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On the critical dimension of a fourth order elliptic problem with negative exponent

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ABSTRACT

We study the regularity of the extremal solution of the semilinear biharmonic equation $\beta\Delta^2 u - \tau\Delta u = \frac{\lambda}{(1-u)^2}$ on a ball $B \subset \mathbb{R}^N$, under Navier boundary conditions $u = \Delta u = 0$ on ∂B, where $\lambda > 0$ is a parameter, while $\tau > 0$, $\beta > 0$ are fixed constants. It is known that there exists λ^* such that for $\lambda > \lambda^*$ there is no solution while for $\lambda < \lambda^*$ there is a branch of minimal solutions. Our main result asserts that the extremal solution u^* is regular ($\sup_B u^* < 1$) for $N \leqslant 8$ and $\beta, \tau > 0$ and it is singular ($\sup_B u^* = 1$) for $N \geqslant 9$, $\beta > 0$, and $\tau > 0$ with $\frac{\tau}{\beta}$ small. Our proof for the singularity of extremal solutions in dimensions $N \geqslant 9$ is based on certain improved Hardy–Rellich inequalities.

1. Introduction

Consider the fourth order elliptic problem

$$\begin{cases} \beta\Delta^2 u - \tau\Delta u = \dfrac{\lambda}{(1-u)^2} & \text{in } \Omega, \\ 0 < u \leqslant 1 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{G_λ}$$

where $\lambda > 0$ is a parameter, $\tau > 0$, $\beta > 0$ are fixed constants, and $\Omega \subset \mathbb{R}^N$ ($N \geqslant 2$) is a bounded smooth domain. This problem with $\beta = 0$ models a simple electrostatic Micro-Electromechanical Sys-

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tems (MEMS) device which has been recently studied by many authors. For instance, see [3,5,7–11, 14–16], and the references cited therein.

Recently, Lin and Yang [18] derived the equation (G_λ) in the study of the charged plates in electrostatic actuators. They showed that there exists $0<\lambda^*<\infty$ such that for $\lambda\in(0,\lambda^*)$ (G_λ) has a minimal regular solutions u_λ ($\sup_B u_\lambda<1$) while for $\lambda>\lambda^*$, (G_λ) does not have any regular solution. Moreover, the branch $\lambda\to u_\lambda(x)$ is increasing for each $x\in B$, and therefore the function $u^*=\lim_{\lambda\nearrow\lambda^*}u_\lambda$ can be considered as a generalized solution that corresponds to the pull-in voltage λ^*. Now the important question is whether the extremal solution u^* is regular or not. In a recent paper Guo and Wei [17] proved that the extremal solution u^* is regular for dimensions $N\leqslant 4$. In this paper we consider the problem (G_λ) on the unit ball in $\mathbb{R}^N$:

$$\begin{cases}\beta\Delta^2u-\tau\Delta u=\dfrac{\lambda}{(1-u)^2} & \text{in } B,\\ 0<u\leqslant 1 & \text{in } B,\\ u=\Delta u=0 & \text{on } \partial B,\end{cases}\qquad (P_\lambda)$$

and show that the critical dimension for (P_λ) is $N=9$. Indeed we prove that the extremal solution of (P_λ) is regular ($\sup_B u^*<1$) for $N\leqslant 8$ and $\beta,\tau>0$ and it is singular ($\sup_B u^*=1$) for $N\geqslant 9$, $\beta>0$, and $\tau>0$ with $\frac{\tau}{\beta}$ small. Our proof of regularity of the extremal solution in dimensions $5\leqslant N\leqslant 8$ is heavily inspired by [4,6]. On the other hand we shall use certain improved Hardy–Rellich inequalities to prove that the extremal solution is singular in dimensions $N\geqslant 9$. Our improve Hardy–Rellich inequalities follow from the recent result of Ghoussoub and Moradifam [12,13] about Hardy and Hardy–Rellich inequalities.

We now start by recalling some of the results from [17] concerning (P_λ) that will be needed in the sequel. Define

$$\lambda^*(B):=\sup\bigl\{\lambda>0\colon\ (P_\lambda)\text{ has a classical solution}\bigr\}.$$

We now introduce the following notion of solution.

Definition 1. We say that u is a *weak solution* of (G_λ), if $0\leqslant u\leqslant 1$ a.e. in Ω, $\frac{1}{(1-u)^2}\in L^1(\Omega)$ and if

$$\int_\Omega u\bigl(\beta\Delta^2\phi-\tau\Delta\phi\bigr)\,dx=\lambda\int_\Omega\frac{\phi}{(1-u)^2}\,dx,\quad \forall\phi\in W^{4,2}(\Omega)\cap H_0^1(\Omega).$$

Say that u is a *weak super-solution* (respectively *weak sub-solution*) of (G_λ), if the equality is replaced with $\geqslant$ (respectively $\leqslant$) for $\phi\geqslant 0$.

We now introduce the notion of stability. First, we equip the function space $\mathcal{H}:=H^2(\Omega)\cap H_0^1(\Omega)=W^{2,2}(\Omega)\cap H_0^1(\Omega)$ with the norm

$$\|\psi\|=\left(\int_\Omega\bigl[\tau|\nabla\psi|^2+\beta|\Delta\psi|^2\bigr]\,dx\right)^{\frac{1}{2}}.$$

Definition 2. We say that a weak solution u_λ of (G_λ) is stable (respectively semi-stable) if the first eigenvalue $\mu_{1,\lambda}(u_\lambda)$ of the problem

$$-\tau\Delta h+\beta\Delta^2h-\frac{2\lambda}{(1-u_\lambda)^3}h=\mu h\quad\text{in }\Omega,\qquad h=\Delta h=0\quad\text{on }\partial\Omega \tag{1}$$

is positive (respectively non-negative).

The operator $\beta\Delta^2 u - \tau\Delta u$ satisfies the following maximum principle which will be frequently used in the sequel.

Lemma 1.1. *(See [17].) Let $u \in L^1(\Omega)$. Then $u \geqslant 0$ a.e. in Ω, provided one of the following conditions hold:*

1. $u \in C^4(\bar{\Omega})$, $\beta\Delta^2 u - \tau\Delta u \geqslant 0$ *on* Ω, *and* $u = \Delta u = 0$ *on* $\partial\Omega$.
2. $\int_\Omega u(\beta\Delta^2\phi - \tau\Delta\phi)\,dx \geqslant 0$ *for all* $0 \leqslant \phi \in W^{4,2}(\Omega) \cap H_0^1(\Omega)$.
3. $u \in W^{2,2}(\Omega)$, $u = 0$, $\Delta u \leqslant 0$ *on* ∂B, *and* $\int_\Omega [\beta\Delta u\Delta\phi + \tau\nabla u\nabla\phi]\,dx \geqslant 0$ *for all* $0 \leqslant \phi \in W^{2,2}(\Omega) \cap H_0^1(\Omega)$.

Moreover, either $u \equiv 0$ or $u > 0$ a.e. in Ω.

2. The pull-in voltage

As in [4,6], we are led here to examine problem (P_λ) with non-homogeneous boundary conditions such as

$$\begin{cases} \beta\Delta^2 u - \tau\Delta u = \dfrac{\lambda}{(1-u)^2} & \text{in } B, \\ \alpha < u \leqslant 1 & \text{in } B, \\ u = \alpha, \quad \Delta u = \gamma & \text{on } \partial B, \end{cases} \tag{$P_\lambda, \alpha, \gamma$}$$

where α, γ are given. Whenever we need to emphasis the parameters β and τ we will refer to problem $(P_{\lambda,\alpha,\gamma})$ as $(P_{\lambda,\beta,\tau,\alpha,\gamma})$. In this section and Section 3 we will obtain several results for the following general form of $(P_\lambda, \alpha, \gamma)$,

$$\begin{cases} \beta\Delta^2 u - \tau\Delta u = \dfrac{\lambda}{(1-u)^2} & \text{in } \Omega, \\ \alpha < u \leqslant 1 & \text{in } \Omega, \\ u = \alpha, \quad \Delta u = \gamma & \text{on } \partial\Omega, \end{cases} \tag{$G_\lambda, \alpha, \gamma$}$$

which are analogous to the results obtained by Gui and Wei for (G_λ) in [17].

Let Φ denote the unique solution of

$$\begin{cases} \beta\Delta^2\Phi - \tau\Delta\Phi = 0 & \text{in } \Omega, \\ \Phi = \alpha, \quad \Delta\Phi = \gamma & \text{on } \partial\Omega. \end{cases} \tag{2}$$

We will say that the pair (α, γ) is admissible if $\gamma \leqslant 0$, $\alpha < 1$, and $\sup_\Omega \Phi < 1$. We now introduce a notion of weak solution.

Definition 3. We say that u is a *weak solution* of $(P_{\lambda,\alpha,\gamma})$, if $\alpha \leqslant u \leqslant 1$ a.e. in Ω, $\frac{1}{(1-u)^2} \in L^1(\Omega)$ and if

$$\int_\Omega (u - \Phi)\big(\beta\Delta^2\phi - \tau\Delta\phi\big) = \lambda \int_\Omega \frac{\phi}{(1-u)^2} \quad \forall\phi \in W^{4,2}(\Omega) \cap H_0^1(\Omega),$$

where Φ is given in (2). We say u is a *weak super-solution* (respectively *weak sub-solution*) of $(P_{\lambda,\alpha,\gamma})$, if the equality is replaced with $\geqslant$ (respectively $\leqslant$) for $\phi \geqslant 0$.

Definition 4. We say a weak solution u of $(P_{\lambda,\alpha,\gamma})$ is regular (respectively singular) if $\|u\|_\infty < 1$ (respectively $\|u\|_\infty = 1$).

We now define

$$\lambda^*(\alpha,\gamma) := \sup\{\lambda > 0\colon\ (P_{\lambda,\alpha,\gamma}) \text{ has a classical solution}\}$$

and

$$\lambda_*(\alpha,\gamma) := \sup\{\lambda > 0\colon\ (P_{\lambda,\alpha,\gamma}) \text{ has a weak solution}\}.$$

Observe that by the Implicit Function Theorem, we can classically solve $(P_{\lambda,\alpha,\gamma})$ for small λ's. Therefore, $\lambda^*(\alpha,\gamma)$ and $\lambda_*(\alpha,\gamma)$ are well defined for any admissible pair (α,γ). To cut down on notations we won't always indicate α and γ. For example, λ_* and λ^* will denote the "weak and strong critical voltages" of $(P_{\lambda,\alpha,\gamma})$.

Now let U be a weak super-solution of $(P_{\lambda,\alpha,\gamma})$ and recall the following existence result.

Theorem 2.1. *(See [17].) For every $0 \leqslant f \in L^1(\Omega)$ there exists a unique $0 \leqslant u \in L^1(\Omega)$ which satisfies*

$$\int_\Omega u\big(\beta\Delta^2\phi - \tau\Delta\phi\big)\,dx = \int_\Omega f\phi\,dx,$$

for all $\phi \in W^{4,2}(\Omega) \cap H_0^1(\Omega)$.

We can introduce the following "weak" iterative scheme: $u_0 = U$ and (inductively) let u_n, $n \geqslant 1$, be the solution of

$$\int_\Omega (u_n - \Phi)\big(\beta\Delta^2\phi - \tau\Delta\phi\big) = \lambda\int_\Omega \frac{\phi}{(1-u_{n-1})^2} \quad \forall\phi \in W^{4,2}(\bar{\Omega}) \cap H_0^1(\Omega)$$

given by Theorem 2.1. Since 0 is a sub-solution of $(P_{\lambda,\alpha,\gamma})$, inductively it is easily shown by Lemma 1.1 that $\alpha \leqslant u_{n+1} \leqslant u_n \leqslant U$ for every $n \geqslant 0$. Since

$$(1-u_n)^{-2} \leqslant (1-U)^{-2} \in L^1(\Omega),$$

by Lebesgue theorem the function $u = \lim_{n\to+\infty} u_n$ is a weak solution of $(P_{\lambda,\alpha,\gamma})$ so that $\alpha \leqslant u \leqslant U$. We therefore have the following result.

Lemma 2.2. *Assume the existence of a weak super-solution U of $(P_{\lambda,\alpha,\gamma})$. Then there exists a weak solution u of $(P_{\lambda,\alpha,\gamma})$ so that $\alpha \leqslant u \leqslant U$ a.e. in Ω.*

In particular, for every $\lambda \in (0,\lambda_*)$, we can find a weak solution of $(P_{\lambda,\alpha,\gamma})$. In the same range of λ's, this is still true for regular weak solutions as shown in the following lemma.

Lemma 2.3. *Let (α,γ) be an admissible pair and u be a weak solution of $(P_{\lambda,\alpha,\gamma})$. Then, there exists a regular solution for every $0 < \mu < \lambda$.*

Proof. Let $\epsilon \in (0,1)$ be given and let $\bar{u} = (1-\epsilon)u + \epsilon\Phi$, where Φ is given in (2). By Lemma 1.1 $\sup_\Omega \Phi < \sup_\Omega u \leqslant 1$. Hence,

$$\sup_\Omega \bar{u} \leqslant (1-\epsilon) + \epsilon\sup_\Omega \Phi < 1, \qquad \inf_\Omega \bar{u} \geqslant (1-\epsilon)\alpha + \epsilon\inf_\Omega \Phi = \alpha,$$

and for every $0 \leqslant \phi \in W^{4,2}(\bar{\Omega}) \cap H_0^1(\Omega)$ there holds:

$$\begin{aligned}\int_\Omega (\bar{u} - \Phi)\big(\beta\Delta^2\phi - \tau\Delta\phi\big) &= (1-\epsilon)\int_\Omega (u - \Phi)\big(\beta\Delta^2\phi - \tau\Delta\phi\big)\\ &= (1-\epsilon)\lambda\int_\Omega \frac{\phi}{(1-u)^2}\\ &= (1-\epsilon)^3\lambda\int_\Omega \frac{\phi}{(1-\bar{u}+\epsilon(\Phi-1))^2}\\ &\geqslant (1-\epsilon)^3\lambda\int_\Omega \frac{\phi}{(1-\bar{u})^2}.\end{aligned}$$

Note that $0 \leqslant (1-\epsilon)(1-u) = 1-\bar{u}+\epsilon(\Phi-1) < 1-\bar{u}$. So $\bar{u}$ is a weak super-solution of $(P_{(1-\epsilon)^3\lambda,\alpha,\gamma})$ so that $\sup_\Omega \bar{u} < 1$. By Lemma 2.2 we get the existence of a weak solution w of $(P_{(1-\epsilon)^3\lambda,\alpha,\gamma})$ so that $\alpha \leqslant w \leqslant \bar{u}$. In particular, $\sup_\Omega w < 1$ and w is a regular weak solution. Since $\epsilon \in (0,1)$ is arbitrarily chosen, the proof is done. □

Lemma 2.3 implies the existence of a regular weak solution U_λ for every $\lambda \in (0,\lambda_*)$. Introduce now a "classical" iterative scheme: $u_0 = 0$ and (inductively) $u_n = v_n + \Phi$, $n \geqslant 1$, where $v_n \in W^{4,2}(\Omega) \cap H_0^1(\Omega)$ is the solution of

$$\beta\Delta^2 v_n - \tau\Delta v_n = \beta\Delta^2 u_n - \tau\Delta u_n = \frac{\lambda}{(1-u_{n-1})^2} \quad \text{in } \Omega \text{ and } \Delta v_n = 0 \text{ on } \partial\Omega. \tag{3}$$

Since $v_n \in W^{4,2}(\Omega) \cap H_0^1(\Omega)$, u_n is also a weak solution of (3), and by Lemma 1.1 we know that $\alpha \leqslant u_n \leqslant u_{n+1} \leqslant U_\lambda$ for every $n \geqslant 0$. Since $\sup_\Omega u_n \leqslant \sup_\Omega U_\lambda < 1$ for $n \geqslant 0$, we get that $(1-u_{n-1})^{-2} \in L^2(\Omega)$ and the existence of v_n is guaranteed. Since v_n is easily seen to be uniformly bounded in $H^2(\Omega)$, we have that $u_\lambda := \lim_{n\to+\infty} u_n$ does hold pointwise and weakly in $H^2(\Omega)$. By Lebesgue theorem, we have that u_λ is a radial weak solution of (P_λ) so that $\sup_\Omega u_\lambda \leqslant \sup_\Omega U_\lambda < 1$. By elliptic regularity theory [1], $u_\lambda \in C^\infty(\bar{\Omega})$ and $u_\lambda = \Delta u_\lambda = 0$ on $\partial\Omega$. So we can integrate by parts to get

$$\int_\Omega \beta\big(\Delta^2 u_\lambda - \tau\Delta u_\lambda\big)\phi\,dx = \int_\Omega u_\lambda\big(\beta\Delta^2\phi - \tau\Delta\phi\big)\,dx = \lambda\int_\Omega \frac{\phi}{(1-u_\lambda)^2}$$

for every $\phi \in W^{4,2}(\Omega) \cap H_0^1(\Omega)$. Hence, u_λ is a classical solution of (P_λ) showing that $\lambda^* = \lambda_*$.

Since the argument above shows that $u_\lambda < U$ for any other classical solution U of (P_μ, α, γ) with $\mu \geqslant \lambda$, we have that u_λ is exactly the minimal solution and u_λ is strictly increasing as $\lambda \uparrow \lambda^*$. In particular, we can define u^* in the usual way: $u^*(x) = \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$.

Lemma 2.4. $\lambda^*(\Omega) < +\infty$.

Proof. Let u be a classical solution of $(P_{\lambda,\alpha,\gamma})$ and let (ψ, μ_1) with $\Delta\psi = 0$ on $\partial\Omega$ denote the first eigenpair of $\beta\Delta^2 - \tau\Delta$ in $H^2(\Omega) \cap H_0^1(\Omega)$ with $\psi > 0$. Now let C be such that

$$\int_{\partial\Omega} \big((\tau\alpha - \beta\gamma)\partial_\nu\psi - \beta\alpha\partial_\nu(\Delta\psi)\big) = C\int_\Omega \psi.$$

Multiplying $(P_{\lambda,\alpha,\gamma})$ by ψ and then integrating by parts one arrives at

$$\int_\Omega \left(\frac{\lambda}{(1-u)^2} - \mu_1 u - C\right)\psi = 0.$$

Since $\psi > 0$ there must exist a point $\bar{x} \in \Omega$ where $\frac{\lambda}{(1-u(\bar{x}))^2} - \mu_1 u(\bar{x}) - C \leqslant 0$. Since $\alpha < u(\bar{x}) < 1$, hence one can conclude that $\lambda \leqslant \sup_{0<u<1}(\mu_1 u + C)(1-u)^2$, which shows that $\lambda^* < +\infty$. □

In conclusion, we have shown the following description of the minimal branch.

Theorem 2.5. $\lambda^* \in (0, +\infty)$ *and the following holds*:

1. *For each* $0 < \lambda < \lambda^*$ *there exists a regular and minimal solution* u_λ *of* $(P_{\lambda,\alpha,\gamma})$.
2. *For each* $x \in \Omega$ *the map* $\lambda \mapsto u_\lambda(x)$ *is strictly increasing on* $(0, \lambda^*)$.
3. *For* $\lambda > \lambda^*$ *there are no weak solutions of* $(P_{\lambda,\alpha,\gamma})$.

3. Stability of the minimal solutions

This section is devoted to the proof of the following stability result for minimal solutions. We shall need the following notion of $\mathcal{H}$-weak solutions, which is an intermediate class between classical and weak solutions.

Definition 5. We say that u is an $\mathcal{H}$-weak solution of $(P_{\lambda,\alpha,\gamma})$ if $u - \Phi \in H^2(\Omega) \cap H^1_0(\Omega)$, $0 \leqslant u \leqslant 1$ a.e. in Ω, $\frac{1}{(1-u)^2} \in L^1(\Omega)$ and

$$\int_\Omega [\beta \Delta u \Delta \phi + \tau \nabla u \nabla \phi]\, dx = \lambda \int_\Omega \frac{\phi}{(1-u)^2}, \quad \forall \phi \in W^{2,2}(\Omega) \cap H^1_0(\Omega),$$

where Φ is given by (2). We say that u is an $\mathcal{H}$-weak super-solution (respectively an $\mathcal{H}$-weak sub-solution) of $(P_{\lambda,\alpha,\gamma})$ if for $\phi \geqslant 0$ the equality is replaced with $\geqslant$ (respectively $\leqslant$) and $u \geqslant 0$ (respectively $\leqslant$), $\Delta u \leqslant 0$ (respectively $\geqslant$) on $\partial\Omega$.

Theorem 3.1. *Suppose that* (α, γ) *is an admissible pair.*

1. *The minimal solution* u_λ *is stable, and is the unique semi-stable* $\mathcal{H}$*-weak solution of* $(P_{\lambda,\alpha,\gamma})$.
2. *The function* $u^* := \lim_{\lambda \nearrow \lambda^*} u_\lambda$ *is a well-defined semi-stable* $\mathcal{H}$*-weak solution of* $(P_{\lambda^*,\alpha,\gamma})$.
3. u^* *is the unique* $\mathcal{H}$*-weak solution of* $(P_{\lambda^*,\alpha,\gamma})$, *and when* u^* *is classical solution, then* $\mu_1(u^*) = 0$.
4. *If* v *is a singular, semi-stable* $\mathcal{H}$*-weak solution of* $(P_{\lambda,\alpha,\gamma})$, *then* $v = u^*$ *and* $\lambda = \lambda^*$.

The main tool is the following comparison lemma which is valid exactly in the class $\mathcal{H}$.

Lemma 3.2. *Let* (α, γ) *be an admissible pair and* u *be a semi-stable* $\mathcal{H}$*-weak solution of* $(P_{\lambda,\alpha,\gamma})$. *Assume* U *is a* $\mathcal{H}$*-weak super-solution of* $(P_{\lambda,\alpha,\gamma})$. *Then*

1. $u \leqslant U$ *a.e. in* Ω;
2. *if* u *is a classical solution and* $\mu_1(u) = 0$ *then* $U = u$.

Proof. (i) Define $w := u - U$. Then by means of the Moreau decomposition for the biharmonic operator (see [2,19]), there exist w_1 and $w_2 \in H^2(\Omega) \cap H^1_0(\Omega)$, with $w = w_1 + w_2$, $w_1 \geqslant 0$ a.e.,

$\beta\Delta^2 w_2 - \tau\Delta w_2 \leqslant 0$ in the $\mathcal{H}$-weak sense and $\int_\Omega \beta\Delta w_1 \Delta w_2 + \tau\nabla w_1.\nabla w_2 = 0$. Lemma 1.1 gives that $w_2 \leqslant 0$ a.e. in Ω.

Given $0 \leqslant \phi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega \beta\Delta w \Delta\phi + \tau\nabla w.\nabla\phi \leqslant \lambda \int_\Omega \big(f(u) - f(U)\big)\phi,$$

where $f(u) := (1-u)^{-2}$. Since u is semi-stable, one has

$$\lambda \int_\Omega f'(u) w_1^2 \leqslant \int_\Omega \beta(\Delta w_1)^2 + \tau|\nabla w_1|^2 = \int_\Omega \beta\Delta w\Delta w_1 + \tau\nabla w.\nabla w_1 \leqslant \lambda \int_\Omega \big(f(u) - f(U)\big) w_1.$$

Since $w_1 \geqslant w$ one has

$$\int_\Omega f'(u) w w_1 \leqslant \int_\Omega \big(f(u) - f(U)\big) w_1,$$

which re-arranged gives

$$\int_\Omega \tilde{f} w_1 \geqslant 0,$$

where $\tilde{f}(u) = f(u) - f(U) - f'(u)(u-U)$. The strict convexity of f gives $\tilde{f} \leqslant 0$ and $\tilde{f} < 0$ whenever $u \neq U$. Since $w_1 \geqslant 0$ a.e. in Ω, one sees that $w \leqslant 0$ a.e. in Ω. The inequality $u \leqslant U$ a.e. in Ω is then established.

(ii) Since u is a classical solution, it is easy to see that the infimum of $\mu_1(u)$ is attained at some ϕ. The function ϕ is then the first eigenfunction of $\beta\Delta^2 - \tau\Delta - \frac{2\lambda}{(1-u)^3}$ in $H^2(\Omega) \cap H_0^1(\Omega)$. Now we show that ϕ is of fixed sign. Using the above decomposition, one has $\phi = \phi_1 + \phi_2$ where $\phi_i \in H^2(\Omega) \cap H_0^1(\Omega)$ for $i = 1, 2$, $\phi_1 \geqslant 0$, $\int_\Omega \beta\Delta\phi_1\Delta\phi_2 + \tau\nabla\phi_1.\nabla\phi_2 = 0$ and $\beta\Delta^2\phi_2 - \tau\Delta\phi_2 \leqslant 0$ in the $\mathcal{H}$-weak sense. If ϕ changes sign, then $\phi_1 \not\equiv 0$ and $\phi_2 < 0$ in Ω (recall that either $\phi_2 < 0$ or $\phi_2 = 0$ a.e. in Ω). We can write now

$$\begin{aligned} 0 = \mu_1(u) &\leqslant \frac{\int_\Omega \beta(\Delta(\phi_1 - \phi_2))^2 + \tau|\nabla(\phi_1 - \phi_2)|^2 - \lambda f'(u)(\phi_1 - \phi_2)^2}{\int_\Omega (\phi_1 - \phi_2)^2} \\ &< \frac{\int_\Omega \beta(\Delta\phi)^2 + \tau|\nabla\phi|^2 - \lambda f'(u)\phi^2}{\int_\Omega \phi^2} \\ &= \mu_1(u), \end{aligned}$$

in view of $\phi_1\phi_2 < -\phi_1\phi_2$ in a set of positive measure, leading to a contradiction.

So we can assume $\phi \geqslant 0$, and by Lemma 1.1 we have $\phi > 0$ in Ω. For $0 \leqslant t \leqslant 1$, define

$$g(t) = \int_\Omega \beta\Delta\big[tU + (1-t)u\big]\Delta\phi + \tau\nabla\big[tU + (1-t)u\big].\nabla\phi - \lambda \int_\Omega f\big(tU + (1-t)u\big)\phi,$$

where ϕ is the above first eigenfunction. Since f is convex one sees that

$$g(t) \geqslant \lambda \int_\Omega \bigl[tf(U) + (1-t)f(u) - f\bigl(tU + (1-t)u\bigr)\bigr]\phi \geqslant 0$$

for every $t \geqslant 0$. Since $g(0) = 0$ and

$$g'(0) = \int_\Omega \beta\Delta(U-u)\Delta\phi + \tau\nabla(U-u).\nabla\phi - \lambda f'(u)(U-u)\phi = 0,$$

we get that

$$g''(0) = -\lambda \int_\Omega f''(u)(U-u)^2\phi \geqslant 0.$$

Since $f''(u)\phi > 0$ in Ω, we finally get that $U = u$ a.e. in Ω. □

A more general version of Lemma 3.2 is available in the following.

Lemma 3.3. *Let (α,γ) be an admissible pair and $\gamma' \leqslant 0$. Let u be a semi-stable $\mathcal{H}$-weak sub-solution of $(P_{\lambda,\alpha,\gamma})$ with $u = \alpha' \leqslant \alpha$, $\Delta u = \beta' \geqslant \beta$ on $\partial\Omega$. Assume that U is a $\mathcal{H}$-weak super-solution of $(P_{\lambda,\alpha,\gamma})$ with $U = \alpha$, $\Delta U = \beta$ on $\partial\Omega$. Then $U \geqslant u$ a.e. in Ω.*

Proof. Let $\tilde{u} \in H^2(\Omega) \cap H_0^1(\Omega)$ denote a weak solution of $\beta\Delta^2\tilde{u} - \tau\Delta\tilde{u} = \beta\Delta^2(u-U) - \tau\Delta(u-U)$ in Ω and $\tilde{u} = \Delta\tilde{u} = 0$ on $\partial\Omega$. Since $\tilde{u} - u + U \geqslant 0$ and $\Delta(\tilde{u} - u + U) \leqslant 0$ on $\partial\Omega$, by Lemma 1.1 one has that $\tilde{u} \geqslant u - U$ a.e. in Ω. By means of the Moreau decomposition (see [2,19]) we write $\tilde{u}$ as $\tilde{u} = w + v$, where $w, v \in H_0^2(\Omega)$, $w \geqslant 0$ a.e. in Ω, $\beta\Delta^2 v - \tau\Delta v \leqslant 0$ in a $\mathcal{H}$-weak sense and $\int_\Omega \beta\Delta w\Delta v + \tau\nabla w.\nabla v = 0$. Then for $0 \leqslant \phi \in W^{4,2}(\bar{\Omega}) \cap H_0^1(\Omega)$, one has

$$\int_\Omega \beta\Delta\tilde{u}\Delta\phi + \tau\nabla\tilde{u}.\nabla\phi \leqslant \lambda \int_\Omega \bigl(f(u) - f(U)\bigr)\phi.$$

In particular, we have

$$\int_\Omega \beta\Delta\tilde{u}\Delta w + \tau\nabla\tilde{u}.\nabla w \leqslant \lambda \int_\Omega \bigl(f(u) - f(U)\bigr)w.$$

Since the semi-stability of u gives that

$$\lambda \int_\Omega f'(u)w^2 \leqslant \int_\Omega \beta(\Delta w)^2 + \tau|\nabla w|^2 = \int_\Omega \beta\Delta\tilde{u}\Delta w + \tau\nabla\tilde{u}.\nabla w,$$

we get that

$$\int_\Omega f'(u)w^2 \leqslant \int_\Omega \bigl(f(u) - f(U)\bigr)w.$$

By Lemma 1.1 we have $v \leqslant 0$ and then $w \geqslant \tilde{u} \geqslant u - U$ a.e. in Ω. So we obtain that

$$0 \leqslant \int_\Omega \big(f(u) - f(U) - f'(u)(u-U)\big) w.$$

The strict convexity of f implies that $U \geqslant u$ a.e. in Ω. □

We need also some a priori estimates along the minimal branch u_λ.

Lemma 3.4. *Let (α, γ) be an admissible pair. Then for every $\lambda \in (0, \lambda^*)$, we have*

$$2 \int_\Omega \frac{(u_\lambda - \Phi)^2}{(1-u_\lambda)^3} \leqslant \int_\Omega \frac{u_\lambda - \Phi}{(1-u_\lambda)^2},$$

where Φ is given by (2)*. In particular, there is a constant $C > 0$ independent of λ so that*

$$\int_\Omega \big(\tau |\nabla u_\lambda|^2 + \beta |\Delta u_\lambda|^2\big)\,dx + \int_\Omega \frac{1}{(1-u_\lambda)^3} \leqslant C, \tag{4}$$

for every $\lambda \in (0, \lambda^*)$.

Proof. Testing $(P_{\lambda,\alpha,\gamma})$ on $u_\lambda - \Phi \in W^{4,2}(\Omega) \cap H_0^1(\Omega)$, we see that

$$\lambda \int_\Omega \frac{u_\lambda - \Phi}{(1-u_\lambda)^2} = \int_\Omega \big(\tau \big|\nabla(u_\lambda - \Phi)\big|^2 + \beta\big(\Delta(u_\lambda - \Phi)\big)^2\big)\,dx \geqslant 2\lambda \int_\Omega \frac{(u_\lambda - \Phi)^2}{(1-u_\lambda)^3}.$$

In the view of $\beta \Delta^2 \Phi - \tau \Delta \Phi = 0$. In particular, for $\delta > 0$ small we have that

$$\int_{\{|u_\lambda| \geqslant \delta\}} \frac{1}{(1-u_\lambda)^3} \leqslant \frac{1}{\delta^2} \int_{\{|u_\lambda - \Phi| \geqslant \delta\}} \frac{(u_\lambda - \Phi)^2}{(1-u_\lambda)^3} \leqslant \frac{1}{\delta^2} \int_\Omega \frac{1}{(1-u_\lambda)^2}$$
$$\leqslant \delta \int_{\{|u_\lambda - \Phi| \geqslant \delta\}} \frac{1}{(1-u_\lambda)^3} + C_\delta$$

by means of Young's inequality. Since for δ small

$$\int_{\{|u_\lambda - \Phi| \leqslant \delta\}} \frac{1}{(1-u_\lambda)^3} \leqslant C,$$

for some $C > 0$, we get that

$$\int_\Omega \frac{1}{(1-u_\lambda)^3} \leqslant C,$$

for some $C > 0$ and for every $\lambda \in (0, \lambda^*)$. Since

$$\int_\Omega \left(\tau|\nabla u_\lambda|^2 + \beta|\Delta u_\lambda|^2\right) dx = \int_\Omega (\beta \Delta u_\lambda \Delta \Phi + \tau \nabla u_\lambda . \nabla \Phi) + \lambda \int_\Omega \frac{u_\lambda - \Phi}{(1-u_\lambda)^2}$$

$$\leqslant \delta \int_\Omega \left(\tau|\nabla u_\lambda|^2 + \beta|\Delta u_\lambda|^2\right) dx + C_\delta + C\left(\int_\Omega \frac{1}{(1-u_\lambda)^3}\right)^{\frac{2}{3}}$$

in view of Young's and Hölder's inequalities, estimate (4) is finally established. □

Proof of Theorem 3.1. (1) Since $\|u_\lambda\|_\infty < 1$, the infimum defining $\mu_1(u_\lambda)$ is achieved at a first eigenfunction for every $\lambda \in (0, \lambda^*)$. Since $\lambda \mapsto u_\lambda(x)$ is increasing for every $x \in \Omega$, it is easily seen that $\lambda \mapsto \mu_1(u_\lambda)$ is a decreasing and continuous function on $(0, \lambda^*)$. Define

$$\lambda_{**} := \sup\left\{0 < \lambda < \lambda^*\colon\ \mu_1(u_\lambda) > 0\right\}.$$

We have that $\lambda_{**} = \lambda^*$. Indeed, otherwise we would have $\mu_1(u_{\lambda_{**}}) = 0$, and for every $\mu \in (\lambda_{**}, \lambda^*)$, u_μ would be a classical super-solution of $(P_{\lambda_{**},\alpha,\gamma})$. A contradiction arises since Lemma 3.2 implies $u_\mu = u_{\lambda_{**}}$. Finally, Lemma 3.2 guarantees the uniqueness in the class of semi-stable $\mathcal{H}$-weak solutions.

(2) It follows from (4) that $u_\lambda \to u^*$ in a pointwise sense and weakly in $H^2(\Omega)$, and $\frac{1}{1-u^*} \in L^3(\Omega)$. In particular, u^* is a H^2-weak solution of $(P_{\lambda^*,\alpha,\gamma})$ which is also semi-stable as the limiting function of the semi-stable solutions $\{u_\lambda\}$.

(3) Whenever $\|u^*\|_\infty < 1$, the function u^* is a classical solution, and by the Implicit Function Theorem we have that $\mu_1(u^*) = 0$ to prevent the continuation of the minimal branch beyond λ^*. By Lemma 3.2, u^* is then the unique $\mathcal{H}$-weak solution of $(P_{\lambda^*,\alpha,\gamma})$.

(4) If $\lambda < \lambda^*$, we get by uniqueness that $v = u_\lambda$. So v is not singular and a contradiction arises. Now, by Theorem 3(3) we have that $\lambda = \lambda^*$. Since v is a semi-stable $\mathcal{H}$-weak solution of $(P_{\lambda^*,\alpha,\gamma})$ and u^* is a $\mathcal{H}$-weak super-solution of $(P_{\lambda^*,\alpha,\gamma})$, we can apply Lemma 3.2 to get $v \leqslant u^*$ a.e. in Ω. Since u^* is also a semi-stable solution, we can reverse the roles of v and u^* in Lemma 3.2 to see that $v \geqslant u^*$ a.e. in Ω. So equality $v = u^*$ holds and the proof is done. □

4. Regularity of the extremal solutions in dimensions $N \leqslant 8$

In this section we shall show that the extremal solution is regular in small dimensions. Let us begin with the following lemma.

Lemma 4.1. *Let $N \geqslant 5$ and (u^*, λ^*) be the extremal pair of (P_λ). If u^* is singular, and he set*

$$\Gamma := \left\{r \in (0,1)\colon\ u_\delta(r) > u^*(r)\right\} \tag{5}$$

is non-empty, where $u_\delta(x) := 1 - C_\delta|x|^{\frac{4}{3}}$ and $C_\delta > 1$ is a constant. Then there exists $r_1 \in (0,1)$ such that $u_\delta(r_1) \geqslant u^*(r_1)$ and $\Delta u_\delta(r_1) \leqslant \Delta u^*(r_1)$.

Proof. Assume by contradiction that for every r with $u_\delta(r_1) \geqslant u^*(r_1)$ one has $\Delta u_\delta(r_1) > \Delta u^*(r_1)$. Since Γ is non-empty and

$$u_\delta(1) = 1 - C_\delta < 0 = u^*(1),$$

there exists $s_1 \in (0,1)$ such that $u_\delta(s_1) = u^*(s_1)$. We claim that

$$u_\delta(s) > u^*(s),$$

for $0 < s < s_1$. Assume that there exist $s_3 < s_2 \leqslant s_1$ such that $u^*(s_2) = u_\delta(s_2)$, $u^*(s_3) = u_\delta(s_3)$ and $u_\delta(s) \geqslant u^*(s)$ for $s \in (s_3, s_2)$. By our assumption $\Delta u_s > \Delta u^*(s)$ for $s \in (s_3, s_2)$ which contradicts the maximum principle and justifies the claim. Therefore $u_\delta(s) > u^*(s)$ for $0 < s < s_1$. Now set $w := u_\delta - u^*$. Then $w \geqslant 0$ on B_{s_1} and $\Delta w \leqslant 0$ in B_{s_1}. Since $w(0) = 0$, by strong maximum principle we get $w \equiv 0$ on B_{s_1}. This is a contradiction and completes the proof. □

Theorem 4.2. *Let $N \geqslant 5$ and (u^*, λ^*) be the extremal pair of (P_λ). When u^* is singular, then*

$$1 - u^* \leqslant C|x|^{\frac{4}{3}} \quad in\ B,$$

where $C := (\frac{\lambda^*}{\beta\bar{\lambda}})^{\frac{1}{3}}$ and $\bar{\lambda} := \frac{8(N-\frac{2}{3})(N-\frac{8}{3})}{9}$.

Proof. For $\delta > 0$, define $u_\delta(x) := 1 - C_\delta |x|^{\frac{4}{3}}$ with $C_\delta := (\frac{\lambda^*}{\beta\bar{\lambda}} + \delta)^{\frac{1}{3}} > 1$. Since $N \geqslant 5$, we have that $u_\delta \in H^2_{\mathrm{loc}}(\mathbb{R}^N)$ and u_δ is a $\mathcal{H}$-weak solution of

$$\beta\Delta^2 u_\delta - \tau\Delta u_\delta = \frac{\lambda^* + \beta\delta\bar{\lambda}}{(1-u_\delta)^2} + \frac{4}{3}\tau C_\delta\left(N - \frac{2}{3}\right)|x|^{-\frac{2}{3}} \quad \text{in } \mathbb{R}^N.$$

We claim that $u_\delta \leqslant u^*$ in B, which will finish the proof by just letting $\delta \to 0$.

Assume by contradiction that the set $\Gamma := \{r \in (0,1)\colon\ u_\delta(r) > u^*(r)\}$ is non-empty. By Lemma 4.1 the set

$$\Lambda := \left\{r \in (0,1)\colon\ u_\delta(r) \geqslant u^*(r) \text{ and } \Delta u_\delta(r) \leqslant \Delta u^*(r)\right\}$$

is non-empty. Let $r_1 \in \Lambda$. Since

$$u_\delta(1) = 1 - C_\delta < 0 = u^*(1),$$

we have that $0 < r_1 < 1$. Define

$$\alpha := u_*(r_1) \leqslant u_\delta(r_1), \qquad \gamma := \Delta u^*(r_1) \geqslant \Delta u_\delta(r_1).$$

Setting $u_{\delta,r_1} = r_1^{-\frac{4}{3}}(u_\delta(r_1 r) - 1) + 1$, we see that u_{δ,r_1} is a $\mathcal{H}$-weak super-solution of $(P_{\lambda^*+\delta\lambda,\beta,r_1^{-2}\tau,\alpha',\gamma'})$, where

$$\alpha' := r_1^{-\frac{4}{3}}(\alpha - 1) + 1, \qquad \gamma' = r_1^{\frac{2}{3}}\gamma.$$

Similarly, define $u^*_{r_1}(r) = r_1^{-\frac{4}{3}}(u^*(r_1 r) - 1) + 1$. Note that $\Delta^2 u^* - \alpha\Delta u^* \geqslant 0$ in B and $\Delta u^* = 0$ on ∂B. Hence, by maximum principle we have $\Delta u^* \leqslant 0$ in B and therefore $\gamma' \leqslant 0$. Also obviously $\alpha' < 1$. So, (α', γ') is an admissible pair and by Theorem 3.1(4) we get that $(u^*_{r_1}, \lambda^*)$ coincides with the extremal pair of $(P_{\lambda,\beta,r_1^{-2}\tau,\alpha',\gamma'})$ in B. Also by Lemma 2.2 we get the existence of a week solution of $(P_{\lambda^*+\delta\lambda,\beta,r_1^{-2}\tau,\alpha',\gamma'})$. Since $\lambda^* + \delta\lambda > \lambda^*$, we contradict the fact that λ^* is the extremal parameter of $(P_{\lambda,\beta,r_1^{-2}\tau,\alpha',\gamma'})$. □

Now we are ready to prove the following result.

Theorem 4.3. *If $5 \leqslant N \leqslant 8$, then the extremal solution u^* of $(P)_\lambda$ is regular.*

Proof. Assume that u^* is singular. For $\epsilon > 0$ define $\varphi(x) := |x|^{\frac{4-N}{2}+\epsilon}$ and note that

$$(\Delta\varphi)^2 = \big(H_N + O(\epsilon)\big)|x|^{-N+2\epsilon}, \quad \text{where } H_N := \frac{N^2(N-4)^2}{16}.$$

Given $\eta \in C_0^\infty(B)$, and since $N \geqslant 5$, we can use the test function $\eta\varphi \in H_0^2(B)$ into the stability inequality to obtain

$$2\lambda^* \int_B \frac{\varphi^2}{(1-u^*)^3} \leqslant \beta \int_B (\Delta\varphi)^2 + \tau \int_B |\nabla\varphi|^2 + O(1),$$

where $O(1)$ is a bounded function as $\epsilon \to 0$. By Theorem 4.2 we find

$$2\bar{\lambda} \int_B \frac{\varphi^2}{|x|^4} \leqslant \int_B (\Delta\varphi)^2 + O(1),$$

and then

$$2\bar{\lambda} \int_B |x|^{-N+2\epsilon} \leqslant \big(H_N + O(\epsilon)\big) \int_B |x|^{-N+2\epsilon} + O(1).$$

Computing the integrals on obtains

$$2\bar{\lambda} \leqslant H_N + O(\epsilon).$$

Letting $\epsilon \to 0$ we get $2\bar{\lambda} \leqslant H_N$. Graphing this relation we see that $N \geqslant 9$. □

5. The extremal solution is singular in dimensions $N \geqslant 9$

In this section we will show that the extremal solution u^* of $(P_{\lambda,\beta,\tau,0,0})$ in dimensions $N \geqslant 9$ is singular for $\tau > 0$ sufficiently small. To do this, first we shall show that the extremal solution of $(P_{\lambda,1,0,0,0})$ is singular in dimensions $N \geqslant 9$. Again to cut down the notation we won't always indicate that $\beta = 1$ and $\tau = 0$.

We have to distinguish between three different ranges for the dimension. For each range, we will need a suitable Hardy–Rellich type inequality that will be established in Appendix A, by using the recent results of Ghoussoub and Moradifam [12].

- *Case* $N \geqslant 16$. To establish the singularity of u^* for these dimensions we shall need the classical Hardy–Rellich inequality, which is valid for all $\phi \in H^2(B) \cap H_0^1(B)$:

$$\int_B (\Delta\phi)^2\,dx \geqslant \frac{N^2(N-4)^2}{16} \int_B \frac{\phi^2}{|x|^4}\,dx. \tag{6}$$

- *Case* $10 \leqslant N \leqslant 16$. For this case, we shall need the following inequality valid for all $\phi \in H^2(B) \cap H_0^1(B)$,

$$\int_B (\Delta\phi)^2 \geqslant \frac{(N-2)^2(N-4)^2}{16}\int_B \frac{\phi^2}{(|x|^2-\frac{N}{2(N-1)}|x|^{\frac{N}{2}+1})(|x|^2-|x|^{\frac{N}{2}})} + \frac{(N-1)(N-4)^2}{4}\int_B \frac{\phi^2}{|x|^2(|x|^2-|x|^{\frac{N}{2}})}. \tag{7}$$

- *Case* $N=9$. This case is the trickiest and will require the following inequality for all $\phi \in H^2(B)\cap H_0^1(B)$, which is valid for $N \geqslant 7$,

$$\int_B |\Delta u|^2 \geqslant \int_B W\big(|x|\big)u^2. \tag{8}$$

where

$$W(r) = K(r)\left(\frac{(N-2)^2}{4(r^2-\frac{N}{2(N-1)}r^{\frac{N}{2}+1})} + \frac{(N-1)}{r^2}\right),$$

$$K(r) = -\frac{\varphi''(r)+\frac{(n-3)}{r}\varphi'(r)}{\varphi(r)},$$

and

$$\varphi(r) = r^{-\frac{N}{2}+2} + 9r^{-2} + 10r - 20.$$

The next lemma will be our main tool to guarantee that u^* is singular for $N \geqslant 9$. The proof is based on an upper estimate by a singular stable sub-solution.

Lemma 5.1. *Suppose there exist $\lambda' > 0$ and a radial function $u \in H^2(B)\cap W^{4,\infty}_{\mathrm{loc}}(B\setminus\{0\})$ such that*

$$\Delta^2 u \leqslant \frac{\lambda'}{(1-u)^2} \quad \text{for } 0<r<1, \tag{9}$$

$$u(1)=0, \qquad \Delta u|_{r=1}=0, \tag{10}$$

$$u \text{ is singular}, \tag{11}$$

and

$$2\beta\int_B \frac{\varphi^2}{(1-u)^3} \leqslant \int_B (\Delta\varphi)^2 \quad \text{for all } \varphi \in H^2(B)\cap H_0^1(B), \tag{12}$$

for some $\beta > \lambda'$. Then u^* is singular and

$$\lambda^* \leqslant \lambda'. \tag{13}$$

Proof. By Lemma 3.3 we have (13). Let $\frac{\lambda'}{\beta} < \gamma < 1$ and

$$\alpha := \left(\frac{\gamma\lambda^*}{\lambda'}\right)^{\frac{1}{3}}, \tag{14}$$

and define $\bar{u} := 1 - \alpha(1-u)$. We claim that

$$u^* \leqslant \bar{u} \quad \text{in } B. \tag{15}$$

To prove this, we shall show that for $\lambda < \lambda^*$,

$$u_\lambda \leqslant \bar{u} \quad \text{in } B. \tag{16}$$

Indeed, we have

$$\Delta^2(\bar{u}) = \alpha\Delta^2(\bar{u}) \leqslant \frac{\alpha\lambda'}{(1-u)^2} = \frac{\alpha^3\lambda'}{(1-\bar{u})^2}.$$

By (13) and the choice of α,

$$\alpha^3\lambda' < \lambda^*.$$

To prove (15) it suffices to prove it for $\alpha^3\lambda' < \lambda < \lambda^*$. Fix such λ and assume that (15) is not true. Then

$$\Lambda = \left\{0 \leqslant R \leqslant 1 \,\middle|\, u_\lambda(R) > \bar{u}(R)\right\},$$

in non-empty. There exists $0 < R_1 < 1$, such that $u_\lambda(R_1) \geqslant u^*(R_1)$ and $\Delta u_\lambda(R_1) \leqslant \Delta u^*(R_1)$, since otherwise we can find $0 < s_1 < s_2 < 1$ so that $u_\lambda(s_1) = \bar{u}(s_1)$, $u_\lambda(s_2) = \bar{u}(s_2)$, $u_\lambda(R) > \bar{u}(R)$, and $\Delta u_\lambda(R_1) > \Delta u^*(R_1)$ which contradict the maximum principle. Now consider the following problem

$$\Delta^2 u = \frac{\lambda}{(1-u)^2} \quad \text{in } B,$$
$$u = u_\lambda(R_1) \quad \text{on } \partial B,$$
$$\Delta u = \Delta u_\lambda \quad \text{on } \partial B.$$

Then u_λ is a solution to the above problem while $\bar{u}$ is a sub-solution to the same problem. Moreover $\bar{u}$ is stable since,

$$\lambda < \lambda^*$$

and hence

$$\frac{2\lambda}{(1-\bar{u})^3} \leqslant \frac{2\lambda^*}{\alpha^3(1-u)^3} = \frac{2\lambda'}{\gamma(1-u)^3} < \frac{2\beta}{(1-u)^3}.$$

Table 1
Summary.

N	λ'_N	β_N
9	249	251
10	320	367
11	405	574
12	502	851
13	610	1211
14	730	1668
15	860	2235
$16 \leqslant N \leqslant 30$	$\frac{H_N}{2} - 1$	$\frac{H_N}{2}$
$N \geqslant 31$	$27\bar{\lambda}$	$\frac{H_N}{2}$

We deduce $\bar{u} \leqslant u_\lambda$ in B_{R_1} which is impossible, since $\bar{u}$ is singular while u_λ is smooth. This establishes (15). From (15) and the above two inequalities we have

$$\frac{2\lambda^*}{(1-u^*)^3} \leqslant \frac{2\lambda'}{\gamma(1-u)^3} < \frac{\beta}{(1-u)^3}.$$

Thus

$$\inf_{\varphi \in C_0^\infty} (B) \frac{\int_B (\Delta\varphi)^2 - \frac{2\lambda^*\varphi^2}{(1-u^*)^3}}{\int_B \varphi^2} > 0.$$

This is not possible if u^* is a smooth solution. □

For any $m > \frac{4}{3}$ define

$$w_m := 1 - a_{N,m} r^{\frac{4}{3}} + b_{N,m} r^m,$$

where

$$a_{N,m} := \frac{m(N+m-2)}{m(N+m-2) - \frac{4}{3}(N-2/3)} \quad \text{and} \quad b_{N,m} := \frac{\frac{4}{3}(N-2/3)}{m(N+m-2) - \frac{4}{3}(N-2/3)}.$$

Now we are ready to prove the main result of this section.

Theorem 5.2. *The following upper bounds on λ^* hold in large dimensions.*

1. *If $N \geqslant 31$, then Lemma* 5.1 *holds with $u := w_2$, $\lambda'_N = 27\bar{\lambda}$ and $\beta = \frac{H_N}{2} > 27\bar{\lambda}$.*
2. *If $16 \leqslant N \leqslant 30$, then Lemma* 5.1 *holds with $u := w_3$, $\lambda'_N = \frac{H_N}{2} - 1$, $\beta_N = \frac{H_N}{2}$.*
3. *If $10 \leqslant N \leqslant 15$, then Lemma* 5.1 *holds with $u := w_3$, $\lambda'_N < \beta_N$ given in Table* 1.
4. *If $N = 9$, then Lemma* 5.1 *holds with $u := w_{2.8}$, $\lambda'_9 := 249 < \beta_9 := 251$.*

The extremal solution is therefore singular for dimensions $N \geqslant 9$.

Proof. (1) Assume first that $N \geqslant 31$, then it is easy to see that $a_{N,2} < 3$ and $a_{N,2}^3\bar{\lambda} \leqslant 27\bar{\lambda} < \frac{H_N}{2}$. We shall show that w_2 is a singular $\mathcal{H}$-weak sub-solution of $(P)_{a_{N,2}^3\bar{\lambda}}$ which is stable.Note that

$w_2 \in H^2(B)$, $\frac{1}{1-w_2} \in L^3(B)$, $0 \leqslant w_2 \leqslant 1$ in B, and

$$\Delta^2 w_2 \leqslant \frac{a_{N,2}^3 \bar{\lambda}}{(1-w_2)^2} \quad \text{in } B \setminus \{0\}.$$

So w_2 is a $\mathcal{H}$-weak sub-solution of $(P)_{27\bar{\lambda}}$. Moreover,

$$w_2 = 1 - |x|^{\frac{4}{3}} + (a_{N,2} - 1)\big(|x|^{\frac{4}{3}} - |x|^2\big) \leqslant 1 - |x|^{\frac{4}{3}}.$$

Since $27\bar{\lambda} \leqslant \frac{H_N}{2}$, we get that

$$54\bar{\lambda} \int_B \frac{\varphi^2}{(1-w_2)^3} \leqslant H_N \int_B \frac{\varphi^2}{(1-w_2)^3} \leqslant H_N \int_B \frac{\varphi^2}{|x|^4} \leqslant \int_B (\Delta\varphi)^2$$

for all $\varphi \in C_0^\infty(B)$. Hence, w_2 is stable. Thus it follows from Lemma 5.1 that u^* is singular and $\lambda^* \leqslant 27\bar{\lambda}$.

(2) Assume $16 \leqslant N \leqslant 30$ and consider

$$w_3 := 1 - a_{N,3} r^{\frac{4}{3}} + b_{N,3} r^3.$$

We show that it is a singular $\mathcal{H}$-weak sub-solution of $(P_{\frac{H_N}{2}-1})$ which is stable. Indeed, we clearly have $0 \leqslant w_3 \leqslant 1$ a.e. in B, $w_3 \in H^2(B)$ and $\frac{1}{1-w_3} \in L^3(B)$. Note that

$$\begin{aligned} H_N \int_B \frac{\varphi^2}{(1-w_3)^3} &= H_N \int_B \frac{\varphi^2}{(a_{N,m} r^{\frac{4}{3}} - b_{N,m} r^m)^3} \\ &\leqslant \sup_{0<r<1} \frac{H_N}{(a_{N,m} - b_{N,m} r^{m-\frac{4}{3}})^3} \int_B \frac{\varphi^2}{r^4} \\ &= H_N \int_B \frac{\varphi^2}{r^4} \leqslant \int_B (\Delta\varphi)^2. \end{aligned}$$

Using maple one can verify that for $16 \leqslant N \leqslant 31$,

$$\Delta^2 w_3 \leqslant \frac{\frac{H_N}{2} - 1}{(1-w_3)^2} \quad \text{on } (0,1).$$

Hence, w_3 is a sub-solution of $(P_{\frac{H_N}{2}-1})$. By Lemma 5.1 u^* is singular and $\lambda^* \leqslant \frac{H_N}{2} - 1$.

(3) Assume $10 \leqslant N \leqslant 15$. We shall show that w_3 satisfies the assumptions of Lemma 5.1 for each dimension $10 \leqslant N \leqslant 15$. Using maple, for each dimension $10 \leqslant N \leqslant 15$, one can verify that inequality (17) holds for λ'_N given by Table 1. Then, by using maple again, we show that there exists $\beta_N > \lambda'_N$ such that

$$\begin{aligned} &\frac{(N-2)^2(N-4)^2}{16} \frac{1}{(|x|^2 - \frac{N}{2(N-1)}|x|^{\frac{N}{2}+1})(|x|^2 - |x|^{\frac{N}{2}})} + \frac{(N-1)(N-4)^2}{4} \frac{1}{|x|^2(|x|^2 - |x|^{\frac{N}{2}})} \\ &\quad \geqslant \frac{2\beta_N}{(1-w_3)^3}. \end{aligned}$$

The above inequality and improved Hardy–Rellich inequality (30) guarantee that the stability condition (20) holds for $\beta_N > \lambda'$. Hence, by Lemma 5.1 the extremal solution is singular for $10 \leqslant N \leqslant 15$. The values of λ_N and β_N are shown in Table 1.

(4) Let $u := w_{2.8}$. Using Maple on can see that

$$\Delta^2 u \leqslant \frac{249}{(1-u)^2} \quad \text{in } B$$

and

$$\frac{502}{(1-u(r))^3} \leqslant W(r) \quad \text{for all } r \in (0,1),$$

where W is given by (32). Since, $502 > 2 \times 249$, by Lemma 5.1 the extremal solution u^* is singular in dimension $N = 9$. □

Remark 5.3. It follows from the proof of Theorem 5.2 that for $N \geqslant 9$ and $\frac{\tau}{\beta}$ sufficiently small, there exists $u \in H^2(B) \cap W^{4,\infty}_{\mathrm{loc}}(B \setminus \{0\})$ such that

$$\Delta^2 u - \frac{\tau}{\beta}\Delta u \leqslant \frac{\lambda''_N}{(1-u)^2} \quad \text{for } 0 < r < 1, \tag{17}$$

$$u(1) = 0, \qquad \Delta u|_{r=1} = 0, \tag{18}$$

$$u \text{ is singular}, \tag{19}$$

and

$$2\beta'_N \int_B \frac{\varphi^2}{(1-u)^3} \leqslant \int_B (\Delta\varphi)^2 + \frac{\tau}{\beta}|\nabla\varphi|^2 \quad \text{for all } \varphi \in H^2(B) \cap H^1_0(B), \tag{20}$$

where $\beta'_N > \lambda''_N > 0$ are constants. Indeed, for each dimension $N \geqslant 9$, it is enough to take u to be the sub-solution we constructed in the proof of Theorem 5.2, $\beta'_N := \beta_N$, $\lambda' < \lambda'' < \beta$. If $\frac{\tau}{\beta}$ is sufficiently small so that $-\frac{\tau}{\beta}\Delta u < \frac{\lambda''-\lambda'}{(1-u)^2}$ on $(0,1)$, then with an argument similar to that of Lemma 5.1 we deduce that the extremal solution u^* of $(P_{\lambda,\beta,\tau,0,0})$ is singular. We believe that the extremal solution of $(P_{\lambda,\beta,\tau,0,0})$ is singular for all $\beta, \tau > 0$ in dimensions $N \geqslant 9$.

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Appendix A. Improved Hardy–Rellich inequalities

We now prove the improved Hardy–Rellich inequalities used in Section 4. They rely on the results of Ghoussoub and Moradifam in [12] which provide necessary and sufficient conditions for such inequalities to hold. At the heart of this characterization is the following notion of a Bessel pair of functions.

Definition 6. Assume that B is a ball of radius R in $\mathbb{R}^N$, $V, W \in C^1(0,1)$, and $\int_0^R \frac{1}{r^{N-1}V(r)}\,dr = +\infty$. Say that the couple (V, W) is a *Bessel pair on* $(0, R)$ if the ordinary differential equation

$$y''(r)+\left(\frac{N-1}{r}+\frac{V_r(r)}{V(r)}\right)y'(r)+\frac{W(r)}{V(r)}y(r)=0 \tag{$\mathrm{B}_{V,W}$}$$

has a positive solution on the interval $(0,R)$.

The needed inequalities will follow from the following two results.

Theorem A.1. *(See Ghoussoub and Moradifam [12].) Let V and W be positive radial C^1-functions on $B\backslash\{0\}$, where B is a ball centered at zero with radius R in $\mathbb{R}^N$ $(N\geqslant 1)$ such that $\int_0^R \frac{1}{r^{N-1}V(r)}\,dr=+\infty$ and $\int_0^R r^{N-1}V(r)\,dr<+\infty$. The following statements are then equivalent:*

1. *(V,W) is a Bessel pair on $(0,R)$.*
2. *$\int_B V(|x|)|\nabla\phi|^2\,dx \geqslant \int_B W(|x|)\phi^2\,dx$ for all $\phi\in C_0^\infty(B)$.*

Theorem A.2. *Let B be the unit ball in $\mathbb{R}^N$ $(N\geqslant 5)$. Then the inequality*

$$\int_B |\Delta u|^2\,dx \geqslant \int_B \frac{|\nabla u|^2}{|x|^2-\frac{N}{2(N-1)}|x|^{\frac{N}{2}+1}}\,dx+(N-1)\int_B \frac{|\nabla u|^2}{|x|^2}\,dx \tag{21}$$

holds for all $u\in C_0^\infty(\bar{B})$.

We shall need the following result to prove (21).

Lemma A.3. *For every $u\in C^1([0,1])$ the following inequality holds*

$$\int_0^1 \left|u'(r)\right|^2 r^{N-1}\,dr \geqslant \int_0^1 \frac{u^2}{r^2-\frac{N}{2(N-1)}r^{\frac{N}{2}+1}}r^{N-1}\,dr-(N-1)\big(u(1)\big)^2. \tag{22}$$

Proof. Let $\varphi := r^{-\frac{N}{2}+1}-\frac{N}{2(N-1)}$ and $k(r):=r^{N-1}$. Define $\psi(r)=u(r)/\varphi(r)$, $r\in[0,1]$. Then

$$\begin{aligned}
\int_0^1 \left|u'(r)\right|^2 k(r)\,dr &= \int_0^1 \left|\psi(r)\right|^2\left|\varphi'(r)\right|^2 k(r)\,dr+\int_0^1 2\varphi(r)\varphi'(r)\psi(r)\psi'(r)k(r)\,dr\\
&\quad+\int_0^1 \left|\varphi(r)\right|^2\left|\psi'(r)\right|^2 k(r)\,dr\\
&=\int_0^1 \left|\psi(r)\right|^2\big(\left|\varphi'(r)\right|^2 k(r)-(k\varphi\varphi')'(r)\big)\,dr+\int_0^1 \left|\varphi(r)\right|^2\left|\psi'(r)\right|^2 k(r)\,dr\\
&\quad+\psi^2(1)\varphi'(1)\varphi(1)\\
&\geqslant \int_0^1 \left|\psi(r)\right|^2\big(\left|\varphi'(r)\right|^2 k(r)-(k\varphi\varphi')'(r)\big)\,dr+\psi^2(1)\varphi'(1)\varphi(1).
\end{aligned}$$

Note that $\psi^2(1)\varphi'(1)\varphi(1) = u^2(1)\frac{\varphi'(1)}{\varphi(1)} = -(N-1)u^2(1)$. Hence, we have

$$\int_0^1 \left|u'(r)\right|^2 k(r)\,dr \geqslant \int_0^1 -u^2(r)\left(\frac{k'(r)\varphi'(r)+k(r)\varphi''(r)}{\varphi}\right)dr - (N-1)u^2(1) \tag{23}$$

Simplifying the above inequality we get (22). □

The decomposition of a function into its spherical harmonics will be one of our tools to prove Theorem A.2. Let $u \in C_0^\infty(\bar{B})$. By decomposing u into spherical harmonics we get

$$u = \Sigma_{k=0}^{\infty} u_k \quad \text{where } u_k = f_k\big(|x|\big)\varphi_k(x)$$

and $(\varphi_k(x))_k$ are the orthonormal eigenfunctions of the Laplace–Beltrami operator with corresponding eigenvalues $c_k = k(N+k-2)$, $k \geqslant 0$. The functions f_k belong to $u \in C^\infty([0,1])$, $f_k(1) = 0$, and satisfy $f_k(r) = O(r^k)$ and $f'(r) = O(r^{k-1})$ as $r \to 0$. In particular,

$$\varphi_0 = 1 \quad \text{and} \quad f_0 = \frac{1}{N\omega_N r^{N-1}} \int_{\partial B_r} u\,ds = \frac{1}{N\omega_N} \int_{|x|=1} u(rx)\,ds. \tag{24}$$

We also have for any $k \geqslant 0$, and any continuous real valued W on $(0,1)$,

$$\int_B |\Delta u_k|^2\,dx = \int_B \left(\Delta f_k\big(|x|\big) - c_k \frac{f_k(|x|)}{|x|^2}\right)^2 dx, \tag{25}$$

and

$$\int_B W\big(|x|\big)|\nabla u_k|^2\,dx = \int_B W\big(|x|\big)|\nabla f_k|^2\,dx + c_k \int_B W\big(|x|\big)|x|^{-2} f_k^2\,dx. \tag{26}$$

Now we are ready to prove Theorem A.2. We shall use the inequality

$$\int_0^1 \left|x'(r)\right|^2 r^{N-1}\,dr \geqslant \frac{(N-2)^2}{4} \int_0^1 \frac{x^2(r)}{r^2 - \frac{N}{2(N-1)} r^{\frac{N}{2}+1}} r^{N-1}\,dr$$
$$\text{for all } x \in C^1\big([0,1]\big) \text{ with } x(1) = 0. \tag{27}$$

Proof of Theorem A.2. For all $N \geqslant 5$ and $k \geqslant 0$, we have

$$\frac{1}{N w_N} \int_B |\Delta u_k|^2\,dx = \frac{1}{N w_N} \int_B \left(\Delta f_k\big(|x|\big) - c_k \frac{f_k(|x|)}{|x|^2}\right)^2 dx$$
$$= \int_0^1 \left(f_k''(r) + \frac{N-1}{r} f_k'(r) - c_k \frac{f_k(r)}{r^2}\right)^2 r^{N-1}\,dr$$

$$= \int_0^1 \big(f_k''(r)\big)^2 r^{N-1}\,dr + (N-1)^2 \int_0^1 \big(f_k'(r)\big)^2 r^{N-3}\,dr$$

$$+ c_k^2 \int_0^1 f_k^2(r) r^{N-5} + 2(N-1) \int_0^1 f_k''(r) f_k'(r) r^{N-2}$$

$$- 2c_k \int_0^1 f_k''(r) f_k(r) r^{N-3}\,dr - 2c_k(N-1) \int_0^1 f_k'(r) f_k(r) r^{N-4}\,dr.$$

Integrate by parts and use (24) for $k=0$ to get

$$\frac{1}{N\omega_N}\int_B |\Delta u_k|^2\,dx \geqslant \int_0^1 \big(f_k''(r)\big)^2 r^{N-1}\,dr + (N-1+2c_k)\int_0^1 \big(f_k'(r)\big)^2 r^{N-3}\,dr$$
$$+ \big(2c_k(n-4)+c_k^2\big)\int_0^1 r^{n-5} f_k^2(r)\,dr + (N-1)\big(f_k'(1)\big)^2. \tag{28}$$

Now define $g_k(r) = \frac{f_k(r)}{r}$ and note that $g_k(r) = O(r^{k-1})$ for all $k \geqslant 1$. We have

$$\int_0^1 \big(f_k'(r)\big)^2 r^{N-3} = \int_0^1 \big(g_k'(r)\big)^2 r^{N-1}\,dr + \int_0^1 2g_k(r) g_k'(r) r^{N-2}\,dr + \int_0^1 g_k^2(r) r^{N-3}\,dr$$
$$= \int_0^1 \big(g_k'(r)\big)^2 r^{N-1}\,dr - (N-3)\int_0^1 g_k^2(r) r^{N-3}\,dr.$$

Thus,

$$\int_0^1 \big(f_k'(r)\big)^2 r^{N-3} \geqslant \frac{(N-2)^2}{4}\int_0^1 \frac{f_k^2(r)}{r^2 - \frac{N}{2(N-1)} r^{\frac{N}{2}+1}} r^{N-3}\,dr - (N-3)\int_0^1 f_k^2(r) r^{N-5}\,dr. \tag{29}$$

Substituting $2c_k \int_0^1 (f_k'(r))^2 r^{N-3}$ in (28) by its lower estimate in the last inequality (29), and using Lemma A.3, we get

$$\frac{1}{N\omega_N}\int_B |\Delta u_k|^2\,dx \geqslant \frac{(N-2)^2}{4}\int_0^1 \frac{(f_k'(r))^2}{r^2 - \frac{N}{2(N-1)} r^{\frac{N}{2}+1}} r^{N-1}\,dr$$
$$+ 2c_k \frac{(N-2)^2}{4}\int_0^1 \frac{f_k^2(r)}{r^2 - \frac{N}{2(N-1)} r^{\frac{N}{2}+1}} r^{n-3}\,dr$$

$$+ (N-1)\int_0^1 \big(f_k'(r)\big)^2 r^{N-3}\,dr + c_k(N-1)\int_0^1 \big(f_k(r)\big)^2 r^{N-5}\,dr$$

$$+ c_k\big(c_k-(N-1)\big)\int_0^1 r^{N-5} f_k^2(r)\,dr + c_k\int_0^1 \frac{(N-2)^2}{4(r^2-\frac{N}{2(N-1)}r^{\frac{N}{2}+1})} - \frac{2}{r^2}\,dr$$

$$\geqslant \frac{(N-2)^2}{4}\int_0^1 \frac{(f_k'(r))^2}{r^2-\frac{N}{2(N-1)}r^{\frac{N}{2}+1}} r^{N-1}\,dr$$

$$+ c_k\frac{(N-2)^2}{4}\int_0^1 \frac{f_k^2(r)}{r^2-\frac{N}{2(N-1)}r^{\frac{N}{2}+1}} r^{n-3}\,dr$$

$$+ (N-1)\int_0^1 \big(f_k'(r)\big)^2 r^{N-3}\,dr + c_k(N-1)\int_0^1 \big(f_k(r)\big)^2 r^{N-5}\,dr.$$

The proof is complete in the view of (26). □

We shall now deduce the following corollary.

Corollary A.4. *Let $N \geqslant 5$ and B be the unit ball in $\mathbb{R}^N$. Then the following improved Hardy–Rellich inequality holds for all $\phi \in H^2(B)\cap H_0^1(B)$:*

$$\int_B (\Delta\phi)^2 \geqslant \frac{(N-2)^2(N-4)^2}{16}\int_B \frac{\phi^2}{(|x|^2-\frac{N}{2(N-1)}|x|^{\frac{N}{2}+1})(|x|^2-|x|^{\frac{N}{2}})} + \frac{(N-1)(N-4)^2}{4}\int_B \frac{\phi^2}{|x|^2(|x|^2-|x|^{\frac{N}{2}})}. \tag{30}$$

Proof. Let $\alpha := \frac{N}{2(N-1)}$ and $V(r) := \frac{1}{r^2-\alpha r^{\frac{N}{2}+1}}$ and note that

$$\frac{V_r}{V} = -\frac{2}{r} + \frac{\alpha(N-2)}{2}\frac{r^{\frac{N}{2}-2}}{1-\alpha r^{\frac{N}{2}-1}} \geqslant -\frac{2}{r}.$$

The function $y(r) = r^{-\frac{N}{2}+2}-1$ is decreasing and is then a positive super-solution on $(0,1)$ for the ODE

$$y'' + \left(\frac{N-1}{r}+\frac{V_r}{V}\right)y'(r) + \frac{W_1(r)}{V(r)}y = 0,$$

where

$$W_1(r) = \frac{(N-4)^2}{4(r^2-r^{\frac{N}{2}})(r^2-\alpha r^{\frac{N}{2}+1})}.$$

Hence, by Theorem A.1 we deduce

$$\int_B \frac{|\nabla\phi|^2}{|x|^2-\alpha|x|^{\frac{N}{2}+1}} \geqslant \left(\frac{N-4}{2}\right)^2 \int_B \frac{\phi^2}{(|x|^2-\alpha|x|^{\frac{N}{2}+1})(|x|^2-|x|^{\frac{N}{2}})}$$

for all $\phi \in H^2(B)\cap H_0^1(B)$. Similarly, for $V(r)=\frac{1}{r^2}$ we have that

$$\int_B \frac{|\nabla\phi|^2}{|x|^2} \geqslant \left(\frac{N-4}{2}\right)^2 \int_B \frac{\phi^2}{|x|^2(|x|^2-|x|^{\frac{N}{2}})}$$

for all $\phi \in H^2(B)\cap H_0^1(B)$. Combining the above two inequalities with (21) we get (30). □

Corollary A.5. *Let $N \geqslant 7$ and B be the unit ball in $\mathbb{R}^N$. Then the following improved Hardy–Rellich inequality holds for all $\phi \in H^2(B)\cap H_0^1(B)$:*

$$\int_B |\Delta u|^2 \geqslant \int_B W\big(|x|\big)u^2, \tag{31}$$

where

$$W(r)=K(r)\left(\frac{(N-2)^2}{4(r^2-\frac{N}{2(N-1)}r^{\frac{N}{2}+1})}+\frac{(N-1)}{r^2}\right), \tag{32}$$

$$K(r)=-\frac{\varphi''(r)+\frac{(n-3)}{r}\varphi'(r)}{\varphi(r)},$$

and

$$\varphi(r)=r^{-\frac{N}{2}+2}+9r^{-2}+10r-20.$$

Proof. Let $\alpha := \frac{N}{2(N-1)}$ and $V(r) := \frac{1}{r^2-\alpha r^{\frac{N}{2}+1}}$. Then φ is a sub-solution for the ODE

$$y''+\left(\frac{N-1}{r}+\frac{V_r}{V}\right)y'(r)+\frac{W_2(r)}{V(r)}y=0,$$

where

$$W_2(r)=\frac{K(r)}{r^2-\alpha r^{\frac{N}{2}+1}}.$$

Hence, by Theorem A.1 we have

$$\int_B \frac{|\nabla u|^2}{|x|^2-\alpha|x|^{\frac{N}{2}+1}} \geqslant \int_B W_2\big(|x|\big)u^2. \tag{33}$$

Similarly

$$\int_B \frac{|\nabla u|^2}{|x|^2} \geqslant \int_B W_3(|x|)u^2, \tag{34}$$

where

$$W_3(r) = \frac{K(r)}{r^2}.$$

Combining the above two inequalities with (22) we get improved Hardy–Rellich inequality (31). □

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