Journal of Algebra 370 (2012) 113–132
Contents lists available at SciVerse ScienceDirect
Journal of Algebra
www.elsevier.com/locate/jalgebra

Derived equivalences between matrix subrings and their applications

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ARTICLE INFO

Article history:
Received 5 September 2011
Available online 9 August 2012
Communicated by Michel Van den Bergh

MSC:
18G20
18E30
18G05

Keywords:
Derived category
Derived equivalence
Finitistic dimension
Global dimension

ABSTRACT

In this paper, we construct derived equivalences between matrix subrings. As applications, we calculate the global dimensions and the finitistic dimensions of some matrix subrings. And we show that the finitistic dimension conjecture holds for a class of Harada algebras and a class of tiled triangular algebras.

1. Introduction

Derived equivalences preserve many homological properties of algebras such as the number of simple modules, the finiteness of global dimension and finitistic dimension, the algebraic K-theory and Hochschild (co)homological groups (see [14,4,10,18,19,17]). Thus, in order to study some homological properties of a given algebra, we can turn to the one which is derived equivalent to it.

Recently, Hu and Xi have exhibited derived equivalent endomorphism rings induced by $\mathcal{D}$-split sequences. We find that $\mathcal{D}$-split sequences give a way to construct derived equivalences between matrix subrings. In this paper, we will study the derived equivalences having a characteristic that one of two rings has relatively simple structure.

As applications, we first investigate the global dimension of a matrix subring. By the definition of global dimension, Kirkman and Kuzmanovich in [12] have calculated the global dimensions of some

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0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2012.07.024

matrix subrings. Cowley extended some of their results by triangular decomposition [3]. As never before, we investigate some cases by the method of derived equivalences.

Second, we study the finitistic dimension of a matrix subring. For a ring A, the finitistic dimensions are defined as follows: $l.\mathrm{Fin.dim}(A)$ is the supremum of the projective dimensions of left A-modules of finite projective dimensions, and $\mathrm{fin.dim}(A)$ is the supremum of the projective dimensions of finitely generated left A-modules of finite projective dimensions. Kirkman, Kuzmanovich and Small compute $l.\mathrm{Fin.dim}(A)$ for a noncommutative noetherian ring A in [13]. By derived equivalences, we calculate $l.\mathrm{Fin.dim}(A)$ for a matrix subring A. This result is helpful to study the finitistic dimension conjecture which states that for an Artin algebra A, $\mathrm{fin.dim}(A)$ is finite. This conjecture is still open. We refer the reader to [20] on some new advances on this conjecture.

Little is known about whether the finitistic dimension conjecture holds for matrix subalgebras. Note that the Artin algebra A and the matrix algebra $M_n(A)$ are Morita equivalent. Thus, in order to prove that $\mathrm{fin.dim}(A)$ is finite, it is equivalent to prove that $\mathrm{fin.dim}(M_n(A))$ is finite. Our ideal in this direction is to investigate the finitistic dimension of a matrix subalgebra. If the finitistic dimension of A is finite, what could we say about the finitistic dimension of a matrix subalgebra?

In order to describe the main result precisely, we fix some notation.

Let A be a noetherian ring with identity. Let A_i $(2 \leqslant i \leqslant n)$ be a family of subrings of A with the same identity with A, and let $I_i, I_{i,j}$, $2 \leqslant i \leqslant n$, $2 \leqslant j \leqslant n-1$ be ideals of A satisfying that $I_n \subseteq I_{n-1} \subseteq \cdots \subseteq I_2$, $I_i \subseteq A_i$, $I_j \subseteq I_{i,j}$, $\sum_{l=j+1}^{i-1} I_{i,l} I_{l,j} \subseteq I_{i,j}$, $i \neq j$, $2 \leqslant i \leqslant n$, $2 \leqslant j \leqslant n-1$. In this way, we can construct two rings

$$\Lambda = \begin{pmatrix} A & I_2 & I_3 & \cdots & I_{n-1} & I_n \\ A & A_2 & I_3 & \cdots & I_{n-1} & I_n \\ A & I_{3,2} & A_3 & \cdots & I_{n-1} & I_n \\ A & I_{4,2} & I_{4,3} & A_4 & \cdots & I_n \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ A & I_{n,2} & \cdots & \cdots & I_{n,n-1} & A_n \end{pmatrix} \quad \text{and}$$

$$\Sigma = \begin{pmatrix} A_2/I_2 & 0 & & \cdots & & 0 \\ I_{3,2}/I_2 & A_3/I_3 & \ddots & & & \vdots \\ I_{4,2}/I_2 & I_{4,3}/I_3 & A_4/I_4 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ I_{n,2}/I_2 & I_{n,3}/I_3 & I_{n,4}/I_4 & \cdots & A_n/I_n & 0 \\ A/I_2 & A/I_3 & A/I_4 & \cdots & A/I_n & A \end{pmatrix}$$

with identities. Unless other stated, throughout this paper, Λ and Σ are rings of this forms.

The main result in this paper is the following:

Theorem 1.1. *The two rings Λ and Σ are derived equivalent.*

As a direct consequence of Theorem 1.1, we have the following corollary.

Corollary 1.2.

(1) *Let Λ be as in Theorem 1.1. Then*

$$l.\mathrm{Fin.dim}(A) - 1 \leqslant l.\mathrm{Fin.dim}(\Lambda) \leqslant n + \sum_{i=2}^{n} l.\mathrm{Fin.dim}(A_i/I_i) + l.\mathrm{Fin.dim}(A).$$

(2) *Let Λ be as in Theorem 1.1. Then*

$$\max\left\{l.\mathrm{gl.dim}(A_i/I_i), l.\mathrm{gl.dim}(A), 2 \leqslant i \leqslant n\right\} - 1$$
$$\leqslant l.\mathrm{gl.dim}(\Lambda) \leqslant \sum_{i=2,3,\ldots,n} l.\mathrm{gl.dim}(A/I_i) + l.\mathrm{gl.dim}(A) + n.$$

We define a class of algebras which are called general block extension of a ring with respect to a decomposition of the identity. And we calculate their global dimensions and finitistic dimensions. We also get a class of Harada algebras and a class of tiled triangular rings which satisfy the finitistic dimension conjecture.

This paper is arranged as follows. In Section 2, we fix some notation and recall some definitions and lemmas needed in this paper. In Section 3, the proof of the main result is given. In Section 4, we give some applications of the main result. The definition of general block extension of a ring with respect to a decomposition of the identity is proposed. We calculate their global dimensions and finitistic dimensions. And we also get some classes of algebras which satisfy the finitistic dimension conjecture. In Section 5, we display some examples to illustrate the applications.

2. Preliminaries

In this section, we shall recall some basic definitions and results needed in this paper.

Let A be a ring with identity. We denote by A-Mod the category of left A-modules and by A-mod the category of all finitely generated left A-modules. Mod-A means the category of right A-modules. Given an A-module M, we denote by proj.dim(M) the projective dimension of M. The left global dimension of M, denoted by $l.\mathrm{gl.dim}(A)$, is the supremum of all proj.dim(M) with $M \in A$-Mod. By add(M), we shall mean the full subcategory of A-Mod, whose objects are direct summands of finite direct sums of copies of M. For two morphisms $f: X \to Y$ and $g: Y \to Z$ in A-Mod, the composition of f and g is written as fg, which is a morphism from X to Z.

Let A be a ring with identity. A *complex* $X^\bullet = (X^i, d_X^i)$ of A-modules is a sequence of A-modules and A-module homomorphisms $d_X^i : X^i \to X^{i+1}$ such that $d_X^i d_X^{i+1} = 0$ for all $i \in \mathbb{Z}$. A *morphism* $f^\bullet : X^\bullet \to Y^\bullet$ between two complexes $X^\bullet$ and $Y^\bullet$ is a collection of homomorphisms $f^i : X^i \to Y^i$ of A-modules such that $f^i d_Y^i = d_X^i f^{i+1}$. The morphism $f^\bullet$ is said to be *null-homotopic* if there exists a homomorphism $h^i : X^i \to Y^{i-1}$ such that $f^i = d_X^i h^{i+1} + h^i d_Y^{i-1}$ for all $i \in \mathbb{Z}$. A complex $X^\bullet$ is called *bounded below* if $X^i = 0$ for all but finitely many $i < 0$, *bounded above* if $X^i = 0$ for all but finitely many $i > 0$, and *bounded* if $X^\bullet$ is bounded below and above. We denote by $\mathscr{C}(A)$ (resp., $\mathscr{C}(A\text{-Mod})$) the category of complexes of finitely generated (resp., all) A-modules. The homotopy category $\mathscr{K}(A)$ is quotient category of $\mathscr{C}(A)$ modulo the ideals generated by null-homotopic morphisms. We denote the derived category of A-mod by $\mathscr{D}(A)$ which is the quotient category of $\mathscr{K}(A)$ with respect to the subcategory of $\mathscr{K}(A)$ consisting of all the acyclic complexes. The full subcategory of $\mathscr{K}(A)$ and $\mathscr{D}(A)$ consisting of bounded complexes over A-mod is denoted by $\mathscr{K}^b(A)$ and $\mathscr{D}^b(A)$, respectively. We denoted by $\mathscr{C}^+(A)$ the category of complexes of bounded below, and by $\mathscr{K}^+(A)$ the homotopy category of $\mathscr{C}^+(A)$. The full subcategory of $\mathscr{D}(A)$ consisting of bounded below complexes is denoted by $\mathscr{D}^+(A)$. Similarly, we have the category $\mathscr{C}^-(A)$ of complexes bounded above, the homotopy category $\mathscr{K}^-(A)$ of $\mathscr{C}^-(A)$ and the derived category $\mathscr{D}^-(A)$ of $\mathscr{C}^-(A)$. If we focus on the category of left A-modules, then we have the homotopy category $\mathscr{K}(A\text{-Mod})$ of $\mathscr{C}(A\text{-Mod})$ and the derived category $\mathscr{D}(A\text{-Mod})$ of A-Mod.

The following result, due to Rickard (see [18, Theorem 6.4]), is Morita theorem of derived categories of rings.

Lemma 2.1. *Let A and B be two rings. The following conditions are equivalent*:

(1) $\mathscr{K}^-(A\text{-proj})$ *and* $\mathscr{K}^-(B\text{-proj})$ *are equivalent as triangulated categories*;
(2) $\mathscr{D}^b(A\text{-Mod})$ *and* $\mathscr{D}^b(B\text{-Mod})$ *are equivalent as triangulated categories*;

(3) $\mathscr{K}^b(A\text{-Proj})$ *and* $\mathscr{K}^b(B\text{-Proj})$ *are equivalent as triangulated categories*;
(4) $\mathscr{K}^b(A\text{-proj})$ *and* $\mathscr{K}^b(B\text{-proj})$ *are equivalent as triangulated categories*;
(5) *B is isomorphic to* $\mathrm{End}_{\mathscr{D}^b(A)}(T^\bullet)$, *where $T^\bullet$ is a complex in* $\mathscr{K}^b(A\text{-proj})$ *satisfying*:
   (a) *$T^\bullet$ is self-orthogonal, that is,* $\mathrm{Hom}_{\mathscr{K}^b(A\text{-proj})}(T^\bullet, T^\bullet[i]) = 0$ *for all $i \neq 0$,*
   (b) $\mathrm{add}(T^\bullet)$ *generates* $\mathscr{K}^b(A\text{-proj})$ *as a triangulated category.*

Two rings A and B are called *derived equivalent* if the above conditions (1)–(5) are satisfied. A complex $T^\bullet$ in $K^b(A\text{-proj})$ as above is called a *tilting complex* over A. It is also equivalent to say that the two rings A and B are derived equivalent if and only if there exists a complex $X^\bullet$ in $D(A\text{-Mod})$, isomorphic to a complex in $K^b(A\text{-proj})$ which satisfies [Lemma 2.1(5), (a) and (b)], such that the two rings B and $\mathrm{End}_{D(A\text{-Mod})}(X^\bullet)$ are isomorphic. In particular, if the tilting complex $T^\bullet$ is isomorphic to a module T in $D^b(A)$, then T is called a *tilting module*.

In [9], Hu and Xi define the $\mathcal{D}$-split sequences which occur in many situations, for instance, Auslander–Reiten sequences.

Definition 2.2. (See [9, Definition 3.1].) Let $\mathcal{C}$ be an additive category and $\mathcal{D}$ a full subcategory of $\mathcal{C}$. A sequence

$$X \xrightarrow{f} M \xrightarrow{g} Y$$

in $\mathcal{C}$ is called a $\mathcal{D}$-split sequence if

(1) $M \in \mathcal{D}$;
(2) f is a left $\mathcal{D}$-approximation of X, and g is a right $\mathcal{D}$-approximation of Y;
(3) f is a kernel of g, and g is a cokernel of f.

A $\mathcal{D}$-split sequence implies a derived equivalence between two endomorphism algebras. The following theorem reveals how to construct derived equivalences from $\mathcal{D}$-split sequences.

Lemma 2.3. *(See [9, Theorem 3.5].) Let $\mathcal{C}$ be an additive category and M an object in $\mathcal{C}$. Suppose*

$$X \xrightarrow{f} M' \xrightarrow{g} Y$$

is an $\mathrm{add}(M)$*-split sequence in $\mathcal{C}$. Then the endomorphism ring* $\mathrm{End}_{\mathcal{C}}(X \oplus M)$ *of $X \oplus M$ and the endomorphism ring* $\mathrm{End}_{\mathcal{C}}(Y \oplus M)$ *of $Y \oplus M$ are derived equivalent.*

3. Results and proofs

To prove our results, we first establish a fact.

Lemma 3.1. *Let R be a ring with identity.*

(1) *Let M be a noetherian left R-module, and let $f : M \to M$ be a surjective homomorphism, then f is injective.*
(2) *Let M be an artinian left R-module, and let $f : M \to M$ be an injective homomorphism, then f is surjective.*

Proof. We only prove the first part of the lemma. The second part of the lemma is similar. Set $f^k = \overbrace{f \cdots f}^{k}$. Note that $\mathrm{Ker}\, f^k$ is a submodule of M and $\mathrm{Ker}\, f^i$ is a submodule of $\mathrm{Ker}\, f^{i+1}$ for any $i \geqslant 1$. Since M is a noetherian module, there exists $i_0 \geqslant 1$, satisfying $\mathrm{Ker}\, f^{i_0} = \mathrm{Ker}\, f^{i_0+1}$. Then we have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker} f^{i_0} \arrow[r] \arrow[d, "id"'] & M \arrow[r, "f^{i_0}"] \arrow[d, "id"'] & M \arrow[r] \arrow[d, "f"'] & 0 \\ 0 \arrow[r] & \operatorname{Ker} f^{i_0+1} \arrow[r] & M \arrow[r, "f^{i_0+1}"] & M \arrow[r] & 0 \end{tikzcd}$$

By the snake lemma, we can get $\operatorname{Ker} f = 0$. So f is injective. $\square$

Now, let us prove the main result in this paper.

Proof of Theorem 1.1. Set $\Gamma = M_n(A)$, the $n \times n$ matrix over A.

Denote by e_i the matrix which has 1_A in the (i,i)-th position and zeros elsewhere for $1 \leqslant i \leqslant n$. So $e_1, e_2, \dots, e_n$ are piecewise orthogonal idempotents in Λ, such that $1_\Lambda = e_1 + e_2 + \cdots + e_n$.

Since Λ is a subring of Γ with the same identity, the ring Γ can be considered as a Λ-module just by restriction of the scalars of Γ to Λ.

Now, we consider the exact sequence

$$0 \to \Lambda \xrightarrow{\lambda} \Gamma \xrightarrow{\pi} L \to 0$$

in Λ-Mod, where λ is the inclusion map and L is the cokernel of λ. To show Theorem 1.1, we prove the following statements.

(1) The sequence

$$0 \to \Lambda \xrightarrow{\lambda} \Gamma \xrightarrow{\pi} L \to 0$$

is an $\operatorname{add}(\Lambda e_1)$-split sequence in Λ-Mod.

In fact, we shall check that all conditions in Definition 2.2 are satisfied.

Since the left Λ-module ${}_\Lambda\Gamma$ is a direct sum of some copies of Λe_1, we have $\Gamma \in \operatorname{add}(\Lambda e_1)$. Clearly, Λe_1 is projective as a left Λ-module, then we have an exact sequence

$$0 \to \operatorname{Hom}_\Lambda(D, \Lambda) \xrightarrow{(-,\lambda)} \operatorname{Hom}_\Lambda(D, \Gamma) \xrightarrow{(-,\pi)} \operatorname{Hom}_\Lambda(D, L) \to 0$$

for any $D \in \operatorname{add}(\Lambda e_1)$.

This means that the homomorphism $\pi : {}_\Lambda\Gamma \to {}_\Lambda L$ is a right $\operatorname{add}(\Lambda e_1)$-approximation of ${}_\Lambda L$. Now, we prove that the homomorphism $\lambda : {}_\Lambda\Lambda \to {}_\Lambda\Gamma$ is a left $\operatorname{add}(\Lambda e_1)$-approximation of Λ. In fact, every left Λ-module homomorphism $g : \Lambda \to \Lambda e_1$ is determined by $g(1)$, the image of 1 under g. Similarly, every left Γ-module homomorphism $h : \Gamma \to \Gamma e_1$ is determined by $h(1)$, the image of 1 under h. Note that Γe_1 and Λe_1 are isomorphic as left Λ-modules, and a left Γ-module homomorphism is also a left Λ-module homomorphism. So we assume that $g_1 : \Gamma \to \Lambda e_1$ is a left Λ-module homomorphism which sends 1 to $g(1)$. Then the homomorphism g_1 satisfies $g = \lambda g_1$. Thus we have proved that the homomorphism $\lambda : {}_\Lambda\Lambda \to {}_\Lambda\Gamma$ is a left $\operatorname{add}(\Lambda e_1)$-approximation. Hence (1) is proved.

Note that Λe_i is included in Γe_i which is isomorphic to Λe_1 as left Λ-modules. Then, for each $2 \leqslant i \leqslant n$, the sequence

$$0 \to \Lambda e_i \xrightarrow{\lambda_i} \Lambda e_1 \xrightarrow{\pi_i} L_i \to 0 \tag{$*$}$$

where λ_i is inclusion map and L_i is the cokernel of λ_i, is an $\operatorname{add}(\Lambda e_1)$-split sequence.

By Lemma 2.3, the ring Λ and the endomorphism ring $\operatorname{End}_\Lambda(L_2 \oplus L_3 \oplus \cdots \oplus L_n \oplus \Lambda e_1)$ are derived equivalent via a tilting module $L_2 \oplus L_3 \oplus \cdots \oplus L_n \oplus \Lambda e_1$.

(2) The ring Σ and the endomorphism ring $\mathrm{End}_\Lambda(L_2 \oplus L_3 \oplus \cdots \oplus L_n \oplus \Lambda e_1)$ are isomorphic as rings.

Indeed, we note that

$$\mathrm{End}_\Lambda(L_2 \oplus L_3 \oplus \cdots \oplus L_n \oplus \Lambda e_1) \cong \begin{pmatrix} (L_2, L_2) & (L_2, L_3) & \cdots & (L_2, \Lambda e_1) \\ (L_3, L_2) & (L_3, L_3) & & (L_3, \Lambda e_1) \\ \vdots & & \ddots & \vdots \\ (\Lambda e_1, L_2) & (\Lambda e_1, L_3) & \cdots & (\Lambda e_1, \Lambda e_1) \end{pmatrix}$$

as rings.

In the following, we calculate the endomorphism ring $\mathrm{End}_\Lambda(L_2 \oplus L_3 \oplus \cdots \oplus L_n \oplus \Lambda e_1)$.

The morphism set $\mathrm{Hom}_\Lambda(L_i, \Lambda e_1) = 0$ for $2 \leqslant i \leqslant n$. Applying the functor $\mathrm{Hom}_\Lambda(-, \Lambda e_1)$ to the exact sequence $0 \to \Lambda e_i \to \Lambda e_1 \to L_i \to 0$, we can get the following exact sequence

$$0 \to \mathrm{Hom}_\Lambda(L_i, \Lambda e_1) \to \mathrm{Hom}_\Lambda(\Lambda e_1, \Lambda e_1) \to \mathrm{Hom}_\Lambda(\Lambda e_i, \Lambda e_1) \to 0$$

in $\mathbb{Z}$-Mod for $2 \leqslant i \leqslant n$.

Note that both of $\mathrm{Hom}_\Lambda(\Lambda e_i, \Lambda e_1)$ and $\mathrm{Hom}_\Lambda(\Lambda e_1, \Lambda e_1)$ are isomorphic to A in A-Mod and A is a noetherian ring. By Lemma 3.1, we have $\mathrm{Hom}_\Lambda(L_i, \Lambda e_1) = 0$ for $2 \leqslant i \leqslant n$.

For simplicity, we denote the set $e_i \Lambda e_j$ by Λ_{ij} for $1 \leqslant i, j \leqslant n$. The morphism μ_x denote the right multiplication by x.

Let b be an element in Λ_{ij} for $2 \leqslant i, j \leqslant n$. Since λ_i is the left Λe_1-approximation of Λe_i, we have a morphism $\mu_a : \Lambda e_1 \to \Lambda e_1$ such that $\lambda_i \mu_a = \mu_b \lambda_j$ where a is an element in A. Thus we can get an element α_b in $\mathrm{Hom}_\Lambda(L_i, L_j)$ such that $\pi_i \alpha_b = \mu_a \pi_j$. It follows from the commutativity of the left square that $a = b$. For a given morphism μ_b, there is a unique α_b satisfying $\pi_i \alpha_b = \mu_b \pi_j$. Note that $\pi_i \mu_b = \mu_b \pi_j$, we can get $\alpha_b = \mu_b$.

$$\begin{tikzcd} 0 \arrow[r] & \Lambda e_i \arrow[r, "\lambda_i"] \arrow[d, "\mu_b"'] & \Lambda e_1 \arrow[r, "\pi_i"] \arrow[d, dashed, "\mu_a"'] & L_i \arrow[r] \arrow[d, dashed, "\alpha_b"'] & 0 \\ 0 \arrow[r] & \Lambda e_j \arrow[r, "\lambda_j"] & \Lambda e_1 \arrow[r, "\pi_j"] & L_j \arrow[r] & 0 \end{tikzcd} \qquad (*)$$

where λ_i, λ_j are the inclusion maps and L_i, L_j are the cokernels of λ_i and λ_j, respectively.

Thus, we can define a set of maps from Λ_{ij} to $\mathrm{Hom}_\Lambda(L_i, L_j)$ for $2 \leqslant i, j \leqslant n$.

Define

$$\phi_{ij} : \Lambda_{ij} \to \mathrm{Hom}_\Lambda(L_i, L_j)$$
$$b \mapsto \alpha_b$$

for $2 \leqslant i, j \leqslant n$.

(a) The map ϕ_{ij} is well-defined.

$$\begin{tikzcd} 0 \arrow[r] & \Lambda e_i \arrow[r, "\lambda_i"] \arrow[d, "\mu_b"'] & \Lambda e_1 \arrow[r, "\pi_i"] \arrow[d, "\mu_b"'] & L_i \arrow[r] \arrow[d, "\alpha_b"] \arrow[dl, dashed, "s_i"'] & 0 \\ 0 \arrow[r] & \Lambda e_j \arrow[r, "\lambda_j"] & \Lambda e_1 \arrow[r, "\pi_j"] & L_j \arrow[r] & 0 \end{tikzcd}$$

Suppose that $b=0$, we have $\lambda_i\mu_b=0$. It follows that there is a morphism $s_i : L_i \to \Lambda e_1$ such that $\mu_b=\pi_i s_i$. Thus we have $\pi_i\alpha_b=\mu_b\pi_j=\pi_i s_i\pi_j$. Since π_i is surjective, we have $\alpha_b=s_i\pi_j$. Note that $\mathrm{Hom}_\Lambda(L_i,\Lambda e_1)=0$ for $2\leqslant i\leqslant n$, we obtain $\alpha_b=0$. Hence ϕ_{ij} is well-defined.

(b) The morphism ϕ_{ij} is surjective.

Let α be an element in $\mathrm{Hom}_\Lambda(L_i,L_j)$. Note that Λe_1 is projective module over Λ, thus there exists a morphism $\mu_a : \Lambda e_1 \to \Lambda e_1$ such that $\mu_a\pi_j=\pi_i\alpha$ where a is an element in Λ_{11}. Thus there is a unique morphism $\mu_b : \Lambda e_i \to \Lambda e_j$ such that $\lambda_i\mu_a=\mu_b\lambda_j$ for $b\in\Lambda_{ij}$. So ϕ_{ij} is surjective.

(c) The description of $\mathrm{Ker}\,\phi_{ij}$, i.e., $\mathrm{Ker}\,\phi_{ij}=I_j$ for $2\leqslant i, j\leqslant n$.

$$\begin{tikzcd}
0 \arrow[r] & \Lambda e_i \arrow[r, "\lambda_i"] \arrow[d, "\mu_b"'] & \Lambda e_1 \arrow[r, "\pi_i"] \arrow[d, "\mu_b"] \arrow[dl, dashed, "t_i"'] & L_i \arrow[r] \arrow[d, "\alpha_b"] & 0 \\
0 \arrow[r] & \Lambda e_j \arrow[r, "\lambda_j"] & \Lambda e_1 \arrow[r, "\pi_j"] & L_j \arrow[r] & 0
\end{tikzcd}$$

Suppose that $\alpha_b=0$. Then we have $\mu_b\pi_j=0$. So there is a morphism $t_i : \Lambda e_1 \to \Lambda e_j$ such that $\mu_b=t_i\lambda_j$. Note that there exist $c\in I_j$, $d\in A$ such that $t_i=\mu_c$, $\lambda_j=\mu_d$. Thus, we have $b=cd$, i.e., $b\in I_j$. Conversely, suppose that b is an element in I_j, we have $\mu_b\pi_j=\pi_i\alpha_b=0$. It follows that $\alpha_b=0$. Hence, $\mathrm{Ker}\,\phi_{ij}=I_j$ for $2\leqslant i, j\leqslant n$.

(d) ϕ_{ij} preserves addition and multiplication.

It is easy to prove that ϕ_{ij} preserves addition.

Now, we turn to prove that ϕ_{ij} preserves multiplication, i.e., $\phi_{ij}(b)\phi_{jk}(b')=\phi_{ik}(bb')$ for $2\leqslant i, j\leqslant n$ where b and b' are elements of Λ_{ij} and Λ_{jk} respectively.

It suffices to prove that $\alpha_{bb'}=\alpha_b\alpha_{b'}$. Since $\mu_b\mu_{b'}=\mu_{bb'}$, we have $\lambda_i\mu_{aa'}=\lambda_i\mu_c$ where μ_c is a morphism induced by $\mu_{bb'}$. Thus $\mu_{aa'}-\mu_c$ factorizes through π_i. Note that $\mathrm{Hom}_\Lambda(L_i,\Lambda e_1)=0$ for $2\leqslant i\leqslant n$, we have $\mu_{aa'}=\mu_c$. Hence we get $\alpha_{bb'}=\alpha_b\alpha_{b'}$.

Now, we can define a map

$$\phi=(\phi_{ij}) : (e_2+\cdots+e_n)\Lambda(e_2+\cdots+e_n) \to \begin{pmatrix} (L_2,L_2) & (L_2,L_3) & \cdots & (L_2,L_n) \\ (L_3,L_2) & (L_3,L_3) & & (L_3,L_n) \\ \vdots & \vdots & \ddots & \vdots \\ (L_n,L_2) & (L_n,L_3) & \cdots & (L_n,L_n) \end{pmatrix}$$

$$(a_{ij})_{i-1,j-1} \mapsto \big(\phi_{ij}(a_{ij})\big)_{i-1,j-1}$$

for $2\leqslant i, j\leqslant n$.

The map ϕ_{ij} is well-defined and surjective, so is the map ϕ. It follows from that ϕ_{ij} preserves addition and multiplication for $2\leqslant i, j\leqslant n$ that ϕ is a ring homomorphism. The kernel of ϕ is

$$\begin{pmatrix} I_2 & I_3 & \cdots & I_n \\ I_2 & I_3 & \cdots & I_n \\ \vdots & \vdots & & \vdots \\ I_2 & I_3 & \cdots & I_n \end{pmatrix}.$$

Thus, we have a ring isomorphism

$$\overline{\phi} : \begin{pmatrix} A_2/I_2 & 0 & \cdots & & 0 \\ I_{3,2}/I_2 & A_3/I_3 & \ddots & & \vdots \\ I_{4,2}/I_2 & I_{4,3}/I_3 & A_4/I_4 & & \\ \vdots & \vdots & \vdots & \ddots & 0 \\ I_{n,2}/I_2 & I_{n,3}/I_3 & I_{n,4}/I_4 & \cdots & A_n/I_n \end{pmatrix} \to \begin{pmatrix} (L_2,L_2) & (L_2,L_3) & \cdots & (L_2,L_n) \\ (L_3,L_2) & (L_3,L_3) & & (L_3,L_n) \\ \vdots & \vdots & \ddots & \vdots \\ (L_n,L_2) & (L_n,L_3) & \cdots & (L_n,L_n) \end{pmatrix}.$$

On the other hand, we have an algebra isomorphism φ_2

$$\varphi_2 : A \to \mathrm{End}_\Lambda(\Lambda e_1)$$
$$a \mapsto (f_a : \lambda e_1 \mapsto \lambda a e_1)$$

and an isomorphism of abelian groups φ_3

$$\varphi_3 : (A/I_2, A/I_3, \ldots, A/I_n) \to \big(\mathrm{Hom}_\Lambda(\Lambda e_1, L_2), \mathrm{Hom}_\Lambda(\Lambda e_1, L_3), \ldots, \mathrm{Hom}_\Lambda(\Lambda e_1, L_n)\big)$$
$$(m_1, m_2, \ldots, m_n) \mapsto (f_{m_2}, f_{m_3}, \ldots, f_{m_n})$$

where $f_{m_i} : \lambda e_1 \mapsto \lambda m_i e_1$ for $2 \leqslant i \leqslant n$.

Now, set

$$\varphi = \begin{pmatrix} \overline{\phi} & 0 \\ \varphi_3 & \varphi_2 \end{pmatrix} : \Sigma \to \begin{pmatrix} \mathrm{End}_\Lambda(L_2) & \mathrm{Hom}_\Lambda(L_2, L_3) & \cdots & \mathrm{Hom}_\Lambda(L_2, \Lambda e_1) \\ \mathrm{Hom}_\Lambda(L_3, L_2) & \mathrm{End}_\Lambda(L_3) & & \mathrm{Hom}_\Lambda(L_3, \Lambda e_1) \\ \vdots & & \ddots & \vdots \\ \mathrm{Hom}_\Lambda(\Lambda e_1, L_2) & \mathrm{Hom}_\Lambda(\Lambda e_1, L_3) & \cdots & \mathrm{End}_\Lambda(\Lambda e_1) \end{pmatrix}$$

$$\begin{pmatrix} r_{22} & r_{23} & \cdots & r_{2n} & 0 \\ r_{32} & r_{33} & \cdots & r_{3n} & \vdots \\ \vdots & \vdots & \ddots & \vdots & \\ r_{n2} & r_{n3} & \cdots & r_{nn} & 0 \\ m_2 & m_3 & \cdots & m_n & a \end{pmatrix} \mapsto \begin{pmatrix} \overline{\phi_{22}}(r_{22}) & \overline{\phi_{23}}(r_{23}) & \cdots & \overline{\phi_{2n}}(r_{2n}) & 0 \\ \overline{\phi_{32}}(r_{32}) & \overline{\phi_{33}}(r_{33}) & \cdots & \overline{\phi_{3n}}(r_{3n}) & \vdots \\ \vdots & \vdots & \ddots & \vdots & \\ \overline{\phi_{n2}}(r_{n2}) & \overline{\phi_{n3}}(r_{n3}) & \cdots & \overline{\phi_{nn}}(r_{nn}) & 0 \\ \varphi_3(m_2) & \varphi_3(m_3) & \cdots & \varphi_3(m_n) & \varphi_2(a) \end{pmatrix}.$$

Clearly, the map φ is an isomorphism of abelian groups. And it is easy to check that φ is a ring isomorphism. The proof is completed. □

As a direct consequence of Theorem 1.1, we have the following corollary.

Corollary 3.2. *Let A be a noetherian ring with identity, and suppose that $I_2, I_3, \ldots, I_n$ are ideals of A.*

(1) *The two rings*

$$\begin{pmatrix} A & I_2 & I_3 & \cdots & \cdots & I_n \\ A & A & I_3 & \cdots & \cdots & I_n \\ A & I_2 & A & I_4 & \cdots & I_n \\ A & I_2 & I_3 & A & \cdots & I_n \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ A & I_2 & I_3 & \cdots & I_{n-1} & I_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A/I_2 & 0 & \cdots & & & 0 \\ 0 & A/I_3 & \ddots & & & \vdots \\ \vdots & \ddots & A/I_4 & & & \\ & & & \ddots & & \\ 0 & \cdots & & 0 & A/I_n & 0 \\ A/I_2 & A/I_3 & A/I_4 & \cdots & A/I_n & A \end{pmatrix}$$

are derived equivalent.

(2) *The two rings*

$$\begin{pmatrix} A & I_2 & I_3 & \cdots & I_{n-1} & I_n \\ A & A & I_3 & \cdots & I_{n-1} & I_n \\ A & A & A & \cdots & I_{n-1} & I_n \\ \vdots & \vdots & \vdots & \ddots & I_{n-1} & I_n \\ \vdots & \vdots & \vdots & & A & I_n \\ A & A & A & A & \dots & A \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A/I_2 & 0 & \cdots & & & 0 \\ A/I_2 & A/I_3 & \ddots & & & \\ A/I_2 & A/I_3 & A/I_4 & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & & \\ \vdots & \vdots & \vdots & & A/I_n & 0 \\ A/I_2 & A/I_3 & A/I_4 & \cdots & A/I_n & A \end{pmatrix}$$

are derived equivalent.

4. Applications

In Section 3, we have constructed derived equivalences of matrix subrings. In this section, we will give some applications of the main result. At first, we define a class of rings which are called general block extension of a ring with respect to a decomposition of the identity. Then we calculate the finitistic dimension of a general block extension of a ring with respect to a decomposition of the identity. At last, we will consider the finiteness of finitistic dimension of a tiled triangular ring. Proposition 4.14, will give a condition under which the finitistic dimension of a tiled triangular algebra is finite.

The following lemmas, which are taken from [17,10], are useful for this section.

Lemma 4.1. *(See [17, Theorem 1.1].) If two left coherent rings A and B are derived equivalent, and if $T^\bullet$ is a tilting complex over A with $n+1$ non-zero terms such that $B \cong \operatorname{End}(T^\bullet)$, then $\operatorname{fin.dim}(A) - n \leqslant \operatorname{fin.dim}(B) \leqslant \operatorname{fin.dim}(A) + n$.*

Remark. In Lemma 4.1, if we replace A and B by arbitrary ring with identity, then $l.\operatorname{Fin.dim}(A) - n \leqslant l.\operatorname{Fin.dim}(B) \leqslant l.\operatorname{Fin.dim}(A) + n$. The proof is similar.

Lemma 4.2. *(See [10, Proposition 1.7].) Let A be a ring with identity, and let $T^\bullet$ be a tilting complex over A with $\operatorname{End}(T^\bullet) \cong B$. If $T^\bullet$ has $n+1$ non-zero terms, where $n \geqslant 0$, then the following statements hold*:

(a) $l.\operatorname{gl.dim}(A) - n \leqslant l.\operatorname{gl.dim}(B) \leqslant l.\operatorname{gl.dim}(A) + n$;
(b) $\operatorname{inj.dim}({}_AA) - n \leqslant \operatorname{inj.dim}({}_BB) \leqslant \operatorname{inj.dim}({}_AA) + n$.

The following lemma about the estimation of global dimension and finitistic dimension can be found in [5, Corollary 4.21, p. 70].

Lemma 4.3. *(See [5].) Let R and S be rings. Let M be an S–R bimodule and $\Lambda := \begin{pmatrix} R & 0 \\ M & S \end{pmatrix}$. Then the following inequalities hold*:

(1) $l.\operatorname{Fin.dim}(S) \leqslant l.\operatorname{Fin.dim}(\Lambda) \leqslant 1 + l.\operatorname{Fin.dim}(R) + l.\operatorname{Fin.dim}(S)$.
(2) $l.\operatorname{Fin.dim}(\Lambda) \geqslant \sup\{pd({}_RA) \leqslant \infty \mid A \in R\text{-Mod}$ *satisfying* $\operatorname{Tor}_i^A(M, A) = 0$ *for all* $i\}$. *If M is flat as a right-R-module, then* $l.\operatorname{Fin.dim}(\Lambda) \geqslant l.\operatorname{Fin.dim}(R)$.
(3) *If* $\operatorname{proj.dim}({}_SM) \leqslant \infty$, *then* $\operatorname{proj.dim}({}_SM) + 1 \leqslant l.\operatorname{Fin.dim}(\Lambda) \leqslant \max\{l.\operatorname{Fin.dim}(R) + \operatorname{proj.dim}({}_SM) + 1, l.\operatorname{Fin.dim}(S)\}$.
(4) $\max\{l.\operatorname{gl.dim}(R), lgl.dim(S), \operatorname{proj.dim}({}_SM) + 1\} \leqslant l.\operatorname{gl.dim}(\Lambda) \leqslant \max\{l.\operatorname{gl.dim}(R) + \operatorname{proj.dim}({}_SM) + 1, l.\operatorname{gl.dim}(S)\}$.

The corresponding statements hold for the right homological dimensions over Λ.

4.1. General block extension of a ring with respect to a decomposition of the identity

In this part, we will define a class of rings which contains hereditary orders, block extensions of basic algebras.

Definition 4.4. Let A be a ring with identity 1_A. And $1_A = e_1 + e_2 + \cdots + e_m$ is a decomposition of the identity where e_i is an idempotent. Then A can be represented as the following matrix form

$$A = \begin{pmatrix} e_1Ae_1 & e_1Ae_2 & \cdots & e_1Ae_m \\ e_2Ae_1 & e_2Ae_2 & \cdots & e_2Ae_m \\ \vdots & \vdots & \ddots & \vdots \\ e_mAe_1 & e_mAe_2 & \cdots & e_mAe_m \end{pmatrix}.$$

Set $A_i = e_iAe_i$ and $A_{ij} = e_iAe_j$. Then A_i is the subring of A with identity element e_i, and A_{ij} is an (A_i, A_j)-bimodule.

Let $n_1, n_2, \ldots, n_m \in \mathbb{N}$. For $1 \leqslant i, s \leqslant m$, $1 \leqslant j \leqslant n_i$ and $1 \leqslant t \leqslant n_s$, we define

$$P = A(n_1, n_2, \ldots, n_m) = \begin{pmatrix} P(1,1) & P(1,2) & \cdots & P(1,m) \\ P(2,1) & P(2,2) & \cdots & P(2,m) \\ \vdots & \vdots & \ddots & \vdots \\ P(m,1) & P(m,2) & \cdots & P(m,m) \end{pmatrix}$$

which is contained in the ring $\mathrm{End}_A((Ae_1)^{n_1} \oplus \cdots \oplus (Ae_m)^{n_m})$ with the restrictions of the binary operations of addition and multiplication of $\mathrm{End}_A((Ae_1)^{n_1} \oplus \cdots \oplus (Ae_m)^{n_m})$.

$$P(i,s) = \begin{pmatrix} P_{i1,s1} & P_{i1,s2} & \cdots & P_{i1,sn_s} \\ P_{i2,s1} & P_{i2,s2} & \cdots & P_{i2,sn_s} \\ \vdots & \vdots & \ddots & \vdots \\ P_{in_i,s1} & P_{in_i,s2}\cdots & \cdots & P_{in_i,sn_s} \end{pmatrix}_{n_i \times n_s}$$

satisfies that $P_{ip,sq}$ is an (A_i, A_s)-bimodule.

For $P(i,s)$, there are three cases:

Case I: $i = s$.

$$P(i,s) := \begin{pmatrix} A_i & I_{i2} & I_{i3} & \cdots & I_{i(n_i-1)} & I_{in_i} \\ A_i & B_{i2} & I_{i3} & \cdots & I_{i(n_i-1)} & I_{in_i} \\ A_i & I_{i32} & B_{i3} & \cdots & I_{i(n_i-1)} & I_{in_i} \\ A_i & I_{i42} & I_{i43} & B_{i4} & \cdots & I_{in_i} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ A_i & I_{in_i2} & \cdots & \cdots & I_{in_i(n_i-1)} & B_{in_i} \end{pmatrix}$$

where B_{il} is the subring of A_i with the same identity, I_{il} and I_{ipq} are ideals of A_i satisfying $I_{in_i} \subseteq I_{i(n_i-1)} \subseteq \cdots \subseteq I_{i2}$, $I_{il} \subseteq I_{ipl}$, $I_{il} \subseteq B_{il}$ for $2 \leqslant l \leqslant n_i$, $p \neq q$, $3 \leqslant p \leqslant n_i$, $2 \leqslant q \leqslant n_i - 1$.

Case II: $i < s$.

$$P(i,s) := \begin{pmatrix} A_{is} & P_{i1,s2} & \cdots & P_{i1,sn_s} \\ A_{is} & P_{i1,s2} & \cdots & P_{i1,sn_s} \\ \vdots & \vdots & \vdots & \vdots \\ A_{is} & P_{i1,s2} & \cdots & P_{i1,sn_s} \end{pmatrix}.$$

Case III: $i > s$.

$$P(i,s) := \begin{pmatrix} A_{is} & P_{i1,s2} & \cdots & P_{i1,sn_s} \\ A_{is} & P_{i2,s2} & \cdots & P_{i2,sn_s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{is} & P_{in_i,s2} & \cdots & P_{in_i,sn_s} \end{pmatrix}.$$

Suppose that $\sum_{l=1}^{m} P(i,l)P(l,j) \subseteq P(i,j)$. Then P is a ring called general block extension of a ring with respect to a decomposition of the identity.

General block extension of A with respect to a decomposition of the identity contains many classes of subrings of $M_n(A)$. In the following, we will give some examples.

Example. (1) In Definition 4.4, we assume that $m = 1$. Then

$$P = \begin{pmatrix} A & I_2 & I_3 & \cdots & I_{n-1} & I_n \\ A & A_2 & I_3 & \cdots & I_{n-1} & I_n \\ A & I_{3,2} & A_3 & \cdots & I_{n-1} & I_n \\ A & I_{4,2} & I_{4,3} & A_4 & \cdots & I_n \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ A & I_{n,2} & \cdots & \cdots & I_{n,n-1} & A_n \end{pmatrix}$$

where A is a ring with identity, A_i is a subring of A with the same identity for $2 \leqslant i \leqslant n$. $I_i, I_{i,j}$ are ideals of A for $2 \leqslant i \leqslant n$, $2 \leqslant j \leqslant n-1$. In particular, set $I_i = a\Omega$, $A = A_i = \Omega$ for $2 \leqslant i \leqslant n$ and $I_{i,j} = \Omega$ for $2 \leqslant j < i \leqslant n$, where Ω is a local R-order and a is a regular element in Ω. Then $\Omega/a \cdot \Omega$ is local and P is a QH-order with associated ideal $J = \omega \cdot P$, where

$$\omega = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ a & 0 & \cdots & \cdots & 0 \end{pmatrix}_{n\times n}.$$

(2) In Definition 4.4, let A be a basic algebra, and let $\{e_1, \dots, e_m\}$ be a complete set of orthogonal primitive idempotents of A.

Set

$$P(i,s)=\begin{pmatrix} P_{i1,s1} & P_{i1,s2} & \cdots & P_{i1,sn_s}\\ P_{i2,s1} & P_{i2,s2} & \cdots & P_{i2,sn_s}\\ \vdots & \vdots & \ddots & \vdots\\ P_{in_i,s1} & P_{in_i,s2}\cdots & P_{in_i,sn_s} & P_{in_i,sn_s}\end{pmatrix}=\begin{cases}\begin{pmatrix} A_i & \cdots & A_i\\ & \ddots & \vdots\\ \operatorname{rad}(A_i) & & A_i\end{pmatrix} & (i=s)\\ \begin{pmatrix} A_{is} & \cdots & A_{is}\\ \vdots & & \vdots\\ A_{is} & \cdots & A_{is}\end{pmatrix} & (i\neq s).\end{cases}$$

Then P is called the *block extension* of A which can be found in [16]. In particular, if A is a basic QF-algebra, then P is a basic Harada algebra (see [16]).

Theorem 4.5. *Let A be a noetherian ring with identity. Let P be a general block extension of A with respect to a decomposition of the identity. Then*

$$l.\mathrm{Fin.dim}(A)-1\leqslant l.\mathrm{Fin.dim}(P)\leqslant l.\mathrm{Fin.dim}(A)+\sum_{j=1}^{m}\sum_{i=2}^{n_j} l.\mathrm{Fin.dim}(A_j/I_{j,i})+\sum_{i=1}^{m}n_i-m+1.$$

Proof. Denote by $e_{\sum_{l=1}^{i}n_l+j}$ the matrix which has $1_{A_{i+1}}$ in the $(\sum_{l=1}^{i}n_l+j,\ \sum_{l=1}^{i}n_l+j)$-th position and zeros elsewhere for $1\leqslant j\leqslant n_{i+1}$, $0\leqslant i\leqslant m-1$. Thus $e_1,\dots,e_{n_1},e_{n_1+1},\dots,e_{n_1+n_2},\dots,e_{\sum_{l=1}^{m-1}n_l+1},\dots,e_{\sum_{l=1}^{m}n_l}$ are piecewise orthogonal idempotents in P such that $1_P=e_1+\cdots+e_{\sum_{l=1}^{m}n_l}$.

Set $\Upsilon=\mathrm{End}_P(\overbrace{Pe_1\oplus\cdots\oplus Pe_1}^{n_1}\oplus\cdots\oplus\overbrace{Pe_{\sum_{l=1}^{m-1}n_l1}\oplus\cdots\oplus Pe_{\sum_{l=1}^{m-1}n_l1}}^{n_m})$.

Since P is a subring of Υ with the same identity, Υ can be viewed as a P-module by restriction of the scalars of Υ to P. There is an exact sequence

$$0\to P\xrightarrow{\lambda}\Upsilon\xrightarrow{\pi}L\to 0$$

in P-Mod, where λ is the inclusion map and L is the cokernel of λ. Thus there is an exact sequence

$$0\to Pe_{\sum_{l=1}^{i-1}n_l+n_j}\xrightarrow{\lambda_{n_in_j}}Pe_{\sum_{l=1}^{i-1}n_l+1}\xrightarrow{\pi_{n_in_j}}L_{n_in_j}\to 0$$

in P-Mod, where $\lambda_{n_in_j}$ is the composite of the inclusion map $Pe_{\sum_{l=1}^{i-1}n_l+n_j}\hookrightarrow \Upsilon e_{\sum_{l=1}^{i-1}n_l+n_j}$ and the isomorphism $\Upsilon e_{\sum_{l=1}^{i-1}n_l+n_j}\to Pe_{\sum_{l=1}^{i-1}n_l+1}$ for $1\leqslant i\leqslant m$, $2\leqslant j\leqslant m$. By Theorem 2.3, two rings P and $\mathrm{End}_P((\bigoplus_{\substack{i=1,\dots,m\\ j=2,\dots,m}}L_{n_in_j})\oplus(\bigoplus_{i=1}^{\sum_{i=1}^{m}n_i}Pe_i))$ are derived equivalent.

Note that

$$\mathrm{End}_P\big((L_{n_12}\oplus\cdots\oplus L_{n_1n_1})\oplus\cdots\oplus(L_{n_m1}\oplus\cdots\oplus L_{n_mn_m})\oplus(Pe_{\sum_{l=1}^{m-1}n_l+1}\oplus\cdots\oplus Pe_1)\big)$$

$$\cong\begin{pmatrix}((L_{n_1i},L_{n_1j}))_{2\leqslant i,j\leqslant n_1} & ((L_{n_1i},L_{n_2j}))_{\substack{2\leqslant i\leqslant n_1\\ 2\leqslant j\leqslant n_2}} & \cdots & ((L_{n_1i},L_{n_mj}))_{\substack{2\leqslant i\leqslant n_1\\ 2\leqslant j\leqslant n_m}} & ((L_{n_1i},Pe_i))_{\substack{2\leqslant i\leqslant n_1\\ 1\leqslant j\leqslant m}}\\ ((L_{n_2i},L_{n_1j}))_{\substack{2\leqslant i\leqslant n_2\\ 2\leqslant j\leqslant n_1}} & ((L_{n_2i},L_{n_2i}))_{2\leqslant i,j\leqslant n_2} & \cdots & ((L_{n_2i},L_{n_mj}))_{\substack{2\leqslant i\leqslant n_2\\ 2\leqslant j\leqslant n_m}} & ((L_{n_2i},Pe_j))_{\substack{2\leqslant i\leqslant n_1\\ 1\leqslant j\leqslant m}}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ ((L_{n_mi},L_{n_1j}))_{\substack{2\leqslant i\leqslant n_m\\ 2\leqslant j\leqslant n_1}} & ((L_{n_mi},L_{n_2j}))_{\substack{2\leqslant i\leqslant n_m\\ 2\leqslant j\leqslant n_2}} & \cdots & ((L_{n_mi},L_{n_mj}))_{2\leqslant i,j\leqslant n_m} & ((L_{n_mi},Pe_j))_{\substack{2\leqslant i\leqslant n_m\\ 1\leqslant j\leqslant m}}\\ ((Pe_i,L_{n_1j}))_{\substack{1\leqslant i\leqslant n_m\\ 2\leqslant j\leqslant n_1}} & ((Pe_i,L_{n_2j}))_{\substack{2\leqslant i\leqslant n_m\\ 2\leqslant j\leqslant n_2}} & \cdots & ((L_{n_mi},L_{n_mj}))_{2\leqslant i,j\leqslant n_m} & ((L_{n_mi},Pe_j))_{\substack{2\leqslant i\leqslant n_m\\ 1\leqslant j\leqslant m}}\end{pmatrix}.$$

(1) $\mathrm{Hom}_P(L_{n_p i}, L_{n_q j}) = 0$ for $2 \leqslant i \leqslant n_p$, $2 \leqslant j \leqslant n_q$, $1 \leqslant p < q \leqslant m$.

There is an exact sequence

$$0 \to Pe_{\sum_{l=1}^{p-1} n_l + i} \to Pe_{\sum_{l=1}^{p-1} n_l + 1} \to L_{n_p i} \to 0 \tag{$**$}$$

in P-Mod. Applying the functor $\mathrm{Hom}_P(-, L_{n_q j})$ to $(**)$, we get an exact sequence

$$0 \to \mathrm{Hom}_P(L_{n_p i}, L_{n_q j}) \to \mathrm{Hom}_P(Pe_{\sum_{l=1}^{p-1} n_l + 1}, L_{n_q j}) \to \mathrm{Hom}_P(Pe_{\sum_{l=1}^{p-1} n_l + i}, L_{n_q j}) \to 0.$$

By calculation, we have $\mathrm{Hom}_P(Pe_{\sum_{l=1}^{p-1} n_l + 1}, L_{n_q j}) = \mathrm{Hom}_P(Pe_{\sum_{l=1}^{p-1} n_l + i}, L_{qj}) = 0$. Thus, $\mathrm{Hom}_P(L_{n_p i}, L_{n_q j}) = 0$ for $2 \leqslant i \leqslant n_p$, $2 \leqslant j \leqslant n_q$, $1 \leqslant p < q \leqslant m$.

(2) $\mathrm{Hom}_P(L_{n_k i}, L_{n_k j}) = 0$ for $2 \leqslant i < j \leqslant n_k$, $1 \leqslant k \leqslant m$.

Apply the functor $\mathrm{Hom}_P(-, L_{n_k j})$ to the exact sequence

$$0 \to Pe_{\sum_{j=1}^{k-1} n_j + i} \to Pe_{\sum_{j=1}^{k-1} n_j + 1} \to L_{n_k i} \to 0$$

in P-Mod. We have an exact sequence

$$0 \to \mathrm{Hom}_P(L_{n_k i}, L_{n_k j}) \to \mathrm{Hom}_P(Pe_{\sum_{j=1}^{k-1} n_j + 1}, L_{n_k j}) \to \mathrm{Hom}_P(Pe_{\sum_{j=1}^{k-1} n_j + i}, L_{n_k j}) \to 0.$$

Note that $\mathrm{Hom}_P(Pe_{\sum_{j=1}^{k-1} n_j + 1}, L_{n_k j})$ and $\mathrm{Hom}_P(Pe_{\sum_{j=1}^{k-1} n_j + i}, L_{n_k j})$ are both isomorphic to $A_k/I_{k,j}$ in $A_k/I_{k,j}$-Mod. It follows from Lemma 3.1 that $\mathrm{Hom}_P(L_{n_k i}, L_{n_k j}) = 0$ for $2 \leqslant i < j \leqslant n_k$, $1 \leqslant k \leqslant m$.

Thus

$$\begin{aligned}
&\mathrm{End}_P\big((L_{n_1 2} \oplus \cdots \oplus L_{n_1 n_1}) \oplus \cdots \oplus (L_{n_m 1} \oplus \cdots \oplus L_{n_m n_m}) \oplus (Pe_{\sum_{l=1}^{m-1} n_l + 1} \oplus \cdots \oplus Pe_1)\big) \\
&\cong \begin{pmatrix}
(L_{n_1 2}, L_{n_1 2}) & 0 & 0 & \cdots & & & & 0 \\
* & \ddots & \ddots & \ddots & & & & \vdots \\
 & & (L_{n_1 n_1}, L_{n_1, n_1}) & 0 & & & & \\
 & & \ddots & (L_{n_2 2}, L_{n_2 2}) & \ddots & & & \\
\vdots & & & & \ddots & & & \\
 & & & & & (L_{n_2 n_2}, L_{n_2 n_n}) & & \\
 & & & & & & \ddots & 0 \\
* & & & \cdots & & & * & \mathrm{End}_A(Ae_m \oplus \cdots \oplus Ae_1)
\end{pmatrix}.
\end{aligned}$$

By Lemmas 4.3 and 4.1, we can get the conclusion. □

Corollary 4.6. *Let Λ be as in Theorem 1.1. Then*

$$l.\mathrm{Fin.dim}(A) - 1 \leqslant l.\mathrm{Fin.dim}(\Lambda) \leqslant n + \sum_{i=2}^{n} l.\mathrm{Fin.dim}(A_i/I_i) + l.\mathrm{Fin.dim}(A).$$

Proof. It follows from Theorem 1.1 and Lemma 4.1. □

As a consequence of Theorem 4.5, we can get the following corollary. By this corollary, we can get a class of algebras which have finite finitistic dimension.

Corollary 4.7. *Let Λ be as in Theorem 1.1 and suppose that A is an Artin algebra. Then*:

(1) *If* $\text{fin.dim}(A) < \infty$ *and* $\text{fin.dim}(A_j/I_{j,i}) < \infty$ *for* $2 \leqslant i \leqslant n_j$, $1 \leqslant j \leqslant m$, *then* $\text{fin.dim}(P) < \infty$.
(2) *If* $\text{fin.dim}(P) < \infty$, *then* $\text{fin.dim}(A) < \infty$.

In [16], K. Yamaura proved that any block extension of a basic QF-algebra is a basic left Harada algebra. And for any basic left Harada algebra T, there exists a basic QF-algebra R such that T is isomorphic to an upper staircase factor algebra of a block extension of R. By Theorem 4.5, we can get that the finitistic dimension is finite for the block extension of a QF-algebra. Thus, the finitistic dimension conjecture holds for the class of left Harada algebras.

Corollary 4.8. *Suppose that R is a QF-algebra and P is the block extension of R. Then*

$$\text{fin.dim}(P) \leqslant \sum_{i=1}^{m} n_i - m + 1 < \infty.$$

Proof. Note that $\text{fin.dim}(R) = 0$, $\text{fin.dim}(A_j/\operatorname{rad} A_j) = 0$ for $2 \leqslant i \leqslant n_j$, $1 \leqslant j \leqslant m$. $\square$

Proposition 4.9. *Let Λ be as in Theorem 1.1. Then*

$$\begin{aligned} \max\bigl\{l.\text{gl.dim}(A_i/I_i), l.\text{gl.dim}(A), 2 \leqslant i \leqslant n\bigr\} - 1 &\leqslant l.\text{gl.dim}(\Lambda) \\ &\leqslant \sum_{i=2,3,\ldots,n} l.\text{gl.dim}(A/I_i) + l.\text{gl.dim}(A) + n. \end{aligned}$$

Proof. By Theorem 1.1, we can get that the two rings Λ and Σ are derived equivalent via a tilting module whose projective dimension is less or equal 1. It follows from 4.2 and Lemma 4.3. $\square$

Corollary 4.10. *Let A be a noetherian ring with identity, $I_2, I_3, \ldots, I_n$ ideals of A. Set*

$$\Gamma = \begin{pmatrix} A & I_2 & I_3 & \cdots & \cdots & I_n \\ A & A & I_3 & \cdots & \cdots & I_n \\ A & I_2 & A & I_4 & \cdots & I_n \\ A & I_2 & I_3 & A & \cdots & I_n \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ A & I_2 & I_3 & \cdots & I_{n-1} & A \end{pmatrix}.$$

Then $\max\{l.\text{gl.dim}(A/I_i) - 1, l.\text{gl.dim}(A) - 1, \text{proj.dim}({}_A I_i) + 1, 2 \leqslant i \leqslant n\} \leqslant l.\text{gl.dim}(\Gamma) \leqslant \max\{l.\text{gl.dim}(A/I_i) + \text{proj.dim}({}_A I_j) + 3, l.\text{gl.dim}(A) + 1, 2 \leqslant i, j \leqslant n\}$.

Proof. By Corollary 3.3(2) and Lemma 4.3, we can get the conclusion. $\square$

4.2. *Tiled triangular rings*

Before we turn to the second topic, we recall the definition of recollement, given by Beilinson, Bernstein and Deligne in their work on perverse sheaves.

Definition 4.11. (See [2].) Let $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$ be triangulated categories. Then a recollement of $\mathcal{D}$ relative to $\mathcal{D}'$ and $\mathcal{D}''$, diagrammatically expressed by

$$\mathcal{D}' \rightleftarrows \mathcal{D} \rightleftarrows \mathcal{D}''$$

is given by six exact functors

$$i_* = i_! : \mathcal{D}' \to \mathcal{D}, \quad j^* = j^! : \mathcal{D} \to \mathcal{D}'', \quad i^*, i^! : \mathcal{D} \to \mathcal{D}', \quad j_!, j_* : \mathcal{D}'' \to \mathcal{D},$$

which satisfy the following four conditions:

(R1) $(i^*, i_* = i_!, i^!)$ and $(j_!, j^* = j^!, j_*)$ are adjoint triples, i.e., i^* is left adjoint to i_* which is left adjoint to $i^!$, etc.,
(R2) $i^! j_* = 0$,
(R3) i_*, $j_!$ and j_* are full embeddings,
(R4) any object X in $\mathcal{D}$ determines distinguished triangles

$$i_! i^! X \to X \to j_* j^* X \to \Sigma i_! i^! X \quad \text{and} \quad j_! j^! X \to X \to i_* i^* X \to \Sigma j_! j^! X$$

where the morphisms $i_! i^! X \to X$, etc., are the adjunction morphisms.

Using the notion of recollement, Happel proved the following result. The next lemma is useful to provide a class of algebras which have finite finitistic dimension.

Lemma 4.12. *(See [7,8].) Let A be a finite-dimensional algebra and assume that $\mathscr{D}^b(A)$ has a recollement relative to $\mathscr{D}^b(A')$ and $\mathscr{D}^b(A'')$ for some finite-dimensional algebras A', A''. Then* $\operatorname{fin.dim}(A) < \infty$ *if and only if* $\operatorname{fin.dim}(A') < \infty$ *and* $\operatorname{fin.dim}(A'') < \infty$.

The following lemma, showing how to construct a recollement, is useful in our proof.

Lemma 4.13. *(See [15, Theorem 3].) Let A, B and C be algebras. The following assertions are equivalent:*

(1) $\mathscr{D}^-(A\text{-Mod})$ *is a recollement of* $\mathscr{D}^-(C\text{-Mod})$ *and* $\mathscr{D}^-(B\text{-Mod})$.
(2) *There are two objects $P, Q \in \mathscr{D}^-(A\text{-Mod})$ satisfying the following properties:*
   (a) *There are isomorphism of algebras* $C \cong \operatorname{Hom}_{\mathscr{D}(A\text{-Mod})}(P, P)$ *and* $B \cong \operatorname{Hom}_{\mathscr{D}(A\text{-Mod})}(Q, Q)$.
   (b) *P is exceptional and isomorphic in $\mathscr{D}(A\text{-Mod})$ to a bounded complex of finitely generated projective A-modules.*
   (c) *For every set Λ and every non-zero integer i we have* $\operatorname{Hom}_{\mathscr{D}(A\text{-Mod})}(Q, Q^{(\Lambda)}[i]) = 0$, *the canonical isomorphism* $\operatorname{Hom}_{\mathscr{D}(A\text{-Mod})}(Q, Q)^{(\Lambda)} \to \operatorname{Hom}_{\mathscr{D}(A\text{-Mod})}(Q, Q^{(\Lambda)})$ *is an isomorphism, and Q is isomorphic in $\mathscr{D}(A\text{-Mod})$ to a bounded complex of projective A-modules.*
   (d) $\operatorname{Hom}_{\mathscr{D}(A\text{-Mod})}(P, Q[i]) = 0$ *for all $i \in \mathbb{Z}$.*
   (e) *$P \oplus Q$ generates* $\mathscr{D}(A\text{-Mod})$.

Now, we turn to consider "tiled triangular ring," i.e., rings of the form

$$\Delta = \begin{pmatrix} A & I_{1,2} & \cdots & I_{1,n} \\ A & A & & \vdots \\ \vdots & & \ddots & I_{n-1,n} \\ A & \cdots & \cdots & A \end{pmatrix}$$

for I_{ij} ideals of A.

Now, let us prove the last result in this paper.

Proposition 4.14. *Set*

$$\Phi=\begin{pmatrix} A & I_{1,2} & I_{1,3} & \cdots & I_{1,n-1} & I_{1,n}\\ A & A & I_{2,3} & \cdots & I_{2,n-1} & I_{2,n}\\ A & A & A & \cdots & I_{3,n-1} & I_{3,n}\\ A & A & A & A & \cdots & I_{4,n}\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\ A & A & \cdots & \cdots & A & A \end{pmatrix}.$$

Suppose that Φ is an Artin algebra, $I_{i,j}$ are ideals of A for $1\leqslant i<j\leqslant n$, $\operatorname{proj.dim}({}_{A/I_{i,i+1}}I_{i+1,j+1}/I_{i,j+1})<\infty$ *for $1\leqslant i<j\leqslant n-1$ and* $\operatorname{fin.dim}(A/I_{i,i+1})<\infty$, $\operatorname{fin.dim}(A)<\infty$ *for $1\leqslant i\leqslant n-1$. Then* $\operatorname{fin.dim}(\Phi)<\infty$.

Proof. Set $\Gamma=M_n(A)$. The exact sequence

$$0\to\Phi\xrightarrow{\lambda}\Gamma\xrightarrow{\pi}L\to 0$$

in Φ-Mod is an $\operatorname{add}(\Gamma)$-split sequence. By the method which is similar to the ones in Theorem 1.1, we can prove that the two rings

$$\Phi=\begin{pmatrix} A & I_{1,2} & I_{1,3} & \cdots & I_{1,n}\\ A & A & I_{2,3} & & \vdots\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ \vdots & \vdots & & \ddots & I_{n-1,n}\\ A & A & A & A & A \end{pmatrix}\quad\text{and}\quad \Sigma_1=\begin{pmatrix} A/I_{1,2} & I_{2,3}/I_{1,3} & \cdots & I_{2,n}/I_{1,n} & 0\\ A/I_{1,2} & A/I_{1,3} & & & \vdots\\ \vdots & \vdots & & \ddots & \vdots\\ A/I_{1,2} & A/I_{1,3} & \cdots & A/I_{1,n} & 0\\ A/I_{1,2} & A/I_{1,3} & \cdots & A/I_{1,n} & A \end{pmatrix}$$

are derived equivalent.

Set

$$\Phi_{n-1}=\begin{pmatrix} A/I_{1,2} & I_{2,3}/I_{1,3} & \cdots & I_{2,n}/I_{1,n}\\ A/I_{1,2} & A/I_{1,3} & \ddots & \vdots\\ \vdots & \vdots & \ddots & I_{n-1,n}/I_{1,n}\\ A/I_{1,2} & A/I_{1,3} & \cdots & A/I_{1,n} \end{pmatrix}.$$

For simplicity, we denote Φ_n by

$$\Gamma=\begin{pmatrix} A_1 & J_{1,2} & J_{1,3} & \cdots & J_{1,n}\\ \vdots & A_2 & J_{2,3} & & \vdots\\ \vdots & \vdots & A_3 & \ddots & \vdots\\ \vdots & \vdots & \vdots & \ddots & J_{n-1,n}\\ A_1 & A_2 & A_3 & \cdots & A_n. \end{pmatrix}.$$

Claim. *Suppose that* $\mathrm{proj.dim}({}_{A_1}J_{1,k}) < \infty$, $2 \leqslant k \leqslant n$ *and* $\mathrm{proj.dim}({}_{A_i/J_{i-1,i}}J_{i,j}/J_{i-1,j}) < \infty$, $2 \leqslant i < j \leqslant n-1$. *Then* $\mathrm{fin.dim}(\Gamma) < \infty$ *if and only if* $\mathrm{fin.dim}(A_1) \leqslant \infty$ *and* $\mathrm{fin.dim}(A_i/J_{i-1,i}) < \infty$*, for* $2 \leqslant i \leqslant n$.

Proof of Claim. Let e be an idempotent of Γ which has 1 in the $(1,1)$-th position and zeros elsewhere. By easy computation,

$$\Gamma e\Gamma = \begin{pmatrix} A_1 & J_{1,2} & J_{1,3} & \cdots & J_{1,n} \\ A_1 & J_{1,2} & J_{1,3} & \cdots & J_{1,n} \\ \vdots & & \vdots & & \vdots \\ A_1 & J_{1,2} & J_{1,3} & \cdots & J_{1,n} \end{pmatrix}.$$

Since $\Gamma e\Gamma$ is projective as a right Γ-module, we have $\mathrm{Tor}_i^{\Gamma}(\Gamma/\Gamma e\Gamma, \Gamma/\Gamma e\Gamma) = 0$ for $i > 0$. Then $\lambda : \Gamma \to \Gamma/\Gamma e\Gamma$ is a homological ring epimorphism [6, Definition 4.5]. By [15, Example 6], there is recollement:

$$\mathscr{D}(\mathrm{Mod}-\Gamma/\Gamma e\Gamma) \underset{\mathbf{R}\mathrm{Hom}_\Gamma(\Gamma/\Gamma e\Gamma,-)}{\overset{-\otimes^{\mathbf{L}}_\Gamma \Gamma/\Gamma e\Gamma}{\leftleftarrows \!\!\! \xrightarrow{i_*}}} \mathscr{D}(\mathrm{Mod}-\Gamma) \underset{j_*}{\overset{-\otimes^{\mathbf{L}}_\Gamma \Gamma e\Gamma}{\leftleftarrows \!\!\! \xrightarrow{j^!}}} \mathrm{Tria}_{\mathscr{D}(\Gamma\text{-Mod})}(\Gamma e\Gamma)$$

where $\mathrm{Tria}_{\mathscr{D}(\Gamma)}(\Gamma e\Gamma)$ is the smallest full triangulated subcategory of $\mathscr{D}(\mathrm{Mod}-\Gamma)$ containing $\Gamma e\Gamma$ and closed under small coproducts, i_* is the inclusion functor, $-\bigotimes^{\mathbf{L}}_\Gamma \Gamma/\Gamma e\Gamma$, is the left derived functor of $-\bigotimes_\Gamma \Gamma/\Gamma e\Gamma$, $-\bigotimes^{\mathbf{L}}_\Gamma \Gamma e\Gamma$ is the left derived functor of $-\bigotimes_\Gamma \Gamma e\Gamma$ and $\mathbf{R}\mathrm{Hom}_\Gamma(\Gamma/\Gamma e\Gamma,-)$ is the right derived functor of $\mathrm{Hom}_\Gamma(\Gamma/\Gamma e\Gamma,-)$.

Note that $\Gamma/\Gamma e\Gamma$ has finite projective dimension as a right Γ-module. Then, by Lemma 4.13 and [15, Corollary 3 and Example 6], there is a recollement:

$$\mathscr{D}^-(\mathrm{Mod}-\Gamma/\Gamma e\Gamma) \underset{\mathbf{R}\mathrm{Hom}_\Gamma(\Gamma/\Gamma e\Gamma,-)}{\overset{-\otimes^{\mathbf{L}}_\Gamma \Gamma/\Gamma e\Gamma}{\leftleftarrows \!\!\! \xrightarrow{i_*}}} \mathscr{D}^-(\mathrm{Mod}-\Gamma) \overset{-\otimes^{\mathbf{L}}_\Gamma \Gamma e\Gamma}{\leftleftarrows \!\!\! \rightarrow} \mathscr{D}^-(C)$$

where C is the dg algebra $\mathcal{C}_{dg}\Gamma(i\Gamma e\Gamma, i\Gamma e\Gamma)$ and $i\Gamma e\Gamma$ is an injective resolution of the right Γ-module $\Gamma e\Gamma$. Note that $\Gamma e\Gamma$ is isomorphic to $(e\Gamma)^n$ as right Γ-module. By [11, Theorem 9.2], there is a triangle equivalence between $\mathscr{D}^-(C)$ and $\mathscr{D}^-(\mathrm{Mod}-H^0(C)) = \mathscr{D}^-(\mathrm{Mod}-e\Gamma e)$. Hence there is a recollement:

$$\mathscr{D}^-(\mathrm{Mod}-\Gamma/\Gamma e\Gamma) \underset{\mathbf{R}\mathrm{Hom}_\Gamma(\Gamma/\Gamma e\Gamma,-)}{\overset{-\otimes^{\mathbf{L}}_\Gamma \Gamma/\Gamma e\Gamma}{\leftleftarrows \!\!\! \xrightarrow{i_*}}} \mathscr{D}^-(\mathrm{Mod}-\Gamma) \underset{j_*}{\overset{-\otimes^{\mathbf{L}}_{e\Gamma e} e\Gamma}{\leftleftarrows \!\!\! \xrightarrow{j^!}}} \mathscr{D}^-(\mathrm{Mod}-e\Gamma e)$$

where i_* is inclusion functor, $j^! = \mathbf{R}\mathrm{Hom}_\Gamma(e\Gamma,-)$ is the right derived functor of $\mathrm{Hom}_\Gamma(e\Gamma,-)$, $j_* = \mathbf{R}\mathrm{Hom}_{e\Gamma e}(\Gamma e,-)$ is the right derived functor of $\mathrm{Hom}_{e\Gamma e}(\Gamma e,-)$, $-\bigotimes^{\mathbf{L}}_\Gamma \Gamma/\Gamma e\Gamma$ is the left derived functor of $-\bigotimes_\Gamma \Gamma/\Gamma e\Gamma$, $-\bigotimes^{L}_{e\Gamma e} e\Gamma$ is the left derived functor of $-\bigotimes_{e\Gamma e} e\Gamma$. Since $\mathrm{proj.dim}({}_{A_1}J_{1,k}) < \infty, 2 \leqslant k \leqslant n$, we have that $\Gamma e\Gamma$ have finite projective dimension as a left Γ-module. Thus, the functors $-\bigotimes^{\mathbf{L}}_\Gamma \Gamma/\Gamma e\Gamma$ and $-\bigotimes^{\mathbf{L}}_{e\Gamma e} e\Gamma$ send complexes of bounded homology to complexes of bounded homology. Note that $e\Gamma$ and $\Gamma/\Gamma e\Gamma$ have finite projective dimension as right Γ-modules. Then the functors $\mathbf{R}\mathrm{Hom}_\Gamma(e\Gamma,-)$ and $\mathbf{R}\mathrm{Hom}_\Gamma(\Gamma/\Gamma e\Gamma,-)$ restrict to the functors

$\mathscr{D}^b(\mathrm{Mod}-\Gamma) \to \mathscr{D}^b(\mathrm{Mod}-e\Gamma e)$ and $\mathscr{D}^b(\mathrm{Mod}-\Gamma) \to \mathscr{D}^b(\mathrm{Mod}-\Gamma/\Gamma e\Gamma)$ respectively. Then, we can get a recollement:

$$\begin{tikzcd}[column sep=huge] \mathscr{D}^b(\mathrm{Mod}-\Gamma/\Gamma e\Gamma) \arrow[r, "i_*"] & \mathscr{D}^b(\mathrm{Mod}-\Gamma) \arrow[l, bend right, "-\otimes^{\mathbf{L}}_{\Gamma}\Gamma/\Gamma e\Gamma"'] \arrow[l, bend left, "\mathbf{R}\mathrm{Hom}_{\Gamma}(\Gamma/\Gamma e\Gamma,-)"] \arrow[r, "j^!"] & \mathscr{D}^b(\mathrm{Mod}-e\Gamma e) \arrow[l, bend right, "-\otimes^{\mathbf{L}}_{e\Gamma e}e\Gamma"'] \arrow[l, bend left, "j_*"] \end{tikzcd} . \qquad (*)$$

Since Γ is an Artin algebra and the functors, appearing in $(*)$, take finitely generated modules to finite generated modules, it follows that the following diagram is a recollement [1].

$$\begin{tikzcd} \mathscr{D}^b(\Gamma/\Gamma e\Gamma) \arrow[r] & \mathscr{D}^b(\Gamma) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}^b(e\Gamma e) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

By Lemma 4.12, $\mathrm{fin.dim}(\Gamma) < \infty$ if and only if $\mathrm{fin.dim}(A_1) < \infty$ and $\mathrm{fin.dim}(\Gamma/\Gamma e\Gamma) < \infty$. Let Γ_{n-1} denote $\Gamma/\Gamma e\Gamma$. By similar discussion, we can get $\mathrm{fin.dim}(\Gamma) < \infty$ if and only if $\mathrm{fin.dim}(A_1) < \infty$ and $\mathrm{fin.dim}(A_i/J_{i-1,i}) < \infty$ for $2 \leqslant i \leqslant n$. $\square$

By Claim, $\mathrm{fin.dim}(\Phi_{n-1}) < \infty$ if and only if $\mathrm{fin.dim}(A/I_{i,i+1}) < \infty$ for $1 \leqslant i \leqslant n-1$. By Lemma 4.3, $\mathrm{fin.dim}(A) \leqslant \mathrm{fin.dim}(\Sigma_1) \leqslant \mathrm{fin.dim}(A) + \mathrm{fin.dim}(\Phi_{n-1}) + 1$. By Claim, $\mathrm{fin.dim}(A/I_{i,i+1}) < \infty$, $\mathrm{fin.dim}(A) < \infty$, $1 \leqslant i \leqslant n-1$ imply $\mathrm{fin.dim}(\Sigma_1) < \infty$. By Lemma 4.1, we can get that $\mathrm{fin.dim}(\Sigma_1) < \infty$ implies $\mathrm{fin.dim}(\Phi) < \infty$. $\square$

The following is a typical case of Theorem 4.14.

Corollary 4.15. *Set*

$$\Phi = \begin{pmatrix} A & \mathrm{rad}\,A & I_{1,3} & \cdots & & I_{1,n} \\ A & A & \mathrm{rad}\,A & \ddots & & \vdots \\ A & A & A & \ddots & & \vdots \\ \vdots & \vdots & \ddots & & \ddots & I_{n-2,n} \\ \vdots & \vdots & & & \ddots & \mathrm{rad}\,A \\ A & A & \cdots & & A & A \end{pmatrix}.$$

Suppose that Φ is an Artin algebra, $I_{i,j}$ is an ideal of A for $i, j = 1, 2, \ldots, n$. If $\mathrm{fin.dim}(A) < \infty$, *then* $\mathrm{fin.dim}(\Phi) < \infty$.

Proof. Let $I_{i,i+1} = \mathrm{rad}\,A$ for $i = 1, 2, \ldots, n-1$. $\square$

5. Examples

In this section, we display several examples to illustrate our theorem.

Example 1. Let A be a noetherian ring with ideal I. Let Γ be the ring

$$\Gamma = \begin{pmatrix} A & I & I^2 & I^3 \\ A & A & I^2 & I^3 \\ A & I & A & I^3 \\ A & I & I & A \end{pmatrix}.$$

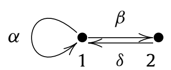
By Proposition 4.9, $\max\{l.\mathrm{gl.dim}(A), l.\mathrm{gl.dim}(A/I), l.\mathrm{gl.dim}(A/I^2), l.\mathrm{gl.dim}(A/I^3)\} - 1 \leqslant l.\mathrm{gl.dim}(\Gamma) \leqslant l.\mathrm{gl.dim}(A) + l.\mathrm{gl.dim}(A/I) + l.\mathrm{gl.dim}(A/I^2) + l.\mathrm{gl.dim}(A/I^3) + 4$.

Example 2. Let $A = k[x]/(x^n)$ for $n \geqslant 1$, and $I = \mathrm{rad}(A)$. Let

$$\Lambda = \begin{pmatrix} A & I & I^2 & I^3 \\ A & A & I & I^3 \\ A & A & A & I \\ A & A & A & A \end{pmatrix}.$$

Since $k[x]/(x^n)$ is representation-finite, the finitistic dimension of $k[x]/(x^n)$ is finite. By Corollary 4.15, $\mathrm{fin.dim}(\Lambda) < \infty$.

Example 3. Let A be a k-algebra given by the following quiver



with relations $\{\alpha^3 = \beta\delta, \alpha\beta = 0, \delta\alpha = 0\}$.

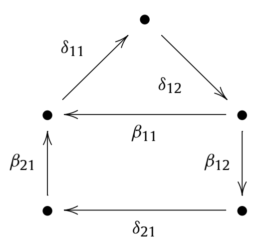
Then A can be represented as the following matrix form:

$$A = \begin{pmatrix} k[\alpha]/(\alpha)^4 & k\beta \\ k\delta & k[\delta\beta]/(\delta\beta)^2 \end{pmatrix}.$$

Suppose that $P(3,2)$ is the block extension of A.

$$P(3,2) = \begin{pmatrix} k[\alpha]/(\alpha)^4 & k[\alpha]/(\alpha)^4 & k[\alpha]/(\alpha)^4 & k\beta & k\beta \\ (\alpha)/(\alpha)^4 & k[\alpha]/(\alpha)^4 & k[\alpha]/(\alpha)^4 & k\beta & k\beta \\ (\alpha)/(\alpha)^4 & (\alpha)/(\alpha)^4 & k[\alpha]/(\alpha)^4 & k\beta & k\beta \\ k\delta & k\delta & k\delta & k[\delta\beta]/(\delta\beta)^2 & k[\delta\beta]/(\delta\beta)^2 \\ k\delta & k\delta & k\delta & (\delta\beta)/(\delta\beta)^2 & k[\delta\beta]/(\delta\beta)^2 \end{pmatrix}.$$

$P(3,2)$ can be described by the quiver



with relations $\{e(\alpha^3) = \beta_{11}\delta_{11}\delta_{12}\beta_{11}\delta_{11}\delta_{12}\beta_{11} = \beta_{12}\delta_{21}\beta_{21} = e(\beta\delta), e(\alpha\beta) = \beta_{11}\delta_{11}\delta_{12}\beta_{12} = 0, e(\delta\alpha) = \beta_{21}\delta_{11}\delta_{12}\beta_{11} = 0\}$, where e is the extension map defined in [21]. By Corollary 4.8, we have $\mathrm{fin.dim}\, P(3,2) \leqslant 4$.

Acknowledgments

I want to express my gratitude to my supervisor Professor Changchang Xi for encouragement and useful suggestions. I also want to express my gratitude to the referee for his (her) careful reading and valuable comments.

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