A PROOF BY GRAPHS THAT $\text{PSL}(2, 7) \cong \text{PSL}(3, 2)$

R.H. JEURISSEN

Mathematisch Instituut, Katholieke Universiteit, Toernooiveld, 6525 ED Nijmegen, The Netherlands

Received 5 March 1987

We give three definitions of the Coxeter graph. By the second one we see that $\text{PSL}(2, 7)$ is contained in the automorphism group of that graph as a subgroup of index 2, and by the third one that the same holds for $\text{PSL}(3, 2)$.

1. Three definitions of the Coxeter graph

1.1. First definition

This probably is the oldest one; it appears in [1] and [3]. There are 28 vertices $t_i, a_i, b_i, c_i, i = 0, 1, \ldots, 6$. For all $i$, $t_i$ is the center of a claw with endpoints $a_i, b_i$ and $c_i$, and $a_i$ is adjacent to $a_{i+1}$, $b_i$ to $b_{i+2}$ and $c_i$ to $c_{i+3}$ (indices mod 7). These are the only adjacencies, so the graph is regular of degree 3. Note that the $t_i$ span a 7-cycle, as well as the $b_i$ and the $c_i$, whereas the $t_i$ form a stable set.

1.2. Second definition

This is the one given in [2], slightly modified. One takes as vertices the 28 unordered pairs of points from $\text{PG}(1, 7)$. Vertices $\{a, b\}$ and $\{c, d\}$ are adjacent if and only if $(a, b, c, d)$ is a harmonic tetrad. It follows that the graph is regular of degree 3. To see that one has the same graph as in 1.1 denote the points of $\text{PG}(1, 7)$ by $0, 1, 2, 3, 4, 5, 6, \infty$ in the obvious way, and identify $t_i$ with $\{i, \infty\}$, $a_i$ with $\{2 + i, 5 + i\}$, $b_i$ with $\{3 + i, 4 + i\}$ and $c_i$ with $\{1 + i, 6 + i\}$, $i = 0, 1, \ldots, 6$, addition mod 7.

It is immediate that $\text{PGL}(2, 7)$, so also $\text{PSL}(2, 7)$, is faithfully represented as a group of automorphisms of the graph.

1.3. Third definition

Label the points of the Fano plane $\text{PG}(2, 2)$ by $0, 1, 2, \ldots, 6$ in such a way that $\{0, 1, 3\}$ and its cyclic shifts mod 7 ($\{1, 2, 4\}, \ldots, \{6, 0, 2\}$) are the collinear triples. As vertices take the other 28 unordered triples (i.e. the triangles). Define adjacency as disjointness. The graph is regular of degree 3. Again the graph is isomorphic to that of 1.1: identify $t_i$ with $\{0 + i, 4 + i, 6 + i\}$, $a_i$ with $\{1 + i, 3 +
It is immediate that the subgroup \( \text{PSL}(3, 2) \) of \( S_7 \) consisting of the collineations of the Fano plane induces a group of automorphisms of the graph. This representation is faithful, e.g. since only the trivial collineation fixes all vertices \( a_i \).

2. The isomorphism

\( \text{PSL}(2, 7) \) and \( \text{PSL}(3, 2) \) are both simple of order 168. To prove that they are isomorphic it now suffices to show that the automorphism group of the Coxeter graph has order \( \leq 336 \). (It then also follows from 1.2 that it is \( \text{PGL}(2, 7) \), the latter having order 336). This we see most easily from the first definition. The stabilizer of \( t_0 \) has index at most 28, in it the point-wise stabilizer of the neighbors \( a_0, b_0 \) and \( c_0 \) of \( t_0 \) has index at most 6, and in that one the subgroup \( H \) fixing both other neighbors \( a_i \) and \( a_6 \) of \( a_0 \) has index at most 2. Since \( 28 \cdot 6 \cdot 2 = 336 \), we have only to prove that \( H \) is trivial. Now \( a_1, a_2, a_3, a_4, a_5, a_6 \) is a path, and any other path from \( a_1 \) to \( a_6 \) of length 5 must contain two \( t \)-vertices, separated by two \( b \)- or two \( c \)-vertices. The only possibility is \( a_1, t_1, b_1, b_6, t_6, a_6 \). Now \( H \) fixes \( a_1 \) and \( a_6 \) and also \( b_0 \), which has distance 3 from \( a_2 \) (path \( b_0, b_2, t_2, a_2 \)) but not from \( t_1 \). So both paths are fixed by \( H \). But if all \( a_i \) are fixed the same holds for all \( t_i \). Moreover \( b_i \) is the unique common neighbor of \( t_i \) and \( b_{i-2} \), while \( b_0 \) is fixed. it follows that all \( b_i \) are fixed, and then the \( c_i \) are fixed, too.

3. Remarks

3.1

In the definition of adjacency in 1.2 we used harmonic tetrads instead of the arithmetic progressions that were used in [2], section 2 (however see also section 4, end). In this way it is less of a “miracle” that the elements of \( \text{PGL}(2, 7) \) induce automorphisms of the graph. It is still remarkable, however, that arithmetic progressions yield harmonic tetrads: this only happens for characteristic 7.

3.2

Since the stabilizer \( H \) of the path \( a_1, a_0, t_0, b_0 \) is the identity (see 2), the automorphism group acts regularly on the set of \( 28 \cdot 3 \cdot 2 \cdot 2 = 336 \) 3-paths. This fact is, conversely, used in [2] to show that the group has order 336.

3.3

In 1.3 the vertices can be considered as triangles in \( \text{PG}(2, 2) \), i.e. as certain sets of 3 points and 3 lines. Since two triangles have a common point iff they have a
common line, also correlations induce automorphisms of the graph. No correlation maps the sides of a triangle onto its points for all triangles, so the group of collineations and correlations is represented faithfully. This shows that it is isomorphic to $\text{PGL}(2,7)$.

References