Discrete Mathematics 83 (1990) 49–62
North-Holland

ON THE NUMBER OF DISTINCT MINIMAL CLIQUE PARTITIONS AND CLIQUE COVERS OF A LINE GRAPH

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Received 10 June 1987

Let G be a line graph. Orlin determined the clique covering and clique partition numbers $\mathrm{cc}(G)$ and $\mathrm{cp}(G)$. We obtain a constructive proof of Orlin's result and in doing so we are able to completely enumerate the number of distinct minimal clique covers and partitions of G, in terms of easily calculable parameters of G.

We apply our results to give a new proof of Whitney's Theorem: if G and H are graphs, neither of which is K_3, then G and H are isomorphic if and only if their line graphs are isomorphic.

1. Introduction

We will be dealing exclusively with finite simple graphs. A *clique* in a graph G is a complete subgraph of G. A *clique covering* (*partition*) of G is a collection of cliques with the property that each edge of G occurs in at least one (exactly one) clique in the collection. The *clique covering number* $\mathrm{cc}(G)$ is the quantity $\min\{|P| : P \text{ is a clique covering of } G\}$; the *clique partition number* $\mathrm{cp}(G)$ is the quantity $\min\{|P| : P \text{ is a clique partition of } G\}$. A *minimal clique* covering (partition) P is a clique covering (partition) with $|P| = \mathrm{cc}(G)$ ($|P| = \mathrm{cp}(G)$). Note that we always have $\mathrm{cc}(G) \leq \mathrm{cp}(G)$ since any clique partition of G is also a clique covering. Also, since any graph has a clique partition (just take P to be the edge set) the above notions are well-defined.

Let G be a graph and let H be the graph defined from G as follows: the vertices of H are the edges of G, and two vertices x and y in H are adjacent if and only if (viewed as edges in G) they intersect. H is called the *line graph* of G and is usually denoted $H = G^*$. To a line graph G^* we can associate a 'canonical' clique partition, as follows. For each vertex v of G let e_v be the set of edges in G which contain v. Then e_v induces a clique in G^*, and the set $P = \{e_v : v \in G\}$ is a clique partition of G^*. Note that each vertex of G^* is contained in exactly two cliques in P. The following result indicates that this property may be used to characterize line graphs.

Theorem 1.1. *Let H be a graph admitting a clique partition P with the property that each vertex in H is contained in at most two cliques in P. Then there is a graph G with $H = G^*$.*

Proof. Let X be the set of vertices in H which are contained in one clique of P and let Y be the set of vertices which are not contained in any clique in P. Let P' be the multiset defined by $P' = P \cup X \cup Y \cup Y$. Let G be the graph whose vertices are the cliques of P', where two vertices C_1 and C_2 of G are adjacent if and only if $C_1 \cap C_2 \neq \emptyset$ (i.e. as cliques in H they contain a common vertex). It is easily verified that $G^* = H$. $\square$

Remarks. The clique partition P' of G^* is the 'canonical' one in the aforementioned sense.

The above results easily generalizes: a graph H admits a clique partition in which each vertex is contained in at most k cliques if and only if H is the line graph (i.e. intersection graph) of a partial block design with block size k.

Let G be a graph. A *wing* in G is a triangle with the property that exactly two of its vertices have degree 2 in G. Orlin proved the following result [3, Corollary 4.10]:

Theorem 1.2 (Orlin). *Let G be a connected graph, $G \neq K_3$, and let v_2 be the number of vertices of degree at least two in G and w be the number of wings in G. Then*

$$\mathrm{cc}(G^*) = v_2 - w \quad \textit{and} \quad \mathrm{cp}(G^*) = v_2.$$

Orlin then makes some remarks (cf. Remarks 4.8 and 4.11 in [3]) concerning the uniqueness of minimal partitions (covers) of certain line graphs. These remarks are not quite correct however; for comparison we refer the reader to our Theorems 2.6 and 2.14. We will obtain an essentially 'direct' proof of Theorem 1.2 (Orlin's proof is inductive) by showing that any minimal cover/partition of G^* is obtained by modifying the canonical partition in one of a few very specific ways. In so doing we will also be able to enumerate the number of distinct (but not necessarily non-isomorphic) minimal covers/partitions of G^* (see esp. Lemma 2.5, and Theorems 2.6 and 2.14). Then in Section 3 we obtain some interesting corollaries of these results, including (see Theorem 3.3) a new proof of a well-known result of Whitney [4]: if G and H are graphs, neither of which is K_3, and G^* is isomorphic to H^* then G is isomorphic to H.

Throughout this paper we will use standard notation. Thus $\bar{G}$ denotes the complement of G; $V(G)$ and $E(G)$ represent, respectively, the vertex set and edge set of G. We will write $|G|$ to mean $|V(G)|$. A clique with two vertices will be called an edge; a clique with three vertices will be called a triangle. A clique with just one vertex will be called trivial. A star in G is a collection of edges which contain a common vertex. Note that a star need not consist of all edges incident with some vertex, but only a subcollection of those edges. We will use the notation S_v^i to indicate a star with i edges, centered at v. K_n denotes the complete graph with n vertices, and T_n denotes the complete graph with n vertices minus the edges of a maximum matching (the so-called 'cocktail party graph' on n vertices). The degree of the vertex v is denoted $d(v)$. The *join* v of two graphs G and H is the graph whose vertex set is the disjoint union of the vertex sets of G and H, where two vertices x and y are adjacent if and only if either

(i) $x, y \in V(G)$ and x is adjacent to y in G, or
(ii) $x, y \in V(H)$ and x is adjacent to y in H, or
(iii) $x \in V(G)$ and $y \in V(H)$.

A graph is always assumed to have at least one vertex.

2. Covering and partitioning the edge set of a line graph

We begin by noting that the cliques in a line graph admit to a simple description.

Lemma 2.1. *Let $H = G^*$ be a line graph. Then any clique in H is induced either by a star or a triangle in G.*

Proof. A clique in H corresponds to a collection of mutually intersecting edges in G. The result follows. □

Example 1. Let $G = K_4$, so that $G^* = T_6$. Write $V(G) = \{1, 2, 3, 4\}$ and $V(G^*) = \{\{1, 2\} = a, \{1, 3\} = b, \{1, 4\} = c, \{2, 3\} = d, \{2, 4\} = e, \{3, 4\} = f\}$. It is easily verified that there are exactly two minimal clique covers of G^* (each of which is also a partition), namely

$$\begin{array}{lcl} a, b, c & & a, b, d \\ a, e, d & \text{and} & a, e, c \\ f, b, d & & f, b, c \\ f, e, c & & f, e, d \end{array}$$

The first cover corresponds to the set of (maximal) stars in G; the second corresponds to the set of triangles in G. (Note: the reader will have observed that the above covers are in fact isomorphic in the usual sense; however, we will not be concerning ourselves with isomorphism in this paper, so that we will consider these covers to be 'different'.)

Thus when $G = K_4$, G^* admits a minimal clique cover induced by triangles in G. The same is clearly true for $G = K_3$; the following result indicates that there are no other (connected) graphs with this property.

Lemma 2.2. *Let G be a connected graph, $G \neq K_4$, P be a minimal clique covering of G^*, and v be a vertex of degree at least 3 in G. Then P contains a clique with either $d(v) - 1$ or $d(v)$ vertices, induced by a star in G centered at v.*

Proof. For $y \in V(G)$ let S_y^i denote a star in G, centered at y, with i edges (so that $i \leq d(y)$). Let $\{x_1, \ldots, x_{d(v)}\}$ be the neighbours of v in G. Suppose first that P does not contain any clique induced by an S_v^i. Then from Lemma 2.1 P must contain $\binom{d(v)}{2}$ triangles $T_1, \ldots, T_{\binom{d(v)}{2}}$ induced by the triangles $\{vx_1x_2, vx_1x_3, \ldots, vx_{d(v)-1}x_{d(v)}\}$ in G. Additionally, P must contain some clique C covering the edge (in G^*) corresponding to the intersecting edges x_1x_2, x_2x_3 in G. But these $1 + \binom{d(v)}{2}$ cliques can now be replaced by the $d(v)+1$ cliques induced by the stars $S_v^{d(v)}, \ldots, S_{x_{d(v)}}^{d(x_{d(v)})}$, producing a smaller cover than P unless $d(v) = 3$ and $G = K_4$ (cf. Example 1).

Now suppose that P contains a clique C induced by an S_v^i where $i \leq d(v) - 2$, say $S_v^i = \{vx_1, \ldots, vx_i\}$. If $i = 1$ then C is a trivial clique and so could be removed from P; thus $i \geq 2$, whence $d(v) \geq 4$. P cannot contain a second clique D induced by an S_v^j, else we could replace C, D by the single clique induced by $S_v^{d(v)}$, producing a smaller cover. Thus from Lemma 2.1 P must contain the triangles $T_1, \ldots, T_{2d(v)-3}$ induced by the triangles $\{vx_1x_{d(v)}, vx_2x_{d(v)}, \ldots, vx_{d(v)-1}x_{d(v)}, vx_1x_{d(v)-1}, \ldots, vx_{d(v)-2}x_{d(v)-1}\}$ in G. As above we can now replace these T_j together with C by the $d(v)+1$ cliques induced by $S_v^{d(v)}, \ldots, S_{x_{d(v)}}^{d(x_{d(v)})}$. Since $d(v) \geq 4$ we have a smaller cover than P, a contradiction. □

Lemma 2.3. *Let G be a connected graph, and P be a minimal clique covering of G^*. Let $v \in V(G)$ and suppose that P contains a clique induced by $S_v^{d(v)-1} = \{vx_1, \ldots, vx_{d(v)-1}\}$. Then P contains the $d(v) - 1$ cliques $T_1, \ldots, T_{d(v)-1}$ induced by the triangles $\{vx_1x_{d(v)}, \ldots, vx_{d(v)-1}x_{d(v)}\}$ in G; furthermore, for each $1 \leq i \leq d(v) - 1$, x_i has degree exactly two in G (with neighbours v and $x_{d(v)}$).*

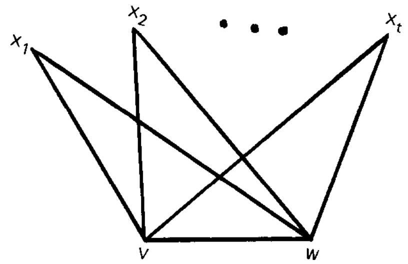
Proof. The first part of the conclusion is a consequence of Lemma 2.1. From the hypothesis we have $d(v) \geq 3$, whence $d(x_{d(v)}) \geq 3$. Now G cannot be K_4 here (cf. Example 1) so that from Lemma 2.2, P contains a clique induced by an $S_{x_{d(v)}}^j$. Thus if for some $1 \leq i \leq d(v) - 1$, x_i has degree greater than two in G then applying Lemma 2.2 again, P would contain a clique induced by an $S_{x_i}^k$; but then we could replace $T_1, \ldots, T_{d(v)-1}$, $S_v^{d(v)-1}$, $S_{x_{d(v)}}^j$ and $S_{x_i}^k$ by the $d(v)+1$ cliques induced by the stars $S_v^{d(v)}, \ldots, S_{x_{d(v)}}^{d(x_{d(v)})}$ obtaining a smaller cover. □

Lemma 2.4. *Let G be a connected graph and P be a minimal clique covering of G^*. Let v have degree two in G with neighbours x_1 and x_2. If x_1 is not adjacent to x_2 then P contains the clique induced by $S_v^2 = \{vx_1, vx_2\}$. If x_1 is adjacent to x_2 and some x_i has degree 2 (so that either $G = K_3$ or vx_1x_2 is a wing) then P contains the clique induced by the triangle $\{vx_1x_2\}$ in G. If x_1 is adjacent to x_2 and no x_i has degree 2 then P contains either the clique induced by S_v^2 or the clique induced by $\{vx_1x_2\}$.*

Proof. This is a direct application of Lemma 2.1. □

We are now in a position to characterize all minimal clique coverings of a given line graph G^*. First we introduce some more definitions. A *semiwing* in a graph G is a triangle with the property that exactly one of its vertices has degree two in G. Let $\{v, w\}$ be an edge in G and let $\{w, x_1, \ldots, x_t\}$ be the neighbours of v in G. If $t \geq 2$ and vwx_i is a semiwing for each i we will say that $\{v, w\}$ is a *t-edge with respect to v.*

Example 2. If a connected graph G has an edge $\{v, w\}$ which is a t-edge with respect to both of its endpoints then G is the following graph (with $t + 2$ vertices):



We denote the above graph W_t, $t \geq 2$.

Lemma 2.5. *Let $G = W_t$, $t \geq 2$. Then* $\mathrm{cc}(G^*) = \mathrm{cp}(G^*) = t + 2$. *There are exactly $2^t + 3$ distinct minimal covers of G^*, 2 of which are partitions.*

Proof. From Lemmas 2.2, 2.3 and 2.4 a minimal cover P will contain cliques induced by stars centered at v and w; we have the following possibilities:

(i) P contains S_v^t and S_w^t. Then P also contains the triangles induced by vwx_i, $i = 1, \ldots, t$. Here P is a partition of G^*.

(ii) P contains S_v^t and S_w^{t+1}, or S_v^{t+1} and S_w^t. Then P contains the triangles as in (i).

(iii) P contains S_v^{t+1} and S_w^{t+1}. Here there are 2^t distinct possibilities. For each subset $A \subseteq \{1, \ldots, t\}$ we construct the cover P_A by assigning to it the cliques induced by S_v^{t+1}, S_w^{t+1}, $\{S_{x_i}^2 : i \in A\}$ and $\{vwx_i : i \notin A\}$. When $A = \{1, \ldots, t\}$, P_A is a partition. $\square$

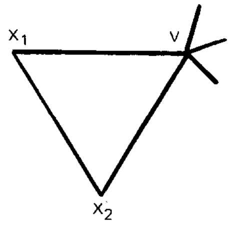
In the following 't-edge' means t-edge with respect to an end-point, as defined above.

Theorem 2.6. *Let G be a connected graph, $G \neq K_3$, K_4 or W_t, $t \geq 2$. Let v_2 denote the number of vertices in G of degree at least two , w denote the number of wings in G and s denote the number of semiwings in G. Suppose that G has a_t t-edges, $t \geq 2$. Then*

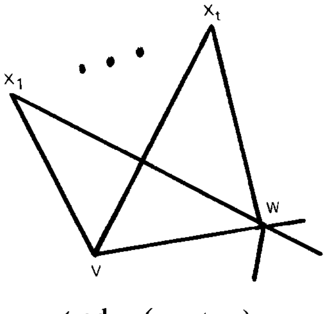
$$\mathrm{cc}(G^*) = v_2 - w$$

and there are $2^{s - \sum a_t t} \cdot \prod (2^t + 1)^{a_t}$ distinct minimal clique covers of G^*. If G has no wings then $\mathrm{cp}(G^*) = \mathrm{cc}(G^*) = v_2$ *and G^* has a unique minimal clique partition.*

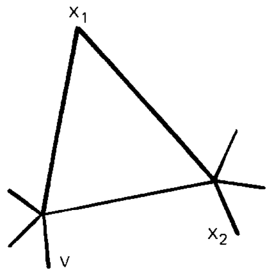
Proof. We apply Lemmas 2.2, 2.3 and 2.4. We note that it is a direct consequence of these lemmas that $\mathrm{cc}(G^*) \geq v_2 - w$. Furthermore in order to enumerate the number of distinct minimal covers of G^* it is necessary only to analyze the following 'pieces' of G:



Wings



t-edge (w.r.t. v)



Semiwings (i.e. not containing t-edges).

To each wing vx_1x_2 we must associate to P the triangle in G^* induced by vx_1x_2 (Lemma 2.4). Note that neither vx_1 nor vx_2 can be t-edges, so that by Lemmas 2.2 and 2.3 P contains a clique induced by an $S_v^{d(v)}$, where $\{vx_1, vx_2\} \subseteq S_v^{d(v)}$. Thus if G has a wing, no minimal clique cover can be a partition.

To a t-edge $\{v, w\}$ we can associate to P any one of $2^t + 1$ distinct collections of cliques (Lemmas 2.2, 2.3, 2.4):

(i) use the clique induced by $S_v^t = \{vx_1, \ldots, vx_t\}$ together with the t cliques induced by the triangles $\{vwx_i : 1 \leq i \leq t\}$, or

(ii) use the clique induced by S_v^{t+1} and then choose a subset $A \subseteq \{1, \ldots, t\}$ and use the cliques induced by $\{S_{x_i}^2 : i \in A\}$ and $\{vwx_i : i \notin A\}$. This gives 2^t more possibilities.

Note that since $G \neq W_t$, it follows from Lemmas 2.2 and 2.3 that P must contain the clique induced by $S_w^{d(w)}$; thus (assuming that G has no wings) the only minimal cover that can be a partition here corresponds to (ii), by choosing $A = \{1, \ldots, t\}$.

Finally, to a semiwing vx_1x_2 not containing any t-edges (there are $s - \sum a_t t$ of these) we can associate to P one of two possibilites (Lemma 2.4), either the clique induced by $S_{x_1}^2$ or the clique induced by the triangle vx_1x_2. Since vx_1x_2 does not contain any t-edges it follows from Lemmas 2.2 and 2.3 that P contains cliques induced by $S_v^{d(v)}$ and $S_{x_2}^{d(x_2)}$. Thus if P is to be a partition, only the choice $S_{x_1}^2$ will do. The result follows. □

We now turn our attention specifically to clique partitions of line graphs G^*. We already have the following partial result which we can isolate from Example 1, Lemma 2.5 and Theorem 2.6.

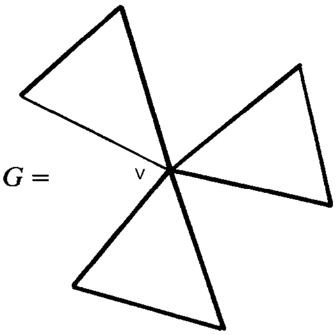
Lemma 2.7. *Let G be a connected graph with no wings, $G \neq K_3$, and let v_2 denote the number of vertices in G with degree at least two. Then*

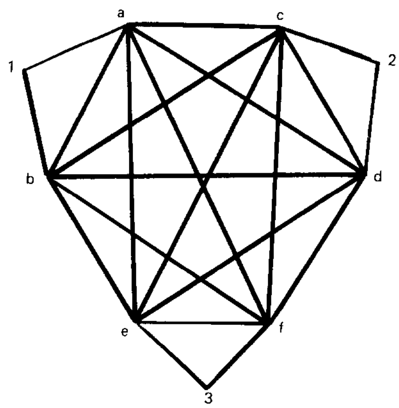
$$\mathrm{cp}(G^*) = v_2.$$

If $G = K_4$ or W_t then G^* admits exactly two minimal partitions, else G^* has a unique minimal clique partition.

We must now derive some properties of clique partitions of line graphs, analogous to Lemmas 2.2, 2.3 and 2.4. Unfortunately these lemmas do not apply if P is a partition, as the following example illustrates.

Example 3.

Let $G =$ 

, $G^* =$ 

G^* admits exactly three minimal partitions:

$1ab$	$1ab$	$1a$
$2cd$	$2cd$	$1b$
$3ef$	$3ef$	$2c$
ace	acf	$2d$
adf	ade	$3e$
bcf	bce	$3f$
bde	bdf	$abcdef.$

Now v has degree 6 in G, yet in the first two partitions above there are no cliques corresponding to S_v^5 or S_v^6. In fact in each of these partitions there are *four* cliques corresponding to S_v^3's, a situation which certainly could not occur when considering minimal covers.

It is easy to see that for any graph G with v_2 vertices of degree at least two, $\text{cp}(G^*) \leq v_2$. We just take the 'canonical' partition of G^* induced by the stars $\{S_v^{d(v)} : v \text{ has degree} \geq 2 \text{ in } G\}$. In what follows this canonical partition will be denoted P^*.

Lemma 2.8. *Let G be a connected graph, and let P be a clique partition of G^*. Let T denote the set of triangles in G that are neither wings nor semiwings, and let $Q = \{C \in P : C$ is induced by a triangle in $T\}$. Let $R = \{v \in V(G) : d(v) \geqslant 3$ and there exist no cliques in P induced by any $S_v^i\}$. Then $|Q| \geqslant |R|$, with equality occurring only if either $|R| = 0$ or $G = K_4$.*

Proof. Let $v \in R$, and let x_1, x_2 be two neighbours of v. Since P contains no cliques induced by any S_v^i it follows from Lemma 2.1 that P contains the clique C induced by the triangle vx_1x_2 in G. Now v has degree at least 3, and since it follows from the preceding argument that the neighbourhood of v is complete, we conclude that both x_1 and x_2 have degree at least 3 as well; that is, $C \in Q$. Therefore since each $v \in R$ has degree at least three, and each $C \in Q$ arises from a triangle in G (which in turn contains at most three vertices of R), we have

$$|Q| \geqslant 3\,|R|/3;$$

assuming that $R \neq \emptyset$ equality can occur here only if each vertex $v \in R$ has degree exactly three and each triangle vx_1x_2 in T inducing a clique $C \in Q$ contains three vertices of R. Since G is connected, G must be K_4. □

Lemma 2.9. *Let G be a connected graph, $G \neq W_t$ and P be a clique partition of G^*. Suppose that P contains a clique induced by a semiwing T in G. Then for some vertex $v \in T$, P contains at least two cliques induced by stars centered at v.*

Proof. Let $T = wx_1x_2$ where w has degree 2 in G. Since $G \neq W_t$ some neighbour of w, say x_1, has a neighbour y which is not adjacent to x_2. Thus P must contain a clique induced by an $S_{x_1}^i$ where $\{x_1y, x_1w\} \subseteq S_{x_1}^i$ and a clique induced by an $S_{x_1}^j$ where $\{x_1y, x_1x_2\} \subseteq S_{x_1}^j$. These two cliques are distinct because x_1w and x_1x_2 occur together in T. □

Using Lemma 2.8 and 2.9 one could recover Lemma 2.7. We must now consider the possibility that G has some wings. We will be making use of the following well-known results (see [1, 2]).

Theorem 2.10 (de Bruijn and Erdős). *If $n \geqslant 1$ then* $\mathrm{cp}(K_n \vee \bar{K}_2) = n + 1$.

Theorem 2.11 (Gregory, McGuinness and Wallis). *If $n \geqslant 4$ then* $\mathrm{cp}(T_{2n}) \geqslant 2n$. *Also,* $\mathrm{cp}(T_4) = \mathrm{cp}(T_6) = 4$.

Corollary 2.12. *If H is any graph and $n \geqslant 1$ then* $\mathrm{cp}(T_{2n} \vee H) \geqslant n + 1$, *with equality occurring if and only if $n = 1$ and $H = K_1$.*

Proof. If $n \geqslant 2$ then from Theorem 2.11 $\mathrm{cp}(T_{2n} \vee H) \geqslant \mathrm{cp}(T_{2n}) \geqslant n + 1$ with equality occurring only if $n = 3$. But if R is a clique partition of $T_6 \vee H$ with

$|R| = 4$ then by considering the cliques in R containing a given vertex $v \in V(H)$ it follows that there is a (minimal) partition of T_6 containing a subcollection of cliques which are vertex disjoint and cover the vertices of T_6. No such minimal partition exists (see Example 1).

If $n = 1$, and H is not complete then by Theorem 2.11 again $\mathrm{cp}(T_2 \vee H) \geqslant \mathrm{cp}(T_4) = 4 > n + 1$. Thus H is complete and the conclusion follows from Theorem 2.10. □

Lemma 2.13. *Let G be a connected graph and P be a clique partition of G^*. Let v be a vertex of degree at least 3 in G and suppose that for some positive integer n P contains cliques induced by n wings, each containing v. Then P contains at least $n + 1$ cliques induced by stars centered at v, with equality occurring only if $n = 1$ and $d(v) = 3$, or $n = 3$ and $G = \bar{T}_6 \vee K_1$ (i.e. the graph G in Example 3).*

Proof. Let v have neighbours $\{x_1, \ldots, x_{2n}, \ldots, x_{d(v)}\}$ where for each $i = 1, \ldots, n$ the wing $vx_{2i-1}x_{2i}$ induces a clique in P. From Lemma 2.1 it follows that for each $1 \leqslant j \leqslant d(v)$ and each $1 \leqslant k \leqslant 2n$ (where $k \neq 2i$ when $j = 2i - 1$, and $k \neq 2i - 1$ when $j = 2i$, $i = 1, \ldots, n$) the edge in G^* corresponding to $\{vx_j, vx_k\}$ must live in a clique in P induced by a star centered at v. Therefore

(i) if $d(v) = 2n$ (so that $n \geqslant 2$) the set S of cliques in P induced by these stars form a clique partition of T_{2n}. Thus by Theorem 2.11 $|S| \geqslant n + 1$ with equality occurring only if $n = 3$. Since G is connected, $G = \bar{T}_6 \vee K_1$.

(ii) if $d(v) > 2n$ the set S of cliques in P induced by these stars form a clique partition of $T_{2n} \vee H$ where H is a graph with $d(v) - 2n$ vertices. From Corollary 2.12 we have $|S| \geqslant n + 1$, with equality occurring only if $n = 1$ and $H = K_1$, i.e. $d(v) = 3$. □

We are now ready to state our main result concerning clique partitions of line graphs. Following Orlin [3] we will define a *3-wing* in a graph G to be a wing containing a vertex of degree 3 in G.

Theorem 2.14. *Let G be a connected graph, $G \neq K_3$, K_4, $\bar{T}_6 \vee K_1$ or W_t, $t \geqslant 2$. Let v_2 denote the number of vertices in G with degree at least two, and let w_3 denote the number of 3-wings in G. Then*

$$\mathrm{cp}(G^*) = v_2$$

and there are exactly 2^{w_3} distinct minimal clique partitions of G^*.

Proof. We have indicated previously that we can construct the 'canonical' partition P^* of G^*, where $|P^*| = v_2$. We will use Lemmas 2.8, 2.9 and 2.13 to show that if P is any clique partition of G^* then $|P| \geqslant |P^*|$. We will do this by constructing a surjection $f: P \rightarrow V_2(G) = \{v \in V(G): v \text{ has degree at least two}\}$.

Let P be a clique partition of G^*. Let us write $V_2(G)$ as the disjoint union $V_2(G) = R \cup NW \cup W$ where $R = \{v \in V(G) : v$ has degree ≥ 3 and there exist in P no cliques induced by stars centered at $v\}$, $NW = \{v \in V_2(G) - R : v$ does not lie in a wing of $G\}$, and $W = \{v \in V_2(G) : v$ lies in a wing of $G\}$. (It is a consequence of Lemmas 2.1 and 2.13 that R and W are disjoint.)

Let $Q \subseteq P$ consists of those cliques in P induces by triangles in G that are neigher wings nor semiwings. From Lemma 2.8 we can construct a surjection

$$f_1 : Q \to R.$$

Let $v \in NW$. If $d(v) \geq 3$, or $d(v) = 2$ and v is not contained in a triangle in G, then P contains a clique C_v induced by a star centered at v; else (since $G \neq K_3$) $d(v) = 2$ and v is contained in a semiwing vx_1x_2 in G. In this case P contains a clique C_v induced either by S_v^2 or by vx_1x_2. Thus we have a bijection

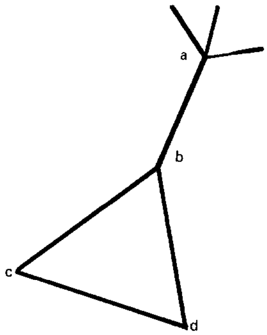
$$f_2 : \{C_v : v \in NW\} \to NW.$$

Finally let $v \in W$, $d(v) \geq 3$ and let $\{vx_1x_2, \ldots, vx_{2t-1}x_{2t}\}$ be the wings in G containing v. Let P contain n cliques induced by the wings $vx_1x_2, \ldots, vx_{2n-1}x_{2n}$. By Lemma 2.13, P will contain (at least) $n+1$ cliques induced by stars centered at v (this is true even if $n = 0$, since $v \notin R$). For each $n+1 \leq i \leq t$ P will contain two more cliques, induced by $S^2_{x_{2i-1}}$ and $S^2_{x_{2i}}$ (Lemma 2.1). We have counted here a set S_v of (at least) $n + n + 1 + 2(t-n) = 2t+1$ cliques in P which we can now associate to the vertices $v, x_1, \ldots, x_{2t}$. In this way we have a surjection

$$f_3 : \bigcup_{\substack{v \in W \\ d(v) \geq 3}} S_v \to W.$$

Since Q, $\{C_v : v \in NW\}$ and $\bigcup_{v \in W, d(v) \geq 3} S_v$ are disjoint subsets of P it follows that there is a surjection $f : P \to V_2(G)$, whence $|P| \geq |P^*|$. Thus $\text{cp}(G^*) = v_2$.

Now suppose that P is minimal, i.e. $|P| = v_2$. Then f_1 must be a bijection. Since $G \neq K_4$ it is an immediate consequence of Lemma 2.8 that $Q = \emptyset$. Similarly f_3 must be a bijection, whence for each $v \in W$, $d(v) \geq 3$ the set $S_v \subseteq P$ contains exactly $2t+1$ cliques. Since $G \neq \bar{T}_6 \vee K_1$ it follows from Lemma 2.13 that if P contains a clique induced by a wing vx_1x_2 then vx_1x_2 is a 3-wing, i.e. $d(v) = 3$. Finally, suppose that P were to contain a clique induced by a semiwing vx_1x_2 in G, where v has degree 2. Since G is not one of the graphs W_t (defined in Example 2) it now follows from Lemma 2.9 that there are two cliques D_1, D_2 in P induced by stars centered at some neighbour of v, say x_1. This is a contradiction since only one of these cliques can appear in the domain of $f_2 \cup f_3$. The foregoing implies that if P contains a clique induced by a triangle T in G, then T is a 3-wing.



A 3-wing.

On the other hand if bcd is a 3-wing we can associate to P either the cliques induced by S_c^2, S_d^2 and S_b^3 or the cliques induced by bcd, $\{ba, bc\}$ and $\{ba, bd\}$.

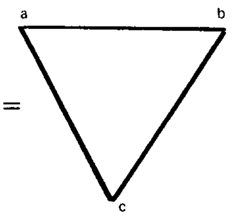
The conclusion follows. □

3. Applications

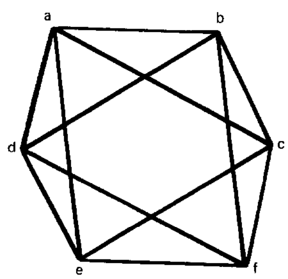
In the introduction we noted (Theorem 1.1) that line graphs can be characterized by their ability to admit a clique partition in which each vertex lies on at most two cliques.

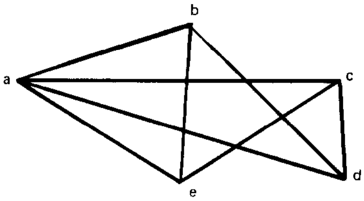
Consider the following set of graphs

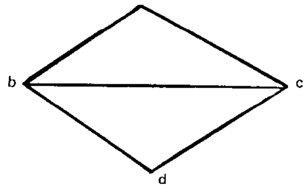
$E_1 = K_3 =$



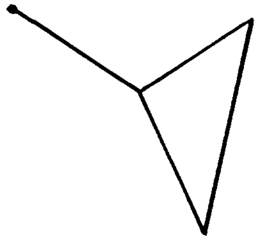
$E_2 = T_6 =$



$E_3 = T_4 \vee K_1 =$ 

$E_4 =$ 

Note that $E_1 = K_3^*$, $E_2 = K_4^*$, $E_3 = W_2^*$ and E_4 is the dual of a 3-wing, i.e.

$E_4 = \Bigg($  $\Bigg)^*$

(If G is the graph in the above figure, we will abuse the definition and say that G *is* a 3-wing.)

We can now prove the following.

Theorem 3.1. *Let H be a graph and suppose that H admits a clique partition P, containing no trivial cliques, in which each vertex of H is contained in at most two cliques. Then if $H \neq E_1$, E_2, E_3 or E_4, we have*

(i) *P is the only partition of H with these properties, and*

(ii) *P is minimal.*

Proof. By Theorem 1.1 we can write $H = G^*$ for some graph G, where P is the 'canonical' partition of H induced by stars centered at vertices of degree at least two in G. Since $H \neq E_1$ (so that $G \neq K_3$) it follows from Example 1, Lemma 2.5, Example 3 and Theorem 2.14 that $\mathrm{cp}(H) = v_2(G)$, whence P is minimal.

Let Q be a clique partition of H, containing no trivial cliques, in which each vertex of H is contained in at most two cliques. By the foregoing, Q is also minimal. Now Example 1, Lemma 2.5, Example 3 and Theorem 2.14 completely characterize minimal clique partitions of G^*. In particular, it is easily seen from these results that we must have $Q = P$, except when $G = K_4$, W_2 or when G is a 3-wing. But G cannot be any of these graphs here since $H \neq E_2$, E_3 or E_4. □

Lemma 3.2. *Let $H = E_1$, E_2, E_3 or E_4. Then there are exactly two distinct clique partitions of H satisfying the hypothesis of Theorem* 3.1.

Proof. We can list the partitions in each case, by direct inspection. We refer the reader to the diagrams above.

H	P_1	P_2	
E_1	*abc*	*ab*	
		ac	
		bc	
E_2	*abc*	*abd*	
	aed	*aec*	
	fbd	*fbc*	
	fec	*fed*	(cf. Example 1)
E_3	*abe*	*abd*	
	acd	*ace*	
	bd	*be*	
	ce	*cd*	
E_4	*abc*	*bcd*	
	bd	*ab*	
	cd	*ac*	□

Remark. Note that the second partition of $E_1 = K_3$ does not satisfy conclusion (ii) of Theorem 3.1, i.e. is not minimal.

Finally, we prove the following result of H. Whitney [4].

Theorem 3.3 (Whitney). *Let $\mathcal{G}$ be the class of all graphs, and $\mathcal{L}$ the subclass of $\mathcal{G}$ consisting of all line graphs. Let $*: \mathcal{G} \to \mathcal{L}$ be defined by $*(G) = G^*$. Then $* \mid \mathcal{G} - \{K_3\}$ is a bijection.*

Proof. Let $H \in \mathcal{L}$, and suppose that $H = G_1^* = G_2^*$. Let P_1, P_2 be the 'canonical' partitions of H induced by the vertices of degree ≥ 2 in G_1, G_2. It is easily verified that G_1 and G_2 are isomorphic (as graphs) if and only if P_1 and P_2 are isomorphic (as partitions); it then follows immediately from Theorem 3.1 that $G_1 \simeq G_2$, except possibly when $H = E_1$, E_2, E_3, or E_4. Now if $H = E_2$, E_3 or E_4 the partitions P_1, P_2 given in Lemma 3.2 are isomorphic (consider the bijection $\alpha: V(H) \to V(H)$ where $\alpha = (af)(b)(c)(d)(e)$ if $H = E_2$, $\alpha = (a)(b)(c)(de)$ if

$H = E_3$ and $\alpha = (ad)(b)(c)$ if $H = E_4$); thus again $G_1 \simeq G_2$. Finally, if $H = E_1 = K_3$ the partitions P_1, P_2 of Lemma 3.2 are not isomorphic; thus when (and only when) $H = K_3$ there are two distinct graphs $G_1, G_2 \in \mathcal{G}$ with $G_1^* = G_2^* = K_3$, namely $G_1 = K_3$ and $G_2 = K_1 \vee \bar{K}_3$ (a star with three edges). Therefore * becomes a bijection if we restrict its domain to $\mathcal{G} - \{K_3\}$. □

Remark. We could of course have restricted the domain of * to $\mathcal{G} - \{K_1 \vee \bar{K}_3\}$, but we have chosen instead to omit K_3 for the following reason: we have seen that K_3 is the only graph in $\mathcal{G}$ whose stars do *not* induce a minimal clique partition of its line graph.

Acknowledgements

We wish to thank U.S.R. Murty for his helpful assistance. The second author's research was supported in part by an NSERC postdoctoral fellowship.

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