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J. Math. Anal. Appl. 313 (2006) 400–418
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Spectral properties of the Cauchy transform on $L_2(\mathbb{C}, e^{-|z|^2}\lambda(z))$

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Received 5 August 2003

Available online 4 November 2005

Submitted by William F. Ames

Abstract

Let $h_{m,p}(z)$, $(m,p) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, be the Landau orthogonal basis of the Hilbert space on $L_2(\mathbb{C}, e^{-|z|^2}\,d\lambda(z))$ where $\lambda(z)$ is the usual Lebesgue measure on the complex plane. In this paper we give some spectral properties of the Cauchy transform on the orthogonal complement of Bargmann space $\Lambda_0(\mathbb{C})$ in $L_2(\mathbb{C}, e^{-|z|^2}\,d\lambda(z))$. In particular for m fixed, we consider the orthogonal projection operator on the Hilbert subspace spanned by $h_{m,p}(z)$, $p = 0, 1, 2, \ldots,$ and we give explicitly the sequence of singular values of its composition with the Cauchy transform in $L_2(\mathbb{C}, e^{-|z|^2}\,d\lambda(z))$. As application of these of the Cauchy transform we get some identities for special functions which could be of independent interest.

Keywords: Cauchy transform; Green transform; Singular values; Gauss hypergeometric functions; Landau basis

1. Introduction and statement of the main results

Let $d\mu(z) = e^{-|z|^2}\,d\lambda(z)$ be the Gaussian density on the complex plane $\mathbb{C} := \{z = x + iy;\ (x, y) \in \mathbb{R}^2\}$, $|z|^2 = x^2 + y^2$ and $d\lambda(z) = dx\,dy$ is the usual Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.09.056

Then in [7] Dostanic has investigated some spectral properties of the Cauchy transform C defined by

$$Cf(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(w)}{z-w}\, d\mu(w), \quad f \in L_2\big(\mathbb{C}, d\mu(z)\big). \tag{1.1}$$

Namely, the sequence of the singular values $s_p(C)$, $p \in \mathbb{Z}_+$, of the compact operator C in $L_2(\mathbb{C}, d\mu(z))$ behaves when $p \to \infty$ as $s_p(C) = \frac{1}{\sqrt{p}}(1 + o(1))$ and that the sequence of singular values $s_p(P_0 C)$, $p \in \mathbb{Z}_+$, behaves when $p \to \infty$ as $s_p(P_0 C) = o(e^{-p \operatorname{Log} \sqrt{3}})$, where P_0 is the orthogonal projection operator from $L_2(\mathbb{C}, d\mu(z))$ onto the Bargmann space [5]:

$$\Lambda_0(\mathbb{C}) = \big\{ f \in L_2\big(\mathbb{C}, d\mu(z)\big),\ f \text{ entire on } \mathbb{C} \big\}.$$

Now our main aim in this paper is to discuss some spectral properties of the Cauchy transform C on the orthogonal complement $\Lambda_0^{\perp}(\mathbb{C})$ of Bargmann space $\Lambda_0(\mathbb{C})$ in $L_2(\mathbb{C}, d\mu(z))$. This will be possible by using the following Hilbertian orthogonal decomposition of $L_2(\mathbb{C}, d\mu(z))$ given by

$$L_2\big(\mathbb{C}, d\mu(z)\big) = \bigoplus_{m \in \mathbb{Z}_+} \Lambda_m(\mathbb{C}) \tag{1.2}$$

where the Hilbert subspaces $\Lambda_m(\mathbb{C})$, $m \in \mathbb{Z}_+$, are defined by

$$\Lambda_m(\mathbb{C}) = \big\{ f \in L_2\big(\mathbb{C}, d\mu(z)\big),\ A^\star A f = m f \big\}. \tag{1.3}$$

In (1.3) the operators $A^\star$ and A are given respectively by $A^\star = -\frac{\partial}{\partial z} + \bar{z}$ and $A = \frac{\partial}{\partial \bar{z}}$ so that

$$A^\star A = -\frac{\partial^2}{\partial z \partial \bar{z}} + \bar{z} \frac{\partial}{\partial \bar{z}}$$

is a second order elliptic differential operator of Laplacian type.

Thus using the Hilbertian orthogonal decomposition (1.2) of the space $L_2(\mathbb{C}, d\mu(z))$, we will be able to give, for every fixed $m \in \mathbb{Z}_+$, explicit formulae for the nonzero eigenvalues $\lambda_{m,p}$, $p = 0, 1, 2, \ldots$, of the positive operator

$$|P_m C| := \sqrt{(P_m C)^\star (P_m C)}$$

where P_m is the orthogonal projection operator from $L_2(\mathbb{C}, d\mu(z))$ onto the space $\Lambda_m(\mathbb{C})$ in $L_2(\mathbb{C}, d\mu(z))$ by (1.3).

Indeed, the main results to which is aimed this paper can be stated as follows.

Theorem. *Let $m \in \mathbb{Z}_+$ and m to be fixed. Let $\lambda_{m,p}$, $p = 0, 1, 2, \ldots$, be the sequence of nonzero eigenvalues of the positive operator*

$$|P_m C| := \sqrt{(P_m C)^\star (P_m C)}$$

where P_m is the orthogonal projection operator from $L_2(\mathbb{C}, d\mu(z))$ onto the space $\Lambda_m(\mathbb{C})$ in $L_2(\mathbb{C}, d\mu(z))$ given by (1.3) *and C is the Cauchy transform in $L_2(\mathbb{C}, d\mu(z))$ as given in* (1.1). *Then we have*:

(i) *The eigenvalues $\lambda_{m,p}$, $p = 0, 1, 2, \ldots$, of $|P_m C|$ are given explicitly via the following formula*:

$$(\lambda_{m,p})^2 = \gamma_{m-1,p}\, {}_2F_1\left(-\min(m-1,p), -\min(m-1,p), |m-1-p|+1, \frac{1}{4}\right) \tag{1.4}$$

where

$$\gamma_{m-1,p} = \left(\frac{2}{3}\right)^{2\min(m-1,p)} 3^{-(|m-1-p|+1)} \frac{(\max(m-1,p)!)^2}{m!p!|m-1-p|!},$$

${}_2F_1(\alpha,\beta;\gamma,x)$ *stands for the usual Gauss hypergeometric function.*

(ii) *For $m \in \mathbb{Z}_+$ and $p = 0,1,2,\ldots$ let $\phi_{m,p}$ be the functions of Hermite type defined on $\mathbb{C}$ by*

$$\phi_{m,p}(z) = \begin{cases} \frac{-1}{p+1}\, {}_1F_1(1, p+2, |z|^2) z^{p+1}, & m = 0, \\ -(-1)^{m+p} \frac{\partial^{m-1+p}}{\partial^{m-1} z \partial^p \bar{z}} e^{-|z|^2}, & m = 1,2,\ldots. \end{cases} \tag{1.5}$$

Then for each $p = 0,1,2,\ldots$, the above function $\phi_{m,p}(z)$ is an eigenvector of the operator $|P_m C|$ with $\lambda_{m,p}$ as eigenvalue.

Remark. (i) The usual Gauss hypergeometric functions ${}_2F_1(\alpha,\beta;\gamma,x)$ and ${}_1F_1(\alpha,\gamma,z)$, see [10] or [11,12], are defined by

$${}_2F_1(\alpha,\beta;\gamma,x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\gamma+k)} \frac{x^k}{k!}, \quad |x| < 1,$$

and

$${}_1F_1(\alpha,\beta,z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)}{\Gamma(\beta+k)} \frac{z^k}{k!}$$

where $\Gamma(\cdot)$ is gamma function.

(ii) For $p \geqslant m-1$, the nonzero-eigenvalues $\lambda_{m,p}$ of the operator $|P_m C|$ are given simply using (1.4) by

$$\begin{aligned}(\lambda_{m,p})^2 = {} & \left(\frac{2}{3}\right)^{2(m-1)} 3^{-(p-m+2)} \frac{(p!)^2}{m!p!(p-m+1)!} \\ & \times {}_2F_1\left(-m+1; -m+1; p-m+2; \frac{1}{4}\right).\end{aligned} \tag{1.6}$$

Corollary (Asymptotic of $\lambda_{m,p}$ when $p \to \infty$). *Let $m \in \mathbb{Z}_+$, m fixed. Then the asymptotic behaviour of the eigenvalues $\lambda_{m,p}$ of the operator $|P_m C|$ is given explicitly by*

$$\lambda_{m,p} = \frac{1}{2\sqrt{m!}} \left(\frac{4}{3}\right)^{m/2} e^{-p \operatorname{Log}\sqrt{3}} \sqrt{\frac{\Gamma(p+1)}{\Gamma(p-m+2)}}. \tag{1.7}$$

Now we give an outline of the content of this paper. In Section 2, we recall the main properties of the Landau basis $h_{m,p}(z)$ of $L_2(\mathbb{C}, d\mu(z))$ that are relevant for our purpose. Section 3 deals with the action of the Cauchy transform in (1.1) on the Landau basis $\{h_{m,p}(z)\}$ of $L_2(\mathbb{C}, d\mu(z))$. In Section 4, we provide for fixed m explicit formulae for the Schwartz kernel functions of $P_m C$ as well of $(P_m C)^\star(P_m C)$ and we give the proof of main results stated in Section 1. Proceeding in the same way, we deal with the Green transform on $L_2(\mathbb{C}, d\mu(z))$ and we state its main spectral properties that correspond to the analog of what we have done for the Cauchy transform C on $L_2(\mathbb{C}, d\mu(z))$.

2. The Landau orthogonal basis $h_{m,p}(z)$, $(m,p)\in\mathbb{Z}_+\times\mathbb{Z}_+$, of the Hilbert space $L_2(\mathbb{C}, d\mu(z))$

In this section,we state some elementary properties of the polynomials $h_{m,p}(z)$, $(m,p)\in\mathbb{Z}_+\times\mathbb{Z}_+$, of Hermite type [15] that were given in the abstract through the formula

$$h_{m,p}(z)=(-1)^{m+p}e^{|z|^2}\frac{\partial^{m+p}}{\partial z^m\partial\bar{z}^p}e^{-|z|^2} \tag{2.1}$$

where $z=x+iy$, $(x,y)\in\mathbb{R}^2$, $\bar{z}=x-iy$, $\frac{\partial}{\partial z}=\frac{1}{2}\left[\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right]$ and $\frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left[\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right]$.

Namely the following proposition summarize the properties of these polynomials $h_{m,p}(z)$ in the following proposition that we will be using in this paper.

Proposition 2.1. *Let $h_{m,p}(z)$, $(m,p)\in\mathbb{Z}_+\times\mathbb{Z}_+$, be the set of polynomials given by*

$$h_{m,p}(z)=(-1)^{m+p}e^{|z|^2}\frac{\partial^{m+p}}{\partial z^m\partial\bar{z}^p}e^{-|z|^2}.$$

Then the following statements hold:

(i) *For fixed $(m,p)\in\mathbb{Z}_+\times\mathbb{Z}_+$, the polynomial $h_{m,p}(z)$ can be expressed as*

$$(\alpha)\qquad h_{m,p}(z)=A^{\star m}(z^p)\quad \text{where } A^\star=-\frac{\partial}{\partial z}+\bar{z}. \tag{2.2}$$

$$(\beta)\qquad h_{m,p}(z)=c_{m,p}\,{}_1F_1\big(-\min(m,p);\ |m-p|+1;\ |z|^2\big)|z|^{|m-p|}e^{-i(m-p)\arg z} \tag{2.3}$$

where $z=|z|e^{i\arg z}$ and $c_{m,p}=\frac{(-1)^{\min(m,p)}\max(m,p)!}{|m-p|!}$.

$(\beta)_{\text{bis}}$ *The polynomials $\{h_{m,p}(z)\}_{m\geqslant 0,\,p\geqslant 0}$ form a complete orthogonal system of $L_2(\mathbb{C}, d\mu(z))$ and their norms $\|h_{m,p}\|$ are given by $\|h_{m,p}\|^2=\pi m!p!$.*

(ii) *Let $\Delta=A^\star A$ where $A^\star=-\frac{\partial}{\partial z}+\bar{z}$ and $A=\frac{\partial}{\partial\bar{z}}$ and for fixed $m\in\mathbb{Z}_+$, let $\Lambda_m(\mathbb{C})=\{f\in L_2(\mathbb{C},d\mu(z)),\ \ A^\star Af=mf\}$ then $\Lambda_m(\mathbb{C})=\operatorname{span}\{h_{m,p}(z),\ \ p=0,1,2,\ldots\}$ in $L_2(\mathbb{C},d\mu(z))$. Further the spaces $\Lambda_m(\mathbb{C})$ are pairwise orthogonal in $L_2(\mathbb{C},d\mu(z))$ and we have the Hilbertian decomposition $L_2(\mathbb{C},d\mu(z))=\bigoplus_{m\geqslant 0}\Lambda_m(\mathbb{C})$.*

(iii) *For fixed $m\in\mathbb{Z}_+$, let $P_m(z,w)$ be the integral Schwartz kernel of the orthogonal projection operator P_m from $L_2(\mathbb{C},d\mu(z))$ on the Hilbert subspace $\Lambda_m(\mathbb{C})$. Then $P_m(z,w)$ can be expressed as*

$$(\gamma)\qquad P_m(z,w)=\sum_{p=0}^{\infty}\frac{h_{m,p}(z)\overline{h_{m,p}(w)}}{\|h_{m,p}\|^2}\quad \text{with } \|h_{m,p}\|^2=\pi m!p!, \tag{2.4}$$

$$(\delta)\qquad P_m(z,w)=\frac{1}{\pi}e^{z\bar{w}}\,{}_1F_1\big(-m;1;|z-w|^2\big). \tag{2.4$_{\text{bis}}$}$$

Remark 2.1. Our wording the $h_{m,p}$'s as Landau basis of $L_2(\mathbb{C}, d\mu(z))$ comes from the fact that operator Δ in (ii) above can be intertwined to give rise to the usual Schrödinger operator H on $\mathbb{R}^2$ in the presence of a constant magnetic field. Namely H is given by

$$H=-\frac{1}{4}\left[\left(\frac{\partial}{\partial x}+iy\right)^2+\left(\frac{\partial}{\partial y}-ix\right)^2\right].$$

The above operator H is called actually the Landau Hamiltonian for, in 1930 Landau was the first to investigate many spectral properties of the operator H in $L_2(\mathbb{R}^2, d\lambda(z))$.

For instance the spectrum $\sigma(H)$ of H in $L_2(\mathbb{R}, d\lambda(z))$ is given by the discrete set

$$\sigma(H) = \left\{\lambda_n = \frac{1}{2} + n;\ n \in \mathbb{Z}_+\right\}.$$

Proof of Proposition 2.1. The proof of (α) in (i) is easy to handle using induction argument on $m = 0, 1, 2, \ldots$. For (β) in (i) we use the result of (α) i.e. $h_{m,p}(z) = (A^\star)^m(z^p)$ and observing that the operators $\frac{\partial}{\partial z}$ and $\bar{z}$ are commuting we can use the binomial formula for $(A^\star)^m = (\frac{\partial}{\partial z} + \bar{z})^m$ as well as the fact that

$$\left(\frac{\partial}{\partial z}\right)^j (z^p) = \frac{\Gamma(p+1)}{\Gamma(p-j+1)} z^{p-j} \quad \text{for } j = 0, 1, 2, \ldots.$$

Thus by applying the Leibnitz binomial formula to the operator $(-\frac{\partial}{\partial z} + \bar{z})^m$ we can write the polynomials $h_{m,p}(z) = (A^\star)^m(z^p)$, i.e. $h_{m,p}(z) = (-\frac{\partial}{\partial z} + \bar{z})^m(z^p)$, as follows:

$$h_{m,p}(z) = \sum_{j=0}^{m\vee p} \frac{m!p!(-1)^j z^{p-j} \bar{z}^{m-j}}{j!\Gamma(p-j+1)(m-j)!} \quad \text{where } m \vee p = \min(m, p).$$

Thus if $p \geqslant m$ in the above, we can reindex the above summation by setting $k = m - j$ and to obtain in this case that we have

$$h_{m,p}(z) = \frac{(-1)^m p!}{(p-m)!} \sum_{k=0}^{m} \frac{(-1)^k m!\Gamma(p-m+1)}{(m-k)!\Gamma(p-m+k+1)} \frac{|z|^{2k}}{k!} z^{p-m}.$$

But the last summation over k is equal to the confluent hypergeometric function ${}_1F_1(-m; p-m+1; |z|^2)$. Hence the formula in (β) of (i) holds. Now to show that (β) holds also when $p \leqslant m$, we reindex the above summation in j by setting this time, $k = p - j$ and we see clearly that (β) holds.

Note that the formulae when $p \leqslant m$ or $p \geqslant m$ can be put together into a closed one formula. Namely, for every $(m, p) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ we have

$$h_{m,p}(z) = \frac{(-1)^{m\vee p}(m \wedge p)!}{(|m-p|)!} {}_1F_1\big(-m \vee p; |m-p| + 1; |z|^2\big)|z|^{|m-p|} e^{-i(m-p)\arg z} \tag{2.5}$$

where $z = |z|e^{i \arg z}$ and $m \wedge p = \max(m, p)$ and $m \vee p = \min(m, p)$.

The result of (β_{bis}) is classic.

(ii) Intissar et al. introduce in [3,4] the Hilbert subspaces $\Lambda_m(\mathbb{C})$ of $L_2(\mathbb{C}, d\mu(z))$ and give a precise description of the expansion of their elements in terms of the appropriate Fourier series on $\mathbb{C}$. As these spaces are realized as the null space of the operator $A^\star A - mI$, $m \in \mathbb{Z}_+$, where $A = \frac{\partial}{\partial \bar{z}}$ and $A^\star = -\frac{\partial}{\partial z} + \bar{z}$ then we can deduce that $\Lambda_m(\mathbb{C}) = \operatorname{span}\{h_{m,p}(z),\ p = 0, 1, 2, \ldots\}$ in $L_2(\mathbb{C}, d\mu(z))$.

Further the spaces $\Lambda_m(\mathbb{C})$ are pairwise orthogonal in $L_2(\mathbb{C}, d\mu(z))$ and we have the Hilbert decomposition $L_2(\mathbb{C}, d\mu(z)) = \bigoplus_{m\geqslant 0} \Lambda_m(\mathbb{C})$. See [4] for more details.

(iii) For fixed $m \in \mathbb{Z}_+$, let $P_m(z, w)$ be the integral Schwartz kernel of the orthogonal projection operator P_m from $L_2(\mathbb{C}, d\mu(z))$ on the Hilbert subspace $\Lambda_m(\mathbb{C})$. Then for $f \in L_2(\mathbb{C}, d\mu(z))$ we have

$$P_m f = \frac{1}{\pi} \sum_{p=0}^{\infty} \left\langle f, \frac{1}{\sqrt{m!p!}} A^{\star m} z^p \right\rangle \frac{1}{\sqrt{m!p!}} A^{\star m} z^p$$

and by using (i) we can deduce the explicit expressions of the $P_m(z,w)$ as

$$P_m(z,w)=\sum_{p=0}^{\infty}\frac{h_{m,p}(z)\overline{h_{m,p}(w)}}{\|h_{m,p}\|^2}\quad\text{with } \|h_{m,p}\|^2=\pi m!p!.$$

But this series can be written in a closed form (see [4, (2.6)]) as

$$P_m(z,w)=\frac{1}{\pi}e^{z\bar{w}}\,{}_1F_1\big(-m;1;\ |z-w|^2\big).\qquad\square$$

3. The Cauchy transform in $L_2(\mathbb{C},d\mu(z))$ and its action on the Landau basis of $L_2(\mathbb{C},d\mu(z))$

In this section, we recall the definition of Cauchy transform in $L_2(\mathbb{C},d\mu(z))$ as well as some of their elementary properties in the Carleman–Schatten classes and we give a precise description of its action on the Landau basis $\{h_{m,p}(z)\}$ of $L_2(\mathbb{C},d\mu(z))$.

3.1. Definition of Cauchy transform in $L_2(\mathbb{C},d\mu(z))$ and their properties in Schatten scale

Let begin by recalling that $L_2(\mathbb{C},d\mu(z))$ is the Hilbert space whose the elements are Lebesgue measurable complex valued functions $f(z)$ on $\mathbb{C}$ that are square integrable with respect to Gaussian density measure $d\mu(z)=e^{-|z|^2}\,d\lambda(z)$, $z=x+iy$, $|z|^2=x^2+y^2$ and $d\lambda(z)=dx\,dy$ is Lebesgue measure on $\mathbb{C}=\mathbb{R}^2$. That is

$$L_2\big(\mathbb{C},d\mu(z)\big):=\left\{f:\mathbb{C}\to\mathbb{C};\ \|f\|^2:=\int_{\mathbb{C}}e^{-|z|^2}\big|f(z)\big|^2\,d\mu(z)<\infty\right\}.$$

Definition 3.1. The Cauchy transform C in $L_2(\mathbb{C},d\mu(z))$ is defined as integral operator given by

$$Cf(z):=\frac{1}{\pi}\int_{\mathbb{C}}\frac{f(\xi)}{z-\xi}e^{-|\xi|^2}\,d\lambda(\xi)\quad\text{where } f\in L_2\big(\mathbb{C},d\mu(z)\big)\text{ and } z\in\mathbb{C}.$$

In below we denote by $N(z,\xi):=\frac{1}{\pi}(z-\xi)^{-1}$ the Schwartz kernel function associated to the above operator C in $L_2(\mathbb{C},d\mu(z))$ and we discuss some properties of the Cauchy transform in the Schatten scale. For this, recall that a compact linear operator T on a Hilbert space H is said to belong to the Schatten ideal $S_p[H]$ for given $p\geqslant 0$ if $\|T\|_p^p:=\operatorname{trace}[(T^*T)^{p/2}]$ is finite. Then with this, we have the following proposition.

Proposition 3.1. *Let C be the Cauchy transform in $L_2(\mathbb{C},d\mu(z))$ as given in Definition* 3.1. *Then the Cauchy transform C belongs to $S_p[L_2(\mathbb{C},d\mu(z))]$ for every $p>2$. In particular C is compact but not Hilbert–Schmidt.*

This proposition can be deduced from Dostanic's results [7]. However, it can also be obtained by combination of Russo's theorem [14] and our following Lemma 3.2. This lemma give explicit informations on the kernel of Cauchy transform and the technique used in its proof can be useful for others concrete kernels. The proof of this proposition will be preceded by two lemmas, the first is Russo's theorem.

Lemma 3.1 (Russo theorem [14]). *Let $(X, d\mu)$ be a finite measure space and let*

$$Tf(x) = \int_X T(x,y) f(y)\, d\mu(y)$$

be an integral operator. Let $1 < q \leqslant 2$ and let p be the conjugate exponent: $1/p + 1/q = 1$.

Suppose that

$$\int_X \left(\int_X \left|T(x,y)\right|^q d\mu(y) \right)^{p/q} d\mu(x) < \infty$$

and

$$\int_X \left(\int_X \left|T(x,y)\right|^q d\mu(x) \right)^{p/q} d\mu(y) < \infty.$$

Then, as an operator on $L_2(X, d\mu)$, T belongs to the Schatten ideal $S_p[L_2(X, d\mu)]$.

Lemma 3.2. *Let $N(z,w) := \frac{1}{\pi}(z-w)^{-1}$ the Schwartz kernel function associated to the Cauchy transform in $L_2(\mathbb{C}, d\mu(z))$.*

Then for every $0 \leqslant q < 2$ we have:

(i) $\int_{\mathbb{C}} |N(z,w)|^q\, d\mu(w) = \pi^{1-q}\Gamma(1-q/2)\, {}_1F_1(1-q/2, 1, |z|^2)e^{-|z|^2}$. *Here ${}_1F_1(a,b,x)$ is the confluent hypergeometric function.*
(ii) $\sup_{z\in\mathbb{C}} \int_{\mathbb{C}} |N(z,w)|^q\, d\mu(w) \leqslant \pi^{1-q}\Gamma(1-q/2)$.

Proof of Proposition 3.1. Note that $|N(z,w)|$ is symmetric in (z,w). Thus the estimate (i) hold also in replacing there z by w. In particular for $q = 1$, the Schur lemma hold from which we deduce that the Cauchy transform C is bounded operator (even a contraction) in $L_2(\mathbb{C}, d\mu(z))$. Further using (ii) of Lemma 3.2 combined with the fact that $\mathrm{vol}((\mathbb{C}, d\mu(z)) = \pi$, it becomes obvious that the assumptions of Lemma 3.1 hold. Hence the operator C belongs to S_p for every $p > 2$. This complete the proof of Proposition 3.1. □

Proof of Lemma 3.2. For (i) we set $\xi = z + \sqrt{x}\omega$, $x > 0$, and $\omega \in S^1 = \{\omega \in \mathbb{C};\ |\omega| = 1\}$. Then the integral in ξ considered in (i) can be written as an integral J over $]0, \infty[$. Namely

$$J = \pi^{1-q} e^{-|z|^2} \int_0^\infty e^{-x} f_q(x) \left(\int_{S^1} e^{\sqrt{x}(\bar{z}\omega + z\bar{\omega})}\, d\sigma(\omega) \right) dx \tag{a}$$

where $f_q(x) = x^{-q/2}$ is integrable near $x = 0$ when $q < 2$.

Now the integral over the circle S^1 can be easily evaluated and the result is

$$\int_{S^1} e^{\sqrt{x}(\bar{z}\omega + z\omega)}\, d\sigma(\omega) = 2\pi \sum_{k=0}^{\infty} \frac{(x|z|^2)^k}{k!^2} = I_0\big(2\sqrt{x}|z|\big)$$

where $I_0(z)$ is the 0th order modified Bessel function. Hence replacing this in (a) we see that for $f_q(x) = x^{-q/2}$, we can integrate term by term to obtain in this case that for the Cauchy kernel $N(z,\xi)$ we have

$$\int_{\mathbb{C}} \left|N(z,\xi)\right|^q d\mu(\xi) = \pi^{1-q} e^{-|z|^2} \sum_{k=0}^{\infty} \frac{\Gamma(1-q/2+k)}{k!} \frac{|z|^{2k}}{k!}. \tag{b}$$

But the last series in (b) is equal to $\Gamma(1-q/2)_1F_1(1-q/2, 1, |z|^2)$ for every $q < 2$. Henceforth we get the desired formula (i), i.e.

$$\int_{\mathbb{C}} \left|N(z,\xi)\right|^q d\mu(\xi) = \pi^{1-q} \Gamma(1-q/2) e^{-|z|^2} {}_1F_1\big((1-q/2), 1, |z|^2\big). \tag{c}$$

Next we prove the estimate (ii) by using the integral representation [8, p. 196] given by

$${}_1F_1(\alpha,\gamma,x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1} e^{tx}\, dt \quad (\Re(\gamma) > \Re(\alpha) > 0)$$

with $\alpha = 1-q/2$, $\gamma = 1$ and $x = |z|^2$, we see that the confluent hypergeometric function involved in (c) can be estimated, for $0 \leqslant q < 2$, as

$${}_1F_1\big(1-q/2, 1, |z|^2\big) \leqslant e^{|z|^2} \quad \text{for every } z \in \mathbb{C}.$$

Thus replacing this into (c) it follows at once that for $q \in [0,2[$ we have

$$\sup_{z\in\mathbb{C}} \int_{\mathbb{C}} \left|N(z,w)\right|^q d\mu(w) = \sup_{w\in\mathbb{C}} \int_{\mathbb{C}} \left|N(z,w)\right|^q d\mu(z) \leqslant \pi^{1-q}\Gamma(1-q/2).$$

Hence (ii) holds and the proof of Lemma 3.2 is complete. $\square$

3.2. Action of Cauchy transform on the Landau basis of $L_2(\mathbb{C}, d\mu(z))$

In this section we establish explicit formula for the action of the Cauchy transform $\mathbb{C}$ on the Landau basis $\{h_{m,p}(z)\}$ of $L_2(\mathbb{C}, d\mu(z))$. We give also explicit evaluations of the norms of $[Ch_{m,p}](z)$ in the Hilbert space $L_2(\mathbb{C}, d\mu(z))$. To begin we recall that hermitian elements $\{h_{m,p}(z)\}$, $(m,p) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, of the Landau basis of $L_2(\mathbb{C}, d\mu(z))$ are given by the following equivalent formulae:

$$h_{m,p}(z) = (-1)^{m+p} e^{|z|^2} \frac{\partial^{m+p}}{\partial z^m \partial \bar{z}^p} e^{-|z|^2} \tag{3.1}$$

or

$$h_{m,p}(z) = \frac{(-1)^{m\vee p}(m\wedge p)!}{(|m-p|)!}\, {}_1F_1\big(-m\vee p;\ |m-p|+1;\ |z|^2\big)|z|^{|m-p|} e^{-i(m-p)\arg z}. \tag{3.2}$$

Note that formula (3.2) permits to extend $h_{m,p}(z)$ as functions in $(m,p) \in \mathbb{Z}\times\mathbb{Z}$ such that for $mp \leqslant 0$, for instance when $m \geqslant 0$ and $p \leqslant 0$ by setting in this case (i.e. $p \in \mathbb{Z}_-$)

$$h_{m,p}(z) = \frac{(-1)^p m!}{(m-p)!}\, {}_1F_1\big(-p; m-p+1; |z|^2\big)(\bar{z})^{m-p}. \tag{3.1_{bis}}$$

In particular the functions $h_{m,-1}(z) = -(\frac{1}{m+1})_1F_1(1; m+2; |z|^2)(\bar{z})^{m+1}$ and $h_{-1,m}(z) = \overline{h_{m,-1}(z)}$ will be used.

Now using the above notations we can state the following results.

Proposition 3.2. *Let C the Cauchy transform in $L_2(\mathbb{C}, d\mu(z))$ as defined in Definition* 3.1. *Then we have*

(i) $[Ch_{m,0}](z) = -e^{-|z|^2} h_{m,-1}(z)$ *for every* $m \in \mathbb{Z}_+$.
(ii) $[Ch_{m,p}](z) = -e^{-|z|^2} h_{m,p-1}(z)$ *and* $\overline{Ch_{m,p}(z)} = -e^{-|z|^2}\overline{h_{m-1,p}(z)}$ *for every* $(m,p) \in \mathbb{Z}_+ \times \mathbb{Z}_+$.
(iii) *For fixed* $m \in \mathbb{Z}_+$ *the system* $\{Ch_{m,p}\}$, $p = 0, 1, \ldots,$ *is orthogonal in* $L_2(\mathbb{C}, d\mu(z))$.
(iv) *Let* $a^{m,p}_{n,q} = \langle Ch_{m,p}, h_{n,q}\rangle$ *be the matrix elements of the Cauchy transform C with respect to the basis* $\{h_{m,p}\}$ *in* $L_2(\mathbb{C}, d\mu(z))$. *Then the coefficients can be specified as follows*:

$$a^{m,p}_{n,q} = \begin{cases} 0 & \text{if } m-p+1 \neq n-q, \\ \pi(-1)^{m+n+1}(m+q)!2^{-(m+q+1)} & \text{if } m = p+n-q-1. \end{cases}$$

(v) $\|Ch_{m,p}\|^2 = \pi\gamma_{m-1,p}\, {}_2F_1\big(-\min(m-1,p), \min(m-1,p); |m-1-p|+1, \frac{1}{4}\big)$

where

$$\gamma_{m-1,p} = \left(\frac{2}{3}\right)^{2\min(m-1,p)} 3^{-(|m-1-p|+1)} \frac{(\max(m-1,p)!)^2}{m!p!|m-1-p|!}.$$

Proof. (i) For $p = 0$, we have $h_{m,0}(z) = \bar{z}^m$ then

$$[Ch_{m,0}](z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{\bar{\xi}^m}{\xi - z}\, d\mu(\xi).$$

By writing $\frac{1}{\xi - z}$ in series one may obtain

$$h_{m,0}(z) = -\bar{z}^{m+1}\int_0^{|z|^2} x^m e^{-x}\, dx = -e^{-|z|^2} h_{m,-1}(z).$$

(ii) For $p \geqslant 1$ and $m \in \mathbb{Z}_+$ we see that

$$[Ch_{m,p}](z) = \frac{1}{\pi}\int_{\mathbb{C}} (-1)^{p+m}\frac{\partial}{\partial\xi}\left[\frac{\partial^{p-1}}{\partial\bar{\xi}^{p-1}}\frac{\partial^m}{\partial\xi^m} e^{-|\xi|^2}\right]\frac{1}{\xi - z}\, d\lambda(\xi)$$

and by using the Green formulae [9, Chapter III, p. 319] or [13, Chapter 20, p. 433] we can deduce that

$$[Ch_{m,p}](z) = -e^{-|z|^2} h_{m,p-1}(z)$$

and also

$$[\overline{Ch_{m,p}}\,](z) = -e^{-|z|^2}\overline{h_{m-1,p}}(z).$$

(iii) By using the explicit expression of $h_{m,p}(z)$ in terms of the confluent hypergeometric function ${}_1F_1$ as given in Proposition 2.1 and (ii) of this proposition, we deduce that the system $\{Ch_{m,p}\}$ is orthogonal in $L_2(\mathbb{C}, d\mu(z))$.

(iv) For $(m,p)\in\mathbb{Z}_+\times\mathbb{Z}_+$ and $(n,q)\in\mathbb{Z}_+\times\mathbb{Z}_+$ we set $a_{m,p}^{n,q}=\langle Ch_{m,p},h_{n,q}\rangle$. Then since $Ch_{m,p}(z)=-c^{-|z|^2}h_{m,p-1}(z)$ we can use Proposition 2.1 together with the notations therein to see that $a_{m,p}^{n,q}$ can be written as

$$a_{m,p}^{n,q}=-\big\langle e^{-|z|^2}(A^\star)^m(z^{p-1}),(A^\star)^n(z^q)\big\rangle$$

where $A^\star=\frac{\partial}{\partial z}-\bar z$ and whose adjoint in $L_2(\mathbb{C},d\mu(z))$ is given by $A=\frac{\partial}{\partial\bar z}$.

Now using the fact that $A(e^{-|z|^2}f(z))=e^{-|z|^2}(\bar A f)(z)$ where $\bar A=\frac{\partial}{\partial\bar z}-z$ together with the fact that $\bar A$ is commuting with $A^\star$ we see that the coefficients $a_{m,p}^{n,q}$ can be written as

$$a_{m,p}^{n,q}=-\big\langle e^{-|z|^2}(A^\star)^m(\bar A)^n(z^{p-1}),z^q\big\rangle.$$

But observing that $(\bar A)^n(z^{p-1})=(-1)^n z^{n+p-1}$ and using the definition of the generalized Hermite polynomials $h_{m,p}$ in terms of the power of $A^\star$ we end that we have

$$a_{m,p}^{n,q}=(-1)^{n+1}\big\langle e^{-|z|^2}h_{m,n+p-1}(z),z^q\big\rangle.$$

Next replacing $h_{m,n+p-1}(z)$ by its expression in terms of the confluent hypergeometric function ${}_1F_1(a,c,x)$ and using polar coordinates in $z\in\mathbb{C}$, say $z=\sqrt{x}e^{i\theta}$, $x>0$ and $\theta\in[0,2\pi[$, we get at once that for (m,p) and (n,q) such that $m-n-p+q+1$ not zero we have $a_{m,p}^{n,q}=0$ and for the values (m,p,n,q) on the hyperplane of $\mathbb{R}^4$ given by the equation $m-n-p+q+1=0$, the coefficients $a_{m,p}^{n,q}$ are given by the following simplified integral:

$$a_{m,p}^{n,q}=\pi(-1)^{n+1}C_{m,n+p-1}\int_0^\infty {}_1F_1(-m,q+1,x)x^q e^{-2x}\,dx$$

where the constant

$$C_{m,n+p-1}=\frac{(-1)^{\min(m,n+p-1)}(\max(m,n+p-1))!}{|m-(n+p-1)|!}$$

as in (β) of Proposition 2.1 and which for the case at hand, i.e. $m-n-p+q+1=0$ it can be simplified and we have $C_{m,n+p-1}=\frac{(-1)^m(m+q)!}{q!}$.

Thus to get the precise value of $a_{m,p}^{n,q}$ when $m-n-p+q+1=0$ we have to evaluate the integral

$$I=\int_0^\infty {}_1F_1(-m,q+1,x)x^q e^{-2x}\,dx.$$

But this easy to handle for it is a classical integral in special functions. I is then given by terms of the Gauss hypergeometric function ${}_2F_1(a,b,c,x)$,

$$I=2^{-(q+1)}q!\,{}_2F_1\big(-m,q+1,q+1;\tfrac12\big).$$

Further knowing that ${}_2F_1(a,c,c,x)=(1-x)^{-a}$, $|x|<1$, and applying this to the above Gauss hypergeometric function involved in I we get, finally,

$$I=q!2^{-(m+q+1)}.$$

That is the coefficients $a_{m,p}^{n,q}$ are given by

$$a_{m,p}^{n,q}=\pi(-1)^{m+n+1}(m+q)!2^{-(m+q+1)}$$

if (m,p,n,q) satisfy $m-n-p+q+1=0$. The proof of (iv) is complete.

(v) We begin by recalling the following integral formula (see [8, p. 708])

$$J=\int_0^\infty e^{-\lambda x}x^{c-1}\,{}_1F_1(a,c,kx)\,{}_1F_1(a',c,k'x)\,dx \text{ can be evaluated to be equal to}$$

$$J=\Gamma(c)\lambda^{a+a'-c}(\lambda-k)^{-a}(\lambda-k')^{-a'}\,{}_2F_1\left(a,a',c;\frac{kk'}{(\lambda-k)(\lambda-k')}\right). \tag{3.3}$$

Now by using Proposition 3.1 we deduce that

$$\|Ch_{m,p}\|^2=\|\overline{Ch_{m,p}}\|^2=\int_{\mathbb{C}} e^{-3|z|^2}\left|h_{m-1,p}(z)\right|^2 d\lambda(z). \tag{3.4}$$

According to the explicit expression of $h_{m-1,p}(z)$ in terms of the confluent hypergeometric function ${}_1F_1$ as given in Proposition 2.1, we know that

$$\left|h_{m-1,p}(z)\right|=|c_{m-1,p}|\,{}_1F_1\left(-\min(m-1,p);|m-1-p|+1;|z|^2\right)|z|^{|m-1-p|}$$

where

$$c_{m-1,p}=\frac{(-1)^{\min(m-1,p)}(\max(m-1,p))!}{|m-1-p|!}.$$

Hence by replacing this into the integral (3.4) we can perform a polar coordinates change of variable to see that we can apply formula (3.3) with $\lambda=3$, $a=a'=-\min(m-1,p)$, $c=|m-1-p|+1$ and $k=k'=1$ and from which we obtain the desired formula as stated in (v) of this proposition. □

Remark 3.1. We can apply also the formula (3.3) to give explicit evaluations of the matrix elements $\langle Ch_{m,p},h_{n,q}\rangle$ of the Cauchy transform C with respect to the Landau basis $\{h_{m,p}\}$ of $L_2(\mathbb{C},d\mu(z))$.

Now, as a consequence of the method of the proof of Proposition 3.2 we have

Corollary 3.1. *Let $m\in\mathbb{Z}_+$ and $q\in\mathbb{Z}_+-\{0\}$ be given. Then the following identity on the confluent hypergeometric functions holds*:

$$\sum_{j=0}^{m}\frac{\partial^j}{\partial x^j}\left[{}_1F_1(-m,q+1,x)\right]=\frac{q}{m+q}\,{}_1F_1(-m,q,x) \tag{3.5}$$

for every real number $x\in\mathbb{R}$.

Sketch of the proof of identity (3.5). That as now stated the identity (3.5) can be shown using induction on m and some recursion formula holds for the ${}_1F_1(a,c;x)$. However, to identity (3.5) as stated above we were led to it in a natural manner by computing in two different ways $Ch_{m,p}(z)=-e^{-|z|^2}h_{m,p-1}(z)$ as stated in Proposition 3.2(i). On the other hand, by expanding $\frac{1}{z-\xi}$ into geometrical series, say

$$\frac{1}{z-\xi}=\begin{cases}\sum_{j=0}^{\infty}\frac{\xi^j}{z^{j+1}} & \text{if } |\xi|<|z|,\\ -\sum_{j=0}^{\infty}\frac{z^j}{\xi^{j+1}} & \text{if } |\xi|>|z|,\end{cases}$$

and using the explicit formula of $h_{m,p}(\xi)$ in terms of the confluent hypergeometric function ${}_1F_1$ as stated in (β) of Proposition 2.1 we can see easily that $Ch_{m,p}(z)$ can be represented by the integral $(p-1 \geqslant m)$ given as follows:

$$Ch_{m,p}(z) = -C_{m,p} z^{p-m-1} \int_{|z|^2}^{\infty} e^{-x}\, {}_1F_1(-m, p-m+1, x)\, dx$$

where the constant $C_{m,p}$ is given in (β) of Proposition 2.1 for $p \geqslant m+1$ by

$$C_{m,p} = \frac{(-1)^m p!}{|m-p|!} = \frac{(-1)^m (m+q)!}{q!}, \quad p = m+q.$$

Further using m integration by parts in the integral

$$\int_{|z|^2}^{\infty} e^{-x}\, {}_1F_1(-m, p-m+1, x)\, dx, \quad p = m+q,$$

with the fact that

$$\begin{aligned} Ch_{m,p}(z) &= -e^{-|z|^2} h_{m,p-1}(z) \\ &= \frac{(-1)^{m+1}(p-1)!}{|m-p+1|!}\, {}_1F_1\big(-m, |m-p+1|+1, |z|^2\big) z^{p-m-1} \end{aligned}$$

and making simplification we get the desired result:

$$\left(\sum_{j=0}^{m} \frac{d^j}{dx^j}\right)\big[{}_1F_1(-m, q+1, x)\big] = \frac{q}{m+q}\, {}_1F_1(-m, q, x)$$

for $m \geqslant 0$ and $q \geqslant 1$ (integers).

4. Expansion of the kernel functions of the operators $P_m C$ and $(P_m C)^\star(P_m C)$ and proof of the main result as stated in the introduction

In this section, we consider the product of the orthogonal projection operator on $\Lambda_m(\mathbb{C})$, $m = 0, 2, \ldots$ with the Cauchy transform.

Namely, for $f \in L_2(C, d\mu(z))$, let

$$P_m f(z) = \int_{\mathbb{C}} P_m(z, w) f(w)\, d\mu(w)$$

with kernel function

$$P_m(z, w) = \sum_{p=0}^{\infty} \frac{h_{m,p}(z)\overline{h_{m,p}(w)}}{\|h_{m,p}\|^2}$$

and

$$Cf(z) = \int_{\mathbb{C}} N(z, w) f(w)\, d\mu(w)$$

with kernel function $N(z,w)=\frac{1}{\pi}\frac{1}{z-w}$ we consider the product

$$P_m Cf(z)=\int\limits_{\mathbb{C}} N_m(z,w)f(w)\,d\mu(w).$$

It is clear (by standard calculations) that the kernel function $N_m(z,w)$ of the operator P_mC is given by the integral

$$N_m(z,w)=\int\limits_{\mathbb{C}} P_m(z,\xi)N(\xi,w)\,d\mu(\xi) \tag{4.1}$$

and the kernel function $B_m(z,w)$ of the operator $(P_mC)^\star P_mC$ is given by the integral

$$B_m(z,w)=\int\limits_{\mathbb{C}} N_m^\star(z,\xi)N_m(\xi,w)\,d\mu(\xi). \tag{4.2}$$

Henceforth, the first aim of this section is to decompose $N_m(z,w)$ and $B_m(z,w)$ into an orthogonal expansion which will be suitable for handling the eigenvalues and eigenvectors of the operator $|P_mC|=\sqrt{(P_mC)^\star P_mC}$.

We start with the following proposition.

Proposition 4.1. *For fixed $m=0,1,2,\dots$ let $N_m(z,w)$ be the Schwartz kernel in $L_2(\mathbb{C},d\mu(z))$ of the operator P_mC and $B_m(z,w)$ the Schwartz kernel in $L_2(\mathbb{C},d\mu(z))$ of the operator $(P_mC)^\star P_mC$. Then the integral kernel functions $N_m(z,w)$ and $B_m(z,w)$ can be expanded into the following "biorthogonal" series in $L_2(\mathbb{C},d\mu(z))$:*

(i) *For $m\geqslant 0$, the kernel $N_m(z,w)$ has the following expression formula*:

$$N_m(z,w)=\sum_{p=0}^{\infty}\frac{h_{m,p}(z)}{\|h_{m,p}\|^2}e^{-|w|^2}\overline{h_{m-1,p}(w)}. \tag{4.3}$$

In particular, for $m=0$ we have

$$N_0(z,w)=-\sum_{p=0}^{\infty}\frac{1}{(p+1)!}z^p e^{-|w|^2}\,{}_1F_1\big(1,p+2;|w|^2\big)\bar{w}^{p+1}. \tag{4.4}$$

(ii) *For $m\geqslant 0$, the kernel $B_m(z,w)$ is given by*

$$B_m(z,w)=\sum_{p=0}^{\infty}\frac{e^{-|z|^2}h_{m-1,p}(z)e^{-|w|^2}\overline{h_{m-1,p}(w)}}{\|h_{m,p}\|^2}. \tag{4.5}$$

Before to give the proof of this proposition, we mention in following remark that the series (4.3)–(4.5) can be written into closed explicit form.

Remark 4.1.

(1) For $m \geqslant 0$ the kernel function $N_m(z,w)$ can be written into following closed explicit form:

(α) For $m=0$ we have

$$N_0(z,w) = \frac{1}{\pi}\frac{1}{\bar{w}(z-w)}\left[1-e^{\bar{w}(z-w)}\right].$$

(β) For $m \geqslant 1$ we have

$$\begin{aligned} N_m(z,w) &= \frac{1}{m}e^{-|w|^2}A^\star\big(P_{m-1}(z,w)\big) \\ &= \frac{1}{m\pi}e^{-|w|^2}\left(\frac{\partial}{\partial z}-\bar{z}\right)\left[e^{z\bar{w}}\,{}_1F_1\big(-m+1,1,|z-w|^2\big)\right] \\ &= \frac{1}{\pi}(\bar{z}-\bar{w})e^{\bar{w}(z-w)} \\ &\quad\times\left[\frac{{}_1F_1(m,1,-|z-w|^2)+(m-1){}_1F_1(m,2,-|z-w|^2)}{m}\right]. \end{aligned} \tag{4.4$_{\text{bis}}$}$$

(2) For $m \geqslant 1$ the kernel function $B_m(z,w)$ can be given into closed explicit form and we have

$$\begin{aligned} B_m(z,w) &= \frac{e^{-(|z|^2+|w|^2)}}{m}P_{m-1}(z,w) \\ &= \frac{e^{-(|z|^2+|w|^2)}}{m\pi}e^{z\bar{w}}{}_1F_1\big(-m+1,1;\ |z-w|^2\big). \end{aligned} \tag{4.5$_{\text{bis}}$}$$

Proof of Proposition 4.1. (i) Let

$$P_mf(z) = \int_{\mathbb{C}} P_m(z,w)f(w)\,d\mu(w) \quad \text{with } P_m(z,w) = \sum_{p=0}^{\infty}\frac{h_{m,p}(z)\overline{h_{m,p}(w)}}{\|h_{m,p}\|^2}$$

and

$$Cf(z) = \int_{\mathbb{C}} N(z,w)f(w)\,d\mu(w) \quad \text{with kernel function } N(z,w) = \frac{1}{\pi(z-w)},$$

then

$$P_mCf(z) = \int_{\mathbb{C}} N_m(z,w)f(w)\,d\mu(w) \quad \text{with } N_m(z,w) = \int_{\mathbb{C}} P_m(z,\xi)N(\xi,w)\,d\mu(\xi).$$

Now using expression (2.4) of $P_m(z,w)$ and an integration term by term we deduce that

$$\begin{aligned} N_m(z,w) &= \int_{\mathbb{C}}\sum_{p=0}^{\infty}\frac{h_{m,p}(z)\overline{h_{m,p}(\xi)}}{\|h_{m,p}\|^2}N(\xi,w)\,d\mu(\xi) \\ &= \sum_{p=0}^{\infty}\frac{h_{m,p}(z)}{\|h_{m,p}\|^2}\int_{\mathbb{C}}N(\xi,w)\overline{h_{m,p}(\xi)}\,d\mu(\xi) = -\sum_{p=0}^{\infty}\left[\frac{h_{m,p}(z)}{\|h_{m,p}\|^2}\overline{Ch_{m,p}(w)}\right] \end{aligned}$$

and as in (ii) of Proposition 3.2, we have

$$\overline{Ch_{m,p}}(z) = -e^{-|z|^2}\overline{h_{m-1,p}(z)} \quad \text{for every } (m,p) \in \mathbb{Z}_+ \times \mathbb{Z}_+,$$

then we obtain

$$N_m(z,w) = \sum_{p=0}^{\infty} \frac{h_{m,p}(z)}{\|h_{m,p}\|^2} e^{-|w|^2}\overline{h_{m-1,p}(w)}.$$

For $m = 0$ we use the fact that

$$\overline{h_{-1,p}(w)} = h_{p,-1}(w), \qquad \|h_{m,p}\|^2 = \pi m!p! \quad \text{and}$$
$$h_{p,-1}(w) = -\frac{1}{p+1}\, {}_1F_1\big(1; p+2; |z|^2\big)(\bar{w})^{p+1}$$

to deduce that

$$\begin{aligned} N_0(z,w) &= \sum_{p=0}^{\infty} \frac{h_{0,p}(z)}{p!} e^{-|w|^2}\overline{h_{-1,p}(w)} = \sum_{p=0}^{\infty} \frac{h_{0,p}(z)}{p!} e^{-|w|^2} h_{p,-1}(w) \\ &= -\sum_{p=0}^{\infty} \frac{1}{(p+1)!} z^p e^{-|w|^2}\, {}_1F_1\big(1, p+2; |w|^2\big)\bar{w}^{p+1}. \end{aligned}$$

(ii) The kernel function $B_m(z,w)$ of the operator $(P_mC)^\star P_mC$ is given by the integral (4.2) with

$$N_m(z,w) = \sum_{p=0}^{\infty} \frac{h_{m,p}(z)}{\|h_{m,p}\|^2} e^{-|w|^2}\overline{h_{m-1,p}(w)}$$

and

$$N_m^\star(z,w) = \sum_{p=0}^{\infty} \frac{e^{-|z|^2} h_{m-1,p}(z)}{\|h_{m,p}\|^2}\overline{h_{m,p}(w)}$$

i.e.

$$B_m(z,w) = \int_{\mathbb{C}} N_m^\star(z,\xi) N_m(\xi,w)\, d\mu(\xi)$$

we deduce that

$$\begin{aligned} B_m(z,w) &= \int_{\mathbb{C}} \sum_{p=0}^{\infty} \frac{e^{-|z|^2} h_{m-1,p}(z)}{\|h_{m,p}\|^2}\overline{h_{m,p}(\xi)} \cdot \sum_{k=0}^{\infty} \frac{h_{m,k}(\xi)}{\|h_{m,k}\|^2} e^{-|w|^2}\overline{h_{m-1,k}(w)}\, d\mu(\xi) \\ &= \sum_{p=0}^{\infty} \frac{e^{-|z|^2} h_{m-1,p}(z)}{\|h_{m,p}\|^2} \sum_{k=0}^{\infty} \frac{e^{-|w|^2}\overline{h_{m-1,k}(w)}}{\|h_{m,k}\|^2} \int_{\mathbb{C}} h_{m,k}(\xi)\overline{h_{m,p}(\xi)}\, d\mu(\xi). \end{aligned}$$

By using the orthogonality of $\{h_{m,p}\}$, we deduce that

$$B_m(z,w) = \sum_{p=0}^{\infty} e^{-|z|^2} h_{m-1,p}(z) \frac{e^{-|w|^2}\overline{h_{m-1,p}(w)}}{\|h_{m,p}\|^2} = \sum_{p=0}^{\infty} \frac{Ch_{m-1,p+1}(z)\overline{Ch_{m,p}(w)}}{\|h_{m,p}\|^2}$$

where $\|h_{m,p}\|^2 = \pi m!p!$. $\square$

Therefore for $m \geqslant 1$, we can use (2.6) to deduce that we have

Corollary 4.1. *Let $m \in \mathbb{Z}_+$, m to be fixed and $p = 0, 1, 2, \ldots$, let $\phi_{m,p}$ be the functions of Hermite type defined on $\mathbb{C}$ by*

$$\phi_{m,p}(z) = \begin{cases} \frac{-1}{p+1}\, {}_1F_1(1, p+2, |z|^2) z^{p+1}, & m = 0, \\ -(-1)^{m+p} \frac{\partial^{m-1+p}}{\partial^{m-1} z \partial^p \bar{z}} e^{-|z|^2}, & m = 1, 2, \ldots. \end{cases} \tag{4.6}$$

Then for each $p = 0, 1, 2, \ldots$ the above function $\phi_{m,p}(z)$ is an eigenvector of the operator $(P_m C)^\star (P_m C)$ where P_m is the orthogonal projection operator from $L_2(\mathbb{C}, d\mu(z))$ onto the space $\Lambda_m(\mathbb{C})$ as defined in (1.3) *and C is the Cauchy transform in $L_2(\mathbb{C}, d\mu(z))$ as given in* (1.1)*. This eigenvector is associated to μ_p^m as eigenvalue for $(P_m C)^\star (P_m C)$ where μ_p^m is given explicitly via the following formula*:

$$\mu_p^m = \gamma_{m-1,p} \cdot {}_2F_1\left(-\min(m-1, p); -\min(m-1, p); |m-1-p| + 1; \frac{1}{4}\right) \tag{4.7}$$

where

$$\gamma_{m-1,p} = \left(\frac{2}{3}\right)^{2\min(m-1,p)} 3^{-(|m-1-p|+1)} \frac{(\max(m-1, p)!)^2}{m!p!|m-1-p|!}$$

and ${}_2F_1(\alpha, \beta; \gamma, x)$ stands for the usual Gauss hypergeometric function.

Proof. For f in $L_2(\mathbb{C}, d\mu(z))$, we have

$$(P_m C)^\star (P_m C) f(z) = \int_{\mathbb{C}} B_m(z, w) f(w)\, d\mu(w).$$

In particular

$$\begin{aligned}(P_m C)^\star (P_m C) e^{-|z|^2} h_{m-1,p}(z) &= \int_{\mathbb{C}} B_m(z, w) e^{-|w|^2} h_{m-1,p}(w)\, d\mu(w) \\ &= \int_{\mathbb{C}} \sum_{k=0}^{\infty} \frac{C h_{m-1,k+1}(z) \overline{C h_{m,k}(w)}}{\|h_{m,k}\|^2} e^{-|w|^2} h_{m-1,p}(w)\, d\mu(w).\end{aligned}$$

By using the orthogonality of the system $\{e^{-|z|^2} h_{m-1,p}(z)\}$, $p = 0, 1, 2, \ldots$, i.e. of the system $\{C h_{m,p}\}$, $p = 0, 1, 2, \ldots$, we deduce that

$$(P_m C)^\star (P_m C)\left\{e^{-|z|^2} h_{m-1,p}(z)\right\} = \frac{\|C h_{m,p}\|^2}{\|h_{m,p}\|^2} \left\{e^{-|z|^2} h_{m-1,p}(z)\right\}.$$

Consequently the eigenvalues of operator $(P_m C)^\star (P_m C)$ are given by

$$\mu_p^m = \frac{\|C h_{m,p}\|^2}{\|h_{m,p}\|^2}. \tag{4.8}$$

As $\|h_{m,p}\|^2 = \pi m! p!$ and

$$\|C h_{m,p}\|^2 = \pi \gamma_{m-1,p}\, {}_2F_1\left(-\min(m-1, p), -\min(m-1, p); |m-1-p| + 1, \frac{1}{4}\right)$$

(see Proposition 3.2(v)) where

$$\gamma_{m-1,p} = \left(\frac{2}{3}\right)^{2\min(m-1,p)} 3^{-(|m-1-p|+1)} \frac{(\max(m-1,p)!)^2}{m!p!|m-1-p|!}$$

we deduce that

$$\mu_p^m = \gamma_{m-1,p}\, {}_2F_1\left(-\min(m-1,p), \min(m-1,p); |m-1-p|+1, \frac{1}{4}\right) \tag{4.9}$$

with $\gamma_{m-1,p}$ as above in (4.7). $\square$

Now by using the above proposition, the system $\{e^{-|z|^2} h_{m-1,p}(z)\}$, $p = 0, 1, 2, \ldots,$ can be written in this form:

$$e^{-|z|^2} h_{m-1,p}(z) = (-1)^{m-1+p} \frac{\partial^{m-1+p}}{\partial z^{m-1} \partial \bar{z}^p} e^{-|z|^2} \quad \text{for } m \geqslant 1 \text{ and } p = 0, 1, 2, \ldots,$$

and

$$e^{-|z|^2} h_{-1,p}(z) = -\frac{1}{p+1} e^{-|z|^2}\, {}_1F_1\big(1; p+2; |z|^2\big) z^{p+1} \quad \text{for } m = 0 \text{ and } p = 0, 1, 2, \ldots.$$

Remark 4.2. (i) For $m = 1$ the nonzero eigenvalues of operator $(P_m C)^\star (P_m C)$ are given by $\frac{1}{3p+1}$, $p \geqslant 0$, and of multiplicity one.

(ii) To establish the asymptotic behaviour of μ_p^m when $p \to \infty$ we recall that

$$\mu_p^m = \gamma_{m-1,p} \cdot {}_2F_1\left(-\min(m-1,p); -\min(m-1,p); |m-1-p|+1; \frac{1}{4}\right)$$

where

$$\gamma_{m-1,p} = \left(\frac{2}{3}\right)^{2\min(m-1,p)} 3^{-(|m-1-p|+1)} \frac{(\max(m-1,p)!)^2}{m!p!|m-1-p|!}$$

and as m is fixed we have for $p \geqslant m-1$

$$\begin{aligned}
&{}_2F_1\left(-\min(m-1,p); -\min(m-1,p); |m-1-p|+1; \frac{1}{4}\right) \\
&\quad = {}_2F_1\left(-m+1; -m+1; p-m+2; \frac{1}{4}\right) \\
&\quad = \sum_{k=0}^{m-1} \frac{[(-m+1)_k]^2}{k!(p-m+2)_k} \left(\frac{1}{4}\right)^k \to 1 \quad \text{when } p \to \infty.
\end{aligned}$$

Now by using the above polynomial formulae we deduce that

$$\mu_p^m \simeq \frac{1}{4} \left(\frac{4}{3}\right)^m e^{-p \operatorname{Log} 3} \frac{\Gamma(p+1)}{m!\Gamma(p-m+2)}, \quad p \to \infty. \tag{4.10}$$

(iii) From (4.10) we deduce the corollary given in the introduction.

(iv) It is well known that the spectral theory of the Cauchy transform and its product with Bergman's projection on a bounded domain has been fully studied, see, for example, [1,2,6].

(v) Proceeding in the same way what we have done for the Cauchy transform in this work, we state the main spectral properties of the Green transform on $L_2(\mathbb{C}, d\mu(z))$, i.e. the logarithmic transform given by

$$Gf(z) = \frac{1}{\pi} \int_{\mathbb{C}} \operatorname{Log} |z - \xi|^2 f(\xi)\, d\mu(\xi). \tag{4.11}$$

We give the final proposition whose proof should be omitted.

Proposition 4.2.

(i) *Let* $G(z, w) = \frac{1}{\pi} \operatorname{Log} |z - \xi|^2$ *the Schwartz kernel function associated to the operator* G *in* $L_2(\mathbb{C}, d\mu(z))$ *then we have*

$$\int_{\mathbb{C}} \left|G(z, w)\right|^q d\mu(w) = \pi^{1-q} e^{-|z|^2} \int_0^\infty I_0\big(2\sqrt{x}\,|z|\big) |\operatorname{Log} x|^q e^{-x}\, dx \quad \text{for every } q \geqslant 0,$$

here $I_0(z)$ *stands the modified Bessel function of index zero.*

(ii) *Let* G *be the Green transform in* $L_2(\mathbb{C}, d\mu(z))$ *as given in* (4.11) *then* G *belongs to Schatten space* $S_p[L_2(\mathbb{C}, d\mu(z))]$ *for all* $p > 1$. *In particular* G *is a Hilbert–Schmidt operator.*

(iii) *The functions* $[Gh_{m,p}](z)$ *that characterise the action of the operator* G *on Landau basis* $\{h_{m,p}\}$ *are given explicitly by*
  (a) $(Gh_{0,0}](z) = e^{-|z|^2} \int_0^\infty I_0(2\sqrt{x}\,|z|) \operatorname{Log} x e^{-x}\, dx$.
  (b) $[Gh_{m,p}](z) = e^{-|z|^2} h_{m-1,p-1}(z)$ *for* $(m, p) > (0, 0)$ *where* $(m, p) > (0, 0)$ *stands for lexicographic order.*

(iv) *Let* $a_{m,p}^{n,q} = \langle Gh_{m,p}, h_{n,q}\rangle$ *be the matrix coefficients elements of the Green transform* G *with respect to the basis* $\{h_{m,p} \in L_2(\mathbb{C}, d\mu(z))\}$. *Then the coefficients can be specified as follows*:

$$a_{m,p}^{n,q} = \begin{cases} 0 & \text{if } m - p \neq n - q, \\ (-1)^{m+n}(m + q - 1)!2^{-(m+q)} & \text{if } m = p + n - q. \end{cases}$$

(v) *The square norms of* $Gh_{m,p}$ *are given by*

$$\begin{aligned} &\|Gh_{m,p}\|^2 \\ &\quad = \gamma_{m-1,p-1}\, {}_2F_1\left(-\min(m-1, p-1), \min(m-1, p-1); |m-p|+1, \frac{1}{4}\right) \end{aligned}$$

where

$$\gamma_{m-1,p-1} = \left(\frac{2}{3}\right)^{2\min(m-1,p-1)} 3^{-|m-p|-1} \frac{((\max(m-1, p-1))!)^2}{|m-p|!}.$$

(vi) *Let* $f \in L_2(\mathbb{C}, d\mu(z))$ *and*

$$P_m f(z) = \int_{\mathbb{C}} P_m(z, w) f(w)\, d\mu(w)$$

the orthogonal projection operator of $L_2(\mathbb{C}, d\mu(z))$ *on* $\Lambda_m(\mathbb{C})$ *with kernel function*

$$P_m(z, w) = \sum_{p=0}^\infty \frac{h_{m,p}(z)\overline{h_{m,p}(w)}}{\|h_{m,p}\|^2}, \quad m = 0, 1, 2, \ldots.$$

Then we consider the product of the orthogonal projection operator P_m with the Green transform G to obtain

$$P_m Gf(z) = \int_{\mathbb{C}} G_m(z,w) f(w)\, d\mu(w)$$

where

$$G_m(z,w) = \sum_{p=0}^{\infty} \frac{h_{m,p}(z) e^{-|w|^2} \overline{h_{m-1,p-1}(w)}}{\|h_{m,p}\|^2}, \quad m = 0, 1, 2, \ldots.$$

(vii) *For $p \geqslant m-1$, the nonzero-eigenvalues $\lambda_{m,p}$ of the operator $|P_m G|$ are given by*

$$(\lambda_{m,p})^2 = \left(\frac{2}{3}\right)^{2(m-2)} 3^{-(p-m+3)} \frac{p!}{(m-1)!(p-m+2)!} \times {}_2F_1\left(-m; -m; p-m+3; \frac{1}{4}\right).$$

Acknowledgments

The authors thank Jean-Karim and Soufiane.

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