Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Stochastic Processes and their Applications 117 (2007) 1642–1662
stochastic processes and their applications
www.elsevier.com/locate/spa

Stability of utility-maximization in incomplete markets

Kasper Larsen[a,*], Gordan Žitković[b]

[a] *Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, United States*
[b] *Department of Mathematics, University of Texas at Austin, Austin, TX 78712, United States*

Received 27 June 2006; received in revised form 2 October 2006; accepted 3 October 2006
Available online 2 June 2007

Abstract

The effectiveness of utility-maximization techniques for portfolio management relies on our ability to estimate correctly the parameters of the dynamics of the underlying financial assets. In the setting of complete or incomplete financial markets, we investigate whether small perturbations of the market coefficient processes lead to small changes in the agent's optimal behavior, as derived from the solution of the related utility-maximization problems. Specifically, we identify the topologies on the parameter process space and the solution space under which utility-maximization is a continuous operation, and we provide a counterexample showing that our results are best possible, in a certain sense. A novel result about the structure of the solution of the utility-maximization problem, where prices are modeled by continuous semimartingales, is established as an offshoot of the proof of our central theorem.
Published by Elsevier B.V.

MSC: 91B16; 91B28

Keywords: Appropriate topologies; Continuous semimartingales; Convex duality; Market price of risk process; Mathematical finance; Utility-maximization; V-relative compactness; Well-posedness

1. Introduction

The central problem. Financial theory in general, and mathematical finance in particular, aim to describe and understand the behavior of rational agents faced with an uncertain evolution of asset prices. In the simplest, yet most widespread models of such behavior, the agent has a fixed and immutable assessment of various probabilities related to the future evolution of the prices

* Corresponding author. Tel.: +1 4122683307.
E-mail addresses: kasperl@andrew.cmu.edu (K. Larsen), gordanz@math.utexas.edu (G. Žitković).

0304-4149/$ - see front matter Published by Elsevier B.V.
doi:10.1016/j.spa.2006.10.012

in the financial market. Taking her views as correct, the agent proceeds to implement a dynamic trading strategy which is chosen so as to maximize a certain nonlinear functional of the terminal wealth — the utility functional. Often, the utility functional is of the "expected-utility" type, i.e., the agent's objective is to maximize $\mathbb{U}(X_T) = \mathbb{E}[U(X_T)]$ over all possible random variables X_T she can generate through various investment strategies on a trading horizon $[0, T]$, starting from a given initial wealth x. $U(\cdot)$ is generally a concave and strictly increasing real-valued function defined on the positive semi-axis $(0, \infty)$, and is used as a model of the agent's risk preferences. In order to implement this program in practice, the agent chooses a particular model of the evolution of asset prices, estimates its parameters using the available market data, and combines the obtained market specification with the particular idiosyncratic form of the utility functional $\mathbb{U}$. Having seen how the choice of the market model requires imperfect measurement and estimation, the natural question to ask is then the following:

> *"How are the agent's behavior and its optimality affected by (small) misspecifications of the underlying market model?"*

Unless we can answer this question by a decisive *"Not much!"*, the utility-maximization framework as described above loses its practical applicability.

In the classical setting of the theory of partial differential equations, and applied mathematics in general, similar questions have been posed early in the literature. It is by now a classical methodological requirement to study the following three aspects of every new problem one encounters:

1. existence,
2. uniqueness,
3. sensitivity of the solution with respect to changes of the problem's input parameters.

These criteria are generally known as *Hadamard's well-posedness requirements* (see [11]). The present paper adopts the view that the market model's specification is one of the most important input data in the utility-maximization problem, and focuses on the third requirement with that in mind.

Existing research. In the general setting of the semimartingale stock-price model, the first two of the Hadamard's requirements (existence and uniqueness) have been settled completely by a long line of research reaching at least back to Robert Merton, and continuing with the work of Chuang, Cox, He, Karatzas, Kramkov, Lehoczky, Pearson, Pliska, Schachermayer, Shreve, Xu, etc. (see [20,21,3,12,16,17], merely to scratch the surface). Tight conditions are now known on practically all aspects of the problem which guarantee existence and uniqueness of the optimal investment strategy. The question of sensitivity has been studied to a much lesser degree and, compared to the model-specification issues, much more effort has been devoted to the perturbations of the shape of the utility function or the initial wealth (see, e.g., [14,4]). Related questions of stability of option pricing (under market perturbations) have been studied by El Karoui et al. [8], for the case of the Samuelson (also know as Black–Scholes–Merton) market, and several authors have studied the phase transition "from discrete- to continuous-time models", see e.g., [13] and the monograph [22].

The concept of robust portfolio optimization, which has been studied extensively in the financial and mathematical literature, is related to our notion of stability. The main goal of robust portfolio optimization is to create decision rules that work well – at least up to some degree – under each of several model specifications, or under several probability measures (sets of beliefs) $\mathcal{Q} \in \mathcal{P}$, where $\mathcal{P}$ is a family of financial models. A popular way of approaching this problem consists of allowing for multiple model specifications, and considering investors who care about expected utility, but in a different way in each of the possible models. The starting point for this approach is the celebrated paper [10], where the authors show how to relax the classical von Neumann–Morgenstern preference axioms by introducing $X \mapsto \inf_{\mathcal{Q}\in\mathcal{P}} \left(\mathbb{E}^{\mathcal{Q}}[U(X)] + \varrho(\mathcal{Q})\right)$ as the numerical representation for the robust utility functional (see also [19]). Here X typically represents the terminal value of some admissible trading strategy, and ϱ assigns penalization weights to the different possible model specifications $\mathcal{Q} \in \mathcal{P}$. We cannot give a complete overview of this theory and its many aspects (one interesting property is how model ambiguity interacts with the coefficient of risk aversion, see e.g., [27]), but refer the reader to the textbook [9] and the references therein. We emphasize, though, that while superficially similar to the robust optimization approach, our analysis is based on the assumption that our investor firmly believes that the original probability measure $\mathbb{P}$ is correctly specified, and does not incorporate any model ambiguity into her optimal decision. If we view the perturbations of the model as the perturbations of the underlying probability measure $\mathbb{P}$ (via Girsanov's theorem), one of the facets of our question of stability can be reformulated as follows: Is the $\mathbb{P}$-optimal strategy approximately optimal for all elements in some small-enough set of "nearby" models $\mathcal{Q} \in \mathcal{P}$? In other words, our problem deals with the evaluation of the optimality properties of one prespecified strategy in various market models, while the robust optimization seeks a strategy with good properties under different market models.

Our results. In the present paper, we investigate the stability properties of utility-maximization in a wide class of *complete or incomplete* financial models. Specifically, we develop a methodology which can deal with any financial market with continuous asset prices, without restrictions on the underlying filtration. In the setting of such models (described in detail below, and including Samuelson's model as well as stochastic volatility models), the concept of the market-price-of-risk can be defined in an unambiguous way. Moreover, one of our main technical results states that in these models, the maximal dual elements (in the sense of [17]) are *local martingales* and admit a multiplicative decomposition into a "minimal local martingale density", and an "orthogonal part". As a consequence, we show that in the setting of the dual approach to utility-maximization, the dual optimizer is always a local martingale when the stock price is continuous. This extends a similar result from [18] stated in the more restrictive milieu of Itô-process models.

When the model under scrutiny allows for a notion of volatility, the market-price-of-risk can be interpreted as the drift, weighted by a negative power of the volatility. In particular, misspecifications of the market-price-of-risk translate into homothetic misspecifications in the drift process. [25] discusses the practical difficulties related to estimating the drift, and points out that the magnitude of the error attached to the drift estimate is significant. The continuity of the value function, as well as the optimal terminal wealth of a utility-maximizing agent – seen as functions of the market-price-of-risk – constitute the center of our attention. Therefore, our analysis is to be seen as stability with respect to small drift misspecifications, and hopefully provides some insight also into the more complicated problem of large misspecifications that [25] points at.

The value function of our utility-maximization problem takes values in the Euclidean space $\mathbb{R}$, and there is little discussion about the proper notion of continuity there. However, the market-price-of-risk (in the domain), and the optimal terminal wealths (in the co-domain), are more complicated objects (a stochastic process and a random variable), and present us with a variety of choices for the topology under which the notion of "perturbation" can be interpreted. One of the contributions of this paper is to identify a class of topologies on the domain, and a particular topology (of convergence in probability) on the co-domain, under which utility-maximization becomes a continuous operation when a simple condition of V-relative compactness is satisfied. Under the additional assumption that all the markets under consideration are complete, we show that V-relative compactness is, in fact, both necessary and sufficient. Moreover, we provide an example, set in a complete Itô-process financial market, in which a very strong convergence requirement imposed on the market-price-of-risk processes still fails to lead to any kind of convergence of the corresponding optimal terminal wealths.

On the technical side, the proof of our main stability result requires an analysis of the structure of the solution of the utility-maximization problem. Specifically, a recourse to convex-duality techniques is of great importance; most of the intermediate steps leading to the final result deal with the dual optimization problem and its properties, and for every continuity result in the primal problem, there is a corresponding continuity result in the dual. It is in the heart of the duality approach in convex optimization that one can choose whether to work on the primal or the dual problem – depending on which one is more amenable to analyze in a particular situation – and easily translate the obtained results to the other one. In our case, the advantage of the dual problem is that certain close substitutes for *compactness* (such as the use of Komlos' lemma) bring a number of topological techniques into play. One of the mathematical messages of this paper is that the use of duality theory is not restricted to the existence results only, but can be put to a more versatile use.

The structure of the paper follows a simple template: The next section describes the modeling framework, poses the problem and states the main results. Section 3 invokes some important facts about the convex-duality treatment of utility-maximization problems and provides a proof of the main result through a sequence of lemmas. Appendix A contains an auxiliary result exemplifying the notion of appropriate topologies.

2. The problem formulation and the main results

2.1. The model framework

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and let $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$, be a filtration satisfying the usual conditions. For a continuous $\mathbf{F}$-local martingale $M = (M_t)_{t\in[0,T]}$, let Λ denote the set of all predictable processes $\lambda = (\lambda_t)_{t\in[0,T]}$ with the property that

$$\int_0^T \lambda_u^2 \mathrm{d}\langle M\rangle_u < \infty, \quad \text{a.s.},$$

where, as usual, $\langle M\rangle = (\langle M\rangle_t)_{t\in[0,T]}$ denotes the quadratic variation of the local martingale M. Each $\lambda \in \Lambda$ defines a continuous semimartingale S^λ, where

$$S_t^\lambda = 1 + M_t + \int_0^t \lambda_u \mathrm{d}\langle M\rangle_u, \quad t \in [0, T]. \tag{2.1}$$

Together with the trivial bond-price process $B_t \equiv 1$, S^λ constitutes a financial market. In the sequel, we will simply write *the market* S^λ.

Example 2.1. The proto-example for the family $\{S^\lambda : \lambda \in \Lambda\}$ is the class of Itô-process markets of the form

$$\mathrm{d}S_t^\lambda = S_t^\lambda \left(\mu_t^\lambda \mathrm{d}t + \sigma_t \mathrm{d}B_t\right), \quad S_0^\lambda = 1, \quad \text{where } \mu_t^\lambda = \lambda_t\sigma_t^2,$$

defined on the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$, generated either by a Brownian motion B, or by a pair (B, W) of independent Brownian motions. In the first case the market is complete, but in the second case the market is incomplete. The important continuous models of financial markets such as the Samuelson's model, or the class of stochastic volatility models, fall within this framework.

Remark 2.2. The choice of the special form for the model class in (2.1) is not arbitrary. In fact, it is a consequence of the main result of [6], that *any* continuous arbitrage-free (numéraire-denominated) model of a stochastic market admits the representation (2.1).

2.2. Absence of arbitrage and its consequences

For $\lambda \in \Lambda$, the stochastic exponential process $Z^\lambda = (Z^\lambda_t)_{t\in[0,T]}$, given by

$$Z^\lambda_t = \mathcal{E}(-\lambda \cdot M)_t = \exp\left(-\int_0^t \lambda_u \, \mathrm{d}M_u - \frac{1}{2}\int_0^t \lambda_u^2 \mathrm{d}\langle M\rangle_u\right), \quad t \in [0, T]$$

is a strictly positive local martingale and acts as a state-price-deflator for S^λ. More precisely, Itô's formula implies that the process $Z^\lambda X$ is a local martingale for each semimartingale X of the form $X = H \cdot S^\lambda$, whenever H is a predictable and S^λ-integrable (i.e., $H \in L(S^\lambda)$). When Z^λ is a genuine martingale, the measure $\mathbb{Q}^\lambda \sim \mathbb{P}$ defined by

$$\frac{\mathrm{d}\mathbb{Q}^\lambda}{\mathrm{d}\mathbb{P}} = Z^\lambda_T$$

is a probability measure under which the stock-price process S^λ is a local martingale. In that case, the market S^λ satisfies the condition of No Free Lunch with Vanishing Risk (NFLVR). It is customary to call $\mathbb{Q}^\lambda$ the *minimal local martingale measure*. In general, the set $\mathcal{M}^\lambda$ of equivalent local martingale measures (i.e., all probability measures $\mathbb{Q}$, equivalent to $\mathbb{P}$, under which the process S^λ is a local martingale) is larger than just a singleton. The following result is a direct consequence of Theorem 1 in [26], which, in turn, is a generalization of the results in [1,2].

Proposition 2.3 (*Schweizer, Ansel, Stricker*). *When $\mathcal{M}^\lambda \neq \emptyset$, every probability measure $\mathbb{Q} \in \mathcal{M}^\lambda$ has the form*

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = Z^\lambda_T \mathcal{E}(L)_T,$$

for some local martingale L strongly orthogonal to M, meaning $\langle L, M\rangle \equiv 0$.

It is an unexpected result of [7] that the market S^λ can satisfy NFLVR without the density process Z^λ having the martingale property. In that case, the minimal martingale measure does not exist.

We do not postulate that the process Z^λ is a (uniformly integrable) martingale. Instead, we restrict our attention to the set $\Lambda_M \subseteq \Lambda$, containing all $\lambda \in \Lambda$ such that the financial market S^λ admits NFLVR. The existence of an equivalent martingale measure for the process S^λ, $\lambda \in \Lambda_M$, now follows from the celebrated Fundamental Theorem of Asset Pricing of Delbaen and Schachermayer [5].

Remark 2.4. Even though we will only consider $\lambda \in \Lambda_M$, all the results in the sequel can be extended to the most general case $\lambda \in \Lambda$. Admittedly, in this general case, the markets under consideration will not be arbitrage free in the sense of NFLVR, but the existence of a strictly positive state-price-deflator Z^λ turns out to be enough. This (mild) generalization would add to the technicalities of the proofs without adding much to the content, so we have chosen not to pursue it.

2.3. The utility-maximization problem

Definition 2.5. A strictly concave, strictly increasing C^1-function $U : (0,\infty) \to \mathbb{R}$ satisfying the Inada conditions:

$$\lim_{x\to 0} U'(x) = +\infty, \qquad \lim_{x\to\infty} U'(x) = 0,$$

as well as the reasonable asymptotic elasticity condition $AE[U] < 1$, where

$$AE[U] = \begin{cases} \limsup\limits_{x\to\infty} \dfrac{xU'(x)}{U(x)}, & \text{if } \lim\limits_{x\to\infty} U(x) = +\infty, \\ 0, & \text{otherwise} \end{cases}$$

is called a *reasonably elastic utility function*.

Remark 2.6. As usual, we extend the utility function U to the negative semi-axis by defining $U(x) = -\infty$ for negative x-values.

Given a financial market S^λ with $\lambda \in \Lambda$, the *utility-maximization problem* for a financial agent with initial wealth $x > 0$ (and the risk attitude described by the utility function U) is to maximize the expected utility $\mathbb{E}[U(X_T)]$ over all terminal values of the wealth processes obtainable by trading in the stock S^λ and investing in the risk-free security in a self-financing manner. More precisely, the utility-maximization problem is posed through its value function $u^\lambda : \mathbb{R}_+ \to \mathbb{R}$, where

$$u^\lambda(x) = \sup_{X\in\mathcal{X}^\lambda(x)} \mathbb{E}[U(X_T)] \tag{2.2}$$

and $\mathcal{X}^\lambda(x)$ is the usual class of wealth processes constrained by an admissibility requirement in order to rule out the doubling strategies

$$\mathcal{X}^\lambda(x) = \{x + H\cdot S^\lambda \;:\; H \in L(S^\lambda),\; x + H\cdot S^\lambda \text{ is a non-negative process}\}.$$

A number of authors have studied the problem (2.2) on various levels of generality. Culminating with [17], this line of research has established a natural set of regularity assumptions on the market and on the utility function, under which (2.2) admits a unique solution $(\hat{X}^{x,\lambda}_t)_{t\in[0,T]} \in \mathcal{X}^\lambda(x)$, and the value function $x \mapsto u^\lambda(x) = \mathbb{E}[U(\hat{X}^{x,\lambda}_T)]$ is finite-valued and continuously differentiable.

2.4. The central problem

Now that we have introduced all the needed elements, we can pose our stability problem for the utility-maximization problem

Problem 2.1. Given an initial wealth $x > 0$, let the sequence $\{\lambda^n\}_{n\in\mathbb{N}}$ in Λ_M converge to $\lambda^0 \in \Lambda_M$ in some topology. Under which conditions on the sequence $\{\lambda^n\}_{n\in\mathbb{N}}$ and the topology in which it converges to λ^0, will

1. the value functions $u^{\lambda^n}(x)$,
2. the optimal terminal wealths $\hat{X}_T^{x,\lambda^n}$,

converge to the corresponding value function $u^{\lambda^0}(x)$ and the corresponding optimal terminal wealth $\hat{X}_T^{x,\lambda^0}$?

2.5. *Appropriate topologies*

Before we give a precise statement of our main result, we introduce and comment on a class of topologies in the present subsection, as well as the concept of V-relative compactness, in the following subsection. Standardly, $\mathbb{L}^0$ denotes the set of all (equivalence classes) of $\mathcal{F}$-measurable finite-valued random variables, and $\mathbb{L}^0_+$ denotes its positive cone.

Definition 2.7. A metrizable topology τ on Λ is said to be *appropriate* if the mapping $\lambda \mapsto Z_T^\lambda$ of Λ into $\mathbb{L}^0_+$ is continuous when Λ is endowed with τ, and $\mathbb{L}^0$ with the topology of convergence in probability.

Remark 2.8. The requirement of metrizability in the Definition 2.7 is imposed only to simplify the analysis below as it allows us to circumvent the use of nets. Any topology for which $\lambda \mapsto Z_T^\lambda$ is continuous can be weakened to a metrizable topology with the same property.

The following example describes two natural appropriate topologies.

Example 2.9. 1. Let the positive measure μ^M, defined on the predictable σ-algebra on the product space $[0, T] \times \Omega$, be given by

$$\mu^M(A) = \mathbb{E} \int_0^T \mathbf{1}_A(t) \, \mathrm{d}\langle M \rangle_t. \tag{2.3}$$

Proposition A.1 in Appendix A states that the restriction of the $\mathbb{L}^2(\mu^M)$-norm

$$\|\lambda\|^2_{\mathbb{L}^2(\mu^M)} = \mathbb{E} \int_0^T \lambda_u^2 \, \mathrm{d}\langle M \rangle_u,$$

onto $\{\lambda \in \Lambda : \|\lambda\|_{\mathbb{L}^2(\mu^M)} < \infty\}$ induces an appropriate topology.

2. Another example of an appropriate topology is the so-called *ucp*-topology (uniform convergence on compact sets in probability), when restricted to left-continuous processes in Λ_M. In other words, a sequence $\{\lambda^n\}_{n\in\mathbb{N}}$ converges to λ in ucp if the sequence

$$\sup_{t\in[0,T]} \left|\lambda_t^n - \lambda_t\right|$$

of random variables converges to 0 in probability. For more information about the ucp-topology, see Section II.4 in [23].

2.6. *The* log-*example*

In order to acquire a better understanding of our main result, we provide a simple example that illustrates the use of appropriate topologies in the form of the $\mathbb{L}^2(\mu^M)$-class, i.e., the sequence of models with square integrable market-price-of-risk processes, $\lambda^n \in \mathbb{L}^2(\mu^M)$ for $n \in \mathbb{N}$. We consider an investor with $U(x) = \log(x)$ (the so-called *log-investor*). It is well known that her behavior is myopic, and that the optimal terminal wealth is given by

$$\hat{X}_T^{x,\lambda^n} = \frac{x}{Z_T^{\lambda^n}}. \tag{2.4}$$

Thanks to Proposition A.1, if $\lambda^n \to \lambda^0$ in $\mathbb{L}^2(\mu^M)$, then $Z_T^{\lambda^n} \to Z_T^{\lambda^0}$. Consequently, for the optimal wealths, given by (2.4), we have $\hat{X}_T^{x,\lambda^n} \to \hat{X}_T^{x,\lambda^0}$ in probability. Furthermore, inserting (2.4) into (2.2) yields the following expression for the value function

$$u^n(x) = \mathbb{E}\left[\log\left(\hat{X}_T^{x,\lambda^n}\right)\right] = \log(x) + \mathbb{E}\left[\int_0^T \lambda_u^n \mathrm{d}M_u + \frac{1}{2}\int_0^T \left(\lambda_u^n\right)^2 \mathrm{d}\langle M\rangle_u\right].$$

Since $\lambda^n \in \mathbb{L}^2(\mu^M)$ for all $n \in \mathbb{N}$, the stochastic integral in the expression above is a genuine martingale, and the following representation holds

$$u^n(x) = \log(x) + \frac{1}{2}\|\lambda^n\|^2_{\mathbb{L}^2(\mu^M)}.$$

This relation shows that the requirement $\lambda^n \in \mathbb{L}^2(\mu^M)$ grants the finiteness of the value function u^n. It also implies that the convergence $\lambda^n \to \lambda^0$ in $\mathbb{L}^2(\mu^M)$ implies pointwise convergence of the value functions $u^n(\cdot)$ to $u^0(\cdot)$.

For an investor with a general utility function $U(\cdot)$, the corresponding optimizer $\hat{X}_T^{x,\lambda}$ can be a lot more complicated than (2.4), and, as we illustrate, more regularity needs to be imposed in order to obtain positive results. This is the content of the next subsection.

2.7. *V-relative compactness*

A reasonably elastic utility function U (as in Definition 2.5) is linked via conjugacy to its *Legendre–Fenchel transform* $V : (0, \infty) \to \mathbb{R}$, given by

$$V(y) = \sup_{x>0}(U(x) - xy).$$

Definition 2.10. A subset Λ' of Λ is said to be V-*relatively compact* if the following family of random variables

$$\{V(Z_T^\lambda) : \lambda \in \Lambda'\} \tag{2.5}$$

is uniformly integrable.

Remark 2.11. It is enough to replace $V(Z_T^\lambda)$ by $V^+(Z_T^\lambda) = \max(V(Z_T^\lambda), 0)$ in (2.5). Indeed, the family $\{Z_T^\lambda : \lambda \in \Lambda\}$ is contained in the unit ball of $\mathbb{L}^1$, and concavity properties of the function $V^-(\cdot) = \max(0, -V(\cdot))$ can be used to conclude that $\{V^-(Z_T^\lambda) : \lambda \in \Lambda\}$ is uniformly integrable (see the first part of the proof of Lemma 3.2, p. 914 in [17] for more details).

2.8. *The main result*

Theorem 2.12. *Let Λ' be a V-relatively compact subset of Λ_M, and let τ be an appropriate topology. Then for any $\lambda \in \Lambda'$, the function $u^\lambda : (0, \infty) \to \mathbb{R}$ is finite-valued, and for each*

$x > 0$, there exists an a.s.-unique optimal terminal wealth $\hat{X}_T^{x,\lambda}$ (the last element of the wealth process $\hat{X}^{x,\lambda} \in \mathcal{X}^\lambda(x)$) for the utility maximization problem (2.2). Moreover, the mappings

$$\Lambda' \times (0,\infty) \ni (\lambda, x) \mapsto u^\lambda(x) \in \mathbb{R}, \quad \text{and}$$
$$\Lambda' \times (0,\infty) \ni (\lambda, x) \mapsto \hat{X}_T^{x,\lambda} \in \mathbb{L}^0_+$$

are jointly continuous when Λ' is equipped with τ, and $\mathbb{L}^0$ with the topology of convergence in probability.

In the special case of complete markets, we have the following converse of Theorem 2.12.

Proposition 2.13. *Let $\{\lambda^n\}_{n\in\mathbb{N}_0}$ be a sequence in Λ_M such that each λ^n defines a complete market, i.e., $\mathcal{M}^{\lambda^n} = \{\mathbb{Q}^{\lambda^n}\}$. Suppose that $u^{\lambda^n}(x) \to u^{\lambda^0}(x)$ and $\hat{X}_T^{x,\lambda^n} \to \hat{X}_T^{x,\lambda^0}$ in probability, for all $x > 0$. Then the sequence $\{\lambda^n\}_{n\in\mathbb{N}}$ is V-relatively compact, and $\lambda^n \to \lambda^0$ in an appropriate topology.*

2.9. *On the conditions in the main Theorem 2.12*

The purpose of this subsection is provide some intuition about the requirement of V-relative compactness in connection with Theorem 2.12 and Proposition 2.13. We consider an investor whose preferences are of the "power" type, i.e.,

$$U(x) = \frac{1}{\gamma} x^\gamma, \quad V(y) = \frac{1}{\gamma'} y^{-\gamma'} \quad \text{where } \gamma' = \frac{\gamma}{1-\gamma}$$

for some $\gamma \in (-\infty, 1) \setminus \{0\}$. For $\gamma < 0$, $V^+ \equiv 0$, so the V-relative compactness property holds automatically. For $\gamma \in (0,1)$, however, this is not always the case.

Specializing further, let us assume that all the markets $\lambda \in \Lambda' \subseteq \Lambda$ under consideration are complete, and that the value functions in (2.2) are finite. Convex duality theory (also known as the martingale method in the financial literature) relates the optimal terminal wealth $\hat{X}_T^{x,\lambda} = \hat{X}_T^\lambda$ to the state-price-deflator Z_T^λ via

$$(\hat{X}_T^\lambda)^{\gamma-1} = y Z_T^\lambda, \tag{2.6}$$

where the Lagrange multiplier $y = y(x,\lambda)$ corresponding to the agent's budget constraint is uniquely determined by the equation $x = \mathbb{E}[Z_T^\lambda \hat{X}_T^\lambda]$, with $\hat{X}_T^\lambda$ as given by (2.6). Indeed, solving for $\hat{X}_T^\lambda$ allows us to compute y explicitly:

$$y = \left(\frac{1}{x} \mathbb{E}\left[(Z_T^\lambda)^{\frac{\gamma}{\gamma-1}} \right] \right)^{\gamma-1}. \tag{2.7}$$

Eq. (2.6) implies that $\hat{X}_T^\lambda$ varies continuously with λ, essentially if and only if the Lagrange multiplier $y = y(x,\lambda)$ does. This observation leads naturally to the concept of V-relative compactness. More precisely, let $\{Z_T^{\lambda^n}\}_{n\in\mathbb{N}_0}$, be a sequence of state-price-deflators with $Z_T^{\lambda^n} \to Z_T^{\lambda^0}$ in probability. The uniform integrability of the sequence $\{(Z_T^{\lambda^n})^{\frac{\gamma}{\gamma-1}}\}_{n\in\mathbb{N}}$ (which is, up to a constant, equal to $\{V(Z_T^{\lambda^n})\}_{n\in\mathbb{N}}$) implies the convergence of $\mathbb{E}[(Z^n)^{\frac{\gamma}{\gamma-1}}]$ to $\mathbb{E}[(Z^0)^{\frac{\gamma}{\gamma-1}}]$. Hence, the Lagrange multipliers $y_n = y(x,\lambda^n)$ converge to $y_0 = y(x,\lambda^0)$.

The following example illustrates that even when both the state-price-deflators $Z_T^{\lambda^n}$ and the market-price-of-risk processes converge in $\mathbb{L}^2$, the lack of the V-relative compactness leads to a serious breakdown in continuity. Even more important for applications is that fact that both the value functions and the optimal terminal wealths *do converge*, but not to the value function or to the optimal terminal wealth in the limiting market.

Example 2.14. Let $\mathbf{F} = \mathcal{F}_{t\in[0,1]}$ be the augmented filtration generated by a single Brownian motion B, and let $\{f^n\}_{n\in\mathbb{N}}$ be the sequence of positive, $\mathcal{F}_1$-measurable random variables, given by

$$f^n(\omega) \triangleq \begin{cases} n & \text{if } B_1(\omega) \geq \alpha_n \\ 1 & \text{if } B_1(\omega) \in (\beta_n, \alpha_n) \\ n^{-1} & \text{if } B_1(\omega) \leq \beta_n \end{cases}$$

where the increasing sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ and the decreasing sequence $\{\beta_n\}_{n\in\mathbb{N}}$ are given implicitly by $\Phi(\alpha_n) = 1 - \frac{1}{2}n^{-5}$ and $\Phi(\beta_n) = \frac{1}{2}n^{-3}$, where $\Phi(\cdot)$ denotes the distribution function of the standard normal random variable. It follows by a direct computation that $f^n \to 1$ almost surely and $\mathbb{E}[f^n] \to 1$. By the Martingale Representation Theorem and since $f^n(\omega) \in [n^{-1}, n]$, it also follows that there exist a sequence $\{\lambda^n\}_{n\in\mathbb{N}}$ of predictable processes in $\mathbb{L}^2(\mathbb{P}\times\text{Leb})$ (Leb denotes the Lebesgue measure on [0, 1]), such that

$$\mathrm{d}Z_t^{\lambda^n} = -Z_t^{\lambda^n}\lambda_t^n \mathrm{d}B_t, \quad t \in [0, 1], \quad \text{and} \quad Z_1^{\lambda^n} = c_n f^n, \text{ a.s.,}$$

where $c_n = 1/\mathbb{E}[f^n]$, so that $Z_0^{\lambda^n} = 1$, for all $n \in \mathbb{N}$. The financial market with the risky asset S^{λ^n}, where

$$\mathrm{d}S_t^{\lambda^n} = S_t^{\lambda^n}(\lambda_t^n\,\mathrm{d}t + \mathrm{d}B_t), \quad t \in (0, 1], \quad S_0^{\lambda^n} = 1,$$

admits an equivalent martingale measure $\mathbb{Q}^n$ with $\frac{\mathrm{d}\mathbb{Q}^n}{\mathrm{d}\mathbb{P}} = Z_1^{\lambda^n}$. By the Itô-isometry, we have

$$\begin{aligned} \|\lambda^n\|^2_{\mathbb{L}^2(\mathbb{P}\times\text{Leb})} &= \mathbb{E}\left[\int_0^1 \left(\lambda_u^n\right)^2 \mathrm{d}u\right] \leq \mathbb{E}\left[\int_0^1 \left(\lambda_u^n\right)^2 n^2 \left(Z_u^{\lambda^n}\right)^2 \mathrm{d}u\right] \\ &= n^2\mathbb{E}\left[\left(\int_0^1 \lambda_u^n Z_u^{\lambda^n}\,\mathrm{d}B_u\right)^2\right] = n^2\mathbb{E}\left[\left(Z_1^{\lambda^n} - 1\right)^2\right] \\ &= n^2\mathbb{E}[(c_n f^n - 1)^2] = n^2\{(nc_n - 1)^2(1 - \Phi(\alpha_n)) \\ &\quad + (c_n - 1)^2(\Phi(\alpha_n) - \Phi(\beta_n)) + (n^{-1}c_n - 1)^2\Phi(\beta_n)\} \to 0, \end{aligned}$$

by the construction of α_n and β_n, and thanks to the fact that $c_n \to 1$. Thus, $\lambda^n \to 0$ in $\mathbb{L}^2(\mathbb{P}\times\text{Leb})$ and $Z_1^{\lambda^n} \to 1$ in $\mathbb{L}^2(\mathbb{P})$ and in probability, showing that $\lambda^n \to \lambda^0 \equiv 0$ *appropriately* (see Definition 2.7 and Example 2.9).

The optimal terminal wealth $\hat{X}_1^{\lambda^n}$, in the market with the risky asset S^{λ^n}, and for an investor with unit initial wealth and the power utility $U_{3/4}(x) = \frac{4}{3}x^{3/4}$, is given by the first order condition $U'(\hat{X}_1^{\lambda^n}) = y_n Z_1^{\lambda^n}$, or equivalently

$$\hat{X}_1^{\lambda^n} = y_n^{-4}(Z_1^{\lambda^n})^{-4},$$

where $y_n > 0$ is the Lagrange multiplier determined by the budget-constraint

$$1 = \mathbb{E}^{\mathbb{Q}^n}[\hat{X}_1^{\lambda^n}] = y_n^{-4}\mathbb{E}[(c_n f^n)^{-3}].$$

An explicit computation yields

$$y_n^4 = c_n^{-3}\{n^{-3}(1-\Phi(\alpha_n)) + (\Phi(\alpha_n)-\Phi(\beta_n)) + n^3\Phi(\beta_n)\} \to \frac{3}{2}.$$

Since $Z_1^{\lambda^n} \to 1$ in probability, the sequence $\hat{X}_1^{\lambda^n}$ converges in probability towards the constant random variable with value $\frac{2}{3}$. On the other hand, the optimal strategy in the limiting market (where the risky security evolves as $\mathrm{d}S_t = S_t \mathrm{d}B_t$), is not to invest in the risky asset at all, making $\hat{X}_1 = 1$ the optimal terminal wealth. It is clear now that no convergence of the optimal terminal wealths can take place, even though the convergence $\lambda^n \to \lambda^0 = 0$ is appropriate, and even in $\mathbb{L}^2(\mathbb{P}\times \mathrm{Leb})$. One could obtain a number of similar counterexamples (oscillatory behavior, convergence of the Lagrange multipliers to $+\infty$ or to 0) by a different choice of parameters.

3. Proofs

The strategy behind the proof of our main Theorem 2.12 is to place the utility-maximization problem (2.2) in an appropriate functional-analytic framework and to exploit the dual representation of the value function u^λ and the optimal terminal wealth $\hat{X}_T^{x,\lambda}$. The steps of this program are the content of this section and some of the techniques we apply are inspired by the proof of Berge's Maximum Theorem.

3.1. The dual approach to utility maximization

The results of [17] guarantee the existence and uniqueness of the optimal terminal wealth in each market S^λ, $\lambda \in \Lambda_M$, under mild regularity conditions. Moreover, building on the work of [16] and others, the authors of [17] have established a strong duality relationship between the primal utility-maximization problem (2.2) and a suitable dual problem posed over the set of martingale measures $\mathcal{M}^\lambda$, or its enlargement $\mathcal{Y}^\lambda$. It is this last formulation that is most suited to our purposes. More precisely, with the dual value function v^λ being defined by

$$v^\lambda(y) = \inf_{\mathbb{Q}\in\mathcal{M}^\lambda} \mathbb{E}\left[V\left(y\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right], \tag{3.1}$$

the main result of [17] is the content in Theorem 3.1 below. We state it for the reader's convenience, since its content will be used extensively in the sequel.

Theorem 3.1 (*Kramkov, Schachermayer, ...*). *Let $\lambda \in \Lambda_M$ be arbitrary, but fixed, and let $u^\lambda(\cdot)$ and $v^\lambda(\cdot)$ be the value functions of the primal and the dual problems defined above in (2.2) and (3.1). Then, if $u^\lambda(\cdot)$ does not identically equal $+\infty$, the following statements hold:*

(a) *Both $u^\lambda : (0,\infty) \to \mathbb{R}$ and $v^\lambda : (0,\infty) \to \mathbb{R}$ are finite-valued and continuously differentiable. Furthermore, u^λ is strictly concave and increasing, v^λ is strictly convex and decreasing, and the following conjugacy relation holds between them*

$$v^\lambda(y) = \sup_{x>0}\left(u^\lambda(x) - xy\right), \quad \forall y > 0.$$

(b) *Alternatively, the dual value function is given as*

$$v^\lambda(y) = \inf_{Y\in\mathcal{Y}^\lambda} \mathbb{E}[V(yY_T)], \tag{3.2}$$

where the enlarged domain $\mathcal{Y}^\lambda$ is the set of all non-negative càdlàg supermartingales Y with $Y_0 = 1$ such that XY is a supermartingale for each $X \in \mathcal{X}^\lambda(1)$. The infimum in (3.2) is uniquely attained in $\mathcal{Y}^\lambda$ (with the minimizer denoted by $\hat{Y}^{y,\lambda}$).

(c) *For $x > 0$ and $y = (u^\lambda)'(x)$, the random variable $\hat{X}_T^{x,\lambda} = -V'(y\hat{Y}_T^{y,\lambda})$, belongs to $\mathcal{X}^\lambda(x)$, and is the a.s.-unique optimal terminal wealth for an agent with initial wealth x and utility function U.*

3.2. Structure of the dual domain

Some of the central arguments in the proof of our main result 2.12 depend on a precise characterization of the set $\mathcal{Y}^\lambda$ introduced in (b) above. Thanks to the continuity of the paths of our price process S^λ, this can be achieved in a quite explicit manner, as described in the following proposition.

Proposition 3.2. *For $\lambda \in \Lambda_M$, let Y be in $\mathcal{Y}^\lambda$, i.e., Y is a non-negative càdlàg supermartingale such that $Y_0 = 1$, and YX is a supermartingale for each $X \in \mathcal{X}^\lambda(1)$. When $Y_T > 0$ a.s., we have the following multiplicative decomposition:*

$$Y = Z^\lambda \mathcal{E}(L) D,$$

where $Z^\lambda = \mathcal{E}(-\lambda \cdot M)$, L is a càdlàg local martingale, strongly orthogonal to M, meaning $\langle M, L\rangle \equiv 0$, and D is a predictable, non-increasing, càdlàg process with $D_0 = 1$, $D_T > 0$, a.s.

Proof. For the sake of notational clarity, we omit the superscript λ from all expressions in the present proof. Since Y is strictly positive, Y has a multiplicative Doob–Meyer decomposition:

$$Y_t = \mathcal{E}(-\alpha \cdot M + L)_t D_t$$

for some $\alpha \in L(M)$, a local martingale L satisfying $\langle L, M\rangle \equiv 0$, and a predictable, cádlág, non-increasing process D (see Theorem 8.21, p. 138 in [15]). Thanks to the strong orthogonality of L and M, the relationship $\mathcal{E}(-\alpha \cdot M + L) = \mathcal{E}(-\alpha \cdot M)\mathcal{E}(L)$ holds. Therefore, it remains to show that $\alpha = \lambda$ almost everywhere with respect to the measure μ^M defined in (2.3).

By the strict positivity of the process D, we can write $\mathrm{d}D_t = D_{t-}\mathrm{d}F_t$ for a non-increasing predictable process F. Using Theorem 2.1 in [6], F can be split into an integral with respect to $\mathrm{d}\langle M\rangle$ (the absolute continuous part) and a singular part F'. More precisely, there exists a μ^M-null set A with $F' = \int_0^\cdot 1_A(u)\mathrm{d}F'_u$, and a non-negative predictable process β such that

$$F_t = -\int_0^t \beta_u \,\mathrm{d}\langle M\rangle_u + F'_t, \quad t \in [0, T].$$

With this notation, we have $\mathrm{d}D_t = -D_{t-}\beta_t \mathrm{d}\langle M\rangle_t + D_{t-}\mathrm{d}F'_t$, and by Itô's Lemma and the predictability of F we get

$$\mathrm{d}Y_t = Y_{t-}(-\alpha_t \,\mathrm{d}M_t + \mathrm{d}L_t - \beta_t \,\mathrm{d}\langle M\rangle_t + \mathrm{d}F'_t), \quad t \in (0, T], \quad Y_0 = 1.$$

Therefore for any admissible portfolio wealth process $X \in \mathcal{X}^\lambda(1)$ generated by a portfolio H, we have

$$\mathrm{d}(Y_t X_t) = Y_{t-}H_t(\lambda_t \mathrm{d}\langle M\rangle_t + \mathrm{d}M_t) + X_t Y_{t-}(-\alpha_t \mathrm{d}M_t + \mathrm{d}L_t - \beta_t \mathrm{d}\langle M\rangle_t + \mathrm{d}F'_t) \\ - Y_{t-}\alpha_t H_t \mathrm{d}\langle M\rangle_t$$

and given the supermartingale property, the drift in the above has to be non-positive, meaning that for any H we have the inequality

$$(H_t(\lambda_t - \alpha_t) - \beta_t X_t)\mathrm{d}\langle M\rangle_t + X_t \mathrm{d}F'_t \leq 0,$$

in the sense that the measure the left-hand-side generates on the predictable sets is non-positive. Moreover, by the singularity between μ^M and $\mathrm{d}F'$, the following must hold μ^M-a.e.

$$H_t(\lambda_t - \alpha_t) \leq \beta_t X_t$$

for all admissible H. Suppose now, contrary to the claim we are trying to prove, that $\mu^M(\lambda \neq \alpha) > 0$. Without loss of generality, we assume that this implies that there exists a predictable set $A_1 \subseteq [0, T] \times \Omega$ with the property that

1. $\lambda - \alpha \geq \varepsilon$ on A_1 for some $\varepsilon > 0$, and
2. $\mu^M(A_1) > 0$.

Since β and λ are finite-valued predictable process and β is non-negative, we can find a constant $\Sigma > 0$ and a predictable set A_2 such that $\beta, |\lambda| \in [0, \Sigma]$ on A_2 and $\mu^M(A) > 0$, where $A = A_1 \cap A_2$.

For $n \in \mathbb{N}$, let $\tilde{H}^n$ be the predictable process given by $\tilde{H} = n\mathbf{1}_A$, and let τ_n be the first exit time of the process $1 + \tilde{H}^n \cdot S$ from the semi-axis $(0, \infty)$. Define the adjusted predictable process H^n by $H^n = \tilde{H}^n \mathbf{1}_{[0,\tau_n]}$, so that $\mu^M(\{H^n > 0\}) > 0$. For each n, H^n is predictable and $X^n \triangleq 1 + (H^n \cdot S)$ is in $\mathcal{X}(1)$, and so by the above we have

$$\varepsilon/\Sigma \leq X^n \beta_t/\Sigma \leq X^n = (1 + n\mathbf{1}_A \cdot S)^{\tau_n}, \quad \mu^M\text{-a.e. on } A. \tag{3.3}$$

Observe that one of the conclusions of (3.3) is that the stopping time τ_n will not be realized on A (because the process $1 + n\mathbf{1}_A \cdot S$ is continuous). Also, as the process $\mathbf{1}_A \cdot S$ is constant off A, we have the following strengthening of (3.3):

$$\varepsilon/\Sigma \leq 1 + n\mathbf{1}_A \cdot S, \quad \mu^M\text{-a.e.} \tag{3.4}$$

Define the non-decreasing continuous process C by $C_t = \int_0^t \mathbf{1}_A(u)\mathrm{d}\langle M\rangle_u$, and note that $\mu^M(A) > 0$ implies that $\mathbb{P}[C_T > 0] > 0$. Therefore, the right inverse G of C, given by $G_s = \inf\{t \geq 0 : C_t > s\}$, where $\inf \emptyset = +\infty$, is a right-continuous, non-decreasing $[0, \infty]$-valued stochastic process, such that $G_s < \infty$ on the (non-trivial) stochastic interval $[0, C_T)$. Define the process V by

$$V_s = \begin{cases} S_{G_s} - S_0, & \text{when } G_s < \infty, \text{ and} \\ S_T - S_0 + \tilde{B}_{s-C_T} & \text{otherwise,} \end{cases}$$

where $\tilde{B}$ is a Brownian motion, defined on an extension of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and independent of $\mathcal{F}_T$. An application of Lévy's criterion shows that V is a Brownian motion with drift $\lambda_{G_s}\mathbf{1}_{\{G_s<\infty\}}$. Letting $n \to \infty$ in (3.4) yields that $V_s \geq 0$ for $s \in [0, C_T)$. On the other hand, as $\lambda \leq \Sigma$ on A, V is bounded from above by a Brownian motion with a constant drift C. This is, however, a contradiction, as almost every trajectory of a Brownian motion with a constant drift enters the negative semi-axis $(-\infty, 0)$, in every neighborhood of 0. □

Corollary 3.3. *For $\lambda \in \Lambda_M$, v^λ is the value function of the dual optimization problem defined by (3.1). For each $y > 0$, such that $v^\lambda(y) < \infty$, there exists a local martingale $L^{y,\lambda}$, strongly orthogonal to M, such that*

$$v^\lambda(y) = \mathbb{E}\left[V\left(yZ_T^\lambda \mathcal{E}(L^{y,\lambda})\right)\right].$$

Proof. Theorem 3.1(c) implies that the infimum in the definition of the dual value function v^λ is attained at a terminal value Y_T of a supermartingale Y with the property that YX is a supermartingale for each $X \in \mathcal{X}^\lambda(1)$. Proposition 3.2 states that each such supermartingale can be written

$$Y = Z^\lambda \mathcal{E}(L) D, \quad \text{with } \langle L, M\rangle \equiv 0.$$

Thanks to the strictly decreasing nature of the function V, we must have $D \equiv 1$. Indeed, $Z^\lambda \mathcal{E}(L)$ dominates $Z^\lambda \mathcal{E}(L) D$ pointwise, and belongs to $\mathcal{Y}^\lambda$. □

Let $\mathcal{B}$ denote the set of all local martingales L, strongly orthogonal to M, such that the terminal value $\mathcal{E}(L)_T$ of the stochastic exponential $\mathcal{E}(L)$ is bounded from below by a positive constant.

Corollary 3.4. *Let $\lambda \in \Lambda_M$, and suppose that $\mathbb{E}[V^+(Z^\lambda_T)] < \infty$. Then for each $y > 0$, we have the representation*

$$v^\lambda(y) = \inf_{L\in\mathcal{B}} \mathbb{E}\left[V\left(yZ^\lambda_T \mathcal{E}(L)_T\right)\right]. \tag{3.5}$$

Proof. The fact that the infimum on the right-hand side of (3.5) is bounded from below by the value function $v^\lambda(y)$ follows directly from Theorem 3.1 and Proposition 3.2. For the other inequality, let $L^{y,\lambda}$ be the local martingale from the statement of Corollary 3.3. If $\mathcal{E}(L^{y,\lambda})_T$ happened to be bounded from below by a strictly positive constant, there would be nothing else left to prove. However, $\mathcal{E}(L^{y,\lambda})_T$ is, in general, not bounded away from zero, so we employ a limiting argument via a suitably defined sequence $L^n \in \mathcal{B}$. For $n \in \mathbb{N}$, let Y^n be the supermartingale in $\mathcal{Y}^\lambda$ given by

$$Y^n = Z^\lambda \left(\frac{n-1}{n}\mathcal{E}(L^{y,\lambda}) + \frac{1}{n}\right).$$

The process Y^n is a positive local martingale with the property that $Y^n X$ is a supermartingale for each $X \in \mathcal{X}^\lambda(1)$. Therefore, the proposition allows us to write $Y^n = Z^\lambda \mathcal{E}(L^n) D^n$, and since $D^n \le 1$ we have $\mathcal{E}(L^n) \ge \frac{1}{n}$ and so $\mathcal{E}(L^n) \in \mathcal{B}$. Furthermore, since V is decreasing and convex, we have

$$\begin{aligned}
\mathbb{E}\left[V\left(yZ^\lambda_T \mathcal{E}(L^n)_T\right)\right] &\le \mathbb{E}\left[V\left(yY^n_T\right)\right] \\
&\le \frac{n-1}{n}\mathbb{E}\left[V\left(yZ^\lambda_T \mathcal{E}(L^{y,\lambda})_T\right)\right] + \frac{1}{n}\mathbb{E}\left[V\left(yZ^\lambda_T\right)\right] \\
&\le \frac{n-1}{n} v^\lambda(y) + \frac{1}{n}\mathbb{E}\left[V^+\left(yZ^\lambda_T\right)\right] \\
&\le \frac{n-1}{n} v^\lambda(y) + \frac{1}{n}\left(C\,\mathbb{E}\left[V^+\left(Z^\lambda_T\right)\right] + D\right)
\end{aligned}$$

for two constants C and D granted by the asymptotic elasticity of U (see Proposition 6.3(iii) of [17]). Taking the lim inf with respect to n on both sides yields the desired inequality. □

3.3. Joint continuity of the value functions

The following lemmas establish a joint continuity property for the primal and dual value functions and their derivatives. Before we proceed, let us agree that in the sequel $\Lambda' \subseteq \Lambda_M$ is V-relatively compact, and that τ is an appropriate topology. By the inequality

$$U(X_T) \le V\left(Z^\lambda_T\right) + X_T Z^\lambda_T,$$

and the supermartingale property of the process XZ^λ when $X \in \mathcal{X}^\lambda(x)$, it follows that $u^\lambda(x) \le \mathbb{E}[V(Z^\lambda_T)] + x < \infty$, for all $x > 0$ and $\lambda \in \Lambda'$. Therefore, the assumptions of the Theorem 3.1 are satisfied, and its conclusions hold.

Lemma 3.5. *Let Y be a random variable, bounded from below by a strictly positive constant, such that $\sup_{\lambda\in\Lambda'} \mathbb{E}[Z^\lambda_T Y] < \infty$. Then the mapping $(y,\lambda) \mapsto V(yZ^\lambda_T Y)$ is continuous from $(0,\infty) \times \Lambda'$ (with the product topology) into $\mathbb{L}^1$. In particular, the mapping $(y,\lambda) \mapsto \mathbb{E}[V(yZ^\lambda_T Y)]$ is continuous.*

Proof. Given that V is a continuous function, the mapping $(y,\lambda) \mapsto V(yZ^\lambda_T Y)$ is continuous in probability because $(y,\lambda) \mapsto yZ^\lambda_T$ is. It will, therefore, be enough to establish the uniform integrability of the family

$$\left\{V\left(yYZ^\lambda_T\right) : y \in B, \lambda \in \Lambda'\right\},$$

when B is a compact segment of the form $[\varepsilon, 1/\varepsilon]$, $0 < \varepsilon < 1$. The boundedness in $\mathbb{L}^1$ of the family $\{yYZ^\lambda_T : y \in B, \lambda \in \Lambda'\}$ and the fact that $\lim_{y\to\infty} \frac{V^-(y)}{y} = 0$, coupled with the De la Vallée Poussin criterion, imply that the family $\{V^-(yYZ^\lambda_T) : y \in B, \lambda \in \Lambda'\}$ is uniformly integrable. As for the positive parts, it will be enough to note that $V^+(yYZ^\lambda_T) \le V^+(y_0 Z^\lambda_T)$, where $y_0 = \varepsilon \operatorname{essinf} Y > 0$, and invoke the argument concluding the proof of Corollary 3.4 to reach the conclusion that the positive parts

$$\left\{V^+\left(yYZ^\lambda_T\right) : y \in B, \lambda \in \Lambda'\right\}$$

form a uniformly integrable family as well. □

Lemma 3.6. *The function*

$$(y,\lambda) \mapsto v^\lambda(y),$$

mapping $(0,\infty) \times \Lambda'$ into $\mathbb{R}$ is upper semi-continuous (with respect to the product topology).

Proof. By Corollary 3.4, the dual value function v^λ has the following representation

$$v^\lambda(y) = \inf_Y \mathbb{E}\left[V\left(yYZ^\lambda_T\right)\right],$$

where the infimum is taken over Y of the form $Y = \mathcal{E}(L)$, where $L \in \mathcal{B}$, i.e., $\mathcal{E}(L)$ is bounded away from zero. For a such a random variable Y, by Lemma 3.5, the mapping $(y,\lambda) \mapsto \mathbb{E}[V(yYZ^\lambda_T)]$ is continuous. Therefore, $(y,\lambda) \mapsto v^\lambda(y)$ is τ-upper semi-continuous as an infimum of continuous mappings. □

Lemma 3.7. *The mapping $(y,\lambda) \mapsto v^\lambda(y)$ is continuous on $(0,\infty) \times \Lambda'$ (with respect to the product topology).*

Proof. Thanks to the result of Lemma 3.6, it is enough to show that $(y,\lambda) \mapsto v^\lambda(y)$ is lower semi-continuous. Let $\{y_n, \lambda^n\}_{n\in\mathbb{N}}$ in $(0,\infty) \times \Lambda'$ converge to $(y,\lambda) \in (0,\infty) \times \Lambda'$. We need to prove that $v^\lambda(y) \le \liminf v^{\lambda^n}(y_n)$. By passing to a subsequence that realizes the liminf, we can assume that the sequence $\{v^{\lambda^n}(y_n)\}_{n\in\mathbb{N}}$ converges and, furthermore, by passing to yet another subsequence, we may assume that $Z^{\lambda^n}_T \to Z^\lambda_T$ almost surely.

Corollary 3.3 states that $v^{\lambda^n}(y_n) = \mathbb{E}[V(y_n \hat{Y}^{y_n,\lambda^n}_T)]$, where the optimizer $\hat{Y}^{y_n,\lambda^n}_T$ can be written as $\hat{Y}^{y_n,\lambda^n}_T = Z^{\lambda^n}_T \mathcal{E}(L^n)_T$, for some local martingale $L^n = L^{y_n,\lambda^n}$ that is strongly orthogonal to M.

Komlos' lemma grants the existence of an almost surely convergent subsequence of the sequence of the Cesàro sums of the sequence $\{y_n Z_T^{\lambda^n} \mathcal{E}(L^n)_T\}_{n\in\mathbb{N}}$. This, of course, implies that there exist a double array $\{\alpha_k^n\}$ with $n \in \mathbb{N}$, $k \in \{n, \ldots, K(n)\}$ for some $K(n) \in \mathbb{N}$, of non-negative weights, and a random variable $h \in \mathbb{L}_+^0$ such that

$$\sum_{k=n}^{K(n)} \alpha_k^n = 1, \quad \text{for all } n, \quad \text{and} \quad h_n = \sum_{k=n}^{K(n)} \alpha_k^n y_k Z_T^{\lambda^k} \mathcal{E}(L^k)_T \to h, \text{ a.s.}$$

Since also $y_n Z_T^{\lambda^n} \to y Z_T^{\lambda}$ a.s., we have (see Lemma 3.8 below)

$$f_n = \sum_{k=n}^{K(n)} \alpha_k^n \mathcal{E}(L^k)_T \to \frac{h}{yZ}, \text{ a.s.}$$

The random variables f_n are all in $\mathcal{Y}^0 = \mathcal{Y}^{\lambda\equiv 0}$, which is closed with respect to convergence in probability, thanks to Lemma 4.1., p. 926 in [17]. Therefore, the limit of f_n will also be in $\mathcal{Y}^0$, and, consequently, $\frac{h}{y} = Z\frac{h}{yZ} \in \mathcal{Y}^{\lambda}$. By Fatou's Lemma (and keeping in mind the uniform integrability of the family of negative parts $\{V^-(Y) : Y \in \mathbb{L}_+^0, \mathbb{E}[Y] \le c\}$ for any $c > 0$), we have

$$\begin{aligned} v^{\lambda}(y) &\le \mathbb{E}[V(h)] = \mathbb{E}[V(\liminf_n h_n)] \le \liminf_n \mathbb{E}\left[V\left(\sum_{k=n}^{K(n)} \alpha_k^n y_k Z_T^{\lambda^k} \mathcal{E}(L^k)_T\right)\right] \\ &\le \liminf_n \sum_{k=n}^{K(n)} \alpha_k^n \mathbb{E}\left[V\left(y_k \hat{Y}_T^{y_k,\lambda^k}\right)\right] = \liminf_n \sum_{k=n}^{K(n)} \alpha_k^n v^{\lambda^k}(y_k) = \lim_n v^{\lambda^n}(y_n). \quad \Box \end{aligned}$$

Lemma 3.8. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers converging to $a > 0$, and let $\{b_k^n\}$, $n \in \mathbb{N}$, $k \ge n$ be a double array of positive numbers.*

$$\text{If } \lim_n \sum_{k=n}^{\infty} a_k b_k^n = c > 0, \quad \text{then } \lim_n \sum_{k=n}^{\infty} b_k^n = c/a.$$

Proof. Let $\epsilon > 0$ be arbitrary. We can find $N(\epsilon) \in \mathbb{N}$ such that $(1+\epsilon) \ge a_k/a \ge (1-\epsilon)$ for all $k \ge N(\epsilon)$. Therefore, for $n \ge N(\epsilon)$, we have

$$\frac{1}{a(1+\epsilon)} \sum_{k=n}^{\infty} a_k b_k^n \le \sum_{k=n}^{\infty} b_k^n \le \frac{1}{a(1-\epsilon)} \sum_{k=n}^{\infty} a_k b_k^n$$

and hence, letting $n \to \infty$ and $\varepsilon \to 0$ yields the desired conclusion. $\Box$

Proposition 3.9. *The following mappings are continuous on $(0,\infty) \times \Lambda'$:*

$$(y,\lambda) \mapsto v^{\lambda}(y), \quad (y,\lambda) \mapsto (v^{\lambda})'(y), \quad (x,\lambda) \mapsto u^{\lambda}(x), \quad (x,\lambda) \mapsto (u^{\lambda})'(x).$$

Proof. Let $\{\lambda^n\}_{n\in\mathbb{N}}$ be a sequence in Λ' converging appropriately to $\lambda \in \Lambda'$. Thanks to the result of Lemma 3.7 and the convexity of the dual value functions, Theorem 25.7 in [24] states that the derivatives $(v^{\lambda^n})'(\cdot)$ converge towards $(v^{\lambda})'(\cdot)$, and uniformly on compact intervals in $(0,\infty)$. The uniform convergence on compact intervals also holds for the original sequence of functions $v^{\lambda^n}(\cdot)$.

To proceed, pick $x > 0$ and $\varepsilon > 0$, and define $y(\epsilon) \triangleq (u^\lambda)'(x) + \epsilon$. The strict increase of $(v^\lambda)'(\cdot)$ implies that

$$\lim_n (v^{\lambda^n})'(y(\epsilon)) = (v^\lambda)'(y(\epsilon)) = (v^\lambda)'((u^\lambda)'(x) + \epsilon) > (v^\lambda)'((u^\lambda)'(x)) = -x,$$

where the last inequality follows directly from continuous differentiability and conjugacy of u^λ and v^λ. Consequently, for large n, we have $-(v^{\lambda^n})'(y(\epsilon)) < x$. Since $(u^{\lambda^n})'(\cdot)$ is strictly decreasing for each $n \in \mathbb{N}$, we get

$$(u^\lambda)'(x) + \epsilon = y(\epsilon) = (u^{\lambda^n})'\left(-(v^{\lambda^n})'(y(\epsilon))\right) > (u^{\lambda^n})'(x),$$

for large n, implying that $\limsup_n (u^{\lambda^n})'(x) \leq (u^\lambda)'(x)$. The other inequality, namely $\liminf_n (u^{\lambda^n})'(x) \geq (u^\lambda)'(x)$, can be proved similarly. By the results obtained so far, we have

$$u^{\lambda^n}(x) = v^{\lambda^n}\left((u^{\lambda^n})'(x)\right) + x\,(u^{\lambda^n})'(x) \to v^\lambda\left((u^\lambda)'(x)\right) + x\,(u^\lambda)'(x) = u^\lambda(x).$$

Finally, the joint continuity of value functions and their derivatives on $\Lambda' \times (0, \infty)$ is a consequence of the already mentioned uniform convergence from Theorem 25.7 in [24]. □

3.4. Continuity of the optimal terminal wealths

Lemma 3.10. *Let $\Lambda' \subseteq \Lambda_M$ be V-relatively compact, and let τ be an appropriate topology on Λ'. The function*

$$(x, \lambda) \mapsto \hat{X}_T^{x,\lambda},$$

where $\hat{X}_T^{x,\lambda}$ is the unique optimal terminal wealth in the market S^λ, is continuous from $(0, \infty) \times \Lambda'$ to $\mathbb{L}^0$ (equipped with the topology of convergence in probability).

Proof. By Theorem 3.1, the optimal terminal wealth admits a representation with $y = (u^\lambda)'(x)$,

$$U'\left(\hat{X}_T^{x,\lambda}\right) = y\hat{Y}_T^{y,\lambda}$$

where $\hat{Y}_T^{y,\lambda}$ attains the minimum in the dual problem (3.2). Thanks to the continuity of the mappings $(x, \lambda) \mapsto (u^\lambda)'(x)$ and $x \to U(x)$, it suffices to show that $(y, \lambda) \mapsto y\hat{Y}_T^{y,\lambda}$ is continuous in probability. Since the topology τ is assumed to be metrizable, it is enough to show that convergence of any sequence $(y_n, \lambda^n) \in (0, \infty) \times \Lambda'$ to $(y, \lambda) \in (0, \infty) \times \Lambda'$ implies the convergence of $y_n \hat{Y}_T^{y_n,\lambda^n} \to y\hat{Y}_T^{y,\lambda}$ in probability. By Proposition 3.2, each $\hat{Y}_T^{y_n,\lambda^n}$ can be expressed as the product

$$\hat{Y}_T^{y_n,\lambda^n} = Z^n H^n, \quad \text{where } Z^n = Z_T^{\lambda^n} \text{ and } H^n = \mathcal{E}(L^{y_n,\lambda^n})_T,$$

for some local martingale L^{y_n,λ^n}, strongly orthogonal to M. In the same way, we will write $\hat{Y}_T^{y,\lambda} = ZH$, with analogous definitions for Z and H.

We start our analysis by noting that for any $\delta > 0$, with $H^\delta = (1 - \delta)H + \delta$, we have by Markov's inequality

$$\begin{aligned}\mathbb{P}[y_n Z^n |H^n - H| > 2\varepsilon] &\leq \mathbb{P}[y_n Z^n |H^n - H^\delta| > \varepsilon] + \mathbb{P}[y_n Z^n \delta |1 - H| > \varepsilon] \\ &\leq \mathbb{P}[y_n Z^n |H^n - H^\delta| > \varepsilon] + \frac{\delta}{\varepsilon}\mathbb{E}[y_n Z^n |1 - H|] \\ &\leq \mathbb{P}[y_n Z^n |H^n - H^\delta| > \varepsilon] + \frac{2y_n\delta}{\varepsilon}\end{aligned}$$

since $\mathbb{E}[Z^n|1-H|] \le \mathbb{E}[Z^n] + \mathbb{E}[Z^n H] \le 2$. We then pick a constant $N > 0$, and by the strict concavity of V we can find a positive constant $\beta = \beta(\varepsilon, N)$ with the property that

$$V\left(\frac{a+b}{2}\right) < \frac{V(a)+V(b)}{2} - \beta$$

if a and b are positive numbers with $|a-b| > \epsilon$ and $(a+b) \le N$. This property, combined with the convexity of V, leads to the following estimate

$$\begin{aligned}\mathbb{E}\left[V\left(y_n Z^n \frac{H^n + H^\delta}{2}\right)\right] - \frac{1}{2}\left(\mathbb{E}[V(y_n Z^n H^n)] + \mathbb{E}[V(y_n Z^n H^\delta)]\right) \\ \le -\beta \mathbb{P}[y_n Z^n |H^n - H^\delta| > \varepsilon,\ y_n Z^n (H^n + H^\delta) < N].\end{aligned} \tag{3.6}$$

The convex combination $\frac{1}{2} Z^n (H^n + H^\delta)$ belongs to the dual domain $\mathcal{Y}^{\lambda^n}$, and so the following inequality for the first term on the left-hand side of (3.6) holds:

$$\mathbb{E}\left[V\left(y_n Z^n H^n\right)\right] = v^{\lambda^n}(y_n) \le \mathbb{E}\left[V\left(y_n Z^n \frac{H^n + H^\delta}{2}\right)\right].$$

Combining this estimate with (3.6) gives

$$\beta \mathbb{P}[y_n Z^n |H^n - H^\delta| > \varepsilon,\ y_n Z^n (H^n + H^\delta) < N] \le \frac{1}{2}\left(\mathbb{E}[V(y_n Z^n H^\delta)] - v^{\lambda^n}(y_n)\right)$$

which combined with Markov's inequality grants the inequality

$$\begin{aligned}\beta \mathbb{P}[y_n Z^n |H^n - H^\delta| > \varepsilon] &\le \frac{1}{N} \beta \mathbb{E}[y_n Z^n |H^n + H^\delta|] + \frac{1}{2}\left(\mathbb{E}[V(y_n Z^n H^\delta)] - v^{\lambda^n}(y_n)\right) \\ &\le 2\beta \frac{y_n}{N} + \frac{1}{2}\left(\mathbb{E}[V(y_n Z^n H^\delta)] - v^{\lambda^n}(y_n)\right).\end{aligned}$$

We therefore have the overall estimate

$$\mathbb{P}[y_n Z^n |H^n - H| > 2\varepsilon] \le \frac{2 y_n \delta}{\varepsilon} + 2\frac{y_n}{N} + \frac{1}{2\beta}\left(\mathbb{E}[V(y_n Z^n H^\delta)] - v^{\lambda^n}(y_n)\right).$$

By Lemma 3.5, the third term in on the right-hand side converges to $\mathbb{E}[V(yZH^\delta)]$, whereas the fourth term converges to $v^\lambda(y)$, thanks to Proposition 3.9, and so the limit, as $n \to \infty$, of those two terms can be bounded from above by

$$\mathbb{E}[V(yZH^\delta)] - \mathbb{E}[V(yZH)] \le \delta K, \quad \text{where } K = \mathbb{E}[V(yZ)] - \mathbb{E}[V(yZH)],$$

by a straightforward use of V's convexity. To recapitulate, we have

$$\limsup_{n\to\infty} \mathbb{P}[y_n Z^n |H^n - H| > 2\varepsilon] \le \frac{2y\delta}{\varepsilon} + 2\frac{y}{N} + K\frac{\delta}{2\beta}. \tag{3.7}$$

Letting first $\delta \to 0$, and then $N \to \infty$ in (3.7) shows that

$$\lim_{n\to\infty} \mathbb{P}[y_n Z^n |H^n - H| > \varepsilon] = 0, \quad \forall \varepsilon > 0,$$

meaning $y_n Z^n H^n - y_n Z^n H \to 0$ in probability, and since also $y_n Z^n \to yZ$ in probability, the result follows. □

3.5. *Proof of Proposition 2.13*

Convergence of the value functions u^{λ^n} towards u^{λ^0} implies the convergence of the derivatives $(u^{\lambda^n})'$ (towards $(u^{\lambda^0})'$) (see Theorem 25.7 in [24]). Following the ideas from the proof of Proposition 3.9, we can conclude that $v^{\lambda^n} \to v^{\lambda^0}$ and $(v^{\lambda^n})' \to (v^{\lambda^0})'$. Since $\hat{X}_T^{x,\lambda^n} = -V'((u^{\lambda^n})'(x)Z_T^{\lambda^n})$, $n \in \mathbb{N}_0$, and $U' = (-V')^{-1}$ is a continuous function, the sequence $Z_T^{\lambda^n}$ converges towards $Z_T^{\lambda^0}$ in probability, implying that the convergence $\lambda^n \to \lambda^0$ is appropriate.

By the definition of the dual value functions, we have $v^{\lambda^n}(y) = \mathbb{E}[V(yZ_T^{\lambda^n})]$. The family $\{V^-(yZ_T^{\lambda^n}) : \lambda \in \Lambda\}$ is uniformly integrable, and hence $\mathbb{E}[V^+(yZ_T^{\lambda^n})] \to \mathbb{E}[V^+(yZ_T^{\lambda^0})]$ for every $y > 0$. This observation, and the fact that $V^+(yZ_T^{\lambda^n}) \to V^+(yZ_T^{\lambda^0})$ in probability, can be fed into Scheffe's Lemma to conclude that $\{V(yZ_T^{\lambda^n}) : n \in \mathbb{N}_0\}$ is a uniformly integrable sequence. Setting $y = 1$ completes the proof.

Acknowledgments

The authors would like to thank Morten Mosegaard Christensen, Philip Protter, the participants at the third Carnegie Mellon–Columbia–Cornell–Princeton Conference and an anonymous referee for fruitful discussions.

Financial support from the Danish Center of Accounting and Finance (D-CAF) is gratefully acknowledged by the first author.

Appendix A

Proposition A.1. *Suppose that $\lambda^n \to \lambda^0$ in $\mathbb{L}^2(\mu^M)$, for some sequence $\{\lambda^n\}_{n\in\mathbb{N}_0}$ in $\mathbb{L}^2(\mu^M)$. Then $Z_T^{\lambda^n} \to Z_T^{\lambda_0}$ in probability.*

Proof. The Itô-isometry implies that $\int_0^T \lambda_u^n \, dM_u \to \int_0^T \lambda_u^0 \, dM_u$ in $\mathbb{L}^2(\mathbb{P})$, and, hence also in probability. Thanks to the continuity of the exponential function, it will be enough to show that

$$\int_0^T (\lambda_u^n)^2 \, d\langle M\rangle_u \to \int_0^T (\lambda_u^0)^2 \, d\langle M\rangle_u \text{ in probability.} \tag{A.1}$$

Let us recall a well-known characterization of convergence in probability which states that a sequence $\{X^n\}_{n\in\mathbb{N}}$ of random variables converges towards a random variable X^0 in probability if and only if, for any subsequence $\{X^{n_k}\}_{k\in\mathbb{N}}$ of $\{X^n\}_{n\in\mathbb{N}}$, there exists a further subsequence $\{X^{n_{k_l}}\}_{l\in\mathbb{N}}$ which converges to X^0 almost surely. With this in mind, let $\int_0^T (\lambda_u^{n_k})^2 \, d\langle M\rangle_u$ be an arbitrary subsequence of $\int_0^T (\lambda_u^n)^2 \, d\langle M\rangle_u$. Since $\lambda^{n_k} \to \lambda^0$ in $\mathbb{L}^2(\mu^M)$, we can extract a subsequence of $\{\lambda^{n_k}\}_k$ which converges μ^M-almost everywhere to λ^0. We denote this subsequence by $\{\lambda^{n_k}\}_k$, as well. By Fatou's lemma (applied to the $d\langle M\rangle$-integrals), we have

$$\liminf_{k\to\infty} \int_0^T (\lambda_u^{n_k})^2 \, d\langle M\rangle_u \geq \int_0^T (\lambda_u^0)^2 \, d\langle M\rangle_u. \tag{A.2}$$

Another application of Fatou's lemma (this time with respect to the probability $\mathbb{P}$), and the fact that $\|\lambda^{n_k}\|^2_{\mathbb{L}^2(\mu^M)} \to \|\lambda^0\|^2_{\mathbb{L}^2(\mu^M)}$, imply that

$$\mathbb{E}\left[\int_0^T (\lambda_u^0)^2 \,\mathrm{d}\langle M\rangle_u\right] = \lim_{k\to\infty} \mathbb{E}\left[\int_0^T (\lambda_u^{n_k})^2 \,\mathrm{d}\langle M\rangle_u\right]$$
$$\geq \mathbb{E}\left[\liminf_{k\to\infty} \int_0^T (\lambda_u^{n_k})^2 \,\mathrm{d}\langle M\rangle_u\right] \geq \mathbb{E}\left[\int_0^T (\lambda_u^0)^2 \,\mathrm{d}\langle M\rangle_u\right]$$

which shows that we have equality in (A.2), $\mathbb{P}$-almost surely. To extract an a.s.-convergent subsequence from $\int_0^T (\lambda_u^{n_k})^2 \,\mathrm{d}\langle M\rangle_u$ – and finish the proof – all we need to do is apply the result of Lemma A.2 below. □

Lemma A.2. *Any sequence $\{f^k\}_{k\in\mathbb{N}} \subseteq \mathbb{L}^1(\mathbb{P})$ of non-negative random variables which satisfies the two properties*

$$\lim_{n\to\infty} \mathbb{E}[f^k] = \mathbb{E}[f^0], \quad \liminf_{k\to\infty} f^k = f^0 \quad \mathbb{P}\text{-}a.s. \tag{A.3}$$

for some $f^0 \in \mathbb{L}^1(\mathbb{P})$, has a subsequence $\{f^{k_l}\}_{l\in\mathbb{N}}$ converging almost surely to f^0.

Proof. From (A.3) and Lebesgue's theorem of monotone convergence, we have

$$\lim_{k\to\infty} \mathbb{E}\left[\inf_{m\geq k} f^m\right] = \mathbb{E}[f^0] = \lim_{k\to\infty} \mathbb{E}[f^k]$$

which means that $f^k - \inf_{m\geq k} f^m \to 0$ in $\mathbb{L}^1$. We can, therefore, extract a subsequence $\{f^{k_l}\}_{l\in\mathbb{N}}$ of $\{f^k\}_{k\in\mathbb{N}}$ such that $f^{k_l} - \inf_{m\geq l} f^{k_m}$ converges to 0 $\mathbb{P}$-a.s. Thanks to the monotonicity of $\inf_{m\geq k} f^m$, the sequence f^{k_l} must itself converge $\mathbb{P}$-a.s. towards $\lim_{k\to\infty} \inf_{m\geq k} f^m = \liminf_k f^k = f^0$. □

References

[1] J.P. Ansel, C. Stricker, Lois de martingale, densités et décomposition de Föllmer–Schweizer, Ann. Inst. H. Poincaré Probab. Statist. 28 (3) (1992) 375–392.

[2] J.P. Ansel, C. Stricker, Unicité et existence de la loi minimale, in: Séminaire de Probabilités, XXVII, in: Lecture Notes in Math., vol. 1557, Springer, Berlin, 1993, pp. 22–29.

[3] J.C. Cox, C.F. Huang, Optimal consumption and portfolio policies when asset prices follow a diffusion process, J. Econom. Theory 49 (1989) 33–83.

[4] L. Carasus, M. Rásonyi, Optimal strategies and utility-based prices converge when agents' preferences do, 2005 (preprint).

[5] F. Delbaen, W. Schachermayer, A general version of the fundamental theorem of asset pricing, Math. Ann. 300 (3) (1994) 463–520.

[6] F. Delbaen, W. Schachermayer, The existence of absolutely continuous local martingale measures, Ann. Appl. Probab. 5 (4) (1995) 926–945.

[7] F. Delbaen, W. Schachermayer, A simple counterexample to several problems in the theory of asset pricing, Math. Finance 8 (1) (1998) 1–11.

[8] N. El Karoui, M. Jeanblanc-Picqué, S.E. Shreve, Robustness of the Black and Scholes formula, Math. Finance 8 (2) (1998) 93–126.

[9] H. Föllmer, A. Schied, Stochastic Finance, in: de Gruyter Studies in Mathematics, vol. 27, Walter de Gruyter & Co., Berlin, 2002 (An introduction in discrete time).

[10] I. Gilboa, D. Schmeidler, Maxmin expected utility with nonunique prior, J. Math. Econom. 18 (2) (1989) 141–153.

[11] J. Hadamard, Sur les problèmes aux dérivées partielles et leur signification physique, Princeton University Bulletin (1902) 49–52.

[12] H. He, N.D. Pearson, Consumption and portfolio policies with incomplete markets and short-sale constraints: The finite-dimensional case, Math. Finance 1 (1991) 1–10.

[13] F. Hubalek, W. Schachermayer, When does convergence of asset price processes imply convergence of option prices? Math. Finance 8 (4) (1998) 385–403.

[14] E. Jouini, C. Napp, Convergence of utility functions and convergence of optimal strategies, Finance Stoch. 8 (1) (2004) 133–144.
[15] J. Jacod, A.N. Shiryaev, Limit Theorems for Stochastic Processes, second ed., in: Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 288, Springer-Verlag, Berlin, 2003.
[16] I. Karatzas, J.P. Lehoczky, S.E. Shreve, G.L. Xu, Martingale and duality methods for utility maximization in an incomplete market, SIAM J. Control Optim. 29 (3) (1991) 702–730.
[17] D. Kramkov, W. Schachermayer, The asymptotic elasticity of utility functions and optimal investment in incomplete markets, Ann. Appl. Probab. 9 (3) (1999) 904–950.
[18] I. Karatzas, G. Žitković, Optimal consumption from investment and random endowment in incomplete semimartingale markets, Ann. Probab. 31 (4) (2003) 1821–1858.
[19] F. Maccheroni, M. Marinacci, A. Rustichini, Variational representation of preferences under ambiguity, Working paper no. 5, ICER working paper series, 2004.
[20] R.C. Merton, Optimum consumption and portfolio rules in a continuous-time model, J. Econom. Theory 3 (1971) 373–413.
[21] S.R. Pliska, A stochastic calculus model of continuous trading: Optimal portfolio, Math. Oper. Res. 11 (1986) 371–382.
[22] J.-L. Prigent, Weak Convergence of Financial Markets, Springer Finance, Springer-Verlag, Berlin, 2003.
[23] P.E. Protter, Stochastic Integration and Differential Equations, second ed., in: Applications of Mathematics (Stochastic Modelling and Applied Probability, New York), vol. 21, Springer-Verlag, Berlin, 2004.
[24] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, 1970.
[25] L.C.G. Rogers, The relaxed investor and parameter uncertainty, Finance Stoch. 5 (2) (2001) 131–154.
[26] M. Schweizer, On the minimal martingale measure and the Föllmer–Schweizer decomposition, Stoch. Anal. Appl. 13 (5) (1995) 573–599.
[27] F. Trojani, P. Vanini, A note on robustness in Merton's model of intertemporal consumption and portfolio choice, J. Econom. Dynam. Control 26 (2002) 423–435.