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Note

The Minimum Number of Components in 4-Regular Perfect Systems of Difference Sets*

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The number of components m in regular $(m, 5, c)$-systems is given in the literature to date by the inequality $m \geqslant 4c - 2$ (Bermond *et al.*, "Proceedings, 18th Hungarian Combin. Colloq.", North-Holland, Amsterdam, 1976). The case $m = 4c - 2$ is called extremal. It is proved that $(4c - 2, 5, c)$-systems do not exist. An example of a $(4c, 5, c)$-system with $c = 2$, is given. Since, in a 4-regular system, m must be even , loc. cit., it is concluded that the lower bound on the number of components is given by $m \geqslant 4c$. © 1984 Academic Press, Inc.

Perfect systems of difference sets were introduced in [5]. Further results on perfect systems can be found in [1–4, 6]. We use the definitions and notation of [5]. We shall only consider the regular case of size four. It is convenient to display the differences of the ith component in the form of a triangle D_i as follows:

$$\begin{array}{ccccccc} & & & z_i & & & \\ & & x_i & & y_i & & \\ & u_i & & v_i & & w_i & \\ p_i & & q_i & & r_i & & s_i \end{array}$$

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where

$$\begin{aligned} u_i &= p_i + q_i; & x_i &= p_i + q_i + r_i \\ v_i &= q_i + r_i; & y_i &= q_i + r_i + s_i \\ w_i &= r_i + s_i; & z_i &= p_i + q_i + r_i + s_i. \end{aligned} \tag{1}$$

One can assume that $p_i < s_i$, otherwise interchange p_i and s_i, q_i and r_i, u_i and w_i, x_i and y_i.

We define the sets T, where $T \in \{P, Q, R, S, U, V, W, X, Y, Z\}$, as follows: $T = \{t_i \mid i = 1,..., m\}$ where $t_i \in \{p_i, q_i, r_i, s_i, u_i, v_i, w_i, x_i, y_i, z_i\}$ and we put $t = t_1 + \cdots + t_m$.

The equations (1) imply that

$$2z = p + q + r + s + u + w. \tag{2}$$

If we sum over the first $6m$ differences, we obtain

$$2z \geqslant 3m(6m + 2c - 1). \tag{3}$$

Now summing over the m largest differences, we obtain

$$z \leqslant \frac{m}{2}(19m + 2c - 1). \tag{4}$$

Thus, $m(19m + 2c - 1) \geqslant 3m(6m + 2c - 1)$, which implies $m \geqslant 4c - 2$.

We now prove that the supposition $m = 4c - 2$ for an arbitrary given c leads to a contradiction. In a 4-regular perfect system with $m = 4c - 2$, the inequalities in (3) and (4) become equalities. We then have $Z = \{9m + c,..., 10m + c - 1\}$ and $P \cup Q \cup R \cup S \cup U \cup W = \{c,..., 6m + c - 1\}$. Also, $V \cup X \cup Y = \{6m + c,..., 9m + c - 1\}$. In each difference triangle D_i, $i = 1,..., m$, we have $x_i + y_i = u_i + z_i$, $i = 1,..., m$ from which it follows that $V = \{6m + c,..., 7m + c - 1\}$ and $X \cup Y = \{7m + c,..., 9m + c - 1\}$. Also, $q_i = y_i - w_i$ and $r_i = x_i - u_i$, $i = 1,..., m$. Since $x_i \geqslant 7m + c$, $y_i \geqslant 7m + c$, $u_i \leqslant 6m + c - 1$, $w_i \leqslant 6m + c - 1$, we obtain $q_i \geqslant m + 1$ and $r_i \geqslant m + 1$, for $i = 1,..., m$. Furthermore, $p_i + s_i = z_i - v_i \geqslant 9m + c - (7m + c - 1) = 2m$. Thus, $s_i > m$ and $\{c,..., m\} \subset P$. Let $I = \{i \mid p_i \in \{c,..., m\}\}$. We will compute

$$\sum_{i \in I} p_i$$

in two ways. First,

$$\sum_{i \in I} p_i = \sum_{j=c}^{m} j.$$

Second, for $i \in I$, $p_i = z_i - y_i$, with $z_i = 9m + c - 1 + \alpha_i$, and $y_i = 8m + c - 1 + \beta_i$, $1 \leqslant \alpha_i \leqslant \beta_i \leqslant m$. Thus,

$$\sum_{i \in I} p_i = \sum_{i \in I} (z_i - y_i) = m\,|I| + \sum_{i \in I} \alpha_i - \sum_{i \in I} \beta_i.$$

Since

$$\sum_{i \in I} \alpha_i \geqslant \sum_{j=1}^{m+1-c} j \qquad \text{and} \qquad \sum_{i \in I} \beta_i \leqslant \sum_{j=c}^{m} j,$$

we have

$$\sum_{j=c}^{m} j \geqslant m(m+1-c) + \sum_{j=1}^{m+1-c} j - \sum_{j=c}^{m} j.$$

The above implies that

$$(m+1-c)(m+c) \geqslant m(m+1-c) + \frac{(m+1-c)(m+2-c)}{2}$$

or $2c \geqslant m + 2 - c$ which yields for $m = 4c - 2$, that $c \leqslant 0$, a contradiction. We can now state:

THEOREM 1. *No $(4c-2, 5, c)$-system exists for any c.*

Since m must be even [5, Prop. 2.3], we also have

THEOREM 2. *In every $(m, 5, c)$-system, $m \geqslant 4c$.*

To show that this is the best possible lower bound on m, we include an example of an $(8, 5, 2)$-system. Let d_{ij} denote the jth row of the ith difference triangle D_i, $i = 1,\dots, 8$; $j = 1,\dots, 4$. We list only the first row of each difference triangle, as this completely determines the corresponding component

$d_{11} = (13, 23, 27, 18)$; $\quad d_{51} = (3, 34, 25, 15)$

$d_{21} = (10, 28, 26, 16)$; $\quad d_{61} = (4, 31, 21, 20)$

$d_{31} = (8, 24, 33, 14)$; $\quad d_{71} = (2, 44, 7, 22)$

$d_{41} = (9, 39, 19, 11)$; $\quad d_{81} = (5, 12, 43, 6)$.

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