 + ix(b_{11}m_1 + b_{12}m_2 + b_{13}m_3))^{-(r_1 - r)} \Bigg] \Bigg\}
\end{aligned}$$

$$= \lim_{x\to 0} \sum_{\mathbf{m}\in\mathbb{Z}^3}^{*} \{(-x^2)\, b_{12} b_{13}\, e^{-2\pi i u_1 m_1} f(m_1 m_2 m_3)$$
$$\cdot [m_1^{-r_1}(m_1 + ix(b_{11}m_1 + b_{12}m_2))^{-1}$$
$$\cdot (m_1 + ix(b_{11}m_1 + b_{12}m_2 + b_{13}m_3))^{-1}$$
$$+ m_1^{-r_1}(m_1 + ix(b_{11}m_1 + b_{13}m_3))^{-1}$$
$$\cdot (m_1 + ix(b_{11}m_1 + b_{12}m_2 + b_{13}m_3))^{-1}]\}; \tag{12}$$

when the necessary estimates are done it turns out to be useful to consider the summands for $t = 0$ and $s = r$ together. Observing

$$m_1^{-r_1}(m_1 + ix(b_{11}m_1 + b_{12}m_2 + b_{13}m_3))^{-1}((m_1 + ix(b_{11}m_1 + b_{12}m_2))^{-1}$$
$$+ (m_1 + ix(b_{11}m_1 + b_{13}m_3))^{-1})$$
$$= m_1^{-(r_1-1)}(1 + ixb_{11})(m_1 + ix(b_{11}m_1 + b_{12}m_2))^{-1}$$
$$\cdot (m_1 + ix(b_{11}m_1 + b_{13}m_3))^{-1}$$
$$\cdot (m_1 + ix(b_{11}m_1 + b_{12}m_2 + b_{13}m_3))^{-1}$$
$$+ m_1^{-r_1}(m_1 + ix(b_{11}m_1 + b_{12}m_2))^{-1}(m_1 + ix(b_{11}m_1 + b_{13}m_3))^{-1}, \tag{13}$$

we get from (12) and (13) by using the Euler summation formula with respect to m_2 and m_3 after some calculations

$$\lim_{x\to 0} S_1(x) = \lim_{x\to 0} \sum_{m_1\in\mathbb{Z}} e^{2\pi i u_1 m_1} \frac{f(m_1)}{m_1^{r_1}} (-x^2 b_{12} b_{13})$$
$$\cdot \int_{-\infty}^{\infty} \frac{dt}{m_1 + ix(b_{11}m_1 + b_{12}t)} \cdot \int_{-\infty}^{\infty} \frac{dt}{m_1 + ix(b_{11}m_1 + b_{13}t)}, \tag{14}$$

the integrals being infinite integrals in the Riemann sense.

From (9), (10), and (14) the assertion easily follows. From (a) we deduce (b) by applying II, Lemma 19, and Lemma 6. ∎

In the following lemma we consider again limiting values of a certain type of functions defined by threefold infinite series:

LEMMA 8. *Let* $A_0 = (a_{ij})_{3,3}$ *be an integral matrix;* $(A_{ij})_{3,3} := \det A_0 \cdot A_0^{-1}$, *if* A_0 *is regular. In this case let* $\prod_{i,j=1}^{3} a_{ij} A_{ij} \neq 0$. *Let* $\mathbf{a}_i := (a_{i1}, a_{i2}, a_{i3})$, $i = 1, 2, 3$, $B = (b_{ij})_{3,3}$ *an integral matrix,* $\prod_{i,j=1}^{3} b_{ij} \neq 0$. *Let* $BA_0 =: (c_{ij})_{3,3}$. *Let* $s \in \mathbb{N} \cup \{0\}$, $\mathbf{c} \in \mathbb{Z}^3$, $\mathbf{c} \neq \mathbf{0}$, $\mathbf{p}_j \in \mathbb{Z}^3$, $j = 1, \ldots, s$, *and* $\mathbf{r} := (r_1, r_2, r_3) \in \mathbb{N}^3$, $r_1 + r_2 + r_3 \geq 5$. *Setting for* $x \in \mathbb{R}$

$$d(\mathbf{m}, x) := f(\det A_0) \prod_{\nu=1}^{3} \left(f(\mathbf{a}_\nu \mathbf{m}^*) \cdot \left(\sum_{j=1}^{3} (a_{\nu j} + ixc_{\nu j}) m_j \right)^{-r_\nu} \right)$$
$$\cdot (1 - f(\mathbf{cm}^*)) \prod_{j=1}^{s} f(\mathbf{p}_j \mathbf{m}^*),$$
$$S(x) := \sum_{\mathbf{m} \in \mathbb{Z}^3}{}^{*} d(\mathbf{m}, x)$$

it follows that the series is absolutely convergent for $x \neq 0$, *convergent for* $x = 0$, *and*

$$\tfrac{1}{2} \lim_{x \to 0} (S(x) + S(-x)) = S(0).$$

Proof. The assertion being trivial for A_0 singular we may assume A_0 to be regular. We first consider the case $s = 0$ where $\prod_{j=1}^{0} f(\mathbf{p}_j \mathbf{m}^*) = 1$. Lemma 5 asserts the absolute convergence of the series for $x \neq 0$, in case $x = 0$ the convergence follows easily by partial fraction decomposition (see [2], p. 290).

Case 1. If $r_j \geqslant 2$, $j = 1, 2, 3$, the assertion is evident.

Case 2. $r_1 = 1 \vee r_2 = 1 \vee r_3 = 1$. We may assume without restriction $r_1 = 1$ and $r_2 + r_3 \geqslant 4$. Setting $\mathbf{c} = d_1 \mathbf{a}_1 + d_2 \mathbf{a}_2 + d_3 \mathbf{a}_3$, $d_j \in \mathbb{Q}$, $j = 1, 2, 3$, we easily see by II, Lemma 19, and [3, Hilfssatz 4], that only for $d_1 = 0 \wedge d_2 \cdot d_3 \neq 0$ the assertion is not self-evident. In that case there exist $t_1 \in \mathbb{Q}$, $t_1 \neq 0$, $t_2, t_3 \in \mathbb{Z}$, $\gcd(t_2, t_3) = 1$, $t_2 \cdot t_3 \neq 0$, such that

$$t_1 \mathbf{c} = t_2 \mathbf{a}_2 + t_3 \mathbf{a}_3. \tag{15}$$

Setting $\mathbf{c}_j := (c_{j1}, c_{j2}, c_{j3})$ we conclude

$$S(x) = \sum_{\mathbf{m} \in \mathbb{Z}^3}{}^{*} \prod_{j=1}^{2} f(\mathbf{a}_j \mathbf{m}^*)((\mathbf{a}_j + ix\mathbf{c}_j)\, \mathbf{m}^*)^{-r_j}$$
$$\cdot f(\mathbf{a}_2 \mathbf{m}^*)((-t_2 t_3^{-1} \mathbf{a}_2 + ix\mathbf{c}_3)\mathbf{m}^*)^{-r_3}(1 - f((t_2 \mathbf{a}_2 + t_3 \mathbf{a}_3)\mathbf{m}^*)). \tag{16}$$

We now apply II, Lemma 19, where A is to be the matrix with the rows $\mathbf{a}_1, \mathbf{a}_2, t_2 \mathbf{a}_2 + t_3 \mathbf{a}_3$ and $D := \det A$.

Setting $\mathbf{b}_j := (b_{j1}, b_{j2}, b_{j3})$, $j = 1, 2$, $\mathbf{b}_3 := (b_{31}, b_{32} - t_2 t_3^{-1} b_{33}, 0)$ we obtain

$$S(x) = |D|^{-3} \sum_{\boldsymbol{\kappa}(D)} \sum_{\mathbf{m} \in \mathbb{Z}^3}{}^{*} f(m_1 m_2)(1 - f(m_3))\, t_2^{-r_3}(-t_3)^{r_3} e^{2\pi i \boldsymbol{\kappa} A^{-1} \mathbf{m}^*}$$
$$\cdot (m_1 + ix\mathbf{b}_1 \mathbf{m}^*)^{-1}(m_2 + ix\mathbf{b}_2 \mathbf{m}^*)^{-r_2}(m_2 + ix\mathbf{b}_3 \mathbf{m}^*)^{-r_3}$$
$$= |D|^{-3} \sum_{\boldsymbol{\kappa}(D)} \sum_{(m_1, m_2) \in \mathbb{Z}^2}{}^{*} f(m_1 m_2)\, t_2^{-r_3}(-t_3)^{r_3} \cdot e^{2\pi i \boldsymbol{\kappa} A^{-1}(m_1, m_2, 0)^*}$$
$$\cdot (m_1 + ix(b_{11} m_1 + b_{12} m_2))^{-1} \prod_{j=2}^{3} (m_2 + ix\mathbf{b}_j (m_1, m_2, 0)^*)^{-r_j}, \tag{17}$$

where the series again converges absolutely. Setting $\alpha_1 m_1 + \alpha_2 m_2 := \boldsymbol{\kappa} A^{-1}(m_1, m_2, 0)^*$ we see for $\alpha_1 \notin \mathbb{Z}$ by partial summation

$$\lim_{x \to 0} \sum_{(m_1, m_2) \in \mathbb{Z}^2}^* f(m_1 m_2)\, e^{2\pi i(\alpha_1 m_1 + \alpha_2 m_2)} (m_1 + ix(b_{11} m_1 + b_{12} m_2))^{-1} \cdot \prod_{j=2}^{3} (m_2 + ix\mathbf{b}_j(m_1, m_2, 0)^*)^{-r_j}$$

$$= \sum_{m_1 \in \mathbb{Z}}^* e^{2\pi i \alpha_1 m_1} \frac{f(m_1)}{m_1} \cdot \sum_{m_2 \in \mathbb{Z}}^* e^{2\pi i \alpha_2 m_2} \frac{f(m_2)}{m_2^{r_2 + r_3}}. \tag{18}$$

If $\alpha_1 \in \mathbb{Z}$ we define

$$T(x) := \sum_{(m_1, m_2) \in \mathbb{Z}^2}^* e^{2\pi i \alpha_2 m_2} \frac{f(m_1)}{(1 + ixb_{11})\, m_1} \prod_{j=2}^{3} \frac{f(m_2)}{(m_2 + ixb_{j1} m_1)^{r_j}};$$

an easy calculation yields

$$\lim_{x \to 0} \left(\sum_{(m_1, m_2) \in \mathbb{Z}^2}^* \frac{e^{2\pi i \alpha_2 m_2} f(m_1 m_2)}{\begin{pmatrix} (m_1 + ix(b_{11} m_1 + b_{12} m_2)) \\ \cdot \prod_{j=2}^{3} (m_2 + ix\mathbf{b}_j(m_1, m_2, 0)^*)^{r_j} \end{pmatrix}} - T(x) \right) = 0. \tag{19}$$

Adding the summands for m_1 and $-m_1$ in the series for $T(x)$ we get

$$T(x) + T(-x) = \frac{-2ib_{11}}{(1 + ixb_{11})(-1 + ixb_{11})} \sum_{m_1 = 1}^{\infty} \left(\sum_{m_2 \in \mathbb{Z}}^* e^{2\pi i \alpha_2 m_2} \frac{f(m_1)}{m_1} \cdot x \cdot \left\{ \prod_{j=2}^{3} \frac{f(m_2)}{(m_2 + ixb_{j1} m_1)^{r_j}} - \prod_{j=2}^{3} \frac{f(m_2)}{(m_2 - ixb_{j1} m_1)^{r_j}} \right\} \right). \tag{20}$$

Because of $b_{21} \neq 0$ we have

$$|x(m_2 + ixb_{21} m_1)^{-1}| \leqslant \sqrt{|x|} \cdot (|m_2|\, |b_{21}|\, |m_1|)^{-1/2}; \tag{21}$$

thus (19) and (20) imply

$$\lim_{x \to 0} (T(x) + T(-x)) = 0. \tag{22}$$

Taking into consideration (17)–(22) and II, Satz 20, we deduce

$$\lim_{x \to 0} (S(x)) + S(-x))$$

$$= |D|^{-3} \sum_{\boldsymbol{\kappa}(D)} \sum_{(m_1, m_2) \in \mathbb{Z}^2}^* t_2^{-r_3} (-t_3)^{r_3} e^{2\pi i \boldsymbol{\kappa} A^{-1}(m_1, m_2, 0)^*} \cdot \frac{f(m_1 m_2)}{m_1 m_2^{r_2 + r_3}}$$

$$= \sum_{\mathbf{m} \in \mathbb{Z}^3}^* \frac{f(\mathbf{a}_1 \mathbf{m}^*) f(\mathbf{a}_2 \mathbf{m}^*)(1 - f((t_2 \mathbf{a}_2 + t_3 \mathbf{a}_3)\, \mathbf{m}^*))}{(\mathbf{a}_1 \mathbf{m}^*)(\mathbf{a}_2 \mathbf{m}^*)^{r_2} (-t_2 t_3^{-1} \mathbf{a}_2 \mathbf{m}^*)^{r_3}}. \tag{23}$$

From the last equation the assertion easily follows. Now let $s>0$. We consider the case $s=1$ only because all other cases can be treated in a similar manner. We can assume without loss of generality $\mathbf{p}_1 \neq \lambda \mathbf{c}$ for all $\lambda \in \mathbb{R}$. On account of

$$f(\mathbf{p}_1 \mathbf{m}^*) = 1 - (1 - f(\mathbf{p}_1 \mathbf{m}^*)) \tag{24}$$

and the absolute convergence of

$$\sum_{\mathbf{m} \in \mathbb{Z}^3} \prod_{j=1}^{3} (f(\mathbf{a}_j \mathbf{m}^*)(\mathbf{a}_j \mathbf{m}^*)^{-r_j})(1 - f(\mathbf{c}\mathbf{m}^*))(1 - f(\mathbf{p}_1 \mathbf{m}^*)) \tag{25}$$

the assertion follows from the case $s=0$. ∎

Proof of Theorem 1. First of all we have to introduce some notations. For $\omega, \xi \in K$, $\boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{Z}^2$, $x \in \mathbb{R}$ we set (see II, Hilfssatz 23)

$$\begin{aligned}
&\tilde{U}_k(\omega + ix\xi, \mathbf{m}, \boldsymbol{\rho}, \boldsymbol{\sigma}) \\
&\quad := (-1)^{k-1}(k-1)!^3 \sum_{\substack{(k-1,k-1,k-1) \\ \kappa_1 = \nu_1 = 0}} \prod_{j=1}^{3} \binom{\kappa_j + \mu_j + \nu_j}{\kappa_j, \mu_j} \\
&\qquad \cdot \varepsilon_1^{\rho_1 \kappa_2 + \sigma_1 \kappa_3} \varepsilon_1'^{\rho_1 \mu_2 + \sigma_1 \mu_3} \varepsilon_1''^{\rho_1(\nu_2+1) + \sigma_1(\nu_3+1)} \\
&\qquad \cdot \varepsilon_2^{\rho_2 \kappa_2 + \sigma_2 \kappa_3} \varepsilon_2'^{\rho_2 \mu_2 + \sigma_2 \mu_3} \varepsilon_2''^{\rho_2(\nu_2+1) + \sigma_2(\nu_3+1)} \\
&\qquad \cdot f(\mathrm{Tr}(\omega \varepsilon_1^{m+\rho_1} \varepsilon_2^{n+\rho_2}))\, f(\mathrm{Tr}(\omega \varepsilon_1^{m+\sigma_1} \varepsilon_2^{n+\sigma_2})) \\
&\qquad \cdot ((\omega' + ix\xi')\, \varepsilon_1'^{m} \varepsilon_2'^{n})^{-(\mu_1+1)} \\
&\qquad \cdot (\mathrm{Tr}(\omega \varepsilon_1^{m+\rho_1} \varepsilon_2^{n+\rho_2}) + ix\, \mathrm{Tr}(\xi \varepsilon_1^{m+\rho_1} \varepsilon_2^{n+\rho_2}))^{-(\kappa_2+\mu_2+\nu_2+1)} \\
&\qquad \cdot (\mathrm{Tr}(\omega \varepsilon_1^{m+\sigma_1} \varepsilon_2^{n+\sigma_2}) + ix\, \mathrm{Tr}(\xi \varepsilon_1^{m+\sigma_1} \varepsilon_2^{n+\sigma_2}))^{-(\kappa_3+\mu_3+\nu_3+1)},
\end{aligned} \tag{26}$$

$$\boldsymbol{\beta} = (\alpha, \beta, \gamma, \delta) := \left(\frac{\varepsilon_1}{\varepsilon_1''}, \frac{\varepsilon_2}{\varepsilon_2''}, \frac{\varepsilon_1'}{\varepsilon_1''}, \frac{\varepsilon_2'}{\varepsilon_2''} \right), \tag{27}$$

$$\begin{aligned}
&U_k(\omega + ix\xi, \mathbf{m}) \\
&\quad := \alpha^{-1}\beta^{-1}(1-\beta)\, \tilde{U}_k(\omega + ix\xi, \mathbf{m}, -1, -1, -1, 0) \\
&\qquad - \alpha(1-\beta)\, \tilde{U}_k(\omega + ix\xi, \mathbf{m}, 1, 0, 1, 1) \\
&\qquad + \beta(1-\alpha)\, \tilde{U}_k(\omega + ix\xi, \mathbf{m}, 0, 1, 1, 1) \\
&\qquad - \alpha^{-1}\beta^{-1}(1-\alpha)\, \tilde{U}_k(\omega + ix\xi, \mathbf{m}, -1, -1, 0, -1) \\
&\qquad + \alpha^{-1}(1-\alpha\beta)\, \tilde{U}_k(\omega + ix\xi, \mathbf{m}, -1, 0, 0, 1) \\
&\qquad - \beta^{-1}(1-\alpha\beta)\, \tilde{U}_k(\omega + ix\xi, \mathbf{m}, 0, -1, 1, 0),
\end{aligned} \tag{28}$$

and (see I, (51))

$$
\begin{aligned}
&V_k(\omega + ix\xi, \mathbf{m}, \boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\tau}) \\
&\quad := \det A'(\boldsymbol{\varepsilon}, \boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\tau}) \cdot (-1)^{k-1} (k-1)!^3 \\
&\qquad \cdot \sum_{(k-1, k-1, k-1)} \prod_{j=1}^{3} \binom{\kappa_j + \mu_j + \nu_j}{\kappa_j, \mu_j} \varepsilon^{\kappa_1\rho_1 + \kappa_2\sigma_1 + \kappa_3\tau_1} \varepsilon_1'^{\mu_1\rho_1 + \mu_2\sigma_1 + \mu_3\tau_1} \\
&\qquad \cdot \varepsilon_1''^{\nu_1\rho_1 + \nu_2\sigma_1 + \nu_3\tau_1} \varepsilon_2^{\kappa_1\rho_2 + \kappa_2\sigma_2 + \kappa_3\tau_2} \varepsilon_2'^{\mu_1\rho_2 + \mu_2\sigma_2 + \mu_3\tau_2} \varepsilon_2''^{\nu_1\rho_2 + \nu_2\sigma_2 + \nu_3\tau_2} \\
&\qquad \cdot f(\mathrm{Tr}(\omega\varepsilon_1^{m+\rho_1}\varepsilon_2^{n+\rho_2}))\, f(\mathrm{Tr}(\omega\varepsilon_1^{m+\sigma_1}\varepsilon_2^{n+\sigma_2}))\, f(\mathrm{Tr}(\omega\varepsilon_1^{m+\tau_1}\varepsilon_2^{n+\tau_2})) \\
&\qquad \cdot (\mathrm{Tr}(\omega\varepsilon_1^{m+\rho_1}\varepsilon_2^{n+\rho_2}) + ix\,\mathrm{Tr}(\xi\varepsilon_1^{m+\rho_1}\varepsilon_2^{n+\rho_2}))^{-(\kappa_1+\mu_1+\nu_1+1)} \\
&\qquad \cdot (\mathrm{Tr}(\omega\varepsilon_1^{m+\sigma_1}\varepsilon_2^{n+\sigma_2}) + ix\,\mathrm{Tr}(\xi\varepsilon_1^{m+\sigma_1}\varepsilon_2^{n+\sigma_2}))^{-(\kappa_2+\mu_2+\nu_2+1)} \\
&\qquad \cdot (\mathrm{Tr}(\omega\varepsilon_1^{m+\tau_1}\varepsilon_2^{n+\tau_2}) + ix\,\mathrm{Tr}(\xi\varepsilon_1^{m+\tau_1}\varepsilon_2^{n+\tau_2}))^{-(\kappa_3+\mu_3+\nu_3+1)}. \qquad (29)
\end{aligned}
$$

The assertion shows that it does not matter which of the three conjugate fields are denoted by K, K', and K'', respectively. A discussion of all possible cases then yields that we can assume without restriction

$$\operatorname{sign}\left(1 - \left|\frac{\varepsilon_j}{\varepsilon_j''}\right|\right) = \operatorname{sign}\left(1 - \left|\frac{\varepsilon_j'}{\varepsilon_j''}\right|\right) \neq 0, \qquad j = 1, 2. \qquad (30)$$

It turns out to be convenient to choose a special basis for W. In the same way as in the proof of II, Hilfssatz 23, we put (see I, (51))

$$
\begin{aligned}
&(b_{ij}(\boldsymbol{\mu}_\kappa))_{3,3} := A'(\boldsymbol{\varepsilon}, \boldsymbol{\mu}_\kappa)\, B_\rho, \\
&(c_{ij}(\boldsymbol{\mu}_\kappa))_{3,3} := \begin{cases} (0)_{3,3} & \text{if } \det A'(\boldsymbol{\varepsilon}, \boldsymbol{\mu}_\kappa) = 0 \\ (A'(\boldsymbol{\varepsilon}, \boldsymbol{\mu}_\kappa)\, B_\rho)^{-1} & \text{if } \det A'(\boldsymbol{\varepsilon}, \boldsymbol{\mu}_\kappa) \neq 0 \end{cases} \qquad \kappa = 1, \ldots, 6,
\end{aligned}
$$

and deduce that there is a $\mathbb{Z}$-basis of W with the property

$$\prod_{i,j=1}^{3} b_{ij}(\boldsymbol{\mu}_\kappa)\, c_{ij}(\boldsymbol{\mu}_\kappa) \neq 0 \quad \text{if} \quad \det A'(\boldsymbol{\varepsilon}, \boldsymbol{\mu}_\kappa) \neq 0, \qquad \kappa = 1, \ldots, 6. \qquad (31)$$

Proceeding as in the proof of II, Hilfssatz 20, we see that we only have to prove (*) for such a basis. We fix such a basis for the rest of the proof.

To be able to apply Theorem 7 we still have to prove the existence of elements of K with suitable properties. Setting $A'(\boldsymbol{\varepsilon}, \boldsymbol{\mu}_\kappa) =: (e_{ij}(\boldsymbol{\mu}_\kappa))_{3,3}$ we denote the adjoint of $e_{ij}(\boldsymbol{\mu}_\kappa)$ by $E_{ij}(\boldsymbol{\mu}_\kappa)$, $\kappa = 1, \ldots, 6$, $i, j = 1, 2, 3$. Let $B(\boldsymbol{\varepsilon}, \boldsymbol{\mu}_\kappa) := (E_{ij}(\boldsymbol{\mu}_\kappa))^*_{3,3}$ and for $\tau \in K$

$$P(\tau, \boldsymbol{\mu}_\kappa) := A'(\boldsymbol{\varepsilon}, \boldsymbol{\mu}_\kappa) \begin{pmatrix} \tau & 0 & 0 \\ 0 & \tau' & 0 \\ 0 & 0 & \tau'' \end{pmatrix} B(\boldsymbol{\varepsilon}, \boldsymbol{\mu}_\kappa). \qquad (32)$$

We define for $\tau \in K$

$$t_1(\tau) := \tau' - \tau, \qquad t_2(\tau) := \tau'' - \tau', \qquad t_3(\tau) := \tau - \tau'',$$

$$s_{ij} := t_j(\varepsilon_i), \qquad i = 1, 2, j = 1, 2, 3, \qquad s_{3j} := t_j(\varepsilon_1 \varepsilon_2), \qquad j = 1, 2, 3.$$

We now choose $\delta_j \in K$, $j = 1, 2, 3$, with the property $|\delta_1| > |\delta_1'|$, $|\delta_1| > |\delta_1''|$, $|\delta_2'| > |\delta_2|$, $|\delta_2'| > |\delta_2''|$, $|\delta_3''| > |\delta_3|$, $|\delta_3''| > |\delta_3'|$. Setting for $n \in \mathbb{N}$, $r = 1, 2, 3$, $\kappa = 1, ..., 6$,

$$P(\delta_r^n, \boldsymbol{\mu}_\kappa) =: (p_{ij}(n, r, \kappa))_{3,3}, \tag{33}$$

an easy calculation yields that there exists $N \in \mathbb{N}$ such that

$$\prod_{r=1}^{3} \prod_{\kappa=1}^{6} \prod_{i,j=1}^{3} p_{ij}(n, r, \kappa) \neq 0 \qquad \text{for} \quad n \geqslant N, \tag{34}$$

and for $\kappa = 1, 2$ and $n \geqslant N$ the following equations hold:

$$\operatorname{sign}(p_{12}(n, 1, \kappa)\, p_{13}(n, 1, \kappa)) = -\operatorname{sign}(s_{32} s_{\kappa 2}) \tag{35}$$

$$\operatorname{sign}(p_{21}(n, 1, \kappa)\, p_{23}(n, 1, \kappa)) = N(\varepsilon_k) \operatorname{sign}(\varepsilon_k s_{12} s_{22}) \tag{36}$$

$$\operatorname{sign}(p_{31}(n, 1, \kappa)\, p_{32}(n, 1, \kappa)) = -N(\varepsilon_\kappa) \cdot \begin{cases} \operatorname{sign}(\varepsilon_1 s_{22} s_{32}) & \kappa = 1 \\ \operatorname{sign}(\varepsilon_2 s_{12} s_{32}) & \kappa = 2 \end{cases} \tag{37}$$

$$\operatorname{sign}(p_{12}(n, 2, \kappa)\, p_{13}(n, 2, \kappa)) = -\operatorname{sign}(s_{33} s_{\kappa 3}) \tag{38}$$

$$\operatorname{sign}(p_{21}(n, 2, \kappa)\, p_{23}(n, 2, \kappa)) = N(\varepsilon_\kappa) \operatorname{sign}(\varepsilon_\kappa' s_{13} s_{23}) \tag{39}$$

$$\operatorname{sign}(p_{31}(n, 2, \kappa)\, p_{32}(n, 2, \kappa)) = -N(\varepsilon_\kappa) \cdot \begin{cases} \operatorname{sign}(\varepsilon_1' s_{23} s_{33}) & \kappa = 1 \\ \operatorname{sign}(\varepsilon_2' s_{13} s_{33}) & \kappa = 2 \end{cases} \tag{40}$$

$$\operatorname{sign}(p_{12}(n, 3, \kappa)\, p_{13}(n, 3, \kappa)) = -\operatorname{sign}(s_{31} s_{11}) \tag{41}$$

$$\operatorname{sign}(p_{21}(n, 3, \kappa)\, p_{23}(n, 3, \kappa)) = N(\varepsilon_\kappa) \operatorname{sign}(\varepsilon_\kappa'' s_{11} s_{21}) \tag{42}$$

$$\operatorname{sign}(p_{31}(n, 3, \kappa)\, p_{32}(n, 3, \kappa)) = -N(\varepsilon_\kappa) \cdot \begin{cases} \operatorname{sign}(\varepsilon_1'' s_{21} s_{31}) & \kappa = 1 \\ \operatorname{sign}(\varepsilon_2'' s_{11} s_{31}) & \kappa = 2. \end{cases} \tag{43}$$

Now we fix $M \in \mathbb{N}$ such that $M \cdot \delta_r^N \rho \omega_j$ is an algebraic integer for $r, j = 1, 2, 3$. This implies

$$\operatorname{Tr}(M \delta_r^N \rho \omega_j) \in \mathbb{Z}, \qquad r, j = 1, 2, 3,$$

and we set in the following: $\lambda_r := M \cdot \delta_r^N$, $r = 1, 2, 3$. We can now proceed to the main part of the proof. Let $\omega \in W$, $x \in \mathbb{R}$, $x \neq 0$, and $a_j(\omega, x) := (\rho\omega(1 + ix\lambda_j),\ \rho'\omega'(1 + ix\lambda_j'),\ \rho''\omega''(1 + ix\lambda_j''))$, $j = 1, 2, 3$. We apply Theorem 4 with $\boldsymbol{\alpha} = a_j(\omega, x)$, $\boldsymbol{\beta} = (\varepsilon_1/\varepsilon_1'', \varepsilon_2/\varepsilon_2'', \varepsilon_1'/\varepsilon_1'', \varepsilon_2'/\varepsilon_2'')$. Using the equation (see I, Hilfssatz 3(e))

$$\sum_{\mathbf{m} \in \mathbb{Z}^2} \alpha^m \beta^n (1 - \varphi_\omega(\mathbf{m}))\, \bar{\varphi}_\omega(\mathbf{m})\, H^{(k)}(\boldsymbol{\alpha}, \mathbf{m})$$
$$= - \sum_{\mathbf{m} \in \mathbb{Z}^2} \boldsymbol{\beta}^{\mathbf{m}} (1 - \varphi_\omega(\mathbf{m}))\, \bar{\varphi}_\omega(\mathbf{m}) \sum_{\nu=3}^{6} g_{2\nu}^{(k)}(\boldsymbol{\alpha}, \mathbf{m}), \tag{44}$$

we get for $j = 1, 2, 3$

$$(-1)^{k-1}(k-1)!^3 \operatorname{sign}(L)[N(\rho)\, N(\omega)(1 + ix\lambda_j)(1 + ix\lambda_j')(1 + ix\lambda_j'')]^{-k}$$
$$= \sum_{\mathbf{m} \in \mathbb{Z}^2} \boldsymbol{\beta}^{\mathbf{m}} (h_{21}^{(k)}(a_j(\omega, x), \mathbf{m}) + h_{22}^{(k)}(a_j(\omega, x), \mathbf{m}))$$
$$+ \sum_{\mathbf{m} \in \mathbb{Z}^2} \alpha^m \beta^n (1 - \varphi_\omega(\mathbf{m}))\, H^{(k)}(a_j(\omega, x), \mathbf{m})$$
$$= \sum_{\mathbf{m} \in \mathbb{Z}^2} \{ V_k(\rho\omega(1 + ix\lambda_j), \mathbf{m}, \boldsymbol{\mu}_1) - V_k(\rho\omega(1 + ix\lambda_j), \mathbf{m}, \boldsymbol{\mu}_2) \}$$
$$- \sum_{\mathbf{m} \in \mathbb{Z}^2} \left\{ (1 - \varphi_\omega(\mathbf{m}))\, \bar{\varphi}_\omega(\mathbf{m}) \sum_{\nu=3}^{6} V_k(\rho\omega(1 + ix\lambda_j), \mathbf{m}, \boldsymbol{\mu}_\nu) \right\}$$
$$+ \sum_{\mathbf{m} \in \mathbb{Z}^2} (1 - \varphi_\omega(\mathbf{m}))(1 - \bar{\varphi}_\omega(\mathbf{m}))\, U_k(\rho\omega(1 + ix\lambda_j), \mathbf{m}). \tag{45}$$

Observing (34), Theorem 7 yields

$$\sum_{\omega \in W_0} \sum_{\mathbf{m} \in \mathbb{Z}^2} |V_k(\rho\omega(1 + ix\lambda_j), \mathbf{m}, \boldsymbol{\mu}_\nu)|$$
$$= \sum_{\omega \in W} |V_k(\rho\omega(1 + ix\lambda_j), \mathbf{0}, \boldsymbol{\mu}_\nu)| < \infty, \qquad j = 1, 2, 3,\ \nu = 1, ..., 6; \tag{46}$$

Lemma 5(b) implies

$$\sum_{\omega \in W_0} \sum_{\mathbf{m} \in \mathbb{Z}^2} (1 - \varphi_\omega(\mathbf{m}))(1 - \bar{\varphi}_\omega(\mathbf{m})) \cdot |U_k(\rho\omega(1 + ix\lambda_j), \mathbf{m})| < \infty,$$
$$j = 1, 2, 3. \tag{47}$$

Thus both series can be rearranged arbitrarily for $x \neq 0$. Then Theorem 7 yields in connection with (35)–(43)

$$\frac{1}{6} \cdot \sum_{j=1}^{3} \lim_{x \to 0} \sum_{\omega \in W} \left(\sum_{\nu=1}^{2} (-1)^{\nu+1} (V_k(\rho\omega(1 + ix\lambda_j), \mathbf{0}, \boldsymbol{\mu}_\nu) \right.$$
$$\left. + V_k(\rho\omega(1 - ix\lambda_j), \mathbf{0}, \boldsymbol{\mu}_\nu)) \right)$$
$$= -T_1 + (-1)^{k-1}(k-1)!^3 \operatorname{sign}(L)(C(k, 1, \varepsilon_1, \varepsilon_2) + C(k, 2, \varepsilon_2, \varepsilon_1)), \tag{48}$$

where T_1 is defined in II, Hilfssatz 23.

From Lemma 8 we deduce by setting $\omega(\mathbf{m}) := \sum_{j=1}^{3} m_j\omega_j$

$$\begin{aligned}
&\frac{1}{6}\cdot\sum_{j=1}^{3}\lim_{x\to 0}\left\{\sum_{\omega\in W_0}\sum_{\mathbf{m}\in\mathbb{Z}^2}(1-\varphi_\omega(\mathbf{m}))\,\bar{\varphi}_\omega(\mathbf{m})\right.\\
&\qquad\left.\cdot\left(\sum_{\nu=3}^{6}(V_k(\rho\omega(1+ix\lambda_j),\mathbf{m},\boldsymbol{\mu}_\nu)+V_k(\rho\omega(1-ix\lambda_j),\mathbf{m},\boldsymbol{\mu}_\nu))\right)\right\}\\
&\quad=\frac{1}{6}\cdot\sum_{j=1}^{3}\lim_{x\to 0}\left\{\sum_{\mathbf{m}\in\mathbb{Z}^3}^{*}(1-f(\mathrm{Tr}(\rho\omega(\mathbf{m}))))\prod_{\boldsymbol{\mu}\in I}f(\mathrm{Tr}(\rho\omega(\mathbf{m})\,\boldsymbol{\varepsilon}^{\boldsymbol{\mu}}))\right.\\
&\qquad\cdot\left(\sum_{\nu=3}^{6}(V_k(\rho\omega(\mathbf{m})(1+ix\lambda_j),0,0,\boldsymbol{\mu}_\nu)\right.\\
&\qquad\left.\left.+V_k(\rho\omega(\mathbf{m})(1-ix\lambda_j),0,0,\boldsymbol{\mu}_\nu)\right)\right\}\\
&\quad=\sum_{\mathbf{m}\in\mathbb{Z}^3}^{*}\left\{(1-f(\mathrm{Tr}(\rho\omega(\mathbf{m}))))\right.\\
&\qquad\left.\cdot\prod_{\boldsymbol{\mu}\in I}f(\mathrm{Tr}(\rho\omega(\mathbf{m})\,\boldsymbol{\varepsilon}^{\boldsymbol{\mu}}))\sum_{j=3}^{6}V_k(\rho\omega(\mathbf{m}),0,0,\boldsymbol{\mu}_j)\right\}\\
&\quad=:T_3.
\end{aligned}\tag{49}$$

From Lemma 5(b) in connection with II, Hilfssatz 26, and I, Hilfssatz 3(e) we conclude using the notations of II, Hilfssatz 26

$$\begin{aligned}
&\frac{1}{6}\cdot\sum_{j=1}^{3}\lim_{x\to 0}\sum_{\omega\in W_0}\sum_{\mathbf{m}\in\mathbb{Z}^2}(1-\varphi_\omega(\mathbf{m}))(1-\bar{\varphi}_\omega(\mathbf{m}))(U_k(\rho\omega(1+ix\lambda_j),\mathbf{m})\\
&\qquad+U_k(\rho\omega(1-ix\lambda_j),\mathbf{m}))\\
&\quad=\sum_{\omega\in W_0}\sum_{\mathbf{m}\in\mathbb{Z}^2}(1-\varphi_\omega(\mathbf{m}))(1-\bar{\varphi}_\omega(\mathbf{m}))\,U_k(\rho\omega,\mathbf{m})\\
&\quad=\sum_{\omega\in W_0}\sum_{\mathbf{m}\in\mathbb{Z}^2}\alpha^m\beta^n(1-\varphi_\omega(\mathbf{m}))(1-\bar{\varphi}_\omega(\mathbf{m}))\,H^{(k)}(\mathbf{a}'(\omega),\mathbf{m})\\
&\quad=\sum_{\omega\in W_0}\left\{\sum_{\mathbf{m}\in\mathbb{Z}^2}\left\{(1-\varphi_\omega(\mathbf{m}))(1-\bar{\varphi}_\omega(\mathbf{m}))\right.\right.\\
&\qquad\left.\cdot\sum_{\mu=1}^{3k-1}\sum_{r=1}^{2}\left[(-1)^{r+1}\det A_r\sum_{j=1}^{3}\boldsymbol{\beta}^{-\boldsymbol{\mu}_{jr}}Q_{k-1}(\mathbf{a}'(\omega),\mu,\boldsymbol{\rho}_{jr},\mathbf{m})\right]\right\}\\
&\qquad+\sum_{\mathbf{m}\in\mathbb{Z}^2}\left\{\sum_{r=1}^{2}(-1)^{r+1}\det A_r(Q_{k-1}(\mathbf{a}'(\omega),0,\boldsymbol{\mu}_r,\mathbf{m})\right.\\
&\qquad\left.\left.+Q_{k-1}(\mathbf{a}'(\omega),0,\boldsymbol{\mu}_{1r},\boldsymbol{\mu}_{3r},\boldsymbol{\mu}_{2r},\mathbf{m})+Q_{k-1}(\mathbf{a}'(\omega),0,\boldsymbol{\mu}_{2r},\boldsymbol{\mu}_{3r},\boldsymbol{\mu}_{1r},\mathbf{m})\right\}\right\}\\
&\quad=:T_4.
\end{aligned}\tag{50}$$

It is easy to show

$$\sum_{\omega \in W_0} \sum_{\mathbf{m} \in \mathbb{Z}^2} (1-\varphi_\omega(\mathbf{m}))(1-\bar{\varphi}_\omega(\mathbf{m}))\,|Q_{k-1}(\mathbf{a}'(\omega), \mu, \mathbf{v}, \mathbf{m})| < \infty \tag{51}$$

for $\mu \in \mathbb{N} \cup \{0\}$, $\mathbf{v} \in \mathbb{Z}^6$. Applying II, Hilfssatz 26(b) to every single summand of T_3 and adding up $-T_3$ and T_4 while observing (51) we get

$$-T_3 + T_4 = T_2, \tag{52}$$

where T_2 was defined in II, Hilfssatz 27(b). From (45) and (48)–(52) we conclude

$$\begin{aligned}(-1)^{k-1}(k-1)!^3\, &\mathrm{sign}(L)\, N(\rho)^{-k}\zeta(k, W, \varepsilon_1, \varepsilon_2)\\ &\cdot \frac{1}{6} \lim_{x \to 0} \sum_{j=1}^{3} \sum_{v=1}^{2} (1+(-1)^v ix\lambda_j)(1+(-1)^v ix\lambda_j')(1+(-1)^v ix\lambda_j'')\\ &= -T_1 + T_2.\end{aligned} \tag{53}$$

As in the proof of II, Satz 28, the assertion follows. ∎

Remark 9. Of course $\zeta(2k+1, W, \varepsilon_1, \varepsilon_2) = 0$ for $k \in \mathbb{N}$; however, if $\chi \not\equiv 1$ (see I, p. 108) or in the still more general case of "Nebenklassenzetafunktionen" (see, e.g., [2, p. 307], where these are defined for real quadratic fields) the methods developed here can be extended to the values at odd positive integers which then are no longer zero in general.

Numerical Results

In the following two tables column 1 contains the discriminant of K. In column 2 we list the coefficients a, b, c of a generating equation $x^3 + ax^2 + bx + c = 0$. Column 3 contains an integral basis of K in case it is not a power integral basis (see remarks on input data). Table I contains only fields of class number one. Hence, the other columns of that table list the values of the Dedekind zeta function of K for the argument $k = 2$ and—for small discriminants—also for $k = 4$ and $k = 6$ (up to the factor $\sqrt{\Delta}^{2k-1}\pi^{-3k}$; see Remark 2(b)).

In Table II the class number h_K of the considered fields is always greater than 1; it is therefore listed in column 4. Column 5 contains generators of an ideal representing the inverse of the ideal class whose zeta value is given in column 7 for $k = 2$ (up to the factor given above). The values of the class zeta functions are added up to obtain the value of $\zeta_K(2)$; this is marked by an asterisk in column 5. Finally, column 6 lists the norm of the ideal in column 5.

We note that the values of the zeta functions listed in both tables distinguish fields of the same discriminant but not necessarily of different ideal classes of a fixed field (compare $\Delta = 21212$).

TABLE I

Zeta Values $\xi_K(k)\sqrt{\Delta}^{2k-1}\pi^{-3k}$ of Fields with Class Number 1

Disc.	Gen. eq.	Int. bas.	$k=2$	$k=4$	$k=6$
49	−1, −2, 1		8/21	2528/2835	473152/212625
81	0, −3, −1		8/9	6368/1215	3222592/91125
148	3, −1, −1		8/3	18464/405	210001984/212625
169	1, −4, 1		8/3	359264/5265	428922304/212625
229	0, −4, 1		16/3	85312/405	2317186688/212625
257	3, −2, −1		16/3	121024/405	4307040128/212625
316	−1, −4, 2		32/3	55936/81	13831672576/212625
321	1, −4, −1		8	5920/9	4883946944/70875
361	−2, −5, −1		8	130784/135	176543630656/1346625
404	1, −5, 1		40/3	627616/405	10531359808/42525
469	4, −2, −1		16	69952/27	13276324736/23625
473	0, −5, 1		40/3	1023328/405	24677996992/42525
564	−1, −5, 3		24	75616/15	
568	−1, −6, −2		80/3	2181056/405	
621	0, −6, 3		80/3	2836544/405	
697	0, −7, 5		64/3	3925504/405	
733	−1, −7, 8		32	1671296/135	
756	0, −6, 2		104/3	5627552/405	
761	−1, −6, −1		80/3	5398976/405	
785	−1, −6, 5		88/3	6035104/405	
788	−1, −7, −3		112/3	1302464/81	
837	0, −6, 1		128/3	1614592/81	
892	−1, −8, 10		160/3	10589824/405	
940	0, −7, 4		176/3	12736832/405	
961	−1, −10, 8	$(\alpha+\alpha^2)/2$	224/3	14631296/405	
985	−1, −6, 1		112/3	13195072/405	
993	−1, −6, 3		136/3	2778464/81	
22356	0, −36, 60	$\alpha^2/2$	6120		
22356	0, −18, 6		5592		
22356	0, −36, 78		5504		

TABLE II

Values $\xi(k, K_0)\sqrt{\Delta}^{2k-1}\pi^{-3k}$ of Class Zeta Functions

Disc.	Gen. eq.	Int. bas.	h_K	Id. basis	Norm	$k=2$
1957	$-1, -9, 10$		2	1	1	304/3
				$2, \alpha$	2	112/3
				*		416/3
2597	$-1, -9, 8$		3	1	1	3152/21
				$2, \alpha$	2	848/21
				$4, \alpha^2$	4	1040/21
				*		240
2777	$-1, -14, 23$		2	1	1	160
				$3, -1+\alpha$	3	32
				*		192
3969	$0, -21, 35$		3	1	1	17048/63
				$3, -1+\alpha$	3	2456/63
				$9, (-1+\alpha)^2$	9	2840/63
				*		7448/21
3969	$0, -21, 28$	$(\alpha+\alpha^2)/3$	3	1	1	20960/63
				$3, 1+\alpha$	3	14816/63
				$9, (1+\alpha)^2$	9	9248/63
				*		2144/3
8069	$-1, -17, -16$		4	1	1	800
				$2, \alpha$	2	160
				$4, \alpha^2$	4	96
				$8, \alpha^3$	8	224
				*		1280
20733	$0, -36, -1$	$(1+\alpha+\alpha^2)/3$	5	1	1	3248
				$2, 1+\alpha$	2	368
				$4, (1+\alpha)^2$	4	304
				$8, (1+\alpha)^3$	8	816
				$16, (1+\alpha)^4$	16	1328
				*		6064
21212	$-1, -36, 18$	$(2\alpha+\alpha^2)/3$	7	1	1	10016/3
				$2, \alpha$	2	1184/3
				$4, \alpha^2$	4	2720/3
				$8, \alpha^3$	8	608/3
				$16, \alpha^4$	16	1376/3
				$32, \alpha^5$	32	1184/3
				$64, \alpha^6$	64	3872/3
				*		20960/3

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