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An anisotropic, superconvergent nonconforming plate finite element ☆

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Abstract

The classical finite element convergence analysis relies on the following regularity condition: there exists a constant c independent of the element K and the mesh such that $h_K/\rho_K \leqslant c$, where h_K and ρ_K are diameters of K and the biggest ball contained in K, respectively. In this paper, we construct a new, nonconforming rectangular plate element by the double set parameter method. We prove the convergence of this element without the above regularity condition. The key in our proof is to obtain the $O(h^2)$ consistency error. We also prove the superconvergence of this element for narrow rectangular meshes. Results of our numerical tests agree well with our analysis.

Keywords: Regularity condition; Double set parameter; Nonconforming plate element; Anisotropic convergence; Superconvergence

1. Introduction

Suppose $\Omega \subset R^n$, $\mathscr{T}_h$ is a family of subdivisions of Ω. $\{\mathscr{T}_h\}=\{K_1, \ldots, K_N\}$, $\Omega=\bigcup_{K_i \in \mathscr{T}_h} K_i$, K_i is the element. The convergence of the classical finite element requires that the mesh satisfies the following regularity [10] or nondegenerate [4] condition: there exists a constant c independent of $K \in \mathscr{T}_h$ and $\mathscr{T}_h$, such that

$$h_K/\rho_K \leqslant c, \tag{1.1}$$

where h_K is the diameter of K, ρ_K is the diameter of the biggest ball contained in K. But recent research shows that (1.1) is not a necessary condition for the convergence of the finite element methods, see for example [2,3,7,8,15,16,22]. Among them, Apel et al. have studied anisotropic Lagrange interpolations. Their fundamental results are of [2, Lemmas 3 and 4] or of [3, Lemmas 2.2 and 2.3], which give the conditions for which the following estimate holds:

$$|D^\alpha(u - Iu)|_{m,K} \leqslant C|D^\alpha u|_{l,K}, \tag{1.2}$$

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where $\alpha=(\alpha_1,\ldots,\alpha_n)$ is a multi-index, $D^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$, $|\alpha|=\sum_{i=1}^n \alpha_i$, $0\leqslant m\leqslant l$, and I is the finite element interpolation operator. We call the above property (1.2) anisotropism of the element.

In [7] we introduced a general theorem for anisotropic interpolation, which yields a new criterion to obtain (1.2), improving Apel's result and which is easier to use. The research on anisotropic elements mainly concentrate on conforming elements. For nonconforming elements, the consistency error should also be estimated. For classical elements, the consistency error estimate still relies on (1.1). As far as we know, only few published articles discuss nonconforming elements on anisotropic meshes.

The superconvergence of finite elements has been widely analyzed mathematically because of its practical importance in engineering computations, the reader is referred to [5,11,21]. The term superconvergence also includes accelerated convergence achieved by means of various recovery (or post-processing) techniques (refer to [14]). However, the studies mainly concentrated on second order problems and conforming C^0 elements. As for the superconvergence results of nonconforming finite elements, there seem to be few articles except Wilson's element for the second order problem (refer to [5,6,18]). Meanwhile, the meshes are required to be regular (sometimes quasi-uniform, namely, satisfy (1.1) and an inverse assumption).

In this paper, a new nonconforming rectangular plate element is presented, and its convergence is proved for anisotropic meshes. In addition, we prove that its anisotropic consistence error is $O(h^2)$, which is the same order as that of Adini's nonconforming element on rectangular meshes (cf. [14]), however, the shape space of our element is only exact for $P_2(K)$. Its natural superconvergence and global superconvergence are derived without the regularity assumption (1.1). As far as the authors' knowledge, this is the first time to find such properties for fourth order problems on anisotropic meshes. Some numerical examples supporting our theoretical analysis are given.

2. Basic theory

Let $\hat{K}$ be the reference element, shape function space $\hat{P}$ be a polynomial space on $\hat{K}$, the dimension of $\hat{P}$ be m, $\hat{P}'$ be the dual space of $\hat{P}$. $\{\hat{p}_1,\ldots,\hat{p}_m\}\subset\hat{P}$ and $\{\hat{N}_1,\ldots,\hat{N}_m\}\subset\hat{P}'$ are a pair of dual basis for $\hat{P}$ and $\hat{P}'$, respectively, i.e.,

$$\hat{N}_i(\hat{p}_j)=\delta_{ij},\quad 1\leqslant i,j\leqslant m.$$

Suppose the interpolation operator $\hat{I}:H^k(\hat{K})\to\hat{P}$, $k\geqslant 1$ is defined such that

$$\hat{N}_i(\hat{I}\hat{v})=\hat{N}_i(\hat{v}),\quad i=1,\ldots,m,\quad \forall\hat{v}\in H^k(\hat{K}). \tag{2.1}$$

Obviously,

$$\hat{I}\hat{v}=\sum_{i=1}^m \hat{N}_i(\hat{v})\hat{p}_i \tag{2.2}$$

and

$$\hat{I}\hat{v}=\hat{v},\quad \forall\hat{v}\in\hat{P}. \tag{2.3}$$

Suppose $\alpha=(\alpha_1,\ldots,\alpha_n)$ is a multi-index, then $\hat{D}^\alpha\hat{P}$ is also a polynomial space on $\hat{K}$. Let

$$\dim \hat{D}^\alpha\hat{P}=r$$

and $\{\hat{q}_i\}_{i=1}^r$ be the set of basis functions of $\hat{D}^\alpha\hat{P}$, suppose $\hat{D}^\alpha\hat{p}_i=\sum_{j=1}^r \alpha_{ij}\hat{q}_j$, $1\leqslant i\leqslant m$, then $\hat{D}^\alpha\hat{I}\hat{v}$ can be expressed by

$$\hat{D}^\alpha\hat{I}\hat{v}\overset{(2.2)}{=}\sum_{i=1}^m \hat{N}_i(\hat{v})D^\alpha\hat{p}_i=\sum_{j=1}^r \beta_j(\hat{v})\hat{q}_j, \tag{2.4}$$

where

$$\beta_j(\hat{v})=\sum_{i=1}^m \alpha_{ij}\hat{N}_i(\hat{v}). \tag{2.5}$$

From (2.5) and (2.1) we have

$$\beta_j(\hat{v}) = \sum_{i=1}^{m} \alpha_{ij} \hat{N}_i(\hat{v}) = \sum_{i=1}^{m} \alpha_{ij} \hat{N}_i(\hat{I}\hat{v}) = \beta_j(\hat{I}\hat{v}). \tag{2.6}$$

Theorem 2.1 (*Chen et al. [7], Chen et al. [8]*). *Let α be a multi-index, $P_l(\hat{K}) \subset \hat{D}^\alpha \hat{P}$ and $\hat{I} : W^{|\alpha|+l+1,p}(\hat{K}) \to \hat{P}$ be the above finite element interpolation operator defined by* (2.1), (2.2) *and satisfy*: $\hat{I} \in L(W^{|\alpha|+l+1,p}(\hat{K}), W^{|\alpha|+m,q}(\hat{K}))$*—the space of continuous linear mappings from* $(W^{|\alpha|+l+1,p}(\hat{K})$ *into* $W^{|\alpha|+m,q}(\hat{K}))$, *and* $W^{l+1,p}(\hat{K}) \hookrightarrow W^{m,q}(\hat{K})$. *If* $\beta_j(\hat{v})$ *of* (2.5) *can be expressed by*

$$\beta_j(\hat{v}) = F_j(\hat{D}^\alpha \hat{v}), \quad 1 \leqslant j \leqslant r, \tag{2.7}$$

where

$$F_j \in (W^{l+1,p}(\hat{K}))', \quad 1 \leqslant j \leqslant r. \tag{2.8}$$

Then

$$|\hat{D}^\alpha(\hat{v} - \hat{I}\hat{v})|_{m,q,\hat{K}} \leqslant C(\hat{I}, \hat{K})|\hat{D}^\alpha \hat{v}|_{l+1,p,\hat{K}}, \quad \forall \hat{v} \in W^{|\alpha|+l+1,p}(\hat{K}). \tag{2.9}$$

Remark 2.1. In [2,3], Apel et al. show that one necessary condition for (2.9) to hold is

$$\hat{D}^\alpha \hat{u} = 0 \Rightarrow \hat{D}^\alpha \hat{I}\hat{u} = 0. \tag{2.10}$$

(2.10) is useful to prove that the finite element interpolation does not have anisotropic behavior (1.2).

3. Anisotropic rectangular plate element with double set parameters

Let the reference element $\hat{K}$ be a square in the (ξ, η) plane, and let its nodes be $\hat{a}_1(-1,-1)$, $\hat{a}_2(1,-1)$, $\hat{a}_3(1,1)$, $\hat{a}_4(-1,1)$, its middle points of 4 sides be $\hat{a}_5(0,-1)$, $\hat{a}_6(1,0)$, $\hat{a}_7(0,1)$, $\hat{a}_8(-1,0)$, and its 4 sides be $\hat{l}_1 = \hat{a}_1\hat{a}_2$, $\hat{l}_2 = \hat{a}_2\hat{a}_3$, $\hat{l}_3 = \hat{a}_3\hat{a}_4$, $\hat{l}_4 = \hat{a}_4\hat{a}_1$.

The degrees of freedom are taken as

$$D(\hat{v}) = (\hat{v}_1, \hat{v}_2, \hat{v}_3, \hat{v}_4, \hat{v}_{5\eta}, \hat{v}_{6\xi}, \hat{v}_{7\eta}, \hat{v}_{8\xi})^\top, \tag{3.1}$$

where $\hat{v}_i = \hat{v}(\hat{a}_i)$, $1 \leqslant i \leqslant 4$ are function values at 4 nodes; $\hat{v}_{i\eta} = \frac{\partial \hat{v}}{\partial \eta}(\hat{a}_i)$, $i = 5, 7$, $\hat{v}_{i\xi} = \frac{\partial \hat{v}}{\partial \xi}(\hat{a}_i)$, $i = 6, 8$, are normal derivatives at the middle points of 4 sides.

The shape function space is taken as

$$\hat{P} = P_2(\hat{K}) \bigoplus \operatorname{span}\{\xi^3, \eta^3\} = \operatorname{span}\{\hat{p}_1, \ldots, \hat{p}_8\}, \tag{3.2}$$

where $\hat{p}_1 = \frac{1}{4}(1-\xi)(1-\eta)$, $\hat{p}_2 = \frac{1}{4}(1+\xi)(1-\eta)$, $\hat{p}_3 = \frac{1}{4}(1+\xi)(1+\eta)$, $\hat{p}_4 = \frac{1}{4}(1-\xi)(1+\eta)$, $\hat{p}_5 = 1-\xi^2$, $\hat{p}_6 = 1-\eta^2$, $\hat{p}_7 = \xi(1-\xi^2)$, $\hat{p}_8 = \eta(1-\eta^2)$.

$\forall \hat{v} \in \hat{P}$, suppose

$$\hat{v} = \sum_{i=1}^{8} \beta_i \hat{p}_i. \tag{3.3}$$

Substituting (3.3) into (3.2) we get

$$\beta_i = \hat{v}_i,\ 1 \leqslant i \leqslant 4,\ \beta_5 = \tfrac{1}{4}(\hat{v}_{8\xi} - \hat{v}_{6\xi}),\ \beta_6 = \tfrac{1}{4}(\hat{v}_{5\eta} - \hat{v}_{7\eta}),$$

$$\beta_7 = \tfrac{1}{8}(-\hat{v}_1 + \hat{v}_2 + \hat{v}_3 - \hat{v}_4 - 2\hat{v}_{8\xi} - 2\hat{v}_{6\xi}),$$

$$\beta_8 = \tfrac{1}{8}(-\hat{v}_1 - \hat{v}_2 + \hat{v}_3 + \hat{v}_4 - 2\hat{v}_{5\eta} - 2\hat{v}_{7\eta}). \tag{3.4}$$

Using the "Double Set Parameter Method" [9], we take another set of nodal parameters:

$$Q(\hat{v}) = (\hat{v}_1, \hat{v}_{1\xi}, \hat{v}_{1\eta}, \ldots, \hat{v}_4, \hat{v}_{4\xi}, \hat{v}_{4\eta})^\top, \tag{3.5}$$

i.e., the function values and the first order partial derivatives at 4 nodes, which are the real degrees of freedom.

Approximating the degrees of freedom $D(\hat{v})$ by the linear combinations of nodal parameters $Q(\hat{v})$ as follows:

$$\begin{cases} \hat{v}_i = \hat{v}_i, 1 \leqslant i \leqslant 4, & \hat{v}_{5\eta} = \frac{1}{2}(\hat{v}_{1\eta} + \hat{v}_{2\eta}), \\ \hat{v}_{6\xi} = \frac{1}{2}(\hat{v}_{2\xi} + \hat{v}_{3\xi}), & \hat{v}_{7\eta} = \frac{1}{2}(\hat{v}_{3\eta} + \hat{v}_{4\eta}), \quad \hat{v}_{8\xi} = \frac{1}{2}(\hat{v}_{1\xi} + \hat{v}_{4\xi}). \end{cases} \tag{3.6}$$

For $\hat{v} \in \hat{P} = P_2(\hat{K}) \cup \{\xi^3, \eta^3\}$, these discretizations are exact.

Substituting (3.6) into (3.4), we get the expressions of β_i of (3.3) in $Q(\hat{v})$:

$$\begin{cases} \beta_i = \hat{v}_i, 1 \leqslant i \leqslant 4, \\ \beta_5 = \frac{1}{8}(\hat{v}_{1\xi} - \hat{v}_{2\xi} - \hat{v}_{3\xi} + \hat{v}_{4\xi}), \quad \beta_6 = \frac{1}{8}(\hat{v}_{1\eta} + \hat{v}_{2\eta} - \hat{v}_{3\eta} - \hat{v}_{4\eta}), \\ \beta_7 = \frac{1}{8}(-\hat{v}_1 + \hat{v}_2 + \hat{v}_3 - \hat{v}_4 - \hat{v}_{1\xi} - \hat{v}_{2\xi} - \hat{v}_{3\xi} - \hat{v}_{4\xi}), \\ \beta_8 = \frac{1}{8}(-\hat{v}_1 - \hat{v}_2 + \hat{v}_3 + \hat{v}_4 - \hat{v}_{1\eta} - \hat{v}_{2\eta} - \hat{v}_{3\eta} - \hat{v}_{4\eta}). \end{cases} \tag{3.7}$$

Let $\mathcal{T}_h$ be a rectangle subdivision of Ω, $\bigcup_{K \in \mathcal{T}_h} K = \overline{\Omega}$. Suppose a general rectangle element K is on (x, y) plane with center (x_K, y_K) and side lengths $2h_{K1}$ and $2h_{K2}$, respectively, its 4 nodes are: $a_1(x_K - h_{K1}, y_K - h_{K2})$, $a_2(x_K + h_{K1}, y_K - h_{K2})$, $a_3(x_K + h_{K1}, y_K + h_{K2})$, $a_4(x_K - h_{K1}, y_K + h_{K2})$. The mapping from $\hat{K}$ to K denoted by $x = F(\hat{x})$ is

$$x = h_{K1}\xi + x_K, \quad y = h_{K2}\eta + y_K. \tag{3.8}$$

The shape function space on K is defined as

$$P_K = \{v = \hat{v} \circ F_K^{-1};\ \hat{v} \in \hat{P} \text{ is determined by (3.3), (3.7)}\}.$$

Let $\hat{I}$ be the finite element interpolation operators deduced by $\hat{P}$, then

$$\hat{I}\hat{v} = \sum_{i=1}^{8} \beta_i \hat{p}_i, \quad \forall \hat{v} \in H^3(\hat{K}), \tag{3.9}$$

where the expressions of $\beta_1, \ldots, \beta_8$ are (3.7).

From (3.3) and (3.7), it is easy to see that

$$\begin{cases} (\hat{I}\hat{v})_i = \hat{v}_i, 1 \leqslant i \leqslant 4, \\ (\hat{I}\hat{v})_{5\eta} = \frac{1}{2}(\hat{v}_{1\eta} + \hat{v}_{2\eta}), \quad (\hat{I}\hat{v})_{6\xi} = \frac{1}{2}(\hat{v}_{2\xi} + \hat{v}_{3\xi}), \\ (\hat{I}\hat{v})_{7\eta} = \frac{1}{2}(\hat{v}_{3\eta} + \hat{v}_{4\eta}), \quad (\hat{I}\hat{v})_{8\xi} = \frac{1}{2}(\hat{v}_{1\xi} + \hat{v}_{4\xi}). \end{cases} \tag{3.10}$$

Define the interpolation on a general element K as

$$I_K v = (\hat{I}\hat{v}) \circ F_K^{-1}, \quad \forall v \in H^3(K).$$

The relations of the nodal parameters on $\hat{K}$ and K are

$$v_i = \hat{v}_i, \quad v_{ix} = \hat{v}_{i\xi} h_{K1}^{-1}, \quad v_{iy} = \hat{v}_{i\eta} h_{K2}^{-1}, \quad 1 \leqslant i \leqslant 4. \tag{3.11}$$

From now on, the sign $\hat{c}$ denotes a general constant which is independent of $h_{K_1}/h_{K_2}, h/h_K, \forall K \in \mathcal{T}_h$, here $h = \max_{K \in \mathcal{T}_h} h_K$, $h_K = \max\{h_{K_1}, h_{K_2}\}$.

Theorem 3.1. *The interpolation operator $\hat{I}$ is anisotropic for the fourth order problem, that is to say*, $\forall \alpha$, $|\alpha| = 2$,

$$\|\hat{D}^\alpha(\hat{v} - \hat{I}\hat{v})\|_{0,\hat{K}} \leqslant \hat{c}|\hat{D}^\alpha \hat{v}|_{1,\hat{K}}, \quad \forall \hat{v} \in H^3(\hat{K}). \tag{3.12}$$

Proof. It is only needed to check the conditions of Theorem 2.1. Let $\alpha = (2, 0)$,

$$\hat{D}^\alpha \hat{I}\hat{v} \overset{(3.9)}{=} \sum_{i=1}^{8} \beta_i \frac{\partial^2 \hat{p}_i}{\partial \xi^2} = -2\beta_5 - 6\xi\beta_7, \tag{3.13}$$

$\{-2, -6\xi\}$ are the bases of $\hat{D}^\alpha \hat{P} = \mathrm{Span}\{1, \xi\}$,

$$\beta_5(\hat{v}) \overset{(3.7)}{=} \frac{1}{8}\left(-\int_{-1}^{1} \frac{\partial^2 \hat{v}}{\partial \xi^2}(\xi, -1)\,\mathrm{d}\xi - \int_{-1}^{1} \frac{\partial^2 \hat{v}}{\partial \xi^2}(\xi, 1)\,\mathrm{d}\xi\right) \triangleq F_1(\hat{D}^\alpha \hat{v}),$$

$$\begin{aligned}\beta_7(\hat{v}) \overset{(3.7)}{=} \frac{1}{16}\Bigg[&\int_{-1}^{1} \mathrm{d}\xi \left(\int_{-1}^{\xi} \frac{\partial^2 \hat{v}}{\partial \xi^2}(\xi, -1)\,\mathrm{d}\xi - \int_{\xi}^{1} \frac{\partial^2 \hat{v}}{\partial \xi^2}(\xi, -1)\,\mathrm{d}\xi\right) \\ &+ \int_{-1}^{1} \mathrm{d}\xi \left(\int_{-1}^{\xi} \frac{\partial^2 \hat{v}}{\partial \xi^2}(\xi, 1)\,\mathrm{d}\xi - \int_{\xi}^{1} \frac{\partial^2 \hat{v}}{\partial \xi^2}(\xi, 1)\,\mathrm{d}\xi\right)\Bigg] \triangleq F_2(\hat{D}^\alpha \hat{v}).\end{aligned} \tag{3.14}$$

From the Hölder inequality and trace theorem [1], we have

$$|F_i(\hat{w})| \leqslant \hat{c}|\hat{w}|_{0,\partial\hat{K}} \leqslant \hat{c}\|\hat{w}\|_{1,\hat{K}}, \quad i = 1, 2. \tag{3.15}$$

For $\alpha = (0, 2)$ and $\alpha = (1, 1)$ the same properties as $\alpha = (2, 0)$ hold. Hence (3.12) follows from (2.8). □

Remark 3.1. (1) If using classical forms, i.e., (3.4) for β_i, (2.9) does not hold, so this 8 degrees of freedom rectangular plate element (denoted by 8-2 element) is not anisotropic and (3.12) does not hold. In fact, for $\alpha = (2.0)$, let $\hat{v} = \xi\eta^2$, then $\hat{D}^\alpha \hat{v} = 0$, but $\beta_5 = 0$, $\beta_7 = \frac{1}{2}$, $\hat{D}^\alpha \hat{I}\hat{v} = -2\beta_5 - 6\xi\beta_7 = -3\xi \neq 0$.

(2) For the initial 8 degrees of freedom rectangular plate element (denoted by 8-1 element), the degrees of freedom are also (3.1), but the shape function space $\hat{P} = P_2(\hat{K}) \cup \{\xi^2\eta, \xi\eta^2\}$. Under the regularity condition (1.1), its convergence is correctly proven in [19]. It is easy to prove that this element does not satisfy (2.9). It should be pointed out that its double set parameter form (denoted by 8-12-1 element) does not satisfy (2.9), either.

4. Anisotropic convergence of 8-12-2 element for the plate bending problems

Consider the plate bending problem [10]: Find $u \in H_0^2(\Omega)$ such that

$$a(u, v) = (f, v), \quad \forall v \in H_0^2(\Omega), \tag{4.1}$$

where Ω is a rectangle domain, $f \in L^2(\Omega)$ and

$$a(u, v) = \int_\Omega A(u, v)\,\mathrm{d}x\,\mathrm{d}y, \; (f, v) = \int_\Omega f v\,\mathrm{d}x\,\mathrm{d}y,$$

$$A(u, v) = \triangle u \triangle v + (1 - \sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}),$$

$$H_0^2(\Omega) = \{v \in H^2(\Omega), v = \frac{\partial v}{\partial n} = 0, \text{ on } \partial\Omega\},$$

here n is the direction normal to the boundary $\partial\Omega$, σ is the Poisson ratio, $0 < \sigma < \frac{1}{2}$, $u_{xy} = \frac{\partial^2 u}{\partial x \partial y}$, etc.

The corresponding differential equation of (4.1) is

$$\begin{cases} \triangle^2 u = f & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

The finite element space V_h is defined as

$$V_h = \{v_h;\ v_h|_K = \hat{v} \circ F_K^{-1},\ \hat{v} \text{ is defined by (3.3), (3.7)}, \forall K \in \mathcal{T}_h,$$
$$v(a) = v_x(a) = v_y(a) = 0, \text{ for all node } a \text{ on } \partial\Omega\}.$$

The discrete problem of (4.1) is: find $u_h \in V_h$, such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \tag{4.3}$$

where $a_h(u_h, v_h) = \sum_{K\in\mathcal{T}_h} \int_K A(u_h, v_h)\,\mathrm{d}x\,\mathrm{d}y$.

Set

$$|\cdot|_h = \left(\sum_{K\in\mathcal{T}_h} |\cdot|_{2,K}^2\right)^{1/2}.$$

It is easy to prove that $|\cdot|_h$ is a norm of V_h, so the discrete problem (4.2) has an unique solution by Lax–Milgram Lemma [4].

Let u and u_h be the solutions of (4.1) and (4.3), respectively, by Strang's Lemma [4,10],

$$|u - u_h|_h \leqslant \hat{c}\left(\inf_{v_h\in V_h} |u - v_h|_h + \sup_{w_h\in V_h} \frac{|a_h(u, w_h) - (f, w_h)|}{|w_h|_h}\right). \tag{4.4}$$

The finite element interpolation operator $I_h : H^3(\Omega) \cap H_0^2(\Omega) \to V_h$ is defined by

$$I_h|_K \triangleq I_K, \quad \forall K \in \mathcal{T}_h.$$

Let $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = 2$, $h_K^\alpha = h_{K1}^{\alpha_1} h_{K2}^{\alpha_2}$. Obviously, $\hat{D}^\alpha \hat{u} = h_K^\alpha D^\alpha u$, then

$$\begin{aligned} \inf_{v_h\in V_h} |u - v_h|_h &\leqslant |u - I_h u|_h \\ &= \left(\sum_{K\in\mathcal{T}_h} \sum_{|\alpha|=2} h_K^{-2\alpha}(h_{K1}h_{K2})\|\hat{D}^\alpha(\hat{u} - \hat{I}\hat{u})\|_{0,\hat{K}}^2\right)^{1/2} \\ &\overset{(3.12)}{\leqslant} \hat{c}\left(\sum_{K\in\mathcal{T}_h} \sum_{|\beta|=1} h_K^{2\beta}|D^\beta u|_{2,K}^2\right)^{1/2}. \end{aligned} \tag{4.5}$$

Now we are in a position to estimate the consistence error.

Let Π_h be the piecewise bilinear interpolation operator on Ω, $\Pi_h|_K = \Pi_K$, $\Pi_K v = \hat{\Pi}\hat{v} \circ F_K^{-1}$, $\hat{\Pi}$ is the bilinear interpolation operator on $\hat{K}$. It is easy to see that $\hat{\Pi}$ is anisotropic for $|\alpha| = 1$, and from Theorem 2.1 we have

$$\|v - \Pi_K v\|_{0,K} \leqslant \hat{c} h_K^2 |v|_{2,K}, \quad |v - \Pi_K v|_{1,K} \leqslant \hat{c} h_K |v|_{2,K}, \quad \forall v \in H^2(K). \tag{4.6}$$

Obviously $\forall w_h \in V_h$, $\Pi_h w_h \in C_0^0(\overline{\Omega})$, by Green's Formula [10],

$$f(\Pi_h w_h) = \int_\Omega f \Pi_h w_h = \int_\Omega \triangle^2 u \Pi_h w_h = -\int_\Omega \nabla\triangle u \cdot \nabla \Pi_h w_h. \tag{4.7}$$

The well-known result [12,18] gives

$$a_h(u, w_h) = -\sum_{K\in\mathcal{T}_h} \int_K \nabla\triangle u \cdot \nabla w_h + E_1(u, w_h) + E_2(u, w_h), \tag{4.8}$$

where

$$\begin{cases} E_1(u,w_h)=\sum_{K\in\mathscr{T}_h}\int_{\partial K}[\triangle u-(1-\sigma)u_{ss}]\dfrac{\partial w_h}{\partial n}\,\mathrm{d}s,\\ E_2(u,w_h)=\sum_{K\in\mathscr{T}_h}\int_{\partial K}(1-\sigma)u_{sn}\dfrac{\partial w_h}{\partial s}\,\mathrm{d}s,\end{cases} \tag{4.9}$$

where $(\cdot)_s=\frac{\partial}{\partial s}$, $(\cdot)_n=\frac{\partial}{\partial n}$, are the tangential and normal derivatives along element boundaries, respectively.

Combining (4.7) and (4.8) yields

$$\begin{aligned} a_h(u,w_h)-f(w_h)=&-\sum_{K\in\mathscr{T}_h}\left[\int_K\nabla\triangle u\cdot\nabla(w_h-\Pi_K w_h)+\int_K f(w_h-\Pi_K w_h)\right]\\ &+E_1(u,w_h)+E_2(u,w_h). \end{aligned} \tag{4.10}$$

From (4.6) we have

$$\begin{aligned} \left|\int_K\nabla\triangle u\cdot\nabla(w_h-\Pi_K w_h)\right| &\leqslant|\Delta u|_{1,K}|w_h-\Pi_K w_h|_{1,K}\\ &\leqslant\hat{c}h_K|\Delta u|_{1,K}|w_h|_{2,K} \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} \left|\int_K f(w_h-\Pi_K w_h)\right| &\leqslant\|f\|_{0,K}\|w_h-\Pi_K w_h\|_{0,K}\\ &\leqslant\hat{c}h_K^2\|f\|_{0,K}|w_h|_{2,K}. \end{aligned} \tag{4.12}$$

By the definition of the element, it is easy to see that $\forall l_i\in\partial K, w_h\in V_h, \frac{\partial w_h}{\partial n}|_{l_i}\in P_1(l_i)$, $\frac{\partial w_h}{\partial n}$ is continuous at the middle point of l_i, so $\int_{l_i}\frac{\partial w_h}{\partial n}$ is continuous between elements and is zero on $\partial\Omega$. Set

$$P_i w=\frac{1}{|l_i|}\int_{l_i}w\,\mathrm{d}s,$$

$U_1=U_3=\triangle u-(1-\sigma)u_{xx}$ and $U_2=U_4=\triangle u-(1-\sigma)u_{yy}$, we have

$$E_1(u,w_h)=\sum_{K\in\mathscr{T}_h}\sum_{i=1}^4\int_{l_i}U_i\left(\frac{\partial w_h}{\partial n}-P_i\frac{\partial w_h}{\partial n}\right)\mathrm{d}s\triangleq\sum_{K\in\mathscr{T}_h}\sum_{i=1}^4 I_i.$$

From the construction of the element we know that $\frac{\partial^2 w_h}{\partial x\partial y}$ is a constant, hence by the skill in [13,17], we can get

$$I_1+I_3=\int_{x_K-h_{K1}}^{x_K+h_{K1}}\left(w(x)\int_{y_K-h_{K2}}^{y_K+h_{K2}}\frac{\partial U_1(x,y)}{\partial y}\,\mathrm{d}y\right)\mathrm{d}x,$$

where

$$\begin{aligned} |w(x)|&=\frac{1}{4h_{K1}h_{K2}}\left|\int_{x_K-h_{K1}}^{x_K+h_{K1}}\left(\int_t^x\int_{y_K-h_{K2}}^{y_K+h_{K2}}\frac{\partial^2 w_h}{\partial r\partial y}(r,y)\,\mathrm{d}r\,\mathrm{d}y\right)\mathrm{d}t\right|\\ &\leqslant\sqrt{\frac{h_{K1}}{h_{K2}}}|w_h|_{2,K}, \end{aligned} \tag{4.13}$$

from which we have

$$|I_1+I_3|\leqslant 2h_{K1}\left(|\Delta u|_{1,K}+(1-\sigma)\left\|\frac{\partial}{\partial x}(u_{xy})\right\|_{0,K}\right)|w_h|_{2,K}.$$

Similarly,

$$|I_2 + I_4| \leqslant 2h_{K2} \left(|\Delta u|_{1,K} + (1-\sigma) \left\| \frac{\partial}{\partial y}(u_{xy}) \right\|_{0,K} \right) |w_h|_{2,K}.$$

Hence

$$|E_1(u, w_h)| \leqslant \hat{c} \left(\sum_{K \in \mathscr{T}_h} (h_K^2 |\Delta u|_{1,K}^2 + \sum_{|\alpha|=1} h_K^{2\alpha} |D^\alpha u|_{2,K}^2) \right)^{1/2} |w_h|_h. \tag{4.14}$$

In the same way, we get

$$|E_2(u, w_h)| \leqslant c \left(\sum_{K \in \mathscr{T}_h} \sum_{|\alpha|=1} h_K^{2\alpha} |D^\alpha u|_{2,K}^2 \right)^{1/2} |w_h|_h. \tag{4.15}$$

Substituting (4.11), (4.12), (4.14) and (4.15) into (4.10), we have

$$\sup_{w_h \in V_h} \frac{|a_h(u, w_h) - f(w_h)|}{|w_h|_h}$$
$$\leqslant \hat{c} \left(\sum_{K \in \mathscr{T}_h} \left(h_K^2 |\Delta u|_{1,K}^2 + \sum_{|\alpha|=1} h_K^{2\alpha} |D^\alpha u|_{2,K}^2 \right) \right)^{1/2} |w_h|_h. \tag{4.16}$$

A combination of (4.5) and (4.16) yields the main result of this section.

Theorem 4.1. *Using rectangular* 8-12-2 *element to solve the plate bending problem*, *we have*

$$|u - u_h|_h \leqslant \hat{c} \left(\sum_{K \in \mathscr{T}_h} \left(h_K^2 |\Delta u|_{1,K}^2 + \sum_{|\alpha|=1} h_K^{2\alpha} |D^\alpha u|_{2,K}^2 + h_K^4 \|f\|_{0,K}^2 \right) \right)^{1/2}, \tag{4.17}$$

where u and u_h are the solutions of (4.1) and (4.2), *respectively*.

Remark 4.1. (1) The classical method to estimate the consistence error [12] is directly based on (4.9), using coordinate transformation, interpolation theory and trace theorem, through $\partial K \to \partial \hat{K} \to \hat{K} \to K$, then (4.14) and (4.15) are obtained. During $\partial K \to \partial \hat{K} \to \hat{K} \to K$ the factor $h_{Ki}(h_{K1}h_{K2})^{-1/2}, i = 1, 2$ appears due to the Jacobian determinants. This makes the constant in (4.14) and (4.15) dependent on h_K/ρ_K.

(2) Getting the consistence error $O(h)$ by the classical method [12] should suppose $u \in H^4(\Omega)$, here we only suppose $u \in H^3(\Omega)$, this argument came from [20].

5. Anisotropic superconvergence of 8-12-2 element

In this section, we discuss the superconvergence behavior of 8-12-2 element for the biharmonic equation with anisotropic meshes.

We consider the following biharmonic equation

$$\begin{cases} \triangle^2 u = f & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Its weak form is taken as: find $u \in H_0^2(\Omega)$ such that

$$\widetilde{a}(u, v) = (f, v), \quad \forall v \in H_0^2(\Omega), \tag{5.2}$$

where

$$\widetilde{a}(u,v)=\int_\Omega (u_{xx}v_{xx}+2u_{xy}v_{xy}+u_{yy}v_{yy})\,\mathrm{d}x\,\mathrm{d}y. \tag{5.3}$$

Let V_h be the finite element space of 8-12-2 element, then the discrete problem of (5.2) is: find $u_h\in V_h$ such that

$$\widetilde{a}_h(u_h,v_h)=(f,v_h),\quad \forall v_h\in V_h, \tag{5.4}$$

where $\widetilde{a}_h(u_h,v_h)=\sum_{K\in\mathscr{T}_h}\int_K(u_{hxx}v_{hxx}+2u_{hxy}v_{hxy}+u_{hyy}v_{hyy})\,\mathrm{d}x\,\mathrm{d}y$.

First we prove that the consistence error for (5.4) is $O(h^2)$. Similar to (4.20), we can easily prove that [12]: for any $w_h\in V_h$,

$$\widetilde{a}_h(u,w_h)-(f,w_h)=\widetilde{E}_1(u,w_h)+\widetilde{E}_2(u,w_h)+\widetilde{E}_3(u,w_h), \tag{5.5}$$

where $\widetilde{E}_1(u,w_h)=\sum_{K\in\mathscr{T}_h}\int_{\partial K}u_{ss}\frac{\partial w_h}{\partial n}\,\mathrm{d}s$, $\widetilde{E}_2(u,w_h)=\sum_{K\in\mathscr{T}_h}\int_{\partial K}u_{sn}\frac{\partial w_h}{\partial s}\,\mathrm{d}s$, $\widetilde{E}_3(u,w_h)=\sum_{K\in\mathscr{T}_h}\int_{\partial K}-\frac{\partial\Delta u}{\partial n}w_h\,\mathrm{d}s$.

Put $U=u_{ss}$, then we have

$$\widetilde{E}_1(u,w_h)=\sum_{K\in\mathscr{T}_h}\sum_{i=1}^4\int_{l_i}(U-P_iU)\left(\frac{\partial w_h}{\partial n}-P_i\frac{\partial w_h}{\partial n}\right)\mathrm{d}s=\sum_{K\in\mathscr{T}_h}\sum_{i=1}^4\tilde{I}_i,$$

where

$$\begin{aligned}\tilde{I}_1+\tilde{I}_3&=\int_{x_K-h_{K1}}^{x_K+h_{K1}}\left[-(U-P_1U)\left(\frac{\partial w_h}{\partial y}-P_1\frac{\partial w_h}{\partial y}\right)(x,y_K-h_{K2})\right.\\&\qquad\left.+(U-P_3U)\left(\frac{\partial w_h}{\partial y}-P_3\frac{\partial w_h}{\partial y}\right)(x,y_K+h_{K2})\right]\mathrm{d}x\\&=\frac{1}{2h_{K1}}\int_{x_K-h_{K1}}^{x_K+h_{K1}}w(x)Q(x)\,\mathrm{d}x,\end{aligned} \tag{5.6}$$

here

$$\begin{aligned}|Q(x)|&=\left|\int_{x_K-h_{K1}}^{x_K+h_{K1}}\left(\int_t^x\int_{y_K-h_{K2}}^{y_K+h_{K2}}\frac{\partial^2U}{\partial r\partial y}(r,y)\,\mathrm{d}r\,\mathrm{d}y\right)\mathrm{d}t\right|\\&\leqslant\hat{c}h_{K1}\sqrt{h_{K1}h_{K2}}\left|\frac{\partial^2u}{\partial x^2}\right|_{2,K}.\end{aligned}$$

Substituting the above estimate and (4.13) into (5.6) yields

$$\tilde{I}_1+\tilde{I}_3\leqslant\hat{c}h_{K1}^2\left|\frac{\partial^2u}{\partial x^2}\right|_{2,K}|w_h|_{2,K}.$$

Similarly,

$$\tilde{I}_2+\tilde{I}_4\leqslant\hat{c}h_{K2}^2\left|\frac{\partial^2u}{\partial y^2}\right|_{2,K}|w_h|_{2,K}.$$

Then

$$|\tilde{E}_1(u,w_h)|\leqslant\hat{c}\left(\sum_{K\in\mathscr{T}_h}\sum_{|\alpha|=2}h_K^{2\alpha}|D^\alpha u|_{2,K}^2\right)^{1/2}|w_h|_h. \tag{5.7}$$

Following the same argument, we can prove

$$|\tilde{E}_2(u,w_h)|\leqslant\hat{c}\left(\sum_{K\in\mathscr{T}_h}\sum_{|\alpha|=2}h_K^{2\alpha}|D^\alpha u|_{2,K}^2\right)^{1/2}|w_h|_h. \tag{5.8}$$

Similarly,

$$\tilde{E}_3(u, w_h) = \sum_{K\in\mathscr{T}_h}\sum_{i=1}^{4}\int_{l_i} -\frac{\partial\Delta u}{\partial n}(w_h - \Pi_K w_h)\,\mathrm{d}s = \sum_{K\in\mathscr{T}_h}\sum_{i=1}^{4} L_i,$$

where

$$\begin{aligned}|L_1 + L_3| &= \left|\int_{x_K-h_{K1}}^{x_K+h_{K1}} \left(\frac{\partial\Delta u}{\partial y}(x, y_K + h_{K2}) - \frac{\partial\Delta u}{\partial y}(x, y_K - h_{K2})\right)(w_h - \Pi_K w_h)(x, y_K + h_{K2})\,\mathrm{d}x\right|\\ &\leqslant \hat{c} h_K^2 |\Delta u|_{2,K} |w_h|_{2,K}.\end{aligned}$$

As for $L_2 + L_4$, we have the same results, thus

$$|\tilde{E}_3(u, w_h)| \leqslant \hat{c}\left(\sum_{K\in\mathscr{T}_h} h_K^4 |\Delta u|_{2,K}^2\right)^{1/2} |w_h|_h. \tag{5.9}$$

Then a collection of (5.7)–(5.9) and (5.5) gives

$$\begin{aligned}&|\widetilde{a}_h(u, w_h) - (f, w_h)|\\ &\quad\leqslant \hat{c}\left(\sum_{K\in\mathscr{T}_h}\left(h_K^4|\Delta u|_{2,K}^2 + \sum_{|\alpha|=2} h_K^{2\alpha}|D^\alpha u|_{2,K}^2\right)\right)^{1/2} |w_h|_h.\end{aligned} \tag{5.10}$$

Remark 5.1. The well known nonconforming rectangular plate element with 12 node parameters is Adini's C^0 element [12], but its consistence error is $\mathrm{O}(h^2)$ only for uniform meshes, i.e., all the elements are equal and regular condition (1.1) holds.

Now we prove the following anisotropic superclose result.

Theorem 5.1. *Suppose $u \in H^4(\Omega)$, we have*

$$|I_h u - u_h|_h \leqslant \hat{c}\left(\sum_{K\in\mathscr{T}_h}\left(h_K^4|\Delta u|_{2,K}^2 + \sum_{|\alpha|=2} h_K^{2\alpha}|D^\alpha u|_{2,K}^2\right)\right)^{1/2}. \tag{5.11}$$

Proof. First we prove

$$|\widetilde{a}_h(u - I_h u, v_h)| \leqslant \hat{c}\left(\sum_{K\in\mathscr{T}_h}\sum_{|\alpha|=2} h_K^{2\alpha}|D^\alpha u|_{2,K}^2\right)^{1/2} |v_h|_h, \quad \forall v_h \in V_h. \tag{5.12}$$

By the expression of $\widetilde{a}_h(\cdot,\cdot)$, it is only need to prove the following three inequalities,

$$\int_K (u - I_h u)_{xx} v_{hxx}\,\mathrm{d}x\,\mathrm{d}y \leqslant \hat{c}\left(\sum_{|\alpha|=2} h_K^{2\alpha}|D^\alpha u|_{2,K}^2\right)^{1/2} |v_h|_{2,K}, \tag{5.13}$$

$$\int_K (u - I_h u)_{yy} v_{hyy}\,\mathrm{d}x\,\mathrm{d}y \leqslant \hat{c}\left(\sum_{|\alpha|=2} h_K^{2\alpha}|D^\alpha u|_{2,K}^2\right)^{1/2} |v_h|_{2,K}, \tag{5.14}$$

$$\int_K (u - I_h u)_{xy} v_{hxy}\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{5.15}$$

Firstly, by the scaling argument,

$$\int_K (u - I_h u)_{xx} v_{hxx}\,\mathrm{d}x\,\mathrm{d}y = h_{K1}^{-4}(4h_{K1}h_{K2}) \int_{\hat{K}} (\hat{u}_{\xi\xi} - (\hat{I}\hat{u})_{\xi\xi})\hat{v}_{h\xi\xi}\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{5.16}$$

Put $\hat{w} = \hat{u}_{\xi\xi}$, then from (3.13) and (3.14), we have

$$\hat{u}_{\xi\xi} - (\hat{I}\hat{u})_{\xi\xi} = \hat{w} - F_1(\hat{w}) - F_2(\hat{w})\xi \triangleq l(\hat{w}).$$

It is easy to see that

$$l(\hat{w}) = 0, \quad \forall \hat{w} \in P_1(\hat{K}).$$

Thus by Bramble–Hilbert Lemma [4,10],

$$\begin{aligned} |l(\hat{w})| &\leqslant \hat{c}\|\hat{v}_{h\xi\xi}\|_{0,\hat{K}} |\hat{w}|_{2,\hat{K}} = \hat{c}\|\hat{v}_{h\xi\xi}\|_{0,\hat{K}} |\hat{u}_{\xi\xi}|_{2,\hat{K}} \\ &\leqslant \hat{c} h_{K1}^4 (4h_{K1}h_{K2})^{-1} \left(\sum_{|\alpha|=2} h_K^{2\alpha} \|D^\alpha u_{xx}\|_{0,K}^2 \right)^{1/2} |v_h|_{2,K}. \end{aligned}$$

Substituting the above result into (5.16) implies (5.13).

Similarly, (5.14) can be proved.

Since v_{hxy} is a constant on K, then $\int_K (u - I_h u)_{xy} v_{hxy}\,\mathrm{d}x\,\mathrm{d}y = 0$. Thus we can obtain (5.12).

Finally,

$$\begin{aligned} |I_h u - u_h|_h^2 &= \widetilde{a}_h(I_h u - u_h, I_h u - u_h) \\ &= \widetilde{a}_h(I_h u - u, I_h u - u_h) + \widetilde{a}_h(u, I_h u - u_h) - (f, I_h u - u_h) \\ &\overset{(5.12)(5.10)}{\leqslant} \hat{c} \left(\sum_{K \in \mathcal{T}_h} \left(h_K^4 |\Delta u|_{2,K}^2 + \sum_{|\alpha|=2} h_K^{2\alpha} |D^\alpha u|_{2,K}^2 \right) \right)^{1/2} |I_h u - u_h|_h, \end{aligned}$$

which completes the proof of the theorem. □

Now we will discuss the natural superconvergence results about the second order derivatives of 8-12-2 element.

Theorem 5.2. *Under the assumption in Theorem* 5.1, *we have the following anisotropic superconvergence results at the central points*,

$$\begin{aligned} &\left(\sum_{K \in \mathcal{T}_h} \sum_{|\alpha|=2} |D^\alpha (u - u_h)(x_K, y_K)|^2 h_{K1} h_{K2} \right)^{1/2} \\ &\leqslant \hat{c} \left(\sum_{K \in \mathcal{T}_h} \left(h_K^4 |\Delta u|_{2,K}^2 + \sum_{|\alpha|=2} h_K^{2\alpha} |D^\alpha u|_{2,K}^2 \right) \right)^{1/2}. \end{aligned} \tag{5.17}$$

Proof. First, we focus on $\alpha = (2, 0)$, due to the triangle inequality,

$$|D^{(2,0)}(u - u_h)(x_K, y_K)|^2 \leqslant 2(|D^{(2,0)}(u - I_h u)(x_K, y_K)|^2 + |D^{(2,0)}(I_h u - u_h)(x_K, y_K)|^2). \tag{5.18}$$

By the scaling technique,

$$\begin{aligned} |D^{(2,0)}(u - I_h u)(x_K, y_K)| &\overset{(3.8)}{=} h_{K1}^{-2} |\hat{D}^{(2,0)}(\hat{u} - \hat{I}\hat{u})(0,0)| \\ &\overset{(3.13)(3.14)}{=} h_{K1}^{-2} |\widetilde{l}(\hat{D}^{(2,0)}\hat{u})|, \end{aligned}$$

where $\widetilde{l}(\hat{w}) = \hat{w}(0,0) + 2F_1(\hat{w})$.

From (3.14) it can be easily checked that for all $\hat{w} \in P_1(\hat{K}), \widetilde{l}(\hat{w}) = 0$, then

$$|\widetilde{l}(\hat{w})| \leqslant \hat{c}|\hat{w}|_{2,\hat{K}}, \quad \forall \hat{w} \in H^2(\hat{K})$$

and

$$|D^{(2,0)}(u - I_h u)(x_K, y_K)| \overset{(3.8)(3.11)}{\leqslant} \hat{c}(h_{K1}h_{K2})^{-1/2} \sum_{|\beta|=2} h_K^{\beta} |D^{\beta} u|_{2,K}. \tag{5.19}$$

Thanks to $\hat{D}^{(2,0)}(\hat{I}\hat{u} - \hat{u}_h) \in P_1(\hat{K})$ and the equivalent norms on the finite dimensional space, we have

$$|D^{(2,0)}(I_h u - u_h)(x_K, y_K)| \overset{(3.8)(3.11)}{\leqslant} \hat{c}(h_{K1}h_{K2})^{-1/2}|Iu - u_h|_{2,K}. \tag{5.20}$$

Substituting (5.19), (5.20) and (5.11) into (5.18), we obtain

$$\left(\sum_{K\in\mathscr{T}_h} |D^{(2,0)}(u - u_h)(x_K, y_K)|^2 h_{K1}h_{K2}\right)^{1/2}$$
$$\leqslant \hat{c}\left(\sum_{K\in\mathscr{T}_h}\left(h_K^4|\Delta u|_{2,K}^2 + \sum_{|\alpha|=2} h_K^{2\alpha}|D^{\alpha}u|_{2,K}^2\right)\right)^{1/2}. \tag{5.21}$$

Similarly, we can get the same results for $\alpha = (0.2)$ and $\alpha = (1, 1)$. Hence (5.17) holds. □

Furthermore, we can obtain the global superconvergence results of 8-12-2 element by virtue of a proper postprocessing technique. For simplicity, we assume that the mesh J_h is obtained by dividing every element K of the coarser mesh J_{3h} into 9 congruent elements $K_1, K_2, \ldots, K_9$, the vertices of $K_1, K_2, \ldots, K_9$ are denote by $Z_{ij}, i, j = 1, 2, 3, 4$. We consider the conventional bicubic Lagrange interpolation operator $\Pi_{3h}^3 : H^2(K) \longrightarrow Q_3(x, y)$ characterized by

$$\Pi_{3h}^3 u(Z_{ij}) = u(Z_{ij}), \quad i, j = 1, 2, 3, 4,$$

where $Q_3(x, y)$ is the space of all polynomials which are of degree $\leqslant 3$ with respect to x and y, respectively.

According to [2], we have

$$|\Pi_{3h}^3 u - u|_{2,\Omega} \leqslant \hat{c}\left(\sum_{K\in\mathscr{T}_h}\sum_{|\alpha|=2} h_K^{2\alpha}|D^{\alpha}u|_{2,K}^2\right)^{1/2}. \tag{5.22}$$

Obviously,

$$|\beta_i| \leqslant \hat{c}\|\hat{D}^{\alpha}\hat{v}\|_{2,\hat{K}} \leqslant \hat{c}\|\hat{D}^{\alpha}\hat{v}\|_{0,\hat{K}}, \quad \forall v \in V_h,$$

where the inverse inequality [1] is used. Then

$$\|D^{\alpha}\Pi_{3h}^3 v\|_{0,K} = h_K^{-\alpha}h_{K1}h_{K2}\|\hat{D}^{\alpha}\hat{\Pi}^3 v\|_{0,\hat{K}}$$
$$\leqslant \hat{c}h_K^{-\alpha}h_{K1}h_{K2}\sum_{i=1}^{r}|\beta_i| \leqslant \hat{c}\|D^{\alpha}v\|_{0,K}, \quad \forall v \in V_h.$$

Hence

$$\|\Pi_{3h}^3 v\|_h = \left(\sum_{K\in\mathscr{T}_h}\sum_{|\alpha|=2}\|D^{\alpha}\Pi_{3h}^3 v\|_{0,K}^2\right)^{1/2} \leqslant \hat{c}\|v\|_h, \quad \forall v \in V_h. \tag{5.23}$$

Then we can get the following superconvergence theorem following the standard technique, cf. [5,14].

Theorem 5.3. *Under the same assumptions as in Theorem* 5.1, *we have*

$$|u - \Pi_{3h}^3 u_h|_h \leqslant \hat{c} \left(\sum_{K \in \mathscr{T}_h} \left(h_K^4 |\Delta u|_{2,K}^2 + \sum_{|\alpha|=2} h_K^{2\alpha} |D^\alpha u|_{2,K}^2 \right) \right)^{1/2}. \tag{5.24}$$

6. Numerical experiments

In order to examine the numerical performance of the above element for anisotropic rectangular meshes, we carry out numerical tests for the following two models:

Model 1. The classical unit square plate bending problem with clamped boundaries under a uniform load. The Poisson ratio is chosen $\sigma = 0.3$ and $f = 1$. The analytic value of deflection at the center is 0.00126532, the analytic value of bending moment at the center is 0.022905. This experiment is used to investigate the convergence for the classical plate bending problem under anisotropic meshes.

Model 2. A biharnomic differential equation with f chosen such that the exact solution of problem (4.2) is $u(x, y) = \sin^2(\pi x)\sin^2(\pi y)$. This experiment is used to investigate the convergence and superconvergence for an ordinary biharmonic problem under anisotropic meshes.

In order to obtain anisotropic meshes, the unit square $\Omega = [0, 1] \times [0, 1]$ is subdivided in the following fashion: each edge of Ω is divided into n segments with $n + 1$ points $(1 - \cos(\frac{i\pi}{n}))/2$, $i = 0, 1, \ldots, \frac{n}{2}$, $(1 + \sin(\frac{i\pi}{n} - \frac{\pi}{2}))/2$, $i = \frac{n}{2} + 1, \ldots, n$. The mesh obtained in this way for $n = 16$ is illustrated in Fig. 1. The aspect ratio of this mesh is demonstrated by Table 1.

For Model 1, Fig. 2 gives the deflection error and the moment error at the central point (i.e., $|(u - u_h)(\frac{1}{2}, \frac{1}{2})|$, $|(M - M_h)(\frac{1}{2}, \frac{1}{2})|$), which shows the anisotropic convergence of 8-12-2 element for model 1.

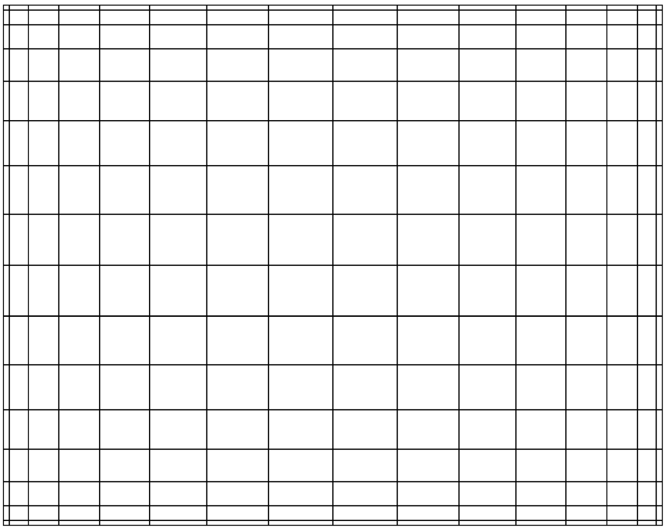

Fig. 1. The anisotropic mesh for the case $n = 16$.

Table 1
The aspect ratio of mesh 2

$n \times n$	16×16	32×32	64×64	128×128
$\max_{K \in \mathscr{T}_h} \{h_K/\rho_K\}$	14.358751	28.786978	57.608674	115.234703
$\max_{K \in \mathscr{T}_h} \{h/h_K\}$	10.53170	20.355408	40.735484	81.483240

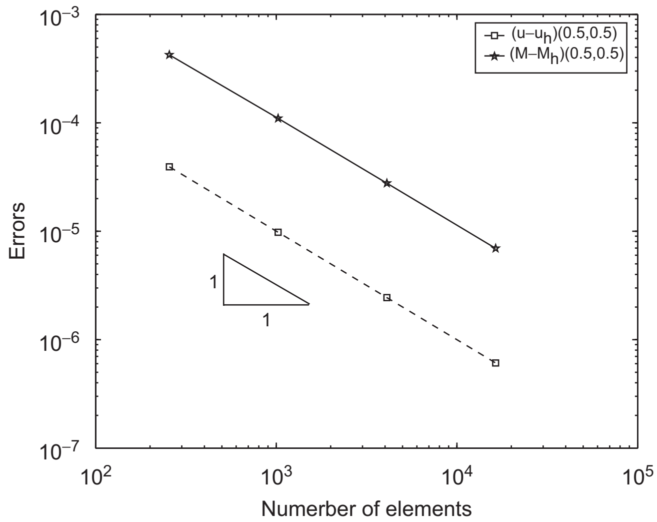

Fig. 2. The errors for model 1.

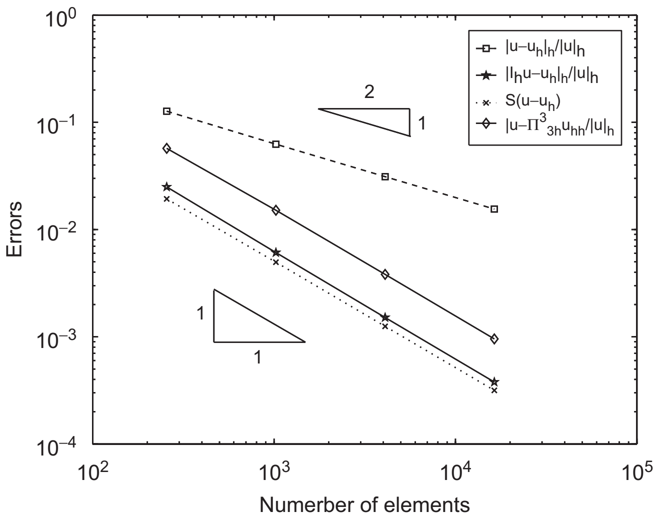

Fig. 3. The errors for model 2.

For Model 2, we compute the errors $|u - u_h|_h/|u|_h$, $|I_h u - u_h|_h/|u|_h$, $S(u - u_h) \triangleq (\sum_{K\in\mathscr{T}_h} \sum_{|\alpha|=2} |D^\alpha(u - u_h)(x_K, y_K)|^2 h_{K1}h_{K2})^{1/2}/|u|_h$ and $|u - \Pi^3_{3h}u_h|_h/|u|_h$. The numerical results are listed in Fig. 3. These results agree well with the theoretical analysis.

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