



The power of digraph products applied to labelings

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ABSTRACT

The \otimes_h -product was introduced in 2008 by Figueroa-Centeno et al. [15] as a way to construct new families of (super) edge-magic graphs and to prove that some of those families admit an exponential number of (super) edge-magic labelings. In this paper, we extend the use of the product \otimes_h in order to study the well known harmonious, sequential, partitioned and (a, d) -edge antimagic total labelings. We prove that if a (p, q) -digraph with $p \leq q$ is harmonious and $h : E(D) \rightarrow S_n$ is any function, then $und(D \otimes_h S_n)$ is harmonious. We obtain analogous results for sequential and partitioned labelings. We also prove that if G is a (super) (a, d) -edge-antimagic total tripartite graph, then nG is (super) (a', d) -edge-antimagic total, where $n \geq 3$, and $d = 0, 2$ and n is odd, or $d = 1$. We finish the paper providing an application of the product \otimes_h to an arithmetic classical result when the function h is constant.

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1. Introduction

For most of the graph theory terminology and notation used in this paper we follow either [10] or [32], unless otherwise specified. For two integers m, n with $m \leq n$ we denote by $[m, n]$ the set $\{m, m + 1, \dots, n\}$ unless otherwise specified. We say that a graph $G = (V, E)$ is a (p, q) -graph when $|V| = p$ and $|E| = q$. Kotzig and Rosa introduced in [25] the concept of edge-magic labeling. A bijective function $f : V \cup E \rightarrow [1, p + q]$ is an *edge-magic labeling* of G if there exists an integer k such that the sum $f(x) + f(xy) + f(y) = k$ for all $xy \in E$. In 1998, Enomoto et al. [12] defined the concepts of super edge-magic graphs and super edge-magic labelings. A *super edge-magic labeling* is an edge-magic labeling with the extra property that $f(V) = [1, p]$. It is worthwhile mentioning that an equivalent labeling had already appeared in the literature in 1991 under the name of *strongly indexable labeling* [1]. A graph that admits a (super) edge-magic labeling is called a (*super*) *edge-magic graph*.

In 2000, Figueroa et al. [13] provided a very useful characterization of super edge-magic graphs that we state in the next lemma.

Lemma 1.1. *A (p, q) -graph $G = (V, E)$ is super edge-magic if and only if there is a bijective function $\bar{f} : V \rightarrow [1, p]$ such that the set $S_E = \{\bar{f}(u) + \bar{f}(v) : uv \in E\}$ is a set of q consecutive integers.*

When we say that a digraph has a labeling we mean that its underlying graph has such labeling, see [15]. For instance, a digraph is super edge-magic if its underlying graph is super edge-magic. We will use the notation $und(D)$ in order to denote the underlying graph of a digraph D .

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In [15], Figueroa et al., defined the following product: let $D = (V, E)$ be a digraph with adjacency matrix $A(D) = (a_{i,j})$ and let $\Gamma = \{\Gamma_i\}_{i=1}^m$ be a family of m digraphs, all of them with the same set of vertices V' . Assume that $h : E \rightarrow \Gamma$ is any function that assigns elements of Γ to the arcs of D . Then the digraph $D \otimes_h \Gamma$ is defined by $V(D \otimes_h \Gamma) = V \times V'$ and $((a_1, b_1), (a_2, b_2)) \in E(D \otimes_h \Gamma) \iff [(a_1, a_2) \in E(D) \wedge (b_1, b_2) \in E(h(a_1, a_2))]$. An alternative way of defining the same product is through adjacency matrices, since we can obtain the adjacency matrix of $D \otimes_h \Gamma$ as follows: if $a_{i,j} = 0$ then $a_{i,j}$ is multiplied by the $p' \times p'$ 0-square matrix, where $p' = |V'|$. If $a_{i,j} = 1$ then $a_{i,j}$ is multiplied by $A(h(i, j))$ where $A(h(i, j))$ is the adjacency matrix of the digraph $h(i, j)$.

Note that when h is constant, $D \otimes_h \Gamma$ is the Kronecker product. From now on, let S_n denote the set of all super edge-magic 1-regular labeled digraphs of order n where each vertex takes the name of the label that has been assigned to it. The main result found in [15] is the following one:

Theorem 1.1. *Let D be a (super) edge-magic digraph and let $h : E(D) \rightarrow S_n$ be any function. Then $und(D \otimes_h S_n)$ is (super) edge-magic.*

The \otimes_h -product was introduced in [15] as a way to construct new families of (super) edge-magic graphs and to prove that some of those families admit an exponential number of (super) edge-magic labelings. For instance, it was proved that if F is an acyclic (super) edge-magic graph of order m with p components then nF admits at least $|S_n|^{(m-p)}$ nonisomorphic (super) edge-magic labelings, where n is any odd number. Ahmad et al., in [2], used the \otimes_h -product to study the super edge-magicness of an odd union of non-necessarily isomorphic acyclic graphs. To produce new families of super edge-magic graphs, it is not only interesting for super edge-magic researches but also for its connections with other types of labelings, see for instance [13]. For 2-regular graphs to have a super edge-magic labeling it is equivalent to having a strong vertex-magic total labeling. Thus, by a result of Gray [19], if an even regular graph of order n has a 2-regular spanning subgraph H which possesses a super edge-magic labeling then the graph has a strong vertex-magic total labeling.

The power of the \otimes_h -product lies in the large number of connections among labelings that emerge from it. Indeed, applications to different types of labeling that included sum and difference labeling can be found in [2,7,26–28]. The main goal of this paper is to extend the use of the \otimes_h product to the well known harmonious, sequential and (a, d) -edge antimagic total labelings, as well as to the recent concept of partitional labeling. We prove that if a (p, q) -digraph with $p \leq q$ is harmonious and $h : E(D) \rightarrow S_n$ is any function, then $und(D \otimes_h S_n)$ is harmonious. We obtain analogous results for sequential and partitional labelings. We also prove that if G is a (super) (a, d) -edge-antimagic total tripartite graph, then nG is (super) (a', d) -edge-antimagic total, where $n \geq 3$, and $d = 0, 2$ and n is odd, or $d = 1$. We also provide an application of the product \otimes_h to classical number theory when the function h is constant. The necessary definitions and references for all the different types of labelings discussed in this paper are provided in the corresponding sections. However, for more information about graph labelings, the interested reader is referred to [16].

2. Harmonious

A (p, q) -graph with $p \leq q$ is called *harmonious* [18] if it is possible to label the vertices with distinct integers (mod q) in such a way that the edge sums are also distinct (mod q). A tree is harmonious if there is a labeling of the vertices in which exactly two vertices have the same label (mod q) and that the condition on the edge sums holds.

The next theorem is an adaptation of Theorem 1.1 for harmonious graphs.

Theorem 2.1. *Let D be a harmonious (p, q) -digraph with $p \leq q$ and let $h : E(D) \rightarrow S_n$ be any function. Then $und(D \otimes_h S_n)$ is harmonious.*

Proof. We rename the vertices of D and each element of S_n after the labels of their corresponding harmonious and super edge-magic labelings respectively. We consider a slight modification of the labels introduced in the proof of Theorem 3.1 of [15]: if $(i, j) \in V(D \otimes_h S_n)$ we assign to the vertex the label $ni + j - 1 \pmod{nq}$.

Given an arc $((i, j)(i', j')) \in E(D \otimes_h S_n)$, coming from an arc $e = (i, i') \in E(D)$ and an arc $(j, j') \in E(h(i, i'))$, the induced arc label is equal to:

$$n(i + i') + j + j' - 2 \pmod{nq}. \tag{1}$$

Since D is harmonious, the set $\{i + i' \pmod{q} | (i, i') \in E(D)\}$ covers all elements in \mathbb{Z}_q . Since each element Γ of S_n is labeled with a super edge-magic labeling, by Corollary 1.1 in [15] (which states that if $G = (V, E)$ is a 2-regular super edge magic graph of order p and f is any super edge-magic labeling of G then $\min\{f(u) + f(v) : uv \in E\} = (p + 3)/2$ and $\max\{f(u) + f(v) : uv \in E\} = (3p + 1)/2$) we have that

$$\{(j + j') | (j, j') \in E(\Gamma)\} = \left[\frac{n + 3}{2}, n + \frac{n + 1}{2} \right].$$

Thus, let us see that the set of arc labels covers all the elements in \mathbb{Z}_{nq} . Let α be the least non-negative residue of $i + i' \pmod{q}$. If $0 \leq \alpha \leq q - 2$ then the set of least non-negative residues of $n(i + i') + j + j' - 2 \pmod{nq}$ covers all integers in $[n\alpha + (n - 1)/2, n(\alpha + 1) + (n - 3)/2]$. Whereas, if $\alpha = q - 1$ then the set of least non-negative residues of $n(i + i') + j + j' - 2 \pmod{nq}$ covers all integers in $[n(q - 1) + (n - 1)/2, nq - 1] \cup [0, (n - 3)/2]$. \square

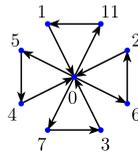


Fig. 1. A friendship digraph D with a harmonious labeling.

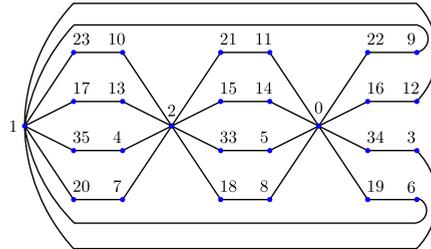


Fig. 2. The graph $und(D \otimes_h S_3)$ with a harmonious labeling.

Note that, since the property harmonious of a labeling is invariant by translations, we can also consider the labeling that assigns to a vertex (i, j) the label $n(i - 1) + j \pmod{nq}$.

Example 2.2. Let D be the friendship digraph with the harmonious labeling given in [18] (see Fig. 1).

Let $S_3 = \{A = 1 \rightarrow 2 \rightarrow 3 \rightarrow 1, B = 1 \rightarrow 3 \rightarrow 2 \rightarrow 1\}$ be the family of all regular digraphs of order 3. Assume that $h : E(D) \rightarrow S_3$ is the function defined by: $h(11, 1) = h(5, 4) = h(7, 3) = h(6, 2) = B$ and $h(e) = A$ otherwise. Then the graph $und(D \otimes_h S_3)$ is harmonious. A harmonious labeling of $und(D \otimes_h S_3)$ can be obtained by assigning the label $3i + j - 1 \pmod{36}$ to the vertex (i, j) (see Fig. 2).

3. Sequential and partitional labelings

The notion of sequential labeling was introduced by Grace in [17]. A *sequential* labeling of a graph G of size q is an injective function $f : V(G) \rightarrow [0, q - 1] \subset \mathbb{Z}$ such that when each edge uv is labeled $f(u) + f(v)$, the resulting edge labels are $[m, m + q - 1]$ for some positive integer m .

A particular case of a sequential labeling was introduced by Ichishima and Oshima in [22]. When G is a bipartite graph of size $2t + s$ with stable sets U and V of the same cardinality s , we say that a sequential labeling of G is *partitional* if: (a) $f(u) \leq t + s - 1$ for each $u \in U$ and $f(v) \geq t - s$ for each $v \in V$, (b) there is a positive integer m such that the induced edge labels are partitioned into three sets: $[m, m + t - 1] \cup [m + t, m + t + s - 1] \cup [m + t + s, m + 2t + s - 1]$, and there is an involution π (automorphism) of G such that

- (i) π exchanges U and V ,
- (ii) $u\pi(u) \in E(G)$, for all $u \in U$, and
- (iii) $\{f(u) + f(\pi(u)) | u \in U\} = [m + t, m + t + s - 1]$.

A graph that admits a sequential (partitional) labeling is respectively a *sequential (partitional) graph*. The family of partitional graphs include (see [22,23]) the n -dimensional cube Q_n for $n \geq 4$, and for every positive integer m the book $S_{2m} \times Q_1$ and the ladder $P_{2m+1} \times Q_1$, where S_i is the star of order $i + 1$ and P_i the path of order i .

The next two theorems show that the set of labelings in which we can use the \otimes_h -product to generate new families of labeled graphs includes sequential and partitional labelings.

Theorem 3.1. *Let D be a sequential digraph and let $h : E(D) \rightarrow S_n$ be any function. Then $und(D \otimes_h S_n)$ is sequential.*

Proof. We rename the vertices of D and each element of S_n after the labels of their corresponding sequential and super edge-magic labelings respectively. Similarly to the proof of Theorem 2.1 if $(i, j) \in V(D \otimes_h S_n)$ we assign to the vertex the label $ni + j - 1$.

Given an arc $((i, j)(i', j')) \in E(D \otimes_h S_n)$, coming from an arc $e = (i, i') \in E(D)$ and an arc $(j, j') \in E(h(i, i'))$, the induced arc label is equal to: $n(i + i') + j + j' - 2$.

Since D is sequential, the set $\{i + i' | (i, i') \in E(D)\}$ covers all elements in $[m, m + |E(D)| - 1]$, for some positive integer m . Since each element Γ of S_n is labeled with a super edge-magic labeling, by Corollary 1.1 in [15] we have $\{(j + j') | (j, j') \in E(\Gamma)\} = [(n + 3)/2, n + (n + 1)/2]$.

Thus, an easy checking shows that the set of arc labels covers all elements in

$$\left[nm + \frac{n-1}{2}, n(m + (|E(D)| - 1)) + \frac{3n-3}{2} \right] = [m', m' + n|E(D)| - 1], \tag{2}$$

where $m' = nm + (n - 1)/2$. Hence, since $|E(D \otimes_h S_n)| = n|E(D)|$ the result follows. \square

Theorem 3.2. *Let D be a partitional graph and let $h : E(\vec{D}) \rightarrow S_n$ be any function, where \vec{D} is the digraph obtained by orienting all edges from one stable set to the other one. Then $\text{und}(\vec{D} \otimes_h S_n)$ is partitional.*

Proof. Assume that U, V are the stable sets of D of the same cardinality s and that $|E(D)| = 2t + s$. We rename the vertices of D and each element of S_n after the labels of their corresponding sequential and super edge-magic labelings respectively. In particular, (a) $u \leq t + s - 1$ for each $u \in U$ and $v \geq t - s$ for each $v \in V$, (b) there is a positive integer m such that the induced edge labels in D are partitioned into three sets: $[m, m + t - 1] \cup [m + t, m + t + s - 1] \cup [m + t + s, m + 2t + s - 1]$, and there is an involution π (automorphism) of D such that

- (i) π exchanges U and V ,
- (ii) $u\pi(u) \in E(D)$, for all $u \in U$, and
- (iii) $\{u + \pi(u) | u \in U\} = [m + t, m + t + s - 1]$.

Assume that the arcs of \vec{D} are oriented from U to V . Clearly, $\vec{D} \otimes_h S_n$ is a bipartite digraph with stable sets $U \times [1, n]$ and $V \times [1, n]$ and $|E(\vec{D} \otimes_h S_n)| = n|E(\vec{D})| = 2nt + ns$. By the proof of [Theorem 3.1](#) we know that the labeling f induced on $\vec{D} \otimes_h S_n$ by assigning to the vertex (i, j) the label $ni + j - 1$ is sequential. Let us see now that f is also partitional.

Condition (a): For each $(u, j) \in U \times \{j\}$ we have

$$f(u, j) = nu + j - 1 \leq n(t + s - 1) + j - 1 \leq nt + ns - 1.$$

Similarly, for each $(v, j') \in V \times \{j'\}$ we obtain $f(v, j') \geq nt - ns$.

Condition (b): Let $m' = nm + (n - 1)/2$. The induced arc labels (2) can be partitioned into three sets:

$$[m', m' + nt - 1] \cup [m' + nt, m' + nt + ns - 1] \cup [m' + nt + ns, m' + 2nt + ns - 1].$$

Let $\tilde{\pi}$ be the automorphism of $\vec{D} \otimes_h S_n$ defined by $\tilde{\pi}_h(u, j) = (\pi(u), j')$, where $(j, j') \in E(h(u, \pi(u)))$. By construction,

- (i) $\tilde{\pi}_h$ exchanges $U \times [1, n]$ and $V \times [1, n]$,
- (ii) $(u, j)\tilde{\pi}_h(u, j) \in E(\vec{D} \otimes_h S_n)$, for all $u \in U$ and $j \in [1, n]$.

Finally, we prove the equality

$$\{(u, j)\tilde{\pi}_h(u, j) | (u, j) \in U \times [1, n]\} = [m' + nt, m' + nt + ns - 1].$$

This clearly holds since D is labeled with a partitional labeling. In particular, we have that $\{u + \pi(u) | u \in U\} = [m + t, m + t + s - 1]$ for some positive integer m and each arc $(u, j)\tilde{\pi}_h(u, j)$ receives the label $nu + j - 1 + n\pi(u) + j' - 1 = n(u + \pi(u)) + j + j' - 2$. \square

4. (Super) (a, d)-edge-antimagic total labelings

Recently, a lot of interest has emerged in relation to labelings of the antimagic type. A good proof for this is the book [8], and for instance the following papers [3–6,9,11,30] that have recently appeared in the literature. In this section we concentrate on (a, d) -edge-antimagic total labelings that were introduced by Simanjuntak et al. in [30]. An (a, d) -edge-antimagic total (EAT) labeling of a (p, q) -graph G is a one to one mapping $f : V(G) \cup E(G) \rightarrow [1, p + q]$ such that the set $\{f(u) + f(uv) + f(v) | uv \in E(G)\}$ is an arithmetic progression starting at a and of difference d . Such a label is called *super* if the smallest possible labels appear on the vertices. A graph that admits a (super) edge-antimagic total labeling is called a (super) edge-antimagic total graph. An example of an EAT labeling is shown in [Fig. 3](#).

Dafik et al. formulated in [11] the following question: if a graph G is super (a, d) -EAT, is the disjoint union of multiple copies of the graph $G(a, d)$ -EAT as well? They answered this question when the graph G is either a cycle or a path.

It was first proved in an unpublished paper by Kotzig [24] (see also [31]) and later, independently and unaware of Kotzig's work, it was reproved by Figueroa et al. [14] that if G is a tripartite graph which admits a (super) $(a, 0)$ -edge-antimagic total labeling and n is odd then the graph nG also admits a (super) $(a, 0)$ -edge-antimagic total labeling. Following the same line of research, Bača et al. [9] have shown that if G is a tripartite graph which admits a (super) $(a, 2)$ -edge-antimagic total labeling then the graph nG also admits a (super) $(a, 2)$ -edge-antimagic total labeling. The main goals in this section are to generalize the results established so far to the case when $d = 1$ and to introduce new proofs of these results based on the Kronecker product of digraphs, that we feel that give more inside information towards the problem than the proofs known so far.

Bača et al. proved in [3] the following result.

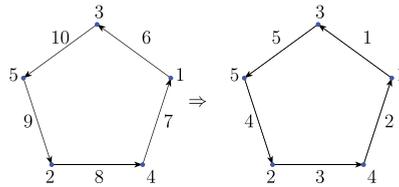


Fig. 3. A super (10, 2)-edge-antimagic total labeling of \vec{C}_5 (on the left).

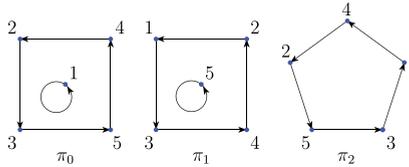


Fig. 4. The three permutations coming from the labeling of Fig. 3.

Theorem 4.1. *The cycle C_n has a super (a, d) -edge-antimagic total labeling if and only if either*

- (i) $d = 0, 2$ and n is odd, $n \geq 3$, or
- (ii) $d = 1$ and $n \geq 3$.

The next lemma shows the existence of three permutations in the symmetric group of n elements that can be obtained from a super (a, d) -edge-antimagic total labeling of the cycle. It will be used in the proof of the main result of the section. We denote by $+_k$ the sum of integers (mod k) and by \mathfrak{S}_n the symmetric group of n elements.

Lemma 4.1. *Let C_n be a super (a, d) -edge-antimagic total graph where the vertices are renamed after the labels of a super (a, d) -edge-antimagic total labeling. Then there exist $\pi_0, \pi_1, \pi_2 \in \mathfrak{S}_n$ such that:*

- The set $\Sigma_k = \{j + \pi_k(j) + \pi_{k+3}(\pi_k(j)) \mid j = 1, \dots, n\}$ is an arithmetic progression of difference d starting at the same number for each $k = 0, 1, 2$.
- $\pi_2 \circ \pi_1 \circ \pi_0 = \text{id}$,

where id denotes the identity permutation.

Proof. Let \vec{C}_n be a strong orientation of C_n . We rename the vertices and the arcs of \vec{C}_n after the labels of a super (a, d) -EAT labeling. Let e_u be the label assigned to the arc (u, v) . We define the following permutations:

$$\pi_0(u) = e_u - n, \quad \pi_1(e_u - n) = v \quad \text{and} \quad \pi_2(v) = u.$$

Clearly, $\Sigma_0 = \{u + \pi_0(u) + \pi_1(\pi_0(u)) = u + e_u - n + v \mid (u, v) \in E(\vec{C}_n)\}$, defines an arithmetic progression starting at $a - n$ and with difference d . The same works for Σ_1 and Σ_2 . \square

Example 4.2. Let us see an example of the previous lemma for $n = 5$. From the $(10, 2)$ -edge-antimagic total labeling of \vec{C}_5 that appears in Fig. 3, we obtain the three permutations that appear in Fig. 4.

Next we prove the following result found in [9,14,24,31] using a different argument. It sheds some new light on the reasons why the theorem is true. Furthermore the proof allows us to construct many different (a, d) -EAT labelings of the resulting graph.

Theorem 4.3. *If G is a (super) (a, d) -EAT tripartite graph, then nG is (super) (a', d) -EAT, where $n \geq 3$,*

- (i) $d = 0, 2$ and n is odd, or
- (ii) $d = 1$.

Proof. For the values of n considered in the statement of Theorem 4.3, we know by Theorem 4.1 that the cycle C_n admits a super (a, d) -edge-antimagic total labeling. Thus by Lemma 4.1 there exist three permutations π_0, π_1 and π_2 in \mathfrak{S}_n such that the set

$$\Sigma_k = \{j + \pi_k(j) + \pi_{k+3}(\pi_k(j)) \mid j = 1, \dots, n\}$$

is an arithmetic progression with difference d for each $k = 0, 1, 2$. Let us denote by F_k the 1-regular digraphs whose adjacency matrix correspond to the graphic representation of each of the permutations π_k , for $k = 0, 1, 2$. We let $P(\vec{C}_n) = \{F_0, F_1, F_2\}$.

We rename the vertices and the edges of G after the labels of a super (a, d) -EAT labeling. Let V_0, V_1 and V_2 be the stable sets of the graph G and let us denote by \vec{G} the digraph obtained from G by orienting each edge from V_k to V_{k+3}^1 . Let $h : E(\vec{G}) \rightarrow P(\vec{C}_n)$ be the function defined by:

$$h((u, v)) = F_k \quad \text{if } u \in V_k.$$

Let us see that $und(\vec{G} \otimes_h P(\vec{C}_n)) = nG$.

For each $j \in [0, n - 1]$ the subdigraph of $\vec{G} \otimes_h P(\vec{C}_n)$ induced by

$$(V_0 \times \{j\}) \cup (V_1 \times \{\pi_0(j)\}) \cup (V_2 \times \{\pi_1(\pi_0(j))\})$$

is isomorphic to \vec{G} . This is clear since, by Lemma 4.1, we know that $\pi_2 \circ \pi_1 \circ \pi_0 = \text{id}$. Next we claim that the graph nG is (super) (a', d) -edge-antimagic total. To prove this, we only have to consider the following induced labeling f :

1. If $(i, j) \in V(\vec{G} \otimes_h P(\vec{C}_n))$ we assign to the vertex the label: $n(i - 1) + j$.
2. If $((i, j), (i', j')) \in E(\vec{G} \otimes_h P(\vec{C}_n))$ we assign to the arc the label: $n(e - 1) + \pi_{k+3}^1(j')$, where e is the label of (i, i') in \vec{G} and $i \in V_k$.

Let us see now that the set $\{f(u) + f(uv) + f(v) \mid uv \in E(\vec{G} \otimes_h P(\vec{C}_n))\}$ is an arithmetic progression with difference d . Let $((i, j), (i', j'))$ be an arc in $E(\vec{G} \otimes_h P(\vec{C}_n))$ coming from arcs $e = (i, i') \in E(\vec{G})$ and $(j, j') \in E(h(i, i'))$. Assume that $i \in V_k$, thus by definition $j' = \pi_k(j)$. Then the corresponding sum $f(u) + f(uv) + f(v)$ is equal to:

$$n(i + i' + e - 3) + j + \pi_k(j) + \pi_{k+3}^1(\pi_k(j)). \tag{3}$$

Since \vec{G} is labeled with an (a, d) -EAT labeling we have that $i + i' + e = a + \mu(e)d$ where $\{\mu(e) \mid e \in E(\vec{G})\} = [0, |E(\vec{G})| - 1]$. Whereas, by Lemma 4.1 there exists $b \in \mathbb{Z}$ such that $j + \pi_k(j) + \pi_{k+3}^1(\pi_k(j)) = b + \nu_k(j)d$, where $\{\nu_k(j) \mid j \in [0, n - 1]\} = [0, n - 1]$ for each $k = 0, 1, 2$. Thus we obtain that

$$n(i + i' + e - 3) + j + \pi_k(j) + \pi_{k+3}^1(\pi_k(j)) = n(a - 3) + b + (n\mu(e) + \nu_k(j))d.$$

Therefore, the set of sum labels of $\vec{G} \otimes_h P(\vec{C}_n)$ is an arithmetic progression starting at $n(a - 3) + b$ and with difference d .

Notice that, if the digraph \vec{G} is super EAT then the vertices of $\vec{G} \otimes_h P(\vec{C}_n)$ receive the smallest labels. \square

As a corollary we obtain a result that is contained in [11].

Corollary 4.1. *Let $m, n \geq 3$. The graph mC_n has a super (a, d) -edge-antimagic total labeling when*

- (i) $d = 0, 2$ and n is odd, or
- (ii) $d = 1$.

Bača et al. showed in [6] that $P_n, n \geq 2$, has a super (a, d) -EAT labeling if and only if $d \in \{0, 1, 2, 3\}$. Using this result and Theorem 4.3 we obtain the next result that also appears in [11].

Corollary 4.2. *Let $m \geq 2$ and $n \geq 3$. The graph mP_n has a super (a, d) -edge-antimagic total labeling when*

- (i) $d = 0, 2$ and n is odd, or
- (ii) $d = 1$.

5. An arithmetic application

So far we have seen applications of super edge-antimagic labelings of cycles together with the product \otimes_h , to different labelings. The next lines are devoted to showing an application of the Kronecker product of oriented cycles to the concepts of greatest common divisor and least common multiple.

Let \vec{C}_n denote the cycle oriented in a cyclic way, say for instance, clockwise. If a_1, \dots, a_n are positive integers, then we will use the notation (a_1, \dots, a_n) and $[a_1, \dots, a_n]$ to denote the greatest common divisor and the least common multiple of a_1, \dots, a_n respectively.

The next result was first proved by Figueroa et al. in [15] and will be of great help through the rest of this section:

Theorem 5.1. $\vec{C}_m \otimes \vec{C}_n = (m, n) \vec{C}_{[m,n]}$.

Since the order of $\vec{C}_m \otimes \vec{C}_n$ and $(m, n) \vec{C}_{[m,n]}$ are being set equal to each other, we get immediately the classical formula

$$mn = (m, n)[m, n] \quad \text{for all } m, n \in \mathbb{N}.$$

The goal of this section is to use the Kronecker product of cyclically oriented cycles in order to generalize this formula to the case of arbitrary many numbers. Furthermore, we also obtain as a corollary that the Kronecker product of cyclically oriented cycles is associative. We start our task with the following technical lemma.

Lemma 5.1. Let a_1, \dots, a_n be positive integers. Then

$$\begin{aligned}
 & \text{(i)} \\
 & \vec{c}_{a_1} \otimes \left(\vec{c}_{a_2} \otimes \left(\vec{c}_{a_3} \otimes \dots \otimes \left(\vec{c}_{a_{n-2}} \otimes \left(\vec{c}_{a_{n-1}} \otimes \vec{c}_{a_n} \right) \right) \dots \right) \right) \\
 & = (a_{n-1}, [a_n])(a_{n-2}, [a_n, a_{n-1}]) \dots (a_1, [a_n, a_{n-1}, \dots, a_2]) \vec{c}_{[a_1, a_2, \dots, a_n]}. \\
 & \text{(ii)} \\
 & \left(\dots \left(\left(\left(\vec{c}_{a_1} \otimes \vec{c}_{a_2} \right) \otimes \vec{c}_{a_3} \right) \otimes \vec{c}_{a_4} \right) \otimes \dots \otimes \vec{c}_{a_n} \right) \\
 & = (a_2, [a_1])(a_3, [a_1, a_2]) \dots (a_n, [a_1, a_2, \dots, a_{n-1}]) \vec{c}_{[a_1, a_2, \dots, a_n]}.
 \end{aligned}$$

Proof. We only prove (i) since the proof of (ii) is similar. In order to prove (i) we use induction on n . For $n = 2$, $\vec{c}_{a_1} \otimes \vec{c}_{a_2} = (a_1, [a_2]) \vec{c}_{[a_1, a_2]}$, by **Theorem 5.1**. Assume that, for $n = k$

$$\begin{aligned}
 & \vec{c}_{a_1} \otimes \left(\vec{c}_{a_2} \otimes \left(\vec{c}_{a_3} \otimes \dots \otimes \left(\vec{c}_{a_{k-2}} \otimes \left(\vec{c}_{a_{k-1}} \otimes \vec{c}_{a_k} \right) \right) \dots \right) \right) \\
 & = (a_{k-1}, [a_k])(a_{k-2}, [a_k, a_{k-1}]) \dots (a_1, [a_k, a_{k-1}, \dots, a_2]) \vec{c}_{[a_1, a_2, \dots, a_k]}.
 \end{aligned}$$

For $n = k + 1$, using the inductive hypothesis for the first equality and the distributive property of the Kronecker product with respect to the union for the second, we have

$$\begin{aligned}
 & \vec{c}_{a_1} \otimes \left(\vec{c}_{a_2} \otimes \left(\vec{c}_{a_3} \otimes \dots \otimes \left(\vec{c}_{a_{k-1}} \otimes \left(\vec{c}_{a_k} \otimes \vec{c}_{a_{k+1}} \right) \right) \dots \right) \right) \\
 & = \left(\vec{c}_{a_1} \otimes \left((a_k, [a_{k+1}]) (a_{k-1}, [a_{k+1}, a_k]) \dots (a_2, [a_{k+1}, a_k, \dots, a_3]) \vec{c}_{[a_2, a_3, \dots, a_{k+1}]} \right) \right) \\
 & = (a_k, [a_{k+1}]) (a_{k-1}, [a_{k+1}, a_k]) \dots (a_2, [a_{k+1}, a_k, \dots, a_3]) \left(\vec{c}_{a_1} \otimes \vec{c}_{[a_2, a_3, \dots, a_{k+1}]} \right) \\
 & = (a_k, [a_{k+1}]) (a_{k-1}, [a_{k+1}, a_k]) \dots (a_2, [a_{k+1}, a_k, \dots, a_3]) (a_1, [a_{k+1}, a_k, \dots, a_2]) \vec{c}_{[a_1, a_2, \dots, a_{k+1}]}. \quad \square
 \end{aligned}$$

Since the order of the graphs in (i) and (ii) of **Lemma 5.1** are on one side:

$$[a_1, a_2, \dots, a_n] \prod_{i=1}^{n-1} (a_{n-i}, [a_n, a_{n-1}, \dots, a_{n-i+1}])$$

and $[a_1, a_2, \dots, a_n] \prod_{i=2}^n (a_i, [a_1, a_2, \dots, a_i])$, respectively, and on the other side $\prod_{i=1}^n a_i$, we get the following corollary:

Corollary 5.1. Let a_1, a_2, \dots, a_n be positive integers. Then

$$\begin{aligned}
 [a_1, a_2, \dots, a_n] \prod_{i=1}^{n-1} (a_{n-i}, [a_n, a_{n-1}, \dots, a_{n-i+1}]) &= \prod_{i=1}^n a_i \\
 &= [a_1, a_2, \dots, a_n] \prod_{i=2}^n (a_i, [a_1, a_2, \dots, a_i]).
 \end{aligned}$$

Hence, when we deal with cycles we obtain the associative property of the Kronecker product. We formalize this fact in the next corollary.

Corollary 5.2. Let l, m, n be positive integers. Then

$$\vec{c}_l \otimes (\vec{c}_m \otimes \vec{c}_n) = (\vec{c}_l \otimes \vec{c}_m) \otimes \vec{c}_n.$$

In order to conclude this section, let us recall the definition of a monoid. Let A be a set of elements and let \circ denote a binary operation defined on the elements of A . Then, the ordered pair (A, \circ) is a monoid if the following three conditions hold:

1. $x \circ y \in A$ for all $x, y \in A$.
2. $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in A$.
3. For all $x \in A$, there exists $e \in A$ such that $e \circ x = x = x \circ e$.

Therefore, it is easy to check that the set of all 1-regular digraphs, including an oriented loop, together with the Kronecker product constitutes a monoid.

6. Conclusions

The fact that super edge-magic labelings have a close relationship with many other types of labelings is well known, established first in [13]. However, what has been recently discovered is that with the help of the Kronecker product and of the product \otimes_h , many other relations among super edge-magic labelings and other types of well studied labelings can be established, and this has been the scope of this paper. The following papers have been devoted to the same goal [7,15,26–28]. In particular, super edge-magic labelings of 2-regular graphs are very useful to develop these relations. Hence, we want to conclude the paper with the following two questions that we feel that are important.

Question 6.1. *Can the techniques developed so far be applied in order to get further relations among labelings?*

Super edge-magic labelings of 2-regular graphs, as we already mentioned before, are a key point in order to develop this type of result. Furthermore, the existence of super edge-magic labelings of 2-regulars graphs has been studied in different papers [15,19–21]. However, we feel that much more needs to be done. Therefore, we propose the following question.

Question 6.2. *Find as many nonisomorphic super edge-magic labelings of 2-regulars graphs as possible.*

Regarding Question 6.2, we want to mention that the work conducted in [29] may be useful in order to improve what is known so far about this question.

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