Permanence, extinction and periodic solution of predator–prey system with Beddington–DeAngelis functional response

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Abstract
In this paper we study permanence, extinction and periodic solution of periodic predator–prey system with Beddington–DeAngelis functional response. We provide a sufficient and necessary condition to guarantee the predator and prey species to be permanent. In addition, sufficient condition is derived for the existence of positive periodic solution. This paper improves some main results obtained by Fan and Kuang [M. Fan, Y. Kuang, Dynamics of nonautonomous predator–prey system with the Beddington–DeAngelis functional response, J. Math. Anal. Appl. 295 (2004) 15–39].

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1. Introduction

Recently, Fan and Kuang considered the following predator–prey model with Beddington–DeAngelis functional response

\begin{align}
\dot{x} &= x \left( a(t) - b(t)x - \frac{c(t)y}{\alpha(t) + \beta(t)x + \gamma(t)y} \right), \\
\dot{y} &= y \left( -d(t) + \frac{f(t)x}{\alpha(t) + \beta(t)x + \gamma(t)y} \right),
\end{align}

(1.1)

where \( x \) and \( y \) represent the number of individuals in the prey and predator population, respectively. For general nonautonomous case, they studied its permanence, extinction and globally asymptotic stability. For the periodic case, they established two sufficient criteria for the existence of a positive periodic solution by using Brouwer fixed point theorem and continuation theorem in coincidence degree, respectively. These criteria are easy to be verified for the given system in the form of (1.1). At the same time, authors pointed that these criteria have room for further improvement. They presented numerical simulation to indicate that (1.1) may admit positive periodic solutions when the conditions in the theorems failed.

On the basis of these obtained results for the system (1.1) with periodic coefficients, we continue the study on the periodic solution and permanence of system (1.1). We get some new conditions for permanence and existence of a positive periodic solution of periodic system (1.1). These results improve some main results obtained by Fan and Kuang [15].

Some works have been done with both autonomous predator–prey model with Beddington–DeAngelis functional response (see [1,3,6,7,13,17]), and some predator–prey models of periodic nature (see [2,5,8,10–12,14–16,18–21,24,25]). Cushing [10–12] considered Lotka–Volterra predator–prey models with periodic coefficients, both with and without time delays. He derived conditions for the existence of a periodic solution for a predator–prey system and discussed the stability of such solutions. Ding, Huang and Zanolin [14], Lopez-Gomez, Ortega and Tineo [18] considered the positive periodic solution of the following Lotka–Volterra predator–prey model with periodic coefficients:

\begin{align}
\dot{x} &= x \left( a(t) - b(t)x - c(t)y \right), \\
\dot{y} &= y \left( -d(t) + e(t)x - f(t)y \right).
\end{align}

(1.2)

They established sufficient and necessary conditions for the existence of a positive periodic solution, respectively. In [19], Teng studied permanence (or uniformly persistence) of (1.2). Butler and Freedman [5] studied the following general Kolmogorov-type model:

\begin{align}
\dot{x} &= xf(t, x, y), \\
\dot{y} &= yg(t, x, y)
\end{align}

(1.3)

for one prey and one predator, where \( f(t + \omega, x, y) = f(t, x, y), g(t + \omega, x, y) = g(t, x, y) \) for all \( t \). They got sufficient conditions for the existence of a positive periodic solution of (1.3). Tineo [23] considered (1.3) and derived sufficient conditions for permanence of (1.3). Burton and Hutson [4] studied the permanence of the nonautonomous system (1.3) by using an average function technique. They got some sufficient conditions for the permanence. But their assumption (H3) does not hold for model (1.1) because \( a(t) \) may become negative at some points.

This paper is organized as follows. In the next section, we introduce notation and state the main results of this paper. These results are proved in Section 3, and a discussion follows in Section 4.
2. Statement of main results

Throughout this paper we assume that the functions \(a(t), b(t), c(t), d(t), f(t), \alpha(t), \beta(t), \gamma(t)\) and \(\omega\) are continuous and periodic with common period \(\omega\); \(b(t)\) is nonnegative, \(c(t), d(t), f(t), \alpha(t), \beta(t), \gamma(t)\) are positive.

For any continuous \(\omega\)-periodic function \(f(t)\) defined on \(\mathbb{R}\) we denote
\[
A_{\omega}(f(t)) = \omega^{-1} \int_{0}^{\omega} f(t) \, dt, \quad f^M = \max_{t \in [0, \omega]} f(t), \quad f^L = \min_{t \in [0, \omega]} f(t).
\] (2.1)

In order to describe our main result, we need first to discuss system (1.1) in the absence of the predator, namely
\[
\dot{x} = x(a(t) - b(t)x).
\] (2.2)

**Lemma 2.1.** [22] If \(b(t) \geq 0\) for all \(t \in \mathbb{R}\) and \(A_{\omega}(b(t)) > 0\), then (2.2) has a unique non-negative \(\omega\)-periodic solution \(x^*(t)\) which is globally asymptotically stable with respect to the positive \(x\)-axis. Moreover, if \(A_{\omega}(a(t)) > 0\) then \(x^*(t) > 0\) for all \(t \in \mathbb{R}\) and if \(A_{\omega}(a(t)) \leq 0\) then \(x^*(t) = 0\).

**Theorem 2.1.** Suppose that
\[
A_{\omega}(a(t)) > 0, A_{\omega}(b(t)) > 0.
\] (2.3)

System (1.1) is permanent and has at least one positive \(\omega\)-periodic solution provided
\[
A_{\omega} \left( -d(t) + \frac{f(t)x^*(t)}{\alpha(t) + \beta(t)x^*(t)} \right) > 0,
\] (2.4)

where \(x^*(t)\) is the unique periodic solution of (2.2) given by Lemma 2.1.

**Remark 2.1.** One may think it is not an easy task to verify the condition (2.4) for a given system in the form of (1.1). But it is an easy task in point of fact. Because (2.2) is a Riccati equation, we can integrate it and obtain its \(\omega\) periodic solution [19]
\[
x^*(t) = \frac{1 - \exp \left( -\int_{0}^{\omega} a(s) \, ds \right)}{\int_{0}^{\omega} b(t-s) \exp \left( -\int_{0}^{s} a(t-\tau) \, d\tau \right) ds},
\] (2.5)

which is globally stable under the assumptions of Lemma 2.1. Hence condition (2.4) is easy to be verified for the given system in the form of (1.1).

**Remark 2.2.** Sometimes one cannot judge the existence of the periodic solution, for some models in the form of (1.1), by [15, Theorem 3.2]. However it can be done by the result in the present theorem. We give the following example to illustrate this point in detail.

**Example.** Let \(a(t) = 3, b(t) = 2 + \cos t, c(t) = 2, d(t) = (1 + 1/2 \sin t)/10, f(t) = 1, \alpha(t) = 1/8 + 1/10 \sin t, \beta(t) = 8 + 4 \sin t, \gamma(t) = 2 + \cos t\), then (1.1) becomes
\[
\dot{x} = x \left( 3 - (2 + \cos t)x - \frac{2y}{1/8 + 1/10 \sin t + (8 + 4 \sin t)x + (2 + \cos t)y} \right),
\]
\[
\dot{y} = y \left( -(1 + 1/2 \sin t)/10 + \frac{x}{1/8 + 1/10 \sin t + (8 + 4 \sin t)x + (2 + \cos t)y} \right).
\] (2.6)
We get
\[ x^*(t) = \frac{1 - \exp(-6\pi)}{\int_0^{2\pi} (2 + \cos(t - s)) \exp(-3s) ds} = \frac{30}{20 + 9 \cos t + 3 \sin t}. \]

By simple computation of numerical integration, we have
\[ A_\omega \left( -d(t) + \frac{f(t) x^*(t)}{\alpha(t) + \beta(t) x^*(t)} \right) \approx 0.043. \]
Hence the predator prey system is permanent by Theorem 2.1.

However, the second assumption of Theorem 3.1 and the second assumption of Theorem 3.2 in [15] do not hold for (2.6) because \( f^L - d^M \beta^M = -0.8 \) and \( A_\omega(f) - A_\omega(d) \beta^M = -0.2 \). Therefore one cannot judge the existence of the positive periodic solution of (2.6) by the theorems in [15].

Suppose \( a(t), b(t), c(t), d(t), \alpha(t), \beta(t) \) and \( \gamma(t) \) are positive and periodic functions. Because \( (a/b)^L \leq x^*(t) \) by [22] (see [9, Lemma 2.1]), we get the following result immediately.

**Corollary 2.1.** Assume that \( a(t), b(t), c(t), d(t), \alpha(t), \beta(t) \) and \( \gamma(t) \) are positive functions. System (1.1) is permanent and has at least one positive \( \omega \) periodic solution if
\[ (f^L - d^M \beta^M) \left( \frac{a}{b} \right)^L > d^M \alpha^M. \] (2.7)

**Remark 2.3.** Corollary 2.1 improves the result of [15, Theorem 3.1]. Assuming that all the coefficients are positive and \( \omega \) periodic functions, in their Theorem 3.1 [15], Fan and Kuang proved that (1.1) has at least one positive \( \omega \) periodic solution under the conditions
\[ a^L \gamma^L > c^M \] (2.8)
and
\[ (f^L - d^M \beta^M) m_1^\varepsilon > d^M \alpha^M. \] (2.9)
Here \( m_1^\varepsilon = (a^L \gamma^L - c^M)/(b^M \gamma^L) - \varepsilon \), and \( \varepsilon \geq 0 \) is sufficiently small such that \( m_1^\varepsilon > 0 \).

It is not difficult to show that (2.9) implies (2.7). In fact, we have (see [22] or [9])
\[ m_1^\varepsilon < \left( \frac{a}{b} \right)^L \leq x^*(t). \]

Hence
\[ (2.9) \Rightarrow (2.7) \Rightarrow (2.4). \]

In addition, condition (2.8) is not necessary for the existence of positive \( \omega \) periodic solution of (1.1) according to Theorem 2.1 and Corollary 2.1.

**Remark 2.4.** We should mention that the functions \( a(t), b(t) \) are not necessary positive as assumed in [15]. This is important both in mathematics and biology because on some time intervals the food is poor and the death rate may exceed the birth rate for some species.

**Theorem 2.2.** Suppose that (2.3) holds. If (1.1) is permanent, then the inequality (2.4) is true.
From Theorems 2.1 and 2.2 we get the following result immediately.

**Corollary 2.2.** Suppose that (2.3) holds. Then (1.1) is permanent if and only if (2.4) holds.

**Remark 2.5.** If all the parameters in (1.1) are positive constants, then (1.1) becomes the system considered in [6], and the condition (2.4) reduces to $-d + (af)/(b\alpha + a\beta) > 0$, which is the sufficient and necessary condition for the autonomous version of (1.1) to have a unique positive equilibrium.

3. **Proof of the main results**

We need the following three propositions to prove Theorem 2.1.

**Proposition 3.1.** Under the assumption (2.3), there exist positive constants $M_x$ and $M_y$ such that

\[
\lim_{t \to \infty} \sup x(t) \leq M_x, \quad \lim_{t \to \infty} \sup y(t) \leq M_y
\]

for all solution $(x(t), y(t))$ of (1.1) with positive initial values.

**Proof.** Obviously, $\mathbb{R}_+^2 = \{(x, y) : x \geq 0, y \geq 0\}$ is a positively invariant set of system (1.1). Given any positive solution $(x(t), y(t))$ of (1.1), we have

\[
\dot{x} \leq a(t)x - b(t)x.
\]

The following auxiliary equation

\[
\dot{V} = V(a(t) - b(t)V)
\]

has a positive and globally asymptotically stable $\omega$-periodic solution $V^*(t)$ by Lemma 2.1. So, by comparison theorem, there exists $T_1 > 0$ such that

\[
x(t) < V^*(t) + 1(t \geq T_1).
\]

Let $M_x = \max_{0 \leq t < \omega} \{ V^*(t) + 1 \}$, we have

\[
\lim_{t \to \infty} \sup x(t) \leq M_x.
\]

From the second equation of (1.1), we get

\[
\dot{y} \leq -d(t)y + \frac{f(t)x}{\gamma(t)} \leq \frac{fM_x}{\gamma dL} - y, \quad t \geq T_1,
\]

because $d(t), f(t), \alpha(t), \beta(t)$ and $\gamma(t)$ are positive periodic functions.

Obviously, we can obtain positive constant $M_y$ such that

\[
\lim_{t \to \infty} \sup y(t) \leq M_y.
\]

This completes the proof of Proposition 3.1. \qed

**Proposition 3.2.** Under the condition (2.3), there exists a positive constant $\eta_x$ such that

\[
\lim_{t \to \infty} \sup x(t) \geq \eta_x
\]

for all solution $(x(t), y(t))$ of (1.1) with positive initial values.
Proof. Suppose that (3.2) is not true, then there is a sequence \( \{ z_m \} \subset \mathbb{R}^2_+ \) such that

\[
\lim_{t \to \infty} \sup x(t, z_m) < \frac{1}{m}, \quad m = 1, 2, \ldots \tag{3.3}
\]

where \((x(t, z_m), y(t, z_m))\) is the solution of (1.1) with \((x(0, z_m), y(0, z_m)) = z_m\). Choosing sufficiently small positive numbers \( \varepsilon_x < 1 \) and \( \varepsilon_y < 1 \) such that

\[
A_\omega \left( -d(t) + \frac{f(t)\varepsilon_x}{\alpha(t) + \beta(t)\varepsilon_x} \right) < 0, \tag{3.4}
\]

and

\[
A_\omega (\phi_x(t)) > 0, \tag{3.5}
\]

where

\[
\phi_x(t) = a(t) - b(t)\varepsilon_x - \frac{c(t)\varepsilon_y}{\alpha(t) + \gamma(t)\varepsilon_y} \exp(\Lambda\omega),
\]

\[
\Lambda = \max_{0 \leq t \leq \omega} \left( d(t) + \frac{f(t)}{\alpha(t) + \beta(t)} \right).
\]

By (3.3), for the given \( \varepsilon_x > 0 \), there exists a positive integer \( N_0 \) such that

\[
\lim_{t \to \infty} \sup x(t, z_m) < \frac{1}{m - \varepsilon_x}, \quad m > N_0. \tag{3.6}
\]

For the rest of this proof we assume that \( m > N_0 \). (3.6) implies there exists \( t_1^{(m)} > 0 \) such that

\[
x(t, z_m) < \varepsilon_x, \quad t \geq t_1^{(m)}
\]

which, together with the predator equation in (1.1), produces

\[
\dot{y}(t, z_m) \leq y(t, z_m) \left( -d(t) + \frac{f(t)\varepsilon_x}{\alpha(t) + \beta(t)\varepsilon_x} \right)
\]

for \( t \geq t_1^{(m)} \). By (3.4), a standard comparison argument shows that

\[
\lim_{t \to \infty} y(t, z_m) = 0.
\]

Therefore there is \( t_2^{(m)} > t_1^{(m)} \) such that

\[
y(t, z_m) < \varepsilon_y, \quad \text{for } t \geq t_2^{(m)}.
\]

This leads to

\[
\dot{x}(t, z_m) \geq x(t, z_m) \left( a(t) - \frac{c(t)\varepsilon_y}{\alpha(t) + \gamma(t)\varepsilon_y} - b(t)x(t, z_m) \right), \quad t \geq t_2^{(m)}.
\]

By (3.5), the following equation

\[
\dot{x} = x \left( a(t) - \frac{c(t)\varepsilon_y}{\alpha(t) + \gamma(t)\varepsilon_y} - b(t)x(t, z_m) \right)
\]

has a positive and \( \omega \)-periodic solution \( x^*(t) \) which is globally asymptotic stable. Hence

\[
x(t, z_m) > \frac{x^*(t)}{2}
\]

for sufficiently large \( t > 0 \) and \( m > N_0 \), which is a contradiction with (3.3). This completes the proof of Proposition 3.2. \( \square \)
Proposition 3.3. If (2.3) and (2.4) hold, then there exists a positive constant \( \eta_y \) such that

\[
\lim_{t \to \infty} \sup_{y(t)} y(t) > \eta_y
\]  

(3.7)

for all solution \((x(t), y(t))\) of (1.1) with positive initial values.

Proof. By assumption (2.4) we can choose constant \( \varepsilon_0 > 0 \) such that

\[
A_\omega (\psi_{\varepsilon_0}(t)) > 0,
\]  

(3.8)

where

\[
\psi_{\varepsilon_0}(t) = -d(t) + \frac{f(t)(x^*(t) - \varepsilon_0)}{\alpha(t) + \beta(t)(x^*(t) - \varepsilon_0) + \gamma(t)\varepsilon_0}.
\]  

(3.9)

Consider the following equation with a positive parameter \( \sigma \):

\[
\dot{x} = x\left(a(t) - \frac{2\sigma c(t)}{\alpha(t) + 2\sigma \gamma(t)} - b(t)x\right).
\]  

(3.10)

Because of \( A_\omega(a(t)) > 0 \), and \( c(t), \alpha(t), \beta(t), \gamma(t) \) are positive functions, and

\[
\frac{\partial}{\partial \sigma} \left( \frac{2\sigma c(t)}{\alpha(t) + 2\sigma \gamma(t)} \right) > 0,
\]  

we know that

\[
A_\omega \left(a(t) - \frac{2\sigma c(t)}{\alpha(t) + 2\sigma \gamma(t)}\right) := A_\omega(\xi(t)) > 0
\]  

(3.11)

for sufficiently small \( \sigma > 0 \). By Lemma 2.1, (3.10) has a unique positive \( \omega \)-periodic solution \( x_\sigma(t) \), which is globally asymptotically stable. Let \( \bar{x}_\sigma(t) \) be the solution of (3.10) with initial condition \( \bar{x}_\sigma(0) = x^*(0) \) in which \( x^*(t) \) is the unique periodic solution of (2.2) given by Lemma 2.1. Hence, for the above \( \varepsilon_0 \), there exists sufficiently large \( T_2 > T_1 \), such that

\[
|\bar{x}_\sigma(t) - x_\sigma(t)| < \frac{\varepsilon_0}{4} \quad \text{for } t \geq T_2.
\]

By the continuity of the solution in the parameter, we have \( \bar{x}_\sigma(t) \to x^*(t) \) uniformly in \([T_2, T_2 + \omega]\) as \( \sigma \to 0 \). Hence for \( \varepsilon_0 > 0 \), there exists \( \sigma_0 = \sigma_0(\varepsilon_0) > 0 \) such that

\[
|\bar{x}_\sigma(t) - x^*(t)| < \frac{\varepsilon_0}{4} \quad \text{for } t \in [T_2, T_2 + \omega], 0 < \sigma < \sigma_0.
\]

So we have

\[
|x_\sigma(t) - x^*(t)| < \frac{\varepsilon_0}{2}, \quad t \in [T_2, T_2 + \omega].
\]

Note that \( x_\sigma(t) \) and \( x^*(t) \) are all \( \omega \)-periodic, we have

\[
|x_\sigma(t) - x^*(t)| < \frac{\varepsilon_0}{2}, \quad t \geq 0, 0 < \sigma < \sigma_0.
\]

Choosing a constant \( \sigma_1 \) \((0 < \sigma_1 < \sigma_0, 2\sigma_1 < \varepsilon_0)\), then

\[
x_{\sigma_1}(t) \geq x^*(t) - \frac{\varepsilon_0}{2}, \quad t \geq 0.
\]  

(3.12)

Suppose that (3.7) is not true, then there exists \( z \in \mathbb{R}^2_+ \) such that

\[
\lim_{t \to \infty} \sup_{y(t)} y(t, z) < \sigma_1,
\]
where \((x(t, z), y(t, z))\) is the solution of (1.1) with \((x(0, z), y(0, z)) = z\). So there exists \(T_3 \geq T_2\) such that
\[
y(t, z) < 2\sigma_1 < \varepsilon_0, \quad t \geq T_3,
\]
and hence
\[
\dot{x}(t, z) \geq x(t, z) \left( a(t) - \frac{2\sigma_1 c(t)}{\alpha(t) + 2\sigma_1 \gamma(t)} - b(t)x(t, z) \right).
\]
Let \(u(t)\) be the solution of (3.10) with \(\sigma = \sigma_1\) and \(u(T_3) = x(T_3, z)\) then
\[
x(t, z) \geq u(t), \quad t \geq T_3.
\]
By the global asymptotic stability of \(x_{\sigma_1}(t)\), for the given \(\varepsilon = \varepsilon_0/2\), there exists \(T_4 \geq T_3\) such that
\[
\left| u(t) - x_{\sigma_1}(t) \right| < \frac{\varepsilon_0}{2}, \quad t \geq T_4.
\]
So
\[
x(t, z) \geq u(t) > x_{\sigma_1}(t) - \frac{\varepsilon_0}{2}, \quad t \geq T_4,
\]
and hence
\[
x(t, z) > x^*(t) - \varepsilon_0, \quad t \geq T_4,
\]
from (3.12). This implies
\[
\dot{y}(t, z) \geq \psi_{\varepsilon_0}(t)y(t, z), \quad t \geq T_4.
\]
Integrating the above inequality from \(T_4\) to \(t\) yields
\[
y(t, z) \geq y(T_4, z) \exp \int_{T_4}^{t} \psi_{\varepsilon_0}(t) \, dt.
\]
\(y(t, z) \to \infty\) as \(t \to \infty\) from (3.8), it is a contradiction. This completes the proof of Proposition 3.3. \(\Box\)

**Proof of Theorem 2.1.** By Propositions 3.1–3.3, (1.1) is uniform weak persistence and (1.1) has a global attractor. From Theorem 1.3.3 in [26], the system (1.1) is permanent under the assumptions (2.3) and (2.4). Using results given by Zhao [25] or another paper by Teng and Chen in [20], we obtain that (1.1) has a positive \(\omega\)-periodic solution. This completes the proof of Theorem 2.1. \(\Box\)

**Proof of Theorem 2.2.** By permanence of (1.1) and Proposition 3.1, there exist constants \(m > 0\) and \(T_5 > 0\) such that \(m < x(t) \leq M_x, y(t) > m\) for \(t \geq T_5\). Because \(A_\omega(a(t)) > 0\) and \((c(t)m)/(\alpha(t) + \beta(t)M_x + \gamma(t)m) \to 0\) as \(m \to 0\), we can choose \(m\) sufficiently small such that
\[
A_\omega \left( a(t) - \frac{c(t)m}{\alpha(t) + \beta(t)M_x + \gamma(t)m} \right) > 0.
\]
Consider the following auxiliary equation:
\[
\dot{u} = u \left( a(t) - \frac{c(t)m}{\alpha(t) + \beta(t)M_x + \gamma(t)m} - b(t)u \right).
\]
It has a unique positive $\omega$-periodic solution $u^*(t)$ which is globally asymptotically stable. Further, $u^*(t) < x^*(t)$ by comparing the analytic expressions of $u^*(t)$ and $x^*(t)$. Note that $u^*(t)$ and $x^*(t)$ are all $\omega$-periodic functions, we can choose sufficiently small constant $\varepsilon > 0$ such that $0 < u^*(t) < x^*(t) - \varepsilon$. Since
\[
\dot{x}(t) \leq x(t) \left( a(t) - \frac{c(t)m}{\alpha(t) + \beta(t)x^*(t) + \gamma(t)m} - b(t)x(t) \right), \quad t \geq T_5,
\]
for the given $\varepsilon > 0$, there exists $T_6 > T_5$ such that $x(t) < u^*(t) + \varepsilon < x^*(t)$ for $t \geq T_6$. Hence
\[
\dot{y}(t) \leq \left( -d(t) + \frac{f(t)x^*(t)}{\alpha(t) + \beta(t)x^*(t) + \gamma(t)m} \right), \quad t \geq T_6. \tag{3.15}
\]
for $t \geq T_6$. Note that
\[
f(t)x^*(t) \left( \frac{1}{\alpha(t) + \beta(t)x^*(t) + \gamma(t)m} \right) = \frac{mf(t)\gamma(t)x^*(t)}{(\alpha(t) + \beta(t)x^*)^2(\alpha(t) + \beta(t)x^*(t) + \gamma(t)m)} \geq v,
\]
where
\[
v = \min_{0 \leq t < \omega} \left\{ \frac{mf(t)\gamma(t)x^*(t)}{(\alpha(t) + \beta(t)x^*(t))^2(\alpha(t) + \beta(t)x^*(t) + \gamma(t)m)} \right\} \ldots.
\]
v > 0 because $f(t), \alpha(t), \beta(t), \gamma(t)$ are positive functions. Obviously,
\[
\frac{f(t)x^*(t)}{\alpha(t) + \beta(t)x^*(t) + \gamma(t)m} \leq \frac{f(t)x^*(t)}{\alpha(t) + \beta(t)x^*(t)} - \nu.
\]
By (3.15) we have
\[
\dot{y}(t) \leq \left( -d(t) + \frac{f(t)x^*(t)}{\alpha(t) + \beta(t)x^*(t)} - \nu \right), \quad t \geq T_6. \tag{3.16}
\]
Integrating (3.16) from $T_6$ to $t$ ($t > T_6$), we have
\[
y(t) \leq y(T_6) \exp \left\{ \int_{T_6}^{t} \left( -d(t) + \frac{f(t)x^*(t)}{\alpha(t) + \beta(t)x^*(t)} - \nu \right) dt \right\}.
\]
We claim that (2.4) is true. Otherwise we have
\[
A_\omega \left( -d(t) + \frac{f(t)x^*(t)}{\alpha(t) + \beta(t)x^*(t)} \right) \leq 0. \tag{3.17}
\]
y($t$) $\to$ 0 as $t \to \infty$ because of $v > 0$. It is in contradiction with the assumption that (1.1) is permanent. This implies that (2.4) is true. □

4. Discussion

Butler and Freedman [5] have studied the general system (1.3) and got sufficient conditions for the existence of a positive $\omega$-periodic solution. But their assumptions (H3) and (H6) are hard to be verified for system in the form of (1.1). Burton and Hutson [4] have studied nonautonomous
system (1.3) by using an average function technique. They have got some sufficient conditions for the permanence. But their assumption (H3) does not hold for model (1.1) because $a(t)$ may become negative at some points. On the other hand, Tineo [23] has considered (1.3) too, and derived sufficient conditions for its permanence. But his assumption (H1) (see [23]) does not hold for model (1.1) because $b(t)$ may become zero at some points. In the present paper we have established the sufficient and necessary condition (2.4) for permanence of (1.1) satisfying (2.3). On the contrary, (1.1) is extinct if and only if

$$A_{\omega} \left(-d(t) + \frac{f(t)x^*(t)}{\alpha(t) + \beta(t)x^*(t)}\right) \leq 0.$$

(4.1)

In addition, under the assumption (4.1), we know that $x(t) \to x^*(t)$ and $y(t) \to 0$. The proof is similar to that of Propositions 3.1 and 3.3 and we omit it.

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References