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Strict Archimedean t-norms and t-conorms as universal approximators

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Abstract

In knowledge representation, when we have to use logical connectives, various continuous t-norms and t-conorms are used. In this paper, we show that every continuous t-norm and t-conorm can be approximated, to an arbitrary degree of accuracy, by a strict Archimedean t-norm (t-conorm).

1. Introduction

Brief idea. When we represent expert knowledge in expert systems and in intelligent control, it is important to adequately describe not only the experts' statements themselves, but also the experts' degrees of confidence in the corresponding statements. It is also important to adequately describe which operations with these degrees of confidence are best representing the expert's use of logical connectives "and" and "or". The experimental determination of these "and" and "or" operations (known as t-norms and t-conorms) is a very complicated task because, in principle, very complicated operations are possible. Do we really need all these complicated operations, or some simple subclass is sufficient?

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In this paper, we show that operations from a certain known class (*strictly Archimedean* operations) can approximate an arbitrary operation with an arbitrary accuracy. This means that whatever the actual t-norm and t-conorm are, we can, with an arbitrary accuracy, approximate then with strict Archimedean operations.

Thus, strict Archimedean t-norms and t-conorms are sufficient for describing expert knowledge.

t-norms and t-conorms. To design intelligent systems capable of performing complicated tasks on par with the best human experts, we must represent the knowledge of these experts in the computer. This knowledge consists of different statements.

Not all these statements have equal weight to the experts: experts may be absolutely sure in some of them, and much less sure in others. Therefore, when we represent expert knowledge in expert systems and in intelligent control, it is important to adequately describe not only the experts' statement, but also the experts' degrees of confidence in the corresponding statements. These degrees of belief are usually represented by numbers from the interval $[0, 1]$ so that 1 corresponds to "absolutely sure", and 0 to no belief at all (see, e.g., [1,2]).

In human reasoning, we combine different statements by using different logical connectives. For example, we may argue about A and B being true, or about A or B taking place. To be able to adequately deal with such logical combinations, we must be able to estimate degrees of belief in these logical combinations. If we are either absolutely sure, or have absolutely no belief in each of these statements, then we can use the rules of classical 2-valued logic to compute the degree of belief in the composite statements $A\&B$ and $A \vee B$.

In order to handle the frequent situations when we are not 100% sure in A and B, we must be able, given the degrees of belief $d(A)$ and $d(B)$ in A and B, to estimate the degree of belief in the composite statements $d(A\&B)$ and $d(A \vee B)$. In other words, we must have two *functions* $f_\&(a, b)$ and $f_\vee(a, b)$ that, given $d(A)$ and $d(B)$, return an estimate $f_\&(d(A), d(B))$ for the degree of belief in $A\&B$ and and estimate $f_\vee(d(A), d(B))$ for the degree of belief in $A \vee B$.

These functions must satisfy several natural properties: e.g., since $A\&B$ means, intuitively, the same as $B\&A$, it is reasonable to expect that the estimates for the degrees of belief in $A\&B$ and in $B\&A$ be the same, i.e., that $f_\&(d(A), d(B)) = f_\&(d(B), d(A))$ for all A and B. Since the statements A and B can have arbitrary degrees of belief a and b, this property means, in effect, that we must have $f_\&(a, b) = f_\&(b, a)$ for arbitrary a and b.

Similarly, from the fact that $A\&(B\&C)$ and $(A\&B)\&C$ mean the same thing, we conclude that $f_\&(a, f_\&(b, c)) = f_\&(f_\&(b, c))$ for all real numbers a, b, and c.

Functions that satisfy these properties are called t-norms and t-conorms (for completeness, precise definitions are given in Section 2).

It is important to choose *t-norms* and *t-conorms* properly. It is often extremely important to choose t-norm and t-conorm properly:

- Historically the first successful expert system MYCIN became successful when its authors managed to find (after a tremendous effort) "and" and "or" operations that adequately describe medical experts [3,4]. At first, they thought that these operations constitute a universal law of human reasoning, but it turned out that for other applications, e.g., for applications in geophysics, radically different operations are needed.
- Different "and" and "or" operations lead to radically different results in fuzzy control (see, e.g., [5]).

t-norms and t-conorms are difficult to determine; how can we make eliciting them easier? It is rather difficult to determine a t-norm and a t-conorm, for two reasons:

- First, for that, we need to query lots of experts, and then process the resulting data. This takes quite some time [3,4]. This difficulty is, probably, unavoidable.
- Second, in general, t-norms and t-conorms can be very complicated. The task of eliciting t-norms and t-conorms from the experts will be much easier if we were able to show that only simple t-norms and t-conorms have to be considered as possible options.

How complicated are the general t-norms and t-conorms? According to the classification theorem [6], every t-norm (correspondingly, every t-conorm) can be represented as a kind of combination of Archimedean t-norms (t-conorms), strict and non-strict (see Section 2 below). According to this classification result, Archimedean t-norms and t-conorms are the basic tools from which more general ones are built. In this sense, Archimedean t-norms and t-conorms are the simplest.

In this paper, we will show that these simplest (strictly Archimedean) t-norms and t-conorms are sufficient in the sense that every continuous t-norm and t-conorm can be approximated, to an arbitrary degree of accuracy, by a strict Archimedean t-norm (t-conorm). Thus, eliciting t-norms and t-conorms *can* be made easier.

What was known before? It is a well-known result (proven in 1963 [7]) that we can approximate, to an arbitrary degree of accuracy, the t-norm $f_\&(a,b) = \min(a,b)$ by strictly Archimedean t-norms, and $f_\vee(a,b) = \max(a,b)$ by strictly Archimedean t-conorms.

In this paper, we generalize this result by showing that an *arbitrary* continuous t-norm (t-conorm) can be approximated by strictly Archimedean t-norms (t-conorms).

2. Definitions

Definition 2.1 (see, e.g., [1,2]). A function $f_\&: [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following four conditions:

- $f_{\&}(1,a) = a$ for all a;
- $f_{\&}(a,b) = f_{\&}(b,a)$ for all a and b;
- $f_{\&}(a, f_{\&}(b,c)) = f_{\&}(f_{\&}(a,b),c))$ for all a, b, and c;
- if $a \leqslant a'$ and $b \leqslant b'$, then $f_{\&}(a,b) \leqslant f_{\&}(a',b')$.

Definition 2.2 (see, e.g., [1,2]). A function $f_{\vee}\colon [0,1] \times [0,1] \to [0,1]$ is called a t-conorm if it satisfies the following four conditions:
- $f_{\vee}(1,a) = a$ for all a;
- $f_{\vee}(a,b) = f_{\vee}(b,a)$ for all a and b;
- $f_{\vee}(a, f_{\vee}(b,c)) = f_{\vee}(f_{\vee}(a,b),c))$ for all a, b, and c;
- if $a \leqslant a'$ and $b \leqslant b'$, then $f_{\vee}(a,b) \leqslant f_{\vee}(a',b')$.

It is also usually required that a t-norm and a t-conorm are *continuous* functions.

Of all possible continuous t-norms and t-conorms, the most widely used are the *idempotent* operations $f_{\&}(a,b) = \min(a,b)$ and $f_{\vee}(a,b) = \max(a,b)$ and *Archimedean* t-norms and t-conorms that are defined as follows:

Definition 2.3 [1,2].
- A t-norm $f_{\&}(a,b)$ is called Archimedean if it is continuous and $f_{\&}(a,a) < a$ for all $a \in (0,1)$.
- An Archimedean t-norm is called strictly Archimedean if it is strictly increasing in each variable for $a, b \in (0,1)$.

Definition 2.4 [1,2].
- A t-conorm $f_{\vee}(a,b)$ is called Archimedean if it is continuous and $f_{\vee}(a,a) > a$ for all $a \in (0,1)$.
- An Archimedean t-norm is called strictly Archimedean if it is strictly increasing in each variable for $a, b \in (0,1)$.

Strictly Archimedean t-norms and t-conorms are easy to represent:

Proposition 2.1 [1,2,6,7].
- *For every continuous strictly increasing function* $\psi\colon [0,1] \to [0,1]$, *the function* $f_{\&}(a,b) = \psi^{-1}(\psi(a) \cdot \psi(b))$ *is a strictly Archimedean t-norm.*
- *If* $f_{\&}(a,b)$ *is a strictly Archimedean t-norm, then there exists a continuous strictly increasing function* $\psi\colon [0,1] \to [0,1]$ *for which* $f_{\&}(a,b) = \psi^{-1}(\psi(a) \cdot \psi(b))$.

A similar representation exists for strictly Archimedean t-conorms.

3. Main results

Definition 3.1. We say that two functions $f(a,b)$ and $f'(a,b)$ are ε-close if for every a and b, we have $|f(a,b) - f'(a,b)| \leqslant \varepsilon$.

Theorem 3.1. *For every continuous t-norm $f_\&$, and for every $\varepsilon > 0$, there exists a strictly Archimedean t-norm $f'_\&$ that is ε-close to $f_\&$.*

Theorem 3.2. *For every continuous t-conorm $f_\vee$, and for every $\varepsilon > 0$, there exists a strictly Archimedean t-norm $f'_\vee$ that is ε-close to $f_\vee$.*

Since the real data always come with some accuracy, these results mean that whatever empirical data we have about the actual expert's use of "and" and "or", and however accurate these data are, these data can always be explained within an assumption that both the "and"-operation (t-norm) and the "or"-operation (t-conorm) are strictly Archimedean.

Thus, to explain arbitrarily complicated human reasoning, it is quite sufficient to use strictly Archimedean t-norms and t-conorms.

Comment: After this paper was submitted to the journal, we learned that a similar (somewhat weaker) result by Fodor and Janei was announced by in [8]: namely, the main result from that paper states that every continuous t-norm can be approximated, with arbitrary accuracy, by continuous Archimedean t-norms that are *not necessarily strictly* Archimedean, while we prove the possibility of approximating an arbitrary continuous t-norm by *strictly* Archimedean t-norms.

4. Proof

4.1. General idea of the proof

The proof of Theorems 3.1 and 3.2 is based on the *classification theorem* for t-norms and t-conorms that was first proven in [6]. According to this theorem, for every t-norm $f_\&(a,b)$, on the interval $[0,1]$, there exists finitely or countably many (possibly none) non-intersecting intervals I_α such that:
- on each of these intervals I_α, $f_\&(a,b)$ is:
  - either isomorphic to $a \cdot b$, i.e., has the form $\psi^{-1}(\psi(a)\psi(b))$ for some strictly increasing function ψ),
  - or isomorphic to $\max(a+b-1,0)$, i.e., has the form

    $$\psi^{-1}(\max\ (\psi(a) + \psi(b) - 1, 0))$$

    for some strictly increasing function ψ;

- if a and b do not belong to the same interval I_α, or if one of the values a, b does not belong to any of the intervals I_α at all, then $f_\&(a,b) = \min(a,b)$.

  Comment: In particular, if we have no intervals at all, we get a t-norm $f_\&(a,b) = \min(a,b)$; to get a t-norm $f_\&(a,b) = a \cdot b$, we must take the entire interval $[0, 1]$ as the only interval I_α.

A similar classification theorem for t-conorms can be easily deduced from the fact that:
- for every t-norm $f_\&(a,b)$, its *dual* $f_\vee(a,b) = 1 - f_\&(1-a, 1-b)$ is a t-conorm; and
- vice versa, for every t-conorm $f_\vee(a,b)$, its *dual*

$$f_\&(a,b) = 1 - f_\vee(1-a, 1-b)$$

is a t-norm.

The desired approximation result says that an arbitrary (and arbitrarily complicated) t-norm can be approximated, with an arbitrary accuracy, by a strictly Archimedean t-norm. We will prove this result step-by-step:
- First, we will show that an arbitrary t-norm can be approximated, with an arbitrary accuracy, by a t-norm that has only finitely many intervals.
- Then, we will show that an arbitrary t-norm with finitely many intervals can be approximated, with an arbitrary accuracy, by a t-norm in which these intervals constitute the entire interval $[0, 1]$, and in which on each interval, the t-norm is isomorphic to $a \cdot b$.
- Finally, we will show that a t-norm with $k > 1$ intervals on each of which this t-norm is isomorphic to $a \cdot b$, can be approximated, with an arbitrary accuracy, by a t-norm with the same property, but with only $k - 1$ intervals. By repeating the last reduction finitely many times, we will finally get an approximating t-norm that has only one interval: $[0, 1]$, and that is isomorphic to $a \cdot b$, i.e., that is strictly Archimedean.

If, on each of these three mega-steps, we choose an approximation with an accuracy $\delta = \varepsilon/3$, then after these three steps, we get a t-norm that approximates the original one with the desired accuracy ε.

Similarly, to achieve the accuracy $\varepsilon/3$ on the their megastep, we must, on each substep of this mega-step, take an approximation with an accuracy $\varepsilon/(3N)$, where N is the number of intervals at the beginning of this mega-step.

Comment: It is sufficient to be able to approximate t-norms. Indeed, if we can approximate an arbitrary t-norm $f_\&$ by an ε-close strictly Archimedean t-norm $f'_\&$, then, given an arbitrary t-conorm $f_\vee$, we will be able to approximate its dual $f_\&(a,b) = 1 - f_\vee(1-a, 1-b)$ by an ε-close strictly Archimedean t-norm $f'_\&(a,b)$. One can then easily show that the dual $f'_\vee$ to $f'_\&$ is a strictly Archimedean t-conorm that is ε-close to the original t-conorm $f_\vee(a,b)$ (because two t-conorms are ε-close iff their duals are ε-close, and vice versa).

4.2. Step 1: Reduction to finitely many intervals

Let us show how to approximate an arbitrary t-norm $f_{\&}$ with an arbitrary accuracy $\delta > 0$, by a t-norm whose classification requires only finitely many intervals.

Indeed, since the intervals I_α that characterize the original t-norm are all located within the interval $[0, 1]$, and these intervals do not intersect with each other, the total number of intervals I_α whose length is $\geqslant \delta$ is finite ($\leqslant 1/\delta$).

We can thus define a new t-norm $f'_{\&}(a,b)$ as follows:
- if in the characterization of $f_{\&}$, the numbers a and b belong to the same interval I_α of length $\geqslant \delta$, then $f'_{\&}(a,b) = f_{\&}(a,b)$;
- for all other pairs (a,b), $f'_{\&}(a,b) = \min(a,b)$.

It is clear that the new t-norm $f'_{\&}$ can be characterized in the same manner as the original t-norm $f_{\&}(a,b)$, but with only finitely many intervals I'_α. So, to prove that this first step does do the desired approximation, it is sufficient to show that the new t-norm $f'_{\&}(a,b)$ is δ-close to the original one, i.e., that $|f'_{\&}(a,b) - f_{\&}(a,b)| \leqslant \delta$ for all a and b.

Indeed, the only case when the difference $f'_{\&}(a,b) - f_{\&}(a,b)$ is different from 0 (i.e., for which $f'_{\&}(a,b) \neq f_{\&}(a,b)$) is when both a and b belong to one of the original intervals $[a^-, a^+]$ of width $a^+ - a^- < \delta$. In this case, $a^- \leqslant f_{\&}(a,b) \leqslant a^+$. Similarly, $f'_{\&}(a,b) = \min(a,b)$ also belongs to the interval $[a^-, a^+]$. So, $f_{\&}(a,b)$ and $f'_{\&}(a,b)$ are two numbers on the same interval $[a^-, a^+]$ of width $< \delta$. Thus, the difference between these two numbers cannot exceed the width of this interval, and is, therefore $< \delta$.

So, $f_{\&}$ and $f'_{\&}$ are, indeed, δ-close. The first part is proven.

4.3. Step 2: Reduction to t-norms that are strictly Archimedean on each interval

Let us start with a t-norm $f_{\&}$ that has finitely many intervals I_α. Since there are finitely many intervals, the space between and outside these intervals I_α (if there is any space left) is also a union of finitely many intervals, on each of which $f_{\&}(a,b) = \min(a,b)$. Let us add these new intervals to the intervals I_α that characterize the t-norm $f_{\&}(a,b)$. When combined, the intervals from this enlarged set $\{J_\alpha\}$ cover the entire interval $[0, 1]$.

We will now show that it is possible to approximate the t-norm $f_{\&}$ by a new t-norm $f'_{\&}$, with the same (extended) set of intervals $\{J_\alpha\}$, but for which on each of these intervals, the t-norm is isomorphic to $a \cdot b$.

We will approximate the original t-norm interval-by-interval. (This is OK, since the values of the two t-norms that are characterized by the same intervals are only different when both a and b belong to the same interval; otherwise, we have $f_{\&}(a,b) = f'_{\&}(a,b) = \min(a,b)$.) These intervals $[a^-, a^+]$ are of two types:
- intervals on which $f_{\&}(a,b) = \min(a,b)$;
- intervals on which $f_{\&}(a,b)$ is isomorphic to $\max(a + b - 1, 0)$.

Let us show how we can approximate intervals of both types.

First, we reduce a t-norm defined on each interval to a t-norm defined on the interval $[0,1]$. Indeed, there exists an easily computable linear transformation $L(x) = (x - a^-)/(a^+ - a^-)$ that maps the interval $[a^-, a^+]$ onto $[0,1]$:
- if $a \in [a^-, a^+]$, then $L(a) = (a - a^-)/(a^+ - a^-) \in [0,1]$; and, vice versa,
- if $A \in [0,1]$, then $L^{-1}(A) = a^- + A \cdot (a^+ - a^-) \in [a^-, a^+]$.

Thus:
- if $f_\&(a,b)$ is a t-norm on the interval $[a^-, a^+]$ (i.e., a function $[a^-, a^+] \times [a^-, a^+] \to [a^-, a^+]$), then the operation

$$F_\&(A,B) = L(f_\&(L^{-1}(A), L^{-1}(B))$$

  is a t-norm on the interval $[0,1]$; and, vice versa,
- if $F_\&(A,B)$ is a t-norm on the interval $[0,1]$, then the operation $f_\&(a,b) = L^{-1}(F_\&(L(a), L(b))$ is a t-norm on $[a^-, a^+]$.

Hence, if we will be able to approximate the t-norm $F_\&(A,B)$ on the interval $[0,1]$ by a close strictly Archimedean t-norm $F'_\&(A,B)$, then the corresponding operation $f'_\&(a,b) = L^{-1}(F'_\&(L(a), L(b))$ on $[a^-, a^+]$ will be close to the original t-norm.

So, it is sufficient to approximate the t-norm $F_\&(A,B)$ defined on the interval $[0,1]$. Depending on whether $f_\&$ (and, hence, $F_\&$) is isomorphic to min or to $\max(A + B - 1, 0)$, we get two different approximations:
- The function $F_\&(A,B) = \min(A,B)$ can be represented as

$$\exp(-\max(|\ln(A)|, |\ln(B)|).$$

Since $\max(x,y) = \lim_{p\to\infty}(x^p + y^p)^{1/p}$, we can, with an arbitrary accuracy, approximate $\min(A,B)$ by

$$F'_\&(A,B) = \exp(-(|\ln(A)|^p + |\ln(B)|^p)^{1/p}).$$

(this approximation was proposed by Schweizer and Sklar [7]). This new function is isomorphic to $A \cdot B$, with the isomorphism given by a function $\psi(A) = \exp(-|\ln(A)|^p)$. The larger p, the better the approximation. So, for sufficiently large p, we can get an arbitrarily close approximation.
- For operations that are isomorphic to $\max(A + B - 1, 0)$, it is somewhat easier to describe an approximating t-norm by describing a dual approximation: to the dual t-conorm that is isomorphic to $N(A,B) = \min(A + B, 1)$.

  Isomorphic means that we have a function $\psi: [0,1] \to [0,1]$ that implements the desired isomorphism, i.e., for which,

$$F_\&(A,B) = \psi^{-1}(N(\psi(A), \psi(B))) = \psi^{-1}(\min(\psi(A) + \psi(B), 1)).$$

It is easy to see that if we find a sequence $N_n(A,B)$ of strictly Archimedean t-norms that tend to $N(A,B)$ (in the uniform metric), then the corresponding isomorphic operations $\psi^{-1}(N_n(\psi(A), \psi(B)))$ will tend to $\psi^{-1}(N(\psi(A), \psi(B)))$

$= F_{\&}(A, B)$. Thus, to be able to approximate an arbitrary t-norm that is isomorphic to N, it is sufficient to be able to approximate $N(A, B)$ itself.

This can be done as follows: we choose $\alpha \to 0$, and approximate $N(a, b)$ by a strict Archimedean operation $G^{-1}(G(A) + G(B))$, where $G(A) = A/(1 - \alpha)$ for $A \leqslant 1 - \alpha$ and $G(A) = 1 - \alpha + \alpha/(1 - A)$ for $A \geqslant 1 - \alpha$. This operation coincides with $\min(A + B, 1)$ when $A + B \leqslant 1 - \alpha$, and leads to the results between $1 - \alpha$ and 1 when $A + B \geqslant 1 - \alpha$. Thus, when $\alpha \to 0$, this operation tends to $N(A, B)$. From this approximation of a dual operation, we can easily obtain the approximation of the original t-norm.

Step 2 is proven.

4.4. Step 3: Reduction to a t-norm with one fewer interval

We want to get a reduction from a t-norm that has k intervals to a t-norm that has $k - 1$ intervals. To achieve this goal, it is sufficient to show that a t-norm that has two intervals can be approximated by a t-norm that has only one interval. By using this construction, we will be able to "merge" the two neighboring intervals and thus, reduce the number of intervals by one.

Let us consider the case when on two neighboring intervals, we have strictly Archimedean operations. Similarly to Step 2, we can prove that it is sufficient to consider the case when these two intervals form the interval $[0, 1]$, i.e., when the first interval is $[0, p]$ and the second interval is $[p, 1]$ for some boundary point $p \in (0, 1)$.

It is known that every continuous function on a compact is uniformly continuous. In particular, the function $f_{\&}(a, b)$, is uniformly continuous, so, there exists a $v > 0$ such that if $|b - b'| \leq v$, then $|f_{\&}(a, b) - f_{\&}(a, b')| \leq \delta/3$. Let us take $p^- = p - \min(\delta/3, v)$; then, $p - \delta/3 \leq p^- < p$, and for every a, we have $|f_{\&}(a, p^-) - f_{\&}(a, p)| \leq \delta/3$. Since the point p is the endpoint of the first interval, we have $f_{\&}(a, p) = a$, so $|f_{\&}(a, p^-) - a| \leq \delta/3$. As p^+, we will take $p^+ = \min(p + \delta/3, (1 + p)/2)$. Then, $p < p^+ \leq p + \delta/3$.

Since the operation $f_{\&}$ is strictly Archimedean on both subintervals, it is isomorphic to $a \cdot b$ on both of them. In other words, there exist functions $\psi_1 : [0, p] \to [0, 1]$ and $\psi_2 : [p, 1] \to [0, 1]$ such that for a, b from the first interval $[0, p]$, we have $f_{\&}(a, b) = \psi_1^{-1}(\psi_1(a) \cdot \psi_1(b))$, while for a and b from the second interval $[p, 1]$, we have $f_{\&}(a, b) = \psi_2^{-1}(\psi_2(a) \cdot \psi_2(b))$.

We want to "merge" these two representations into a single formula that is close to the original two-part operation. For that merger, we will take into consideration the fact that a function ψ_i is not uniquely determined by the t-norm $f_{\&}$: the same t-norm can be obtained if we use a function $\psi_i'(x) = (\psi_i(x))^{r_i}$ for any positive real number r_i.

When $r_i \to \infty$, we have $(\psi_i(x))^{r_i} \to 0$; when $r_i \to 0$, we have $(\psi_i(x))^{r_i} \to 1$. Thus, to achieve a merger, we choose r_1 large enough so that $(\psi_1(x))^{r_1} \leq 1/3$

for all $x \in [0, p^-]$, and we choose r_2 small enough so that $(\psi_2(x))^{r_2} \geq 2/3$ for all $x \in [p^+, 1]$.

Then, we take a monotonic function $\psi(x)$ that is:
- equal to $(\psi_1(x))^{r_1}$ for $x \in [0, p^-]$,
- equal to $(\psi_2(x))^{r_2}$ for $x \in [p^+, 1]$, and
- linear on the remaining (small) interval $[p^-, p^+]$,

and define the new operation

$$f'_{\&}(a, b) = \psi^{-1}(\psi(a) \cdot \psi(b)).$$

Let us show that for all a and b, the values of $f_{\&}(a, b)$ and $f'_{\&}(a, b)$ are δ-close. To prove this closeness, let us consider all possible cases, when $a, b \in [0, p^-], [p^-, p], [p, p^+], [p^+, 1]$. Due to symmetry of a t-norm, it is sufficient to consider $a \leq b$.

- If a and b belong to the same interval $[0, p^-]$, then the new t-norm coincides with the old one.
- Let a belong to the interval $[0, p^-]$ and let b be from the interval $[p^-, p]$. Then, due to the monotonicity of a t-norm and to the property $f_{\&}(a, b) \leq a$, we have $f_{\&}(a, p^-) \leq f_{\&}(a, b) \leq a$, and due to our choice of p^-, we have $f_{\&}(a, p^-) \geq a - \delta/3$. Thus, $f_{\&}(a, b) \in [a - \delta/3, a]$. Similarly, $f'_{\&}(a, p^-) \leq f'_{\&}(a, b) \leq a$; since $a, p^- \in [0, p^-]$, we have $f'_{\&}(a, p^-) = f_{\&}(a, p^-) \geq a - \delta/3$, so $f'_{\&}(a, b)$ belongs to the same interval $[a - \delta/3, a]$ of width $\delta/3 < \delta$. The difference between the two values $f_{\&}(a, b)$ and $f'_{\&}(a, b)$ from this interval cannot exceed $\delta/3 < \delta$, so these two values are indeed δ-close.
- Let $a \in [0, p^-]$ and $b \in [p, 1]$. In this case, $f_{\&}(a, b) = a$ and $f'_{\&}(a, p^-) \leq f'_{\&}(a, b) \leq a$. Since $a, p^- \in [0, p^-]$, we have $f'_{\&}(a, p^-) = f_{\&}(a, p^-)$. Due to our choice of p^-, we have $f_{\&}(a, p^-) \geq a - \delta/3$. Thus, both values $f_{\&}(a, b)$ and $f'_{\&}(a, b)$ belong to the interval $[a - \delta/3, a]$ and hence, these values are δ-close. We have thus covered all the cases in which $a \in [0, p^-]$.
- Let now $a \in [p^-, p]$ and $b \in [p^-, p]$. Then, $f_{\&}(p^-, p^-) \leq f_{\&}(a, b) \leq a \leq p$. Due to our choice of p^-, we have $f_{\&}(p^-, p^-) \geq p^- - \delta/3$, and $p^- \geq p - \delta/3$. Thus, $f_{\&}(p^-, p^-) \geq p - 2\delta/3$. Thus, $f_{\&}(a, b) \in [p - 2\delta/3, p]$. Similarly, $f'_{\&}(p^-, p^-) \leq f'_{\&}(a, b) \leq p$, and since $f'_{\&}(p^-, p^-) = f_{\&}(p^-, p^-)$, we can also conclude that $f'_{\&}(a, b) \in [p - 2\delta/3, p]$. Thus, both $f_{\&}(a, b)$ and $f'_{\&}(a, b)$ belong to the interval $[p - 2\delta/3, p]$ and hence, they are δ-close.
- Let $a \in [p^-, p]$ and $b \in [p, 1]$. In this case, $f_{\&}(a, b) = a \in [p^-, p] \subseteq [p - \delta/3, p]$ and $f'_{\&}(p^-, p^-) \leq f'_{\&}(a, b) \leq a \leq p$. We already know that $f'_{\&}(p^-, p^-) = f_{\&}(p^-, p^-) \in [p - 2\delta/3, p]$. Thus, both values $f_{\&}(a, b)$ and $f'_{\&}(a, b)$ belong to the same interval $[p - 2\delta/3, p]$ and thus, are δ-close. We have covered all cases in which $a \in [p^-, p]$.
- Let now $a \in [p, p^+]$ and $b \in [p, 1]$. In this case, $p \leq f_{\&}(a, b) \leq a \leq p^- \leq p + \delta/3$, and $f'_{\&}(p^-, p^-) \leq f'_{\&}(a, b) \leq a \leq p + \delta/3$. We already know that $f'_{\&}(p^-, p^-) \geq p - 2\delta/3$. Thus, both values $f_{\&}(a, b)$ and $f'_{\&}(a, b)$ belong to the interval $[p - 2\delta/3, p + \delta/3]$ and hence, they are δ-close.

- The only remaining case is when both a and b belong to the same interval $[p^+, 1]$; then the new t-norm coincides with the old one.

Step 3 is proven, and so is the theorem. □

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