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# On retarded integral inequalities and their applications

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**Abstract**

The main results of [J. Math. Anal. Appl. 285 (2003) 436–443] are here generalized to the following retarded integral inequalities:

$$u^m(t) \leqslant c^{m/(m-n)} + \frac{m}{m-n} \int_0^{\alpha(t)} \left[ f(s)u^n(s)w\big(u(s)\big) + g(s)u^n(s) \right] ds$$

and

$$u^m(t) \leqslant c^{m/(m-n)} + \frac{m}{m-n} \int_0^{\alpha(t)} f(s)u^n(s)w\big(u(s)\big)\, ds + \frac{m}{m-n} \int_0^{t} g(s)u^n(s)w\big(u(s)\big)\, ds,$$

where $m > n > 0$ are constants and $t \in R_+ = [0, \infty)$. The results given here can be applied to the global existence of solutions to differential equations with time delay.


*Keywords:* Integral inequality; Retarded differential equation; Global existence

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## 1. Introduction

Pachpatte [8] generalized an interesting integral inequality due to Ou-Iang [7]. Recently, Lipovan presented a retarded version of the Ou-Iang and Pachpatte inequalities. Some other results can be found in [1,3,6,9]. Let us first recall the main results of [5] in the following

**Theorem A.** *Let $u$, $f$ and $g$ be nonnegative continuous functions defined on $R_+$ and let $c$ be a nonnegative constant. Moreover, let $w \in C(R_+, R_+)$ be nondecreasing with $w(u) > 0$ on $(0, \infty)$ and $\alpha \in C^1(R_+, R_+)$ be nondecreasing with $\alpha(t) \leqslant t$ on $R_+$. If*

$$u^2(t) \leqslant c^2 + 2\int_0^{\alpha(t)} \big[f(s)u(s)w\big(u(s)\big) + g(s)u(s)\big]\,ds, \quad t \in R_+, \tag{1}$$

*then for $0 \leqslant t \leqslant t_1$,*

$$u(t) \leqslant \Omega^{-1}\left[\Omega\left(c + \int_0^{\alpha(t)} g(s)\,ds\right) + \int_0^{\alpha(t)} f(s)\,ds\right],$$

*where*

$$\Omega(r) = \int_1^r \frac{ds}{w(s)}, \quad r > 0, \tag{$*$}$$

*$\Omega^{-1}$ is the inverse of $\Omega$, and $t_1 \in R_+$ is chosen so that*

$$\Omega\left(c + \int_0^{\alpha(t)} g(s)\,ds\right) + \int_0^{\alpha(t)} f(s)\,ds \in \mathrm{Dom}(\Omega^{-1}) \quad \text{for all } 0 \leqslant t \leqslant t_1.$$

**Theorem B.** *Let $u$, $f$ and $g$ be nonnegative continuous functions defined on $R_+$ and let $c$ be a nonnegative constant. Moreover, let $w \in C(R_+, R_+)$ be nondecreasing with $w(u) > 0$ on $(0, \infty)$ and $\int_1^\infty (1/w(s))\,ds = \infty$. If $\alpha \in C^1(R_+, R_+)$ is nondecreasing with $\alpha(t) \leqslant t$ on $R_+$ and*

$$u^2(t) \leqslant c^2 + 2\int_0^{\alpha(t)} f(s)u(s)w\big(u(s)\big)\,ds + 2\int_0^t g(s)u(s)w\big(u(s)\big)\,ds, \quad 0 \leqslant t < T, \tag{2}$$

*then*

$$u(t) \leqslant \Omega^{-1}\left[\Omega(c)\int_0^{\alpha(t)} f(s)\,ds + \int_0^t g(s)\,ds\right], \quad 0 \leqslant t < T,$$

*where $\Omega : (0, \infty) \to (\Omega(0), \infty)$ defined by $(*)$ is a $C^1$-diffeomorphism.*

In this paper, we will consider the following general integral inequalities:

$$u^m(t) \leqslant c^{m/(m-n)} + \frac{m}{m-n} \int_0^{\alpha(t)} \left[f(s)u^n(s)w\big(u(s)\big) + g(s)u^n(s)\right] ds \tag{3}$$

and

$$\begin{aligned} u^m(t) \leqslant c^{m/(m-n)} &+ \frac{m}{m-n} \int_0^{\alpha(t)} f(s)u^n(s)w\big(u(s)\big)\, ds \\ &+ \frac{m}{m-n} \int_0^{t} g(s)u^n(s)w\big(u(s)\big)\, ds, \end{aligned} \tag{4}$$

where $m > n > 0$, $c \geqslant 0$ are constants, $f, g, u, w, \alpha$ are assumed as in Theorems A and B. Our results generalize Theorems A and B, and can be used to give criteria ensuring the global existence of solutions to the generalized Liénard equation with time delay and to a retarded Rayleigh type equation.

## 2. Main results

**Theorem 2.1.** *Let $u$, $f$ and $g$ be nonnegative continuous functions defined on $R_+$ and let $c$ be a nonnegative constant. Moreover, let $w \in C(R_+, R_+)$ be nondecreasing with $w(u) > 0$ on $(0, \infty)$ and $\alpha \in C^1(R_+, R_+)$ be nondecreasing with $\alpha(t) \leqslant t$ on $R_+$. If* (3) *holds for $t \in R_+$, then for $0 \leqslant t \leqslant \xi$,*

$$u(t) \leqslant \left\{\Omega^{-1}\left[\Omega\left(c + \int_0^{\alpha(t)} g(s)\, ds\right) + \int_0^{\alpha(t)} f(s)\, ds\right]\right\}^{1/(m-n)}, \tag{5}$$

*where*

$$\Omega(r) = \int_1^r \frac{ds}{w(s^{1/(m-n)})}, \quad r > 0, \tag{6}$$

*$\Omega^{-1}$ is the inverse of $\Omega$, and $\xi \in R_+$ is chosen so that*

$$\Omega\left(c + \int_0^{\alpha(t)} g(s)\, ds\right) + \int_0^{\alpha(t)} f(s)\, ds \in \mathrm{Dom}(\Omega^{-1}) \quad \textit{for all } 0 \leqslant t \leqslant \xi.$$

**Proof.** Let us first assume that $c > 0$. Set

$$z(t) = c^{m/(m-n)} + \frac{m}{m-n} \int_0^{\alpha(t)} \left[f(s)u^n(s)w\big(u(s)\big) + g(s)u^n(s)\right] ds, \quad t \geqslant 0, \tag{7}$$

and

$$p(t) = c + \int_0^{\alpha(t)} g(s)\,ds, \quad t \geqslant 0. \tag{8}$$

From (3) and (7), we have

$$z(0) = c^{m/(m-n)}, \qquad u(t) \leqslant z^{1/m}(t), \quad t \geqslant 0,$$

and

$$\begin{aligned} z'(t) &= \frac{m}{m-n}\big[f\big(\alpha(t)\big)u^n\big(\alpha(t)\big)w\big(u\big(\alpha(t)\big)\big) + g\big(\alpha(t)\big)u^n\big(\alpha(t)\big)\big]\alpha'(t) \\ &\leqslant \frac{m}{m-n}z^{n/m}\big(\alpha(t)\big)\big[f\big(\alpha(t)\big)w\big(u\big(\alpha(t)\big)\big) + g\big(\alpha(t)\big)\big]\alpha'(t). \end{aligned} \tag{9}$$

Since $\alpha(t) \leqslant t$ and $z(t)$ is nondecreasing, we obtain from (9),

$$\frac{m-n}{m}\frac{z'(t)}{z^{n/m}(t)} \leqslant \big[f\big(\alpha(t)\big)w\big(u\big(\alpha(t)\big)\big) + g\big(\alpha(t)\big)\big]\alpha'(t). \tag{10}$$

Integrating (10) on $[0, t]$, by (8) we obtain

$$\begin{aligned} z^{(m-n)/m}(t) &\leqslant c + \int_0^{\alpha(t)} g(s)\,ds + \int_0^{\alpha(t)} f(s)w\big(u(s)\big)\,ds \\ &\leqslant p(t) + \int_0^{\alpha(t)} f(s)w\big(z^{1/m}(s)\big)\,ds. \end{aligned} \tag{11}$$

Let $T \leqslant \xi$ be an arbitrary number. From (11) we deduce that

$$z^{(m-n)/m}(t) \leqslant p(T) + \int_0^{\alpha(t)} f(s)w\big(z^{1/m}(s)\big)\,ds, \quad 0 \leqslant t \leqslant T. \tag{12}$$

Set

$$v(t) = p(T) + \int_0^{\alpha(t)} f(s)w\big(z^{1/m}(s)\big)\,ds, \quad 0 \leqslant t \leqslant T,$$

then we have $v(0) = p(T)$, $z^{1/m}(t) \leqslant v^{1/(m-n)}(t)$ and

$$v'(t) = f\big(\alpha(t)\big)w\big(z^{1/m}\big(\alpha(t)\big)\big)\alpha'(t) \leqslant f\big(\alpha(t)\big)w\big(v^{1/(m-n)}\big(\alpha(t)\big)\big)\alpha'(t), \quad 0 \leqslant t \leqslant T,$$

i.e.,

$$\frac{v'(t)}{w(v^{1/(m-n)}(t))} \leqslant f\big(\alpha(t)\big)\alpha'(t). \tag{13}$$

Integrating (13) on $[0, t]$, we deduce

$$\Omega\big(v(t)\big) \leqslant \Omega\big(p(T)\big) + \int_0^{\alpha(t)} f(s)\,ds, \quad 0 \leqslant t \leqslant T.$$

Thus, from (12), we obtain

$$z^{(m-n)/m}(t) \leqslant v(t) \leqslant \Omega^{-1}\left(\Omega\big(p(T)\big) + \int_0^{\alpha(t)} f(s)\,ds\right), \quad 0 \leqslant t \leqslant T,$$

from which it follows that

$$z^{1/m}(t) \leqslant \left[\Omega^{-1}\left(\Omega\big(p(T)\big) + \int_0^{\alpha(t)} f(s)\,ds\right)\right]^{1/(m-n)}. \tag{14}$$

Taking $t = T$ in (14) and using the fact that $u(t) \leqslant z^{1/m}(t)$ for $t = T$, we obtain

$$u(T) \leqslant \left[\Omega^{-1}\left(\Omega\big(p(T)\big) + \int_0^{\alpha(T)} f(s)\,ds\right)\right]^{1/(m-n)}.$$

Since $T \leqslant \xi$ is arbitrary and noting that $p(T) = c + \int_0^{\alpha(T)} g(s)\,ds$, we have proved the desired inequality (5).

The case $c = 0$ can be handled by repeating the above procedure with $\epsilon > 0$ and subsequently letting $\epsilon \to 0$. This completes the proof of Theorem 2.1. □

**Remark 1.** Setting $m = 2$ and $n = 1$ in Theorem 2.1, we have Lipovan's generalization [5, Theorem 1] of Pachpatte [8].

**Remark 2.** If

$$\int_1^{\infty} \frac{ds}{w(s^{1/(m-n)})} = \infty,$$

then $\Omega(\infty) = \infty$ and (6) is valid on $R_+$, i.e., we can choose $\xi = \infty$.

If we let $n = m - 1$ in Theorem 2.1, then we have the following corollaries.

**Corollary 2.1.** *Let the constants $m, c$ and the functions $u, f, g, w, \alpha$ be defined as in Theorem 2.1, and*

$$u^m(t) \leqslant c^m + m \int_0^{\alpha(t)} \big[f(s)u^{m-1}(s)w\big(u(s)\big) + g(s)u^{m-1}(s)\big]\,ds, \tag{15}$$

*then for $0 \leqslant t \leqslant \xi$,*

$$u(t) \leqslant \Omega^{-1}\left[\Omega\left(c + \int_0^{\alpha(t)} g(s)\,ds\right) + \int_0^{\alpha(t)} f(s)\,ds\right], \tag{16}$$

*where* $\Omega(r) = \int_1^r (1/w(s))\,ds$ *for* $r > 0$, $\Omega^{-1}$ *is the inverse of* $\Omega$, *and* $\xi \in R_+$ *is chosen so that*

$$\Omega\left(c + \int_0^{\alpha(t)} g(s)\,ds\right) + \int_0^{\alpha(t)} f(s)\,ds \in \mathrm{Dom}(\Omega^{-1}) \quad \textit{for all } 0 \leqslant t \leqslant \xi.$$

**Remark 3.** It is interesting to note that (1) and (15) provide the same bound on the unknown function $u(t)$, respectively.

**Corollary 2.2.** *Let the constants* $m, c$ *and the functions* $u, f, g, \alpha$ *be defined as in Theorem* 2.1, *and*

$$u^m(t) \leqslant c^m + m\int_0^{\alpha(t)} \left[f(s)u^m(s) + g(s)u^{m-1}(s)\right] ds, \quad t \geqslant 0,$$

*then*

$$u(t) \leqslant \left(c + \int_0^{\alpha(t)} g(s)\,ds\right)\exp\left(\int_0^{\alpha(t)} f(s)\,ds\right), \quad t \geqslant 0. \tag{17}$$

**Corollary 2.3.** *Let the constants* $m, n, c$ *and the functions* $u, f, \alpha$ *be defined as in Theorem* 2.1, *and*

$$u^m(t) \leqslant c^{m/(m-n)} + \frac{m}{m-n}\int_0^{\alpha(t)} f(s)u^n(s)\,ds, \quad t \geqslant 0,$$

*then*

$$u(t) \leqslant \left(c + \int_0^{\alpha(t)} f(s)\,ds\right)^{1/(m-n)}, \quad t \geqslant 0. \tag{18}$$

**Remark 4.** For $m = 2$, Corollary 2.2 reduces to Corollary 2 in [5]. For $m = 2$, $n = 1$ and $\alpha(t) = t$, Corollary 2.3 reduces to Ou-Iang's inequality (see [7]).

**Theorem 2.2.** *Let the constants* $m, n, c$ *and the functions* $u, f, g, w, \alpha$ *be the same as in Theorem* 2.1, *and* $\int_1^\infty (1/w(s^{1/(m-n)}))\,ds = \infty$. *If* (4) *holds for* $t \geqslant 0$, *then*

$$u(t) \leqslant \left\{\Omega^{-1}\left[\Omega(c) + \int_0^{\alpha(t)} f(s)\,ds + \int_0^{t} g(s)\,ds\right]\right\}^{1/(m-n)}, \quad t \geqslant 0, \tag{19}$$

*where* $\Omega$ *is defined by* (6) *and* $\Omega^{-1}$ *is the inverse of* $\Omega$.

**Proof.** Similar to the proof of Theorem 2.1, let us first assume that $c > 0$. By denoting the right-hand side of (4) by $z(t)$, the same steps as in the case of Theorem 2.1 lead to

$$\frac{m-n}{m}\frac{z'(t)}{z^{n/m}(t)} \leqslant f\big(\alpha(t)\big)w\big(z^{1/m}\big(\alpha(t)\big)\big)\alpha'(t) + g(t)w\big(z^{1/m}(t)\big).$$

Integrating the above inequality on $[0, t]$, we obtain

$$z^{(m-n)/m}(t) \leqslant c + \int_0^{\alpha(t)} f(s)w\big(z^{1/m}(s)\big)\,ds + \int_0^t g(s)w\big(z^{1/m}(s)\big)\,ds.$$

Set

$$v(t) = c + \int_0^{\alpha(t)} f(s)w\big(z^{1/m}(s)\big)\,ds + \int_0^t g(s)w\big(z^{1/m}(s)\big)\,ds,$$

then we have $v(0) = c$, $z^{1/m}(t) \leqslant v^{1/(m-n)}(t)$ and

$$\begin{aligned} v'(t) &= f\big(\alpha(t)\big)w\big(z^{1/m}\big(\alpha(t)\big)\big)\alpha'(t) + g(t)w\big(z^{1/m}(t)\big) \\ &\leqslant f\big(\alpha(t)\big)w\big(v^{1/(m-n)}(t)\big)\alpha'(t) + g(t)w\big(z^{1/m}(t)\big), \quad t \geqslant 0, \end{aligned}$$

i.e.,

$$\frac{v'(t)}{w(v^{1/(m-n)}(t))} \leqslant f\big(\alpha(t)\big)\alpha'(t) + g(t), \quad t \geqslant 0. \tag{20}$$

Integrating (20) on $[0, t]$, we deduce

$$\Omega\big(v(t)\big) \leqslant \Omega(c) + \int_0^{\alpha(t)} f(s)\,ds + \int_0^t g(s)\,ds, \quad t \geqslant 0.$$

Noting that $\Omega(\infty) = \infty$, we have

$$z^{(m-n)/m}(t) \leqslant v(t) \leqslant \Omega^{-1}\left(\Omega(c) + \int_0^{\alpha(t)} f(s)\,ds + \int_0^t g(s)\,ds\right), \quad t \geqslant 0.$$

It follows that

$$z^{1/m}(t) \leqslant \left[\Omega^{-1}\left(\Omega(c) + \int_0^{\alpha(t)} f(s)\,ds + \int_0^t g(s)\,ds\right)\right]^{1/(m-n)}, \quad t \geqslant 0.$$

Since $u(t) \leqslant z^{1/m}(t)$, we have proved the desired inequality (19). This completes the proof of Theorem 2.2. □

**Remark 5.** Let $m = 2$ and $n = 1$, then Theorem 2.2 reduces to Theorem 2 in [5].

If we let $n = m - 1$ in Theorem 2.2, then we have the following corollaries.

**Corollary 2.4.** *Let the constants $m, c$ and the functions $u, f, g, w, \alpha$ be defined as in Theorem* 2.2*, and*

$$u^m(t) \leqslant c^m + m \int_0^{\alpha(t)} f(s) u^{m-1}(s) w\big(u(s)\big)\, ds + m \int_0^t g(s) u^{m-1}(s) w\big(u(s)\big)\, ds, \qquad (21)$$

*then*

$$u(t) \leqslant \Omega^{-1}\left[\Omega(c) + \int_0^{\alpha(t)} f(s)\, ds + \int_0^t g(s)\, ds\right], \quad t \geqslant 0,$$

*where $\Omega = \int_1^r (1/w(s))\, ds$ for $r > 0$ and $\Omega^{-1}$ is the inverse of $\Omega$.*

**Remark 6.** It is interesting to note that (2) and (21) provide the same bound on the unknown function $u(t)$, respectively.

**Corollary 2.5.** *Let the constants $m, c$ and the functions $u, f, g, \alpha$ be defined as in Theorem* 2.2*, and*

$$u^m(t) \leqslant c^m + m \int_0^{\alpha(t)} f(s) u^m(s)\, ds + m \int_0^t g(s) u^m(s)\, ds,$$

*then*

$$u(t) \leqslant c \exp\left(\int_0^{\alpha(t)} f(s)\, ds + \int_0^t g(s)\, ds\right), \quad t \geqslant 0. \qquad (22)$$

## 3. Some applications

In this section we will show that our results are useful in proving the global existence of solutions to certain differential equations with time delay. These applications are given as examples.

We first recall some basic facts. Consider the functional differential equation

$$\begin{cases} X' = H(t, X(t), X(\alpha(t))), \\ X(0) = X_0, \end{cases} \qquad (23)$$

with $X_0 \in R^n$, $H \in C(R_+ \times R^{2n}, R^n)$, and $\alpha \in C^1(R_+, R_+)$ satisfying $\alpha(t) \leqslant t$ for $t \geqslant 0$. A result in [4] guarantees that for every $X_0 \in R^n$, Eq. (23) has a solution. Without additional hypothesis on $H$, the uniqueness of solutions is not granted. However, every solution of (23) has a maximal time of existence $T > 0$ and if $T < \infty$, then

$$\limsup_{t \to T} \|X(t)\|_{R^n} = \infty.$$

We present now two applications of the results from Section 2.

**Example 1.** Consider the generalized Liénard equation with time delay

$$\begin{cases} x' = y + F(x), \\ y' = G(t, x(t-\tau(t))), \end{cases} \tag{24}$$

where $F \in C^1(R, R)$, $G \in C(R_+ \times R, R)$, $\tau \in C^1(R_+, R_+)$, and $\tau(t) \leqslant t$ on $R_+$. If $\alpha(t) = t - \tau(t)$ is an increasing diffeomorphism of $R_+$ and

$$\bigl|F(x)\bigr| \leqslant v\bigl(|x|\bigr), \quad x \in R, \qquad G^m(t, x) \leqslant f(t)|x|^{m-1}v\bigl(|x|\bigr), \quad (t, x) \in R_+ \times R,$$

where $m$ is an integer larger than 1, $f, v \in C(R_+, R_+)$, $v(t) > 0$ for $t > 0$, and $\int_1^\infty (1/v(s))\,ds = \infty$, then all solutions of (24) have global existence.

Indeed, if $(x(t), y(t))$ is a solution of (24) defined on the maximal existence interval $[0, T)$, let $u(t) = [x^m(t) + y^m(t)]^{1/m}$ for $t \in [0, T)$, then we have $|x(t)| \leqslant u(t)$, $|y(t)| \leqslant u(t)$ for $t \in [0, T)$, and for $0 \leqslant t < T$,

$$\begin{aligned} \frac{d}{dt}u^m(t) &= mx^{m-1}(t)x'(t) + my^{m-1}(t)y'(t) \\ &\leqslant m\bigl|x(t)\bigr|^{m-1}\bigl|y(t)\bigr| + m\bigl|x(t)\bigr|^{m-1}\bigl|F\bigl(x(t)\bigr)\bigr| \\ &\quad + m\bigl|y(t)\bigr|^{m-1}\bigl|G\bigl(t, x\bigl(\alpha(t)\bigr)\bigr)\bigr|. \end{aligned} \tag{25}$$

Using the Hölder inequality, for any constants $a, b \geqslant 0$, we can easily obtain

$$a^{m-1}b \leqslant \frac{(m-1)a^m}{m} + \frac{b^m}{m} \leqslant a^m + b^m.$$

Therefore, from (25) we have for $0 \leqslant t < T$,

$$\begin{aligned} \frac{d}{dt}u^m(t) &\leqslant m\bigl[x^m(t) + y^m(t)\bigr] + m\bigl|x(t)\bigr|^{m-1}\bigl|F\bigl(x(t)\bigr)\bigr| \\ &\quad + m\bigl[y^m(t) + G^m\bigl(t, x\bigl(\alpha(t)\bigr)\bigr)\bigr] \\ &\leqslant 2mu^m(t) + mu^{m-1}(t)v\bigl(x(t)\bigr) + mf(t)\bigl|x\bigl(\alpha(t)\bigr)\bigr|^{m-1}(s)v\bigl(\bigl|x\bigl(\alpha(t)\bigr)\bigr|\bigr) \\ &\leqslant 2mu^m(t) + mu^{m-1}(t)v\bigl(u(t)\bigr) + mf(t)u^{m-1}(s)\bigl(\alpha(t)\bigr)v\bigl(u\bigl(\alpha(t)\bigr)\bigr). \end{aligned}$$

With $w(u) =: 2u + v(u)$, an integration on $[0, t]$ for $t < T$ yields

$$\begin{aligned} u^m(t) &\leqslant u^m(0) + m\int_0^t u^{m-1}(s)w\bigl(u(s)\bigr)\,ds + m\int_0^t f(s)u^{m-1}\bigl(\alpha(s)\bigr)v\bigl(u\bigl(\alpha(s)\bigr)\bigr)\,ds \\ &\leqslant u^m(0) + m\int_0^t u^{m-1}(s)w\bigl(u(s)\bigr)\,ds + m\int_0^t f(s)u^{m-1}\bigl(\alpha(s)\bigr)w\bigl(u\bigl(\alpha(s)\bigr)\bigr)\,ds \\ &= u^m(0) + m\int_0^t u^{m-1}(s)(s)w\bigl(u(s)\bigr)\,ds \\ &\quad + m\int_0^{\alpha(t)} \frac{f(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))}u^{m-1}(r)w\bigl(u(r)\bigr)\,ds \end{aligned}$$

after performing the change of variables $r = \alpha(s)$ at some intermediate step. Above $\alpha^{-1}$ is the inverse of the diffeomorphism $\alpha$. Our hypotheses on $v$ guarantee that $\Omega(\infty) = \int_1^\infty (1/w(r))\,dr = \infty$ (see [2]). By Corollary 2.4 we have

$$\begin{aligned}
u(t) &\leqslant \Omega^{-1}\left[\Omega\bigl(u(0)\bigr) + t + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))}\,dr\right] \\
&= \Omega^{-1}\left[\Omega\bigl(u(0)\bigr) + t + \int_0^{t} f(s)\,ds\right], \quad 0 \leqslant t < T,
\end{aligned}$$

where $\Omega^{-1}$ is the inverse of $\Omega$. This proves that $u(t)$ does not blow-up in finite time. Therefore, $T = \infty$, i.e., all solutions of (24) have global existence.

**Example 2.** Consider the Rayleigh equation with time delay

$$\begin{cases} x' = y, \\ y' = F(y) + G(x(\alpha(t))), \end{cases} \tag{26}$$

where $F, G \in C(R, R)$, $\alpha \in C^1(R_+, R_+)$ with $\alpha(t) \leqslant t$ for $t \geqslant 0$. If $\alpha$ is an increasing diffeomorphism of $R_+$ and

$$\bigl|F(x)\bigr| \leqslant v\bigl(|x|\bigr), \quad G^m(x) \leqslant |x|^{m-1} v\bigl(|x|\bigr), \quad x \in R,$$

where $m$ is an integer larger than 1, $v \in C(R_+, R_+)$ is nondecreasing, $v(u) > 0$ for $u > 0$ and $\int_1^\infty (1/v(s))\,ds = \infty$, then all solutions of (26) have global existence.

In fact, if $(x(t), y(t))$ is a solution of (26) defined on the maximal existence interval $[0, T)$, let $u(t) = [x^m(t) + y^m(t)]^{1/m}$ for $t \in [0, T)$, then we have for $0 \leqslant t < T$,

$$\begin{aligned}
\frac{d}{dt} u^m(t) &= m x^{m-1}(t) x'(t) + m y^{m-1}(t) y'(t) \\
&\leqslant m\bigl|x(t)\bigr|^{m-1}\bigl|y(t)\bigr| + m\bigl|y(t)\bigr|^{m-1}\bigl|F\bigl(y(t)\bigr)\bigr| + m\bigl|y(t)\bigr|^{m-1}\bigl|G\bigl(x\bigl(\alpha(t)\bigr)\bigr)\bigr| \\
&\leqslant 2m u^m(t) + m u^{m-1}(t) v\bigl(u(t)\bigr) + m u^{m-1}\bigl(\alpha(t)\bigr) v\bigl(u\bigl(\alpha(t)\bigr)\bigr).
\end{aligned}$$

Setting $w(t) =: 2u + v(u)$ and integrating the above inequality on $[0, t]$ for $t < T$, we obtain

$$\begin{aligned}
u^m(t) &\leqslant u^m(0) + m\int_0^t u^{m-1}(s) w\bigl(u(s)\bigr)\,ds + m\int_0^t u^{m-1}\bigl(\alpha(s)\bigr) v\bigl(u\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\leqslant u^m(0) + m\int_0^t u^{m-1}(s) w\bigl(u(s)\bigr)\,ds + m\int_0^t u^{m-1}\bigl(\alpha(s)\bigr) w\bigl(u\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&= u^m(0) + m\int_0^t u^{m-1}(s) w\bigl(u(s)\bigr)\,ds
\end{aligned}$$

$$+ m \int_0^{\alpha(t)} \frac{1}{\alpha'(\alpha^{-1}(r))} u^{m-1}(r) w\big(u(r)\big)\, ds,$$

where $r = \alpha(s)$. Similar to the discussion in Example 1, we obtain by Corollary 2.4,

$$\begin{aligned} u(t) &\leqslant \Omega^{-1}\left[\Omega\big(u(0)\big) + t + \int_0^{\alpha(t)} \frac{1}{\alpha'(\alpha^{-1}(r))}\, dr\right] \\ &= \Omega^{-1}\big[\Omega\big(u(0)\big) + 2t\big], \quad 0 \leqslant t < T, \end{aligned}$$

where $\Omega$ is defined by (7) and $\Omega^{-1}$ is the inverse of $\Omega$. This proves that $u(t)$ does not blow-up in finite time. Therefore, $T = \infty$, i.e., all solutions of (26) have global existence.

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## References

[1] I. Bihari, A generalization of a lemma of Bellman and its applications to uniqueness problems of differential equations, Acta Math. Acad. Sci. Hungar. 7 (1956) 71–94.
[2] A. Constantin, Global existence of solutions for perturbed differential equations, Ann. Mat. Pura Appl. 168 (1995) 237–299.
[3] A. Constantin, Solutions globales d'équations différentielles perturbées, C. R. Acad. Sci. Paris 320 (1995) 1319–1322.
[4] R. Driver, Existence and continuous dependence of solutions of neutral functional differential equations, Arch. Rational Mech. Anal. 19 (1965) 149–166.
[5] O. Lipovan, A retarded integral inequality and its applications, J. Math. Anal. Appl. 285 (2003) 436–443.
[6] O. Lipovan, A retarded Gronwall-like inequality and its applications, J. Math. Anal. Appl. 252 (2000) 389–401.
[7] L. Ou-Iang, The boundedness of solutions of linear differential equations $y'' + A(t)y = 0$, Shuxue JinZhan 3 (1957) 409–415.
[8] B.G. Pachpatte, On some new inequalities related to certain inequalities in the theory of differential equations, J. Math. Anal. Appl. 189 (1995) 128–144.
[9] B.G. Pachpatte, Inequalities for Differential and Integral Equations, Academic Press, New York, 1998.