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Planetary geostrophic equations for the atmosphere with evolution of the barotropic flow

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ABSTRACT

Atmospheric phenomena such as the quasi-stationary Rossby waves, teleconnection patterns, ultralong persistent blockings and the polar/subtropical jet are characterized by planetary spatial scales, i.e. scales of the order of the earth's radius. This motivates our interest in the relevant physical processes acting on the planetary scales. Using an asymptotic approach, we systematically derive reduced model equations valid for atmospheric motions with planetary spatial scales and a temporal scale of the order of about 1 week. We assume variations of the background potential temperature comparable in magnitude with those adopted in the classical quasi-geostrophic theory. At leading order, the resulting equations include the planetary geostrophic balance. In order to apply these equations to the atmosphere, one has to prescribe a closure for the vertically averaged pressure. We present an evolution equation for this component of the pressure which was derived in a systematic way from the asymptotic analysis. Relative to the *prognostic* closures adopted in existing reduced-complexity planetary models, this new *dynamical* closure may provide for more realistic increased large-scale, long-time variability in future implementations.

1. Introduction

A considerable part of the atmospheric variability shows spatial structures on planetary scales, i.e. scales of the order of the earth's radius. One important question for the dynamics of the atmosphere is which are the relevant physical processes on these large scales. Burger (1958) pointed out that on the

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planetary scale the vorticity remains quasi-stationary and the vorticity equation reduces to a balance between the horizontal divergence of the wind and the advection of planetary vorticity. Phillips (1963) proposed a reduced system of equations which consists of the geostrophic and hydrostatic balance, of a 3D divergence constraint (where the vertical velocity results from variations of the Coriolis parameter) and of a transport equation for the potential temperature on a time scale of the order of 1 day. These equations are referred to as describing geostrophic motions of type two or also as planetary geostrophic equations (PGEs). The spatial and temporal scales of the equations lie in the gap between the ranges of validity of two well known types of models: the quasi-geostrophic (QG) models (e.g. Holton, 1992) and the energy balance climate models (EBMs) (e.g. North, 1975; Oerlemans and van den Dool, 1978). The former describe the baroclinic generation and the evolution of the synoptic eddies (spatial scales of the order of the internal Rossby deformation radius) and temporally resolve the geostrophic and hydrostatic adjustment process of the vorticity field. The velocity field is completely left out in the latter family of models and they describe the temporally and zonally averaged surface air temperature as a balance between radiative processes and the transport of heat in the atmosphere (e.g. by the synoptic eddies).

The numerical solution of the PGEs is much cheaper than that of the full 3D primitive equations, as this is done in atmospheric general circulation models (GCMs), and even than the solution of the QG equations. Due to this fact and because they represent much more realistically the atmosphere than EBMs, PG type equations have been implemented in the atmospheric module of some earth system models of intermediate complexity (EMICs) (Petoukhov et al., 1998, 2000). The range of applicability of EMICs is very large, e.g. global warming scenarios, paleoclimate and feedback studies (Claussen et al., 2001; Petoukhov et al., 2005; Calov and Ganopolski, 2005). When applying the PGEs to the atmosphere, the parameterization of the effects of the synoptic eddies, which are of great relevance to the mass, momentum and heat transport, remains an important topic of ongoing research.

In the ocean this issue is different: the synoptic eddies, often referred to as mesoscale eddies, are not so effective in the transport as their atmospheric counterparts. The cause can be easily seen: the internal Rossby deformation radius for the ocean is much smaller than the deformation radius for the atmosphere and consequently the effects of the eddies are restricted to smaller scales. For this reason the PGEs for a Boussinesq fluid are widely used for modeling the large-scale ocean circulation (Salmon, 1998). Some of the pioneering works on the subject are from Robinson and Stommel (1959) and Welander (1959), where the authors studied the steady version of the equations as a model of the ocean thermocline. In the former paper the authors could reproduce some features of the thermocline, e.g. the upwelling at the equator and the deepening in the west direction. In their study they included simple source terms representing temperature diffusion and surface wind stress. The PGEs can be solved numerically in a closed domain (e.g. Samelson and Vallis, 1997) when Laplacian and biharmonic diffusion is added to the temperature equation. In Colin de Verdiere (1986) it was shown through linear stability analysis, that in the presence of a vertically sheared zonal wind the PGEs exhibit baroclinic instability, and the growth rates of the disturbances increase linearly with the zonal wave number, leading to an ill posed mathematical problem. But if one includes a diffusive friction, the problem could be overcome. In Mundt et al. (1997) one can find the results of numerical simulations with the shallow water formulation of the PGEs and simulations with other more sophisticated balanced models (e.g. the geostrophic potential vorticity model). Although the PG model does not resolve the synoptic scales, the gyre scale motion was reproduced reasonably well, even better than by the QG model, and the numerical efficiency was at least an order of magnitude larger than all other balanced models. The reason for the former observations is that the PGEs allow large variations of the background stratification and of the Coriolis parameter, which is not the case for the QG model, where these variables are linearized about a constant value. Another difference to the QG equations is the absence of the relative vorticity in the PG formulation of the potential vorticity (PV). Thus the pressure cannot be determined through the solution of a Laplace equation (invertibility principle). One way out of this difficulty is the addition of some friction (e.g. linear Rayleigh friction) which leads to an elliptic pressure equation and the analogy with the QG is reconstructed. Another possibility is to prescribe the surface wind stress which in the absence of horizontal boundaries will uniquely determine the barotropic part of the flow. In Bresch et al. (2006) the pressure is found by solving a barotropic vorticity equation and integrating the hydrostatic equation. Considering a Boussinesq fluid and expressing the

Rossby, Froude and Burger numbers for the planetary scale in terms of a small parameter (the Rossby number), in the leading order asymptotic expansion the authors obtained the PGEs—with temperature diffusion, but without friction. Interested in the existence of unique solutions under periodic boundary conditions, they derived an evolution equation for the barotropic component of the pressure under the assumption of a constant Coriolis parameter f. As stated in remark 3 in Bresch et al. (2006) this assumption is not realistic, even a β plane approximation for f would be here inappropriate, since the PGEs describe motions with order one variations of the Coriolis parameter (Pedlosky, 1987).

Taking into account the full variations of the Coriolis parameter, here we use a unified asymptotic approach to meteorological modeling in order to study the atmospheric dynamics on the planetary scale. The method, presented in Klein (2000, 2004), provides a self-consistent mathematical description of a phenomenon and captures the essential physics. Majda and Klein (2003) applied it successfully for the systematic development of some reduced models for the tropical dynamics. These can be regarded as a promising theoretical tool for the explanation of aspects of the Madden–Julian Oscillation (Majda and Biello, 2004; Biello and Majda, 2005). In a recent paper (Klein and Majda, 2006) the asymptotic approach was extended to include moist processes. New reduced model equations, describing deep mesoscale convection and accounting for the important interactions between different spatial scales, were derived. Here we derive reduced model equations applicable to atmospheric motions with planetary spatial scale and temporal scale of the order of about 1 week. The obtained equations consist of the PGEs and an evolution equation for the barotropic component of the pressure.

The outline of this paper is as follows: in Section 2 we discuss the governing equations and the asymptotic method, in Section 3 the coordinate scalings for the planetary regime. Key steps in the derivation are presented in Section 4. In Section 5 we summarize the PGEs and the evolution equation for the pressure is derived in Section 6. A summary of the new model equations and a conclusion can be found in Section 7.

2. Asymptotic representation of the governing equations

We start from the governing equations in spherical coordinates for a compressible fluid on the rotating earth and non-dimensionalize them using the reference quantities: $p_{\text{ref}} = 1$ bar, $g = 9.8$ m/s^2, $\rho_{\text{ref}} = 1.25$ kg/m^3, $u_{\text{ref}} = 10$ m/s, $h_{\text{sc}} = p_{\text{ref}}/g/\rho_{\text{ref}} \approx 10$ km, $t_{\text{ref}} = h_{\text{sc}}/u_{\text{ref}} \approx 20$ min, then the equations take the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}u - \frac{uv\tan\phi}{r} - \frac{uw}{r} + \frac{1}{\mathrm{Ro}}(w\cos\phi - v\sin\phi) + \frac{1}{\mathrm{M}^2}\frac{1}{r\rho\cos\phi}\frac{\partial p}{\partial\lambda} = S_u, \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}v + \frac{u^2\tan\phi}{r} + \frac{vw}{r} + \frac{1}{\mathrm{Ro}}u\sin\phi + \frac{1}{\mathrm{M}^2}\frac{1}{r\rho}\frac{\partial p}{\partial\phi} = S_v, \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}w - \frac{u^2}{r} - \frac{v^2}{r} - \frac{1}{\mathrm{Ro}}u\cos\phi + \frac{1}{\mathrm{M}^2}\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{1}{\mathrm{Fr}^2} = S_w, \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta = S_\theta, \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho + \frac{\rho}{r\cos\phi}\left(\frac{\partial u}{\partial\lambda} + \frac{\partial v\cos\phi}{\partial\phi}\right) + \rho\frac{\partial w}{\partial r} + \frac{2w\rho}{r} = 0, \tag{5}$$

$$\rho\theta = p^{1/\gamma}, \tag{6}$$

where λ, ϕ and r denote longitude, latitude and the distance from the center of the earth and the corresponding spherical unit vectors are $\boldsymbol{e}_\lambda$, $\boldsymbol{e}_\phi$ and $\boldsymbol{e}_r$. The non-dimensional variables p, ρ, θ, u, v, and w denote pressure, density, potential temperature and the velocity components in the direction of $\boldsymbol{e}_\lambda$, $\boldsymbol{e}_\phi$ and $\boldsymbol{e}_r$, respectively. M, Fr, and Ro stand for the Mach, Froude and Rossby number, $S_{u,v,w}$ and S_θ represent momentum and diabatic source terms and γ is the isentropic exponent.

The operator $\mathrm{d}/\mathrm{d}t$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{u}{r\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{r}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial r}. \tag{7}$$

We introduce a small parameter $\varepsilon = (a\Omega^2/g)^{1/3}$, $\varepsilon \sim 1/8\ldots1/6$, $\varepsilon \ll 1$ with $\Omega \sim 7\times10^{-5}\,\mathrm{s}^{-1}$ denoting the earth's rotation frequency and $a \sim 6\times10^3$ km the earth's radius. Next, the Mach, Froude and Rossby numbers are expressed in terms of ε in a carefully chosen limit (a detailed discussion of this step can be found in Klein (2000, 2004), Majda and Klein (2003); see also Keller and Ting (1951) who already suggested introducing the cubic root of this acceleration ratio as a small expansion parameter)

$$\sqrt{\mathrm{M}} \sim \sqrt{\mathrm{Fr}} \sim \frac{1}{\mathrm{Ro}} \sim \varepsilon. \tag{8}$$

It follows naturally for the radius of the earth: $a = \varepsilon^{-3}a^{*}h_{\mathrm{sc}}$, where a^{*} is a constant of order unity. Since we are interested in motions in the atmosphere, we can introduce a new non-dimensional coordinate z, measuring the altitude from the ground

$$r = \varepsilon^{-3}a^{*} + z. \tag{9}$$

Finally, the governing equations take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}u - \varepsilon^3\left(\frac{uv\tan\phi}{R} - \frac{uw}{R}\right) + \varepsilon(w\cos\phi - v\sin\phi) = -\frac{\varepsilon^{-1}}{R\rho\cos\phi}\frac{\partial p}{\partial\lambda} + S_u, \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}v + \varepsilon^3\left(\frac{u^2\tan\phi}{R} + \frac{vw}{R}\right) + \varepsilon u\sin\phi = -\frac{\varepsilon^{-1}}{R\rho}\frac{\partial p}{\partial\phi} + S_v, \tag{11}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}w - \varepsilon^3\left(\frac{u^2}{R} + \frac{v^2}{R}\right) - \varepsilon u\cos\phi = -\frac{\varepsilon^{-4}}{\rho}\frac{\partial p}{\partial z} - \varepsilon^{-4} + S_w, \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta = S_\theta, \tag{13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho + \frac{\varepsilon^3\rho}{R\cos\phi}\left(\frac{\partial u}{\partial\lambda} + \frac{\partial v\cos\phi}{\partial\phi}\right) + \rho\frac{\partial w}{\partial z} + \frac{\varepsilon^3 2w\rho}{R} = 0, \tag{14}$$

$$\rho\theta = p^{1/\gamma}, \tag{15}$$

where $R = a^{*} + \varepsilon^3 z$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{\varepsilon^3 u}{R\cos\phi}\frac{\partial}{\partial\lambda} + \frac{\varepsilon^3 v}{R}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial z}. \tag{16}$$

3. Coordinates scaling for the planetary regime

The length δs of a path increment on the surface of the earth can be expressed in terms of variations of longitude $\delta\lambda$ if the latitude is fixed at ϕ_0

$$\delta s = a\cos\phi_0\delta\lambda. \tag{17}$$

If we keep the longitude constant and vary the latitude, we have

$$\delta s = a\delta\phi. \tag{18}$$

We are interested in motions with planetary spatial scales, i.e. motions with a reference length of the order of the radius of the earth: $x_{\mathrm{plan}} \sim a$. For planetary scale motions horizontal variations δs_{plan} divided by the reference length have to be order one: $\delta s_{\mathrm{plan}}/x_{\mathrm{plan}} \sim \mathcal{O}(1)$, as $\varepsilon \to 0$.

Taking into account the expressions (17) and (18), we obtain the conditions

$$\cos\phi_0\delta\lambda_{\mathrm{P}},\ \delta\phi_{\mathrm{P}} \sim \mathcal{O}(1), \tag{19}$$

where we denote longitudinal and latitudinal variations on the planetary scale with $\delta\lambda_{\mathrm{P}}$ and $\delta\phi_{\mathrm{P}}$, respectively. This constraint is satisfied if we take variations appropriate for the planetary scale motions: $\delta\lambda_{\mathrm{P}} \sim \pi/2 \ldots \pi \sim \mathcal{O}(1)$ and $\delta\phi_{\mathrm{P}} \sim \pi/2 \sim \mathcal{O}(1)$ and if we assume that the motion is not in the vicinity of the poles: $\cos\,\phi_0 \sim \mathcal{O}(1)$. Thus the non-dimensional coordinates λ and ϕ resolve already motions on a planetary scale, they do not have to be rescaled and we will denote them with λ_{P} and ϕ_{P}.

In order to make the argumentation above more clear, we will give another example where the coordinates have to be rescaled. Suppose, we want to resolve synoptic scale motions, then the reference length scale is given by $x_{\mathrm{syn}} \sim 1000\,\mathrm{km} \sim \varepsilon^{-2} h_{\mathrm{sc}} = \varepsilon a$. Here we denote synoptic scale longitudinal and latitudinal variations with $\delta\lambda_{\mathrm{S}}$ and $\delta\phi_{\mathrm{S}}$, respectively. Substituting for δs_{syn} (17) and (18) and requiring $\delta s_{\mathrm{syn}}/x_{\mathrm{syn}} \sim \mathcal{O}(1)$ to hold, we obtain the conditions:

$$\varepsilon^{-1} \cos\,\phi_0 \delta\lambda_{\mathrm{S}},\, \varepsilon^{-1}\delta\phi_{\mathrm{S}} \sim \mathcal{O}(1). \tag{20}$$

This is satisfied only if we set

$$\delta\lambda_{\mathrm{S}} = \varepsilon\delta\lambda_{\mathrm{P}} \text{ and } \delta\phi_{\mathrm{S}} = \varepsilon\delta\phi_{\mathrm{P}}, \tag{21}$$

here again $\delta\lambda_{\mathrm{P}} \sim \pi/2 \ldots \pi \sim \mathcal{O}(1)$ and $\delta\phi_{\mathrm{P}} \sim \pi/2 \sim \mathcal{O}(1)$. In order to resolve synoptic scale motions, we have to introduce new "stretched" coordinates λ_{S}, ϕ_{S}

$$\lambda_{\mathrm{S}} = \frac{1}{\varepsilon}\lambda_{\mathrm{P}} \text{ and } \phi_{\mathrm{S}} = \frac{1}{\varepsilon}\phi_{\mathrm{P}}. \tag{22}$$

Applying this scaling, the classical quasi-geostrophic (QG) equations on a sphere could be rederived (Dolaptchiev, 2008).

Next we consider the time coordinate. An appropriate planetary advective time scale, based on the reference velocity u_{ref}, is given by:

$$t_{\mathrm{plan}} = \frac{x_{\mathrm{plan}}}{u_{\mathrm{ref}}} = \frac{\varepsilon^{-3} a^* h_{\mathrm{sc}}}{h_{\mathrm{sc}}/t_{\mathrm{ref}}} = \varepsilon^{-3} a^* t_{\mathrm{ref}} = \frac{a}{h_{\mathrm{sc}}} t_{\mathrm{ref}} \sim 7\text{ days}. \tag{23}$$

A suitable time coordinate, resolving motions on the planetary time scale is

$$t_{\mathrm{P}} = \frac{t'}{t_{\mathrm{plan}}} = \varepsilon^3 t, \tag{24}$$

where t' stands for the dimensional time coordinate and t for the time coordinate non-dimensionalized by t_{ref}. In this paper we study solutions of (10)–(15), which depend only on the introduced planetary spatial and temporal scales and can be represented as an asymptotic series in terms of ε

$$\boldsymbol{U}(\lambda, \phi, z, t; \varepsilon) = \sum_i \varepsilon^i \boldsymbol{U}^{(i)}(\lambda_{\mathrm{P}}, \phi_{\mathrm{P}}, z, t_{\mathrm{P}}), \tag{25}$$

where $\boldsymbol{U}$ is a vector containing the dependent variables.

3.1. A priori assumptions for the background stratification

In the asymptotic expansion we suppose that the deviations from a constant reference value of the potential temperature θ are small throughout the troposphere and are of the order ε^2. This was justified in Majda and Klein (2003), where typical values of the dry buoyancy-frequencies of the atmosphere were considered. In this case the expansion for potential temperature takes the form

$$\theta = 1 + \varepsilon^2 \Theta^{(2)}(\lambda_{\mathrm{P}}, \phi_{\mathrm{P}}, z, t_{\mathrm{P}}) + \varepsilon^3 \Theta^{(3)}(\lambda_{\mathrm{P}}, \phi_{\mathrm{P}}, z, t_{\mathrm{P}}) + \mathcal{O}(\varepsilon^4). \tag{26}$$

As pointed out in Klein and Majda (2006), variations of the potential temperature order ε are associated with long term radiative balances. This is confirmed if one considers the large equator-to-pole surface temperature difference: ~40–60 K. If $\mathcal{O}(\varepsilon)$ potential temperature variations are assumed, the large meridional temperature gradients induce through thermal wind balance zonal winds of the order $\varepsilon^{(-1)} u_{\mathrm{ref}} \sim 70\,\mathrm{m/s}$, comparable in magnitude with the atmospheric jets. The associated flow regimes will be discussed elsewhere in the near future, see Dolaptchiev (2008) and the recent discussion of these scalings in Klein (2008).

3.2. Source terms

Before starting with the asymptotic analysis, we consider the source terms in the governing equations. On the planetary scale radiative effects have an important contribution to S_θ, a simple parameterization of these processes is the relaxation ansatz e.g. (Fraedrich et al., 1998)

$$S_\theta = \frac{\Theta_e - \Theta}{\tau}, \tag{27}$$

here τ is the radiative relaxation time scale, Θ_e denotes the radiative equilibrium temperature of the atmosphere. A typical value for the radiative relaxation time scale is $\sim$20 days $\sim \varepsilon^{-3} t_{\rm ref}$. Taking into account (26), we can estimate the magnitude of S_θ to be $\mathcal{O}(\varepsilon^5)$. This is consistent with the values in the literature for the radiative heating/cooling rates of about 1 K day^{-1} (e.g. Gill, 2003; Holton, 1992). In non-dimensional form they give the same order for S_θ as mentioned here (see Dolaptchiev, 2006). Thus we obtain $S_\theta^{(i)} = 0$ for $i = 0, \ldots, 4$ and the first non-trivial term has the form

$$S_\theta^{(5)} = \frac{\Theta_e - \Theta^{(2)}}{\tau}. \tag{28}$$

The source terms in the momentum equation represent effects due to friction. We will show later on that the vertical velocities disappear up to $\mathcal{O}(\varepsilon^3)$, consequently we will set $S_w^{(i)} = 0$ up to this order. For the sinks of horizontal momentum we use the same representation as in Marshall and Molteni (1993)

$$S_u = -ku, \qquad S_v = -kv, \tag{29}$$

with a drag coefficient $k(\lambda_p, \phi_p, z)$ given through

$$k = \frac{1}{\tau_f}(1 + \alpha_1 LS(\lambda_p, \phi_p) + \alpha_2 H(\lambda_p, \phi_p, z)). \tag{30}$$

The function LS describes variations of the drag over land and sea, the function H variations due to the topography. The constants α_1, α_2 are user defined weights between 0 and 1. Taking the proposed value in Marshall and Molteni (1993) of 3 days for the relaxation time, we have $\tau_f \sim \varepsilon^{-3} t_{\rm ref}$ and the magnitude of the momentum source term is $\mathcal{O}(\varepsilon^3)$ (see Dolaptchiev, 2008). The first non-trivial source terms are

$$S_u^{(3)} = -ku^{(0)}, \qquad S_v^{(3)} = -kv^{(0)}. \tag{31}$$

Next, we proceed with the asymptotic derivation of the reduced equations.

4. Derivation of model equations for planetary motions

4.1. Notation

From here on we drop the subscripts of the temporal and spatial variables, keeping in mind that they resolve motions with temporal scales of the order of about 7 days and spatial scales comparable with the radius of the earth

$$t_{\rm P}, \lambda_{\rm P}, \phi_{\rm P} \to t, \lambda, \phi. \tag{32}$$

The superscript of the order one variable a^* will be dropped as well. The following notation for the operators is used

$$\nabla = \frac{\boldsymbol{e}_\lambda}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{\boldsymbol{e}_\phi}{a}\frac{\partial}{\partial\phi}, \tag{33}$$

$$\Delta = \frac{1}{a^2\cos^2\phi}\left(\frac{\partial^2}{\partial\lambda^2} + \cos\phi\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\phi}\right)\right), \tag{34}$$

$$\boldsymbol{e}_r \cdot (\nabla \times \boldsymbol{u}) = \frac{1}{a\cos\phi}\left(\frac{\partial v}{\partial \lambda} - \frac{\partial u \cos\phi}{\partial \phi}\right), \tag{35}$$

$$\boldsymbol{u} = \boldsymbol{e}_\lambda u + \boldsymbol{e}_\phi v. \tag{36}$$

4.2. Key steps of the derivation

We substitute the ansatz (25) in the governing equations and collect terms of the same order in ε. From the vertical momentum balance follows that the atmosphere is hydrostatically balanced up to $p^{(4)}$

$$\frac{\partial}{\partial z} p^{(i)} = -\rho^{(i)}, \quad i = 0, \ldots, 4. \tag{37}$$

From the horizontal momentum balance (10) and (11) we obtain that $p^{(0)}$ and $p^{(1)}$ do not depend on the horizontal coordinates

$$\nabla p^{(i)} = 0, \quad i = 0, 1, \tag{38}$$

where for the expansion of the advection operator (16) we have used the Taylor series

$$\frac{1}{R} = \frac{1}{a + \varepsilon^3 z} = \frac{1}{a} - \frac{1}{a^2}\varepsilon^3 z + \mathcal{O}(\varepsilon^6). \tag{39}$$

We will drop the time dependence in $p^{(0)}$ and $p^{(1)}$, since it is unphysical that the leading orders of the pressure change in time horizontally uniform on the considered scales (it is possible to derive this assumption starting from the thermodynamic equation rewritten as an evolution equation for the pressure). Expanding the equation of state (15) we have

$$\rho^{(0)} = p^{(0)1/\gamma}, \tag{40}$$

$$\rho^{(1)} = p^{(0)1/\gamma} \frac{p^{(1)}}{\gamma p^{(0)}}, \tag{41}$$

$$\rho^{(2)} + \rho^{(0)}\Theta^{(2)} = p^{(0)1/\gamma}\left(\frac{p^{(2)}}{\gamma p^{(0)}} + \frac{(1-\gamma){p^{(1)}}^2}{2\gamma^2 {p^{(0)}}^2}\right). \tag{42}$$

If the pressure $p^{(i)}$ is hydrostatically balanced, we have the following useful relationship

$$\underbrace{-\frac{\rho^{(i)}}{\rho^{(0)}}}_{\frac{1}{\rho^{(0)}}\frac{\partial p^{(i)}}{\partial z}} + \underbrace{\frac{p^{(i)}}{\gamma p^{(0)}}}_{p^{(i)}\frac{\partial}{\partial z}\frac{1}{\rho^{(0)}}} = \frac{\partial}{\partial z}\pi^{(i)}, \tag{43}$$

here we have introduced the variable

$$\pi^{(i)} = \frac{p^{(i)}}{\rho^{(0)}}. \tag{44}$$

We combine (40) and (37) and obtain for the pressure

$$p^{(0)}(z) = p_0\left(1 - \frac{\gamma - 1}{\gamma} z\right)^{\gamma/\gamma - 1}. \tag{45}$$

p_0 is an integration constant. In the Newtonian limit, i.e. $\gamma - 1 = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$ (for details see Klein and Majda (2006)), the leading order pressure and density reads: $p^{(0)} = \rho^{(0)} = \exp(-z)$. Transforming (41) with the help of (43) and integrating over z we have $p^{(1)}(z) = p_1 p^{(0)}$, where p_1 is another constant of integration. Note that in the expansion of the pressure $p^{(0)}$ can now absorb the $p^{(1)}$ term. Consequently the series for the pressure takes the form

$$p(\lambda, \phi, z, t) = (p_0 + \varepsilon p_1)p^{(0)}(z) + \varepsilon^2 p^{(2)}(\lambda, \phi, z, t) + \mathcal{O}(\varepsilon^3), \tag{46}$$

where without loss of generality the constant factor $p_0 + \varepsilon p_1$ can be set to 1 by an appropriate choice of non-dimensionalization. From (42) we can represent the hydrostatic balance of $p^{(2)}$ with the help of $\Theta^{(2)}$

$$\frac{\partial}{\partial z}\pi^{(2)} = \Theta^{(2)}. \tag{47}$$

For the zero order continuity equation we obtain

$$\frac{\partial}{\partial z}\rho^{(0)} w^{(0)} = 0. \tag{48}$$

Integration gives for $z \to \infty$ and $\rho^{(0)}(\infty) \to 0$: $w^{(0)}(\infty) \to \infty$ which is not physical. So we require $w^{(0)} = 0$. Analogously it can be shown from the next two order equations that $w^{(1)} = w^{(2)} = 0$. The higher order equations are

$$\nabla \cdot \boldsymbol{u}^{(0)} + \frac{1}{\rho^{(0)}}\frac{\partial}{\partial z}\rho^{(0)} w^{(3)} = 0, \tag{49}$$

$$\frac{\partial}{\partial z}\rho^{(0)} w^{(4)} + \rho^{(0)}\nabla \cdot \boldsymbol{u}^{(1)} = 0, \tag{50}$$

$$\frac{\partial}{\partial t}\rho^{(2)} + \boldsymbol{u}^{(0)} \cdot \nabla\rho^{(2)} + \frac{\partial}{\partial z}(\rho^{(0)} w^{(5)} + \rho^{(2)} w^{(3)}) + \rho^{(0)}\nabla \cdot \boldsymbol{u}^{(2)} + \rho^{(2)}\nabla \cdot \boldsymbol{u}^{(0)} = 0. \tag{51}$$

The first two terms in the expansion of the velocity field are geostrophically balanced

$$\boldsymbol{u}^{(0)} = \frac{1}{f}\boldsymbol{e}_r \times \nabla\pi^{(2)}, \tag{52}$$

$$\boldsymbol{u}^{(1)} = \frac{1}{f}\boldsymbol{e}_r \times \nabla\pi^{(3)}. \tag{53}$$

The time evolution of $\boldsymbol{u}^{(0)}$ appears in the next order

$$\begin{aligned} &\frac{\partial}{\partial t}u^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla u^{(0)} + w^{(3)}\frac{\partial}{\partial z}u^{(0)} - \frac{u^{(0)} v^{(0)} \tan\phi}{a} - f v^{(2)} \\ &\quad = -\frac{1}{a\rho^{(0)}\cos\phi}\left(\frac{\partial}{\partial\lambda}p^{(4)} - \frac{\rho^{(2)}}{\rho^{(0)}}\frac{\partial}{\partial\lambda}p^{(2)}\right) + S_u^{(3)}, \end{aligned} \tag{54}$$

$$\begin{aligned} &\frac{\partial}{\partial t}v^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla v^{(0)} + w^{(3)}\frac{\partial}{\partial z}v^{(0)} + \frac{u^{(0)} u^{(0)} \tan\phi}{a} + f u^{(2)} \\ &\quad = -\frac{1}{a\rho^{(0)}}\left(\frac{\partial}{\partial\phi}p^{(4)} - \frac{\rho^{(2)}}{\rho^{(0)}}\frac{\partial}{\partial\phi}p^{(2)}\right) + S_v^{(3)}. \end{aligned} \tag{55}$$

From the expansion of the potential temperature equation we obtain

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}^{(0)} \cdot \nabla + w^{(3)}\frac{\partial}{\partial z}\right)\Theta^{(2)} = S_\theta^{(5)}. \tag{56}$$

From the equations of the asymptotic expansion we can derive now some practical relations.

4.3. *The vorticity and the PV equation*

We can combine (52) and (47) in a thermal wind equation

$$\frac{\partial}{\partial z}\boldsymbol{u}^{(0)} = \frac{1}{f}\boldsymbol{e}_r \times \nabla\Theta^{(2)}. \tag{57}$$

The leading order vorticity balance can be obtained by calculating the divergence of (52)

$$f\nabla\cdot\boldsymbol{u}^{(0)} = -\boldsymbol{u}^{(0)}\cdot\nabla f = -\frac{v^{(0)}\cos\phi}{a}. \tag{58}$$

This equation states that the generation of vorticity through stretching is balanced by the advection of planetary vorticity (Sverdrup balance). Applying $(-1/a)\partial/\partial\phi$ on (54) and $(1/a\cos\phi)\partial/\partial\lambda$ on (55), one can derive a vorticity equation

$$\begin{aligned}&\frac{\partial}{\partial t}\zeta^{(0)} + \nabla\cdot\boldsymbol{u}^{(0)}\zeta^{(0)} + w^{(3)}\frac{\partial}{\partial z}\zeta^{(0)} + \boldsymbol{e}_r\cdot(\nabla w^{(3)}\times\frac{\partial}{\partial z}\boldsymbol{u}^{(0)}) + \nabla\cdot f\boldsymbol{u}^{(2)}\\ &\quad = e_r\cdot\left(\frac{1}{{\rho^{(0)}}^2}\nabla\rho^{(2)}\times\nabla p^{(2)}\right) + \boldsymbol{e}_r\cdot\nabla\times\boldsymbol{S}^{(3)},\end{aligned} \tag{59}$$

here $\boldsymbol{S}^{(3)} = (S_u^{(3)}, S_v^{(3)})^T$ and the vorticity $\zeta^{(0)}$ is given through

$$\zeta^{(0)} = \boldsymbol{e}_r\cdot(\nabla\times\boldsymbol{u}^{(0)}) = \frac{1}{f}\Delta\pi^{(2)} + \frac{u^{(0)}\cot\phi}{a}. \tag{60}$$

The first term on the r.h.s of the equation above expresses the generation of vorticity due to the curvature of the isobars, in contrast to the QG vorticity here f is not constant. The second term represents a shear vorticity—even in the presence of a constant meridional pressure gradient, the geostrophic zonal wind has meridional variations because f varies. In the vorticity equation (59) nearly all terms from the general form are present and it is quite complex when compared with its QG counterpart. This is in accordance with the study of Burger (1958), who pointed out that for PG motions it is difficult to gain more precise information than the quasi-stationary character of the vorticity (58).

Eqs. (49), (56) and (57) can be combined in a conservation equation for the potential vorticity (see Appendix A for the exact derivation)

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}^{(0)}\cdot\nabla + w^{(3)}\frac{\partial}{\partial z}\right)\frac{f}{\rho^{(0)}}\frac{\partial\Theta^{(2)}}{\partial z} = S_{pv}^{(5)}, \tag{61}$$

where we have defined $S_{pv}^{(5)} = f/\rho^{(0)}\partial S_\theta^{(5)}/\partial z$. This completes the derivation of the hierarchy of perturbation equations as far as we will need them to construct a closed, leading-order system of planetary scale equations. The system of equations written down up to that point is not closed because of the (usual) appearance of a higher-order velocity – here $u^{(2)}$ – in the relative vorticity transport equation (59). The subsequent derivation in Section 6 of the evolution equation for the barotropic part of the pressure provides the desired closure as it allows us to eliminate this higher-order velocity in a way similar to that encountered in the classical derivation of QG theory. In the next section the PGEs are summarized and we briefly discuss the closure problem.

5. The PGEs for the atmosphere

Eqs. (52), (47), (49) and (56) represent the PGEs for the atmosphere (Phillips (1963), for applications to the ocean see Robinson and Stommel (1959), Welander (1959)). Here we recapitulate them

$$\boldsymbol{u}^{(0)} = \frac{1}{f}\boldsymbol{e}_r\times\nabla\pi^{(2)}, \tag{62}$$

$$\frac{\partial}{\partial z}\pi^{(2)} = \Theta^{(2)}, \tag{63}$$

$$\nabla\cdot\mathbf{u}^{(0)} = -\frac{1}{\rho^{(0)}}\frac{\partial}{\partial z}\rho^{(0)}w^{(3)}, \tag{64}$$

$$\frac{\partial}{\partial t}\Theta^{(2)} + \boldsymbol{u}^{(0)}\cdot\nabla\Theta^{(2)} + w^{(3)}\frac{\partial}{\partial z}\Theta^{(2)} = S_\theta^{(5)}. \tag{65}$$

As shown in the previous section, these equations can be combined in one transport equation for the PV variable $(f/\rho^{(0)})\partial\Theta^{(2)}/\partial z$, see (61).

The energy of the system is only potential—the PV equation contains only the vorticity stretching term and the relative vorticity is absent due to the fact that the momentum equation is inertialess. Consequently, the pressure cannot be found through the solution of an elliptic equation as in the QG theory. Suppose $\pi^{(2)}$ is known, then one can find the horizontal wind from the geostrophic balance, assuming periodic boundary conditions in λ and ϕ, and the vertical velocity from the divergence constraint, applying vanishing $w^{(3)}$ at the bottom of the atmosphere. Once the velocities are known, the potential temperature can be calculated from the evolution equation for it. By integrating vertically the hydrostatic balance, one can determine the pressure. In doing this one needs a boundary condition for the pressure—it has to be specified at some level, e.g. at the ground. In general, the pressure depends on the motion and prescribing it at some level using a closure or parameterization that is not rooted directly in the governing equations will be a considerable limitation of the model.

In the next section we systematically derive a closure condition for the PGEs within the present asymptotic framework, see (82), (80), (81) and (83). In analogy with the classical derivation of the PV transport equation in the QG theory, we eliminate higher-order unknown terms from the transport equation of relative vorticity (59). We obtain a new evolution equation for the vertically averaged second-order pressure $\overline{p^{(2)}}^z$ that may be interpreted again as a planetary barotropic PV transport equation. Knowing the distribution of $\Theta^{(2)}$ and $\overline{p^{(2)}}^z$, the surface pressure $p_0^{(2)}$ (note that $p^{(2)}=\pi^{(2)}$ at $z=0$) can be easily found from the hydrostatic balance

$$\overline{p^{(2)}}^z=\int_0^1\left\{\rho^{(0)}(z')\int_0^{z'}\Theta^{(2)}(\lambda,\phi,z,t)\,\mathrm{d}z\right\}\mathrm{d}z'+p_0^{(2)}\int_0^1\rho^{(0)}(z')\,\mathrm{d}z'. \tag{66}$$

Again using the hydrostatic balance and $p_0^{(2)}$ as a boundary condition, the pressure at any level can be reconstructed.

6. The evolution of the barotropic pressure

Following the discussion from the previous section, we begin with the derivation of an equation for the barotropic pressure. We have performed an asymptotic analysis (not shown here) of the dynamics within a layer above the troposphere with vertical variations of the potential temperature similar to the observed in the stratosphere: of the order ε. It revealed vanishing vertical velocities $w^{(3)}$ at the tropopause. Consequently, we assume a rigid lid as a boundary condition at the top for the equations presented here, since they are valid within the troposphere. Now if we average (64) with respect to z and apply $w^{(3)}=0$ at $z=0, 1$, we obtain

$$\overline{\nabla\cdot\rho^{(0)}\boldsymbol{u}^{(0)}}^z=0, \tag{67}$$

where we have used the averaging operator $\overline{()}^z$, defined for a general function $f(\lambda,\phi,z,t)$ as

$$\overline{f}^z(\lambda,\phi,t)=\int_0^1 f(\lambda,\phi,z,t)\,\mathrm{d}z. \tag{68}$$

From (67) using (58) we can represent the horizontal divergence through the geostrophically balanced meridional component $v^{(0)}$ and we have

$$\frac{\partial}{\partial\lambda}\overline{p^{(2)}}^z=0. \tag{69}$$

Consequently, the pressure can be written in the form

$$p^{(2)}=\overline{p^{(2)}}^z(\phi,t)+p^{(2)'}(\lambda,\phi,z,t),\qquad \overline{p^{(2)'}}^z=0. \tag{70}$$

Next we can multiply the vorticity equation (59) by $\rho^{(0)}$ and average it with respect to z and λ (note that $\overline{p^{(2)}}^{z,\lambda} = \overline{p^{(2)}}^{z}$)

$$\frac{\partial}{\partial t}\overline{\rho^{(0)}\zeta^{(0)}}^{z,\lambda} + \overline{\nabla\cdot \boldsymbol{u}^{(0)}\rho^{(0)}\zeta^{(0)} + \rho^{(0)}w^{(3)}\frac{\partial}{\partial z}\zeta^{(0)}}^{z,\lambda} + \overline{\rho^{(0)}\boldsymbol{e}_r\cdot\left(\nabla w^{(3)}\times\frac{\partial}{\partial z}\boldsymbol{u}^{(0)}\right)}^{z,\lambda}$$
$$+\overline{\boldsymbol{e}_r\cdot\frac{1}{\rho^{(0)}}\nabla\frac{\partial}{\partial z}p^{(2)}\times\nabla p^{(2)}}^{z,\lambda} + \overline{\nabla\cdot\rho^{(0)}f\boldsymbol{u}^{(2)}}^{z,\lambda} = \overline{\boldsymbol{e}_r\cdot\nabla\times\rho^{(0)}\boldsymbol{S}^{(3)}}^{z,\lambda}. \tag{71}$$

We have to eliminate terms containing $\boldsymbol{u}^{(2)}$ in order to have a closed equation. The fifth term on the l.h.s. can be written as

$$\overline{\nabla\cdot\rho^{(0)}f\boldsymbol{u}^{(2)}}^{z,\lambda} = f\overline{\nabla\cdot\rho^{(0)}\boldsymbol{u}^{(2)}}^{z,\lambda} + \beta\overline{\rho^{(0)}v^{(2)}}^{z,\lambda} = -f\left(\frac{\partial}{\partial t}\overline{\rho^{(2)}}^{z,\lambda} + \overline{\nabla\cdot\boldsymbol{u}^{(0)}\rho^{(2)}}^{z,\lambda}\right) + \beta\overline{\rho^{(0)}v^{(2)}}^{z,\lambda}. \tag{72}$$

Here we have used the continuity equation (51) and the notation $\beta = 1/a\partial f/\partial\phi$. Using (47) and $\rho^{(0)} = \exp(-z)$ we can express the density $\rho^{(2)}$ in terms of pressure and potential temperature

$$\overline{\rho^{(2)}}^{z,\lambda} = -\overline{\frac{\partial}{\partial z}p^{(2)}}^{z,\lambda} = -\overline{\frac{\partial}{\partial z}\rho^{(0)}\pi^{(2)}}^{z,\lambda} = -\overline{\rho^{(0)}\frac{\partial}{\partial z}\pi^{(2)}}^{z,\lambda} + \overline{\rho^{(0)}\pi^{(2)}}^{z,\lambda} = -\overline{\rho^{(0)}\Theta^{(2)}}^{z,\lambda} + \overline{p^{(2)}}^{z,\lambda}. \tag{73}$$

If we average the potential temperature equation (65) over z and λ, the temporal evolution of $\rho^{(2)}$ can be written as

$$-\frac{\partial}{\partial t}\overline{\rho^{(2)}}^{z,\lambda} = \frac{\partial}{\partial t}\left(\overline{\rho^{(0)}\Theta^{(2)}}^{z,\lambda} - \overline{p^{(2)}}^{z,\lambda}\right) \tag{74}$$

$$= -\overline{\rho^{(0)}\boldsymbol{u}^{(0)}\cdot\nabla\Theta^{(2)} - \rho^{(0)}w^{(3)}\frac{\partial}{\partial z}\Theta^{(2)}}^{z,\lambda} + \overline{\rho^{(0)}S_\theta^{(5)}}^{z,\lambda} - \frac{\partial}{\partial t}\overline{p^{(2)}}^{z} \tag{75}$$

$$= -\overline{\nabla\cdot\rho^{(0)}\boldsymbol{u}^{(0)}\Theta^{(2)}}^{z,\lambda} - \underbrace{\overline{\frac{\partial}{\partial z}\rho^{(0)}w^{(3)}\Theta^{(2)}}^{z,\lambda}}_{=0} + \overline{\rho^{(0)}S_\theta^{(5)}}^{z,\lambda} - \frac{\partial}{\partial t}\overline{p^{(2)}}^{z}. \tag{76}$$

Applying (73) and (76), (72) takes the form

$$\overline{\nabla\cdot\rho^{(0)}f\boldsymbol{u}^{(2)}}^{z,\lambda} = -f\left(\frac{\partial}{\partial t}\overline{p^{(2)}}^{z} + \underbrace{\overline{\nabla\cdot\boldsymbol{u}^{(0)}p^{(2)}}^{z,\lambda}}_{=0} - \overline{\rho^{(0)}S_\theta^{(5)}}^{z,\lambda}\right) + \beta\overline{\rho^{(0)}v^{(2)}}^{z,\lambda}. \tag{77}$$

The second term on the r.h.s. disappears if periodic boundary conditions in λ are assumed. We can express $\overline{\rho^{(0)}v^{(2)}}^{z,\lambda}$ in terms of known variables, if we use the momentum equation (54)

$$\overline{\rho^{(0)}v^{(2)}}^{z,\lambda} = \frac{1}{f}\left(\overline{\frac{\partial\rho^{(0)}u^{(0)}}{\partial t}}^{z,\lambda} - \overline{\rho^{(0)}S_u^{(3)}}^{z,\lambda}\right.$$
$$\left. + \overline{\rho^{(0)}\left(\boldsymbol{u}^{(0)}\cdot\nabla u^{(0)} + w^{(3)}\frac{\partial}{\partial z}u^{(0)} - \frac{u^{(0)}v^{(0)}\tan\phi}{a} - \frac{\rho^{(2)}}{a\cos\phi\rho^{(0)^2}}\frac{\partial p^{(2)}}{\partial\lambda}\right)}^{z,\lambda}\right). \tag{78}$$

Substituting the last two equations in (71), the vorticity equation takes finally the form

$$\frac{\partial}{\partial t}\left(\overline{\rho^{(0)}\zeta^{(0)}}^{z,\lambda} + \frac{\beta}{f}\overline{\rho^{(0)}u^{(0)}}^{z,\lambda} - f\overline{p^{(2)}}^{z}\right) + \overline{\mathrm{NV}}^{z,\lambda} + \overline{\mathrm{NM}}^{z,\lambda} = \overline{S_p}^{z,\lambda}, \tag{79}$$

where the notation is used

$$\overline{NV}^{z,\lambda} = \overline{\nabla \cdot \boldsymbol{u}^{(0)}\rho^{(0)}\zeta^{(0)} + \rho^{(0)}w^{(3)}\frac{\partial}{\partial z}\zeta^{(0)} + \rho^{(0)}\boldsymbol{e}_r \cdot (\nabla w^{(3)} \times \frac{\partial}{\partial z}\boldsymbol{u}^{(0)}) + \frac{\boldsymbol{e}_r}{\rho^{(0)}} \cdot \nabla \frac{\partial}{\partial z}p^{(2)} \times \nabla p^{(2)}}^{z,\lambda}, \tag{80}$$

$$\overline{NM}^{z,\lambda} = \frac{\beta}{f}\left(\overline{\rho^{(0)}\boldsymbol{u}^{(0)} \cdot \nabla u^{(0)} + \rho^{(0)}w^{(3)}\frac{\partial}{\partial z}u^{(0)} - \frac{\rho^{(0)}u^{(0)}v^{(0)}\tan\phi}{a} - \frac{\rho^{(2)}}{a\cos\phi\rho^{(0)}}\frac{\partial p^{(2)}}{\partial\lambda}}^{z,\lambda}\right), \tag{81}$$

$$\overline{S_p}^{z,\lambda} = \overline{\boldsymbol{e}_r \cdot \nabla \times \rho^{(0)}\boldsymbol{S}^{(3)}}^{z,\lambda} + \frac{\beta}{f}\overline{\rho^{(0)}S_u^{(3)}}^{z,\lambda} - f\overline{\rho^{(0)}S_\theta^{(5)}}^{z,\lambda}. \tag{82}$$

$\overline{NV}^{z,\lambda}$ contains terms from the vorticity equation (71): horizontal advection of relative vorticity by the leading order wind $\boldsymbol{u}^{(0)}$, divergence of this wind multiplied by the relative vorticity, vertical advection of vorticity, the twisting term and the solenoidal term. The terms in $\overline{NM}^{z,\lambda}$ and the second term in the brackets of (79) result from the elimination of the advection of planetary vorticity by the zonally and vertically averaged ageostrophic meridional velocity $\overline{v^{(2)}}^{z,\lambda}$ (see (54)). The last term in the brackets of (79) results from the density tendencies caused by the divergence of $\boldsymbol{u}^{(2)}$ (see (51)). We express the bracketed terms in (79) in terms of $\overline{p^{(2)}}^{z}$ using (60) and the geostrophic balance

$$\frac{\partial}{\partial t}\left(\frac{1}{a^2\cos\phi}\frac{\partial}{\partial\phi}\frac{\cos\phi}{f}\frac{\partial}{\partial\phi}\overline{p^{(2)}}^{z} - \frac{\beta}{f^2 a}\frac{\partial}{\partial\phi}\overline{p^{(2)}}^{z} - f\overline{p^{(2)}}^{z}\right) + \overline{NV}^{z,\lambda} + \overline{NM}^{z,\lambda} = \overline{S_p}^{z,\lambda}. \tag{83}$$

The terms $\overline{NV}^{z,\lambda}$ and $\overline{NM}^{z,\lambda}$ can be calculated if the distribution of $p^{(2)}$ is known, then (83) can be integrated in time and after inverting the Helmholtz operator acting on $\overline{p^{(2)}}^{z}$ the evolution of the barotropic component of the pressure will be determined. From (66) the surface pressure $p_0^{(2)}$ can be calculated which will provide the necessary boundary condition to solve (62)–(65).

As we have already mentioned, some EMICs solve the PGEs presented in Section 5 but use a diagnostic parameterization for $\overline{p^{(2)}}^{z}$ in order to close the system. The closure is based on a linear relationship between the pressure and the temperature (Petoukhov et al., 2000). In this way the model has only one prognostic equation—an advection equation for the temperature. This considerably reduces the computational time but also may be one cause of the limited atmospheric variability observed in simulations based on these models. The presented closure here is an additional evolution equation, which will be added to the PGEs. Since it has only one spatial dimension, because of the averaging in λ and z, it will not severely decrease the numerical efficiency of the model. Nevertheless it will add an additional degree of freedom to the system which can improve the representation of the atmospheric variability in the model. As shown, the evolution equation for the barotropic pressure arises from the vorticity equation, it contains terms such as advection of planetary vorticity and of relative vorticity which are not present in the classical PG model. This gives the possibility to capture additional physical phenomena with the model, e.g. zonal planetary Rossby waves.

Some EMICs are vertically averaged models, other have a very crude vertical resolution, e.g. some universal linear structure for the temperature is assumed (Claussen et al., 2001). This motivates to analyse the special case when the pressure distribution is represented as the product of two functions, one dependent on z and another on the horizontal coordinates and time. From the condition (69) we obtain that there are no variations in λ. As a consequence $v^{(0)}$ and $w^{(3)}$ disappear, the terms $\overline{NV}^{z,\lambda}$ and $\overline{NM}^{z,\lambda}$ are zero and the initial pressure distribution remains constant in time. We conclude that in the presented model one should consider at least two modes in order to have non-trivial evolution of $\overline{p^{(2)}}^{z}$. Such an assumption is implicitly made in the CLIMBER EMIC (Petoukhov et al., 2000), taking into account the atmospheric lapse rate dependence on the surface temperature (e.g. Mokhov and Akperov, 2006).

7. Conclusions and outlook

Using an asymptotic approach, we derived reduced model equations valid for one particular regime of planetary scale atmospheric motions with temporal variations of the order of about 1 week. Such temporal and spatial scales characterize atmospheric phenomena like the quasi-stationary Rossby waves, teleconnection patterns, blockings. Here we summarize the equations

$$\boldsymbol{u}^{(0)} = \frac{1}{f}\boldsymbol{e}_r \times \nabla\pi^{(2)}, \tag{84}$$

$$\frac{\partial}{\partial z}\pi^{(2)} = \Theta^{(2)}, \tag{85}$$

$$\nabla \cdot \boldsymbol{u}^{(0)} = -\frac{1}{\rho^{(0)}}\frac{\partial}{\partial z}\rho^{(0)}w^{(3)}, \tag{86}$$

$$\frac{\partial}{\partial t}\Theta^{(2)} + \boldsymbol{u}^{(0)} \cdot \nabla\Theta^{(2)} + w^{(3)}\frac{\partial}{\partial z}\Theta^{(2)} = S_\theta^{(5)}, \tag{87}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{a^2\cos\phi}\frac{\partial}{\partial\phi}\frac{\cos\phi}{f}\frac{\partial}{\partial\phi}\overline{p^{(2)}}^z - \frac{\beta}{f^2 a}\frac{\partial}{\partial\phi}\overline{p^{(2)}}^z - f\overline{p^{(2)}}^z\right) + \overline{NV}^{z,\lambda} + \overline{NM}^{z,\lambda} = \overline{S_p}^{z,\lambda}, \tag{88}$$

for the definition of $\overline{NV}^{z,\lambda}$, $\overline{NM}^{z,\lambda}$ and $\overline{S_p}^{z,\lambda}$ see (80), (81) and (82). The above equations contain the PGEs and a planetary barotropic vorticity equation (88). The PGEs alone do not represent a closed system, since a boundary condition for the surface pressure, or equivalently for the barotropic pressure, is needed. We derived the additional evolution equation (88) which uniquely determines the barotropic component of the flow and provides the desired closure. Consistent with previous studies on planetary scale motions, it contains terms absent in the classical QG model: the solenoidal, the twisting, vertical advection term (where the vertical velocity results from the variation of f) and the advection of planetary vorticity by the ageostrophic wind. The new equation gives the possibility to capture additional physical phenomena, not included in the models based on the PGEs only. It suggests itself as a prognostic alternative to the temperature-based diagnostic closure adopted in reduced-complexity planetary models (e.g. CLIMBER; Petoukhov et al., 2000) and may provide for more realistic increased large-scale, long-time variability in future implementations. It was shown in Colin de Verdiere (1986) that the PGEs produce baroclinic instability in the presence of a shared wind. The addition of (88) to the system should not affect this property, since the last equation governs the barotropic component of the flow and we regard it as a boundary condition for closing the PGEs. Nevertheless it is important to emphasize that in the presented model there is an important coupling between the barotropic dynamics (88) and the temperature equation (65) trough the surface pressure $p_0^{(2)}$ from (66). The last equation shows that changes in the barotropic pressure $\overline{p^{(2)}}^z$ will alter the $p_0^{(2)}$ distribution and thus the surface wind field, which will change the temperature trough advection. In this way (88) will considerably modify the behavior of the model compared with others based only on the classical PGEs.

Expressed in dimensional units the variations near the ground of $p^{(2)}$ from (88) are of the order $\sim$20 hPa. Such fluctuations are comparable with those associated with meridional variations of the zonal mean surface pressure and with anomalies due to quasi-stationary Rossby waves (Peixoto and Oort, 1992). Since $\pi^{(2)}$ from (63) is defined as $p^{(2)}$ scaled with $\rho^{(0)}$ (44), the fluctuations estimated above will increase with height in accordance with the equivalent barotropic structure of the quasi-stationary anomalies (Hoskins and Pearce, 2001).

We have shown in our analysis that the planetary distribution of the vertically averaged leading order pressure is zonally symmetric. Such property possess some planetary oscillations with dynamical relevance to the atmosphere, e.g. the zonal index (Rossby, 1939) describing the transitions between regimes with blocked or enhanced midlatitude westerly flow. Zonally symmetric are also the leading modes of variability in the extratropical circulation, also known as northern and southern annular modes (AM) (Thompson and Wallace, 2000), they are characterized by planetary time scales of about 1–2 weeks. The derived new reduced equations may help for better understanding of the structure of the AM. Whereas the zonal symmetry here is a direct consequence from the averaged continuity equation, idealized experiments (Cash et al., 2002) have indicated that the zonally symmetric AM structure can be interpreted as the resulting distribution from many zonally localized events with a meridional structure similar to that of the AM. Another zonal phenomenon characterized by planetary scales is the poleward propagation of zonal mean zonal wind anomalies (Riehl et al., 1959) with period of about 60 days (Lee et al., 2007).

We want to compare our approach for the derivation of reduced models with the one based on mode truncation. In the latter the governing equations, e.g. the primitive equations, are projected on suitable basis functions. As basis one can choose some functions motivated from the large-scale flow structures, e.g. the slow Hough harmonics (Kasahara, 1977; Tanaka, 2003) or some empirical orthogonal functions (EOFs) (Schubert, 1985; Achatz and Opsteegh, 2003; Achatz and Branstator, 1999). Such models predict the time evolution only of the leading functions and the effects from the unresolved modes have to be parameterized, e.g. trough some linear regression. Instead of truncating the degrees of freedom of the large-scale solution by considering a small number of horizontal or vertical modes, here we filter the governing equations trough the asymptotic technique so that they are valid only for the planetary scales. In this way phenomena not relevant for the planetary scale dynamics like barotropic acoustic waves or hydrostatic gravity waves (present in the PEs) are neglected, retaining the full 3D structure of the solution. In both approaches, the question about the representation of the unresolved scales (here the synoptic eddies) remains open. We can parameterize the baroclinic instability and apply some linear regression fitting procedure (Tanaka, 1991; Achatz and Branstator, 1999). Or we can make use of the recently introduced stochastic mode reduction strategies (Majda et al., 2003; Franzke et al., 2005; Franzke and Majda, 2006). The unified asymptotic technique applied here gives us another tool for representing the synoptic scales and their interactions with the planetary scales. Using two scale asymptotic expansion, one can capture in a systematic way these interactions deriving coupled reduced equations governing both the planetary scale motion and the synoptic scale flow. First attempts in this direction have revealed equations which can be regarded as the anelastic analogon of Pedlosky's two scale model for the large scale ocean circulation (Pedlosky, 1984). In summary, we consider our approach as an alternative to the one based on mode truncation; it reveals new insights in the atmospheric dynamics and because of its systematic basis it has the potential to be utilized for studies of multiple scales phenomena.

As already mentioned we have used in the derivation of the equation for the barotropic flow the boundary condition of vanishing vertical velocity at the bottom and at the top of the domain. This condition was motivated from the asymptotic analysis of the dynamics within a layer above the troposphere with vertical variations of the potential temperature similar to the observed in the stratosphere: of the order ε. In this case we have shown that the vertical velocities $w^{(3)}$ disappear and we have assumed a rigid lid at the top of the troposphere which is also consistent with the QG theory. An open question here is how other boundary conditions, e.g. vanishing zonally averaged vertical mass flux, will modify the presented prognostic closure.

Of particular interest is the asymptotic regime where the constraint of constant background stratification is relaxed allowing meridional variations of $\Theta^{(1)}$. Under the new assumption strong zonal winds, comparable in magnitude with the atmospheric jets, result. We plan to present the related analysis as well as numerical solutions of the new equations in a separate forthcoming paper.

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Appendix A

A.1. Derivation of the PV equation

We differentiate (56) w.r.t. z and multiply it by $f/\rho^{(0)}$

$$\frac{\partial}{\partial t}\left(\frac{f}{\rho^{(0)}}\frac{\partial\Theta^{(2)}}{\partial z}\right)+f\boldsymbol{u}^{(0)}\cdot\nabla\left(\frac{1}{\rho^{(0)}}\frac{\partial\Theta^{(2)}}{\partial z}\right)+\frac{f}{\rho^{(0)}}\underbrace{\frac{\partial\boldsymbol{u}^{(0)}}{\partial z}\cdot\nabla\Theta^{(2)}}_{=0(57)}+\frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\left(w^{(3)}\frac{\partial}{\partial z}\Theta^{(2)}\right)=0. \tag{A-1}$$

For the last term we can write also

$$\frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\left(w^{(3)}\frac{\partial}{\partial z}\Theta^{(2)}\right)=w^{(3)}\frac{\partial}{\partial z}\frac{f}{\rho^{(0)}}\frac{\partial\Theta^{(2)}}{\partial z}+\frac{f}{\rho^{(0)}}\frac{\partial\Theta^{(2)}}{\partial z}\frac{1}{\rho^{(0)}}\frac{\partial\rho^{(0)}w^{(3)}}{\partial z}. \tag{A-2}$$

From (49) and (58) we obtain

$$\frac{1}{\rho^{(0)}}\frac{\partial\rho^{(0)}w^{(3)}}{\partial z}=\frac{1}{f}\boldsymbol{u}^{(0)}\cdot\nabla f. \tag{A-3}$$

Using the last two equations, the fourth term on the l.h.s. of (A-1) takes the form

$$\frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\left(w^{(3)}\frac{\partial}{\partial z}\Theta^{(2)}\right)=w^{(3)}\frac{\partial}{\partial z}\frac{f}{\rho^{(0)}}\frac{\partial\Theta^{(2)}}{\partial z}+\frac{1}{\rho^{(0)}}\frac{\partial\Theta^{(2)}}{\partial z}\boldsymbol{u}^{(0)}\cdot\nabla f. \tag{A-4}$$

So (A-1) can be finally written in the form

$$\left(\frac{\partial}{\partial t}+\boldsymbol{u}^{(0)}\cdot\nabla+w^{(3)}\frac{\partial}{\partial z}\right)\frac{f}{\rho^{(0)}}\frac{\partial\Theta^{(2)}}{\partial z}=S_{pv}^{(5)}, \tag{A-5}$$

where we have defined $S_{pv}^{(5)}=f/\rho^{(0)}\partial S_{\theta}^{(5)}/\partial z$.

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