The Gagliardo–Nirenberg inequalities and manifolds of non-negative Ricci curvature

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Abstract

This article is devoted to show that complete non-compact Riemannian manifolds with non-negative Ricci curvature of dimension greater than or equal to two in which some Gagliardo–Nirenberg type inequality holds are not very far from the Euclidean space.

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1. Introduction

Let $M$ be an $n$-dimensional complete non-compact Riemannian manifold and denote by $\nabla$ the gradient operator on $M$. Given positive numbers $p$ and $q$, we denote by $\mathcal{D}^{p,q}(M)$ the completion of the space of smooth compactly supported functions on $M$ for the norm $\|\cdot\|_{p,q}$ defined by $\|u\|_{p,q} = \|\nabla u\|_p + \|u\|_q$. Recently, the following optimal Gagliardo–Nirenberg inequality on the Euclidean $n$-space $\mathbb{R}^n$ has been proved by Del Pino and Dolbeault:

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Theorem 1.1 (Del Pino and Dolbeault [DPD1, DPD2]). Assume that $1 < p < n$, $p < q \leq \frac{p(n-1)}{n-p}$ and denote the numbers $\delta$, $r$ and $\theta$ by

$$\delta = np - (n-p)q, \quad r = \frac{q-1}{p-1}, \quad \theta = \frac{(q-p)n}{(q-1)(np - (n-p)q)}.$$

(1.1)

Then for all $u \in \mathcal{D}^{p,q}(\mathbb{R}^n)$,

$$\|u\|_r \leq \Phi \|\nabla u\|_p^\theta \|u\|_{q}^{1-\theta}.$$

(1.2)

Here $\Phi$ is the best constant for inequality (1.2) and takes the explicit form

$$\Phi = \left(\frac{q-p}{p\sqrt{\pi}}\right)^\theta \left(\frac{pq}{n(q-p)}\right)^{\frac{\theta}{p}} \left(\frac{\theta}{pq}\right)^{\frac{1}{q}} \left\{\frac{\Gamma\left(\frac{q-p}{q-p}\right)\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma\left(\frac{p-1}{q-p}\right)\Gamma\left(n\frac{p-1}{p} + 1\right)}\right\}^{\frac{\theta}{p}}.$$

Equality holds in (1.2) if and only if for some $\alpha \in \mathbb{R}$, $\beta > 0$, $x \in \mathbb{R}^n$,

$$u(x) = \alpha \left(1 + \beta|x - x|\right)^{-\frac{(p-1)}{(q-p)}} \quad \forall x \in \mathbb{R}^n.$$

Observe that when $q = p\frac{n-1}{n-p}$, we have $\theta = 1$, and $r = \frac{np}{n-p}$, the critical Sobolev exponent. Inequality (1.2) then becomes the optimal Sobolev inequality, as found by Aubin and Talenti [Au1, T], which has many important applications (see e.g., [Au2, Au3, HLP, H1, H2, L] and the references therein). Complete manifolds with non-negative Ricci curvature in which some Sobolev or Caffarelli–Kohn–Nirenberg type inequality is satisfied were studied in [dCX2, Le, X2]. For the Caffarelli–Kohn–Nirenberg inequalities, we refer to [CKN, CCh, L, CW].

In this paper, we study complete non-compact Riemannian manifolds with non-negative Ricci curvature in which some Gagliardo–Nirenberg type inequality is satisfied. For a Riemannian manifold $M$, we let $dv$ be the Riemannian volume element on $M$, $C_0^\infty(M)$ be the space of smooth functions on $M$ with compact support, $B(x, r)$ be the geodesic ball with center $x \in M$ and radius $r$ and $\text{vol}[B(p, r)]$ be the volume of $B(p, r)$. We shall prove the following result.

Theorem 1.2. Let $p, q, r, \theta, \Phi$ be as in Theorem 1.1 and let $C \geq \Phi$ be a constant. Assume that $M$ is an $n(\geq 2)$-dimensional complete non-compact Riemannian manifold with non-negative Ricci curvature and assume that for any $u \in C_0^\infty(M)$, we have

$$\|u\|_r \leq C\|\nabla u\|_p^\theta \|u\|_{q}^{1-\theta}.$$

(1.3)
Then for any \( x \in M \), we have
\[
\text{vol}[B(x, r)] \geq \left( C^{-1}\Phi \left( \frac{\theta}{p} + \frac{1-\theta}{q} - \frac{1}{r} \right) \right)^{-1} V_0(r), \quad \forall r > 0, \tag{1.4}
\]
where \( V_0(r) \) is the volume of an \( r \)-ball in \( \mathbb{R}^n \).

The case of the critical Sobolev inequality which corresponds to the values
\[
C = \Phi, \quad q = \frac{p(n-1)}{n-p}, \quad r = \frac{np}{n-p}, \quad \text{and} \quad \theta = 1,
\]
was first obtained by Ledoux [Le]. It should be also mentioned that the case of the Nash inequality (one of the Gagliardo–Nirenberg inequality not addressed in Theorem 1.1 and thus in Theorem 1.2) was treated by Druet, Hebey and Vaugon and they proved that a complete manifold with non-negative Ricci curvature in which the optimal Nash inequality holds is flat (cf. [DHV]).

The Bishop–Gromov’s comparison theorem [BC,Ch,GLP] implies that if \( M \) is an \( n \)-dimensional complete Riemannian manifold with non-negative Ricci curvature, then for any \( x \in M \), \( \text{vol}[B(x, r)] \leq V_0(r) \) with equality holding if and only if \( B(x, r) \) is isometric to an \( r \)-ball in \( \mathbb{R}^n \). Thus one gets from Theorem 1.2 the following rigidity theorem:

**Corollary 1.3.** Let \( p, q, r, \Phi \) be as in Theorem 1.1 and assume that \( M \) is an \( n(\geq 2) \)-dimensional complete open Riemannian manifold \( M \) with non-negative Ricci curvature in which the following inequality is satisfied:
\[
\|u\|_r \leq \Phi \|\nabla u\|_p^\theta \|u\|_q^{1-\theta}, \quad \forall u \in C^\infty_0(M). \tag{1.5}
\]
Then \( M \) is isometric to \( \mathbb{R}^n \).

A celebrated theorem due to Cheeger–Colding [CC] states that given integer \( n \geq 2 \) there exists a constant \( \delta(n) > 0 \) such that any \( n \)-dimensional complete Riemannian manifold with non-negative Ricci curvature and \( \text{vol}[B(x, r)] \geq (1 - \delta(n))V_0(r) \) for some \( x \in M \) and all \( r > 0 \) is diffeomorphic to \( \mathbb{R}^n \). Combining this Cheeger–Colding’s theorem with Theorem 1.2, we obtain the following topological uniqueness theorem for complete manifolds with non-negative Ricci curvature.

**Corollary 1.4.** Given an integer \( n \geq 2 \), \( p \in (1, n) \), \( q \in (p, \frac{p(n-1)}{n-p}] \), there exists a positive constant \( \varepsilon = \varepsilon(n, p, q) \) depending only on \( n, p \) and \( q \), such that any \( n \)-dimensional complete non-compact Riemannian manifold \( M \) with non-negative Ricci curvature in which the following inequality is satisfied:
\[
\|u\|_r \leq (\Phi + \varepsilon)\|\nabla u\|_p^\theta \|u\|_q^{1-\theta}, \quad \forall u \in C^\infty_0(M),
\]
is diffeomorphic to \( \mathbb{R}^n \), where \( r \) and \( \theta \) are as in (1.1).
Complete manifolds with non-negative Ricci curvature have been studied extensively. One can find some related results about the geometry and topology of manifolds with non-negative Ricci curvature, e.g., in [AG, A, dCX1, OSY, SS, S1, S2, SS0, SO, X1].

2. A proof of Theorem 1.2

Fix a point \( x_0 \in M \) and denote by \( \rho \) the distance function on \( M \) from \( x_0 \). For any \( \lambda > 0 \), let

\[
F(\lambda) = \int_M \frac{dv}{(\lambda + \rho^{p-1})^{q/p}}.
\]

(2.1)

Clearly, \( F \) is well defined and of class \( C^1 \). Indeed, one obtains by using the Fubini theorem (cf. [SY]) that

\[
F(\lambda) = \int_0^{+\infty} \text{vol}\left\{ x : \frac{1}{(\lambda + \rho^{p-1})^{q/p}} > s \right\} ds.
\]

Making the variable change

\[
s = \frac{1}{(\lambda + t^{p-1})^{(p-1)q/p}}
\]

in the above equality and using the fact that \( \text{vol}[B(x_0, t)] \leq \omega_n t^n \), where \( \omega_n \) is the volume of a unit ball in \( \mathbb{R}^n \), we get

\[
F(\lambda) \leq \frac{pq\omega_n}{q-p} \int_0^{+\infty} \frac{t^{n+\frac{1}{p-1}}}{(\lambda + t^{p-1})^{\frac{p(q-1)}{q-p}}} dt.
\]

(2.2)

Since \( q < \frac{np}{n-p} \), we have

\[
n + \frac{1}{p-1} - \frac{p^2(q-1)}{(p-1)(q-p)} = n + \frac{1}{p-1} - \frac{p^2}{p-1} - \frac{p^2}{q-p} < -1.
\]
Thus $0 \leq F(\lambda) < \infty$, $\forall \lambda > 0$, and $F$ is differentiable. Also, we have

$$F'(\lambda) = -\frac{(p-1)q}{q-p} \int_M \frac{dv}{(\lambda + \rho \frac{p}{q-t})^{(q-1)p}}. \quad (2.3)$$

By an approximation procedure, we can apply (1.3) to $\left(\lambda + \rho \frac{p}{q-t}\right)^{-\frac{p-1}{q-t}}$ for every $\lambda > 0$ to get

$$
\left(\int_M \frac{dv}{(\lambda + \rho \frac{p}{q-t})^{(q-1)p}}\right)^{\frac{1}{r}} \leq C \left(\frac{p}{q-p}\right)^\theta \left(\int_M \frac{\rho \frac{p}{q-t} dv}{(\lambda + \rho \frac{p}{q-t})^{(q-1)p}}\right)^{\frac{\theta}{p}} \left(\int_M \frac{dv}{(\lambda + \rho \frac{p}{q-t})^{(p-1)q}}\right)^{1-\frac{\theta}{q}}.
$$

which, combining with (2.3), gives

$$
\left(\frac{-q-p}{(p-1)q} F'(\lambda)\right)^{\frac{1}{r}} \leq C \left(\frac{p}{q-p}\right)^\theta \left(F(\lambda) + \frac{q-p}{(p-1)q} \lambda F'(\lambda)\right)^{\frac{\theta}{p}} F(\lambda)^{1-\frac{\theta}{q}}.
$$

Thus we have

$$
(-F'(\lambda))^{\frac{p}{p-r}} \leq l \left(F(\lambda) + \frac{q-p}{(p-1)q} \lambda F'(\lambda)\right) F(\lambda)^{\frac{1-\frac{\theta}{q}}{\theta}}, \quad (2.4)
$$

where

$$
l = C \left(\frac{p}{q-p}\right)^p \left(\frac{(p-1)q}{q-p}\right)^{\frac{p}{p-r}}.
$$

Consider the function $G : (0, +\infty) \to \mathbb{R}$ given by

$$G(\lambda) = \int_{\mathbb{R}^n} \frac{dx}{(\lambda + |x|^{\frac{p}{q-t}})^{\frac{(p-1)q}{q-p}}}. \quad (2.5)
$$

It follows from (2.2) that

$$G(\lambda) = \frac{pq \omega_n}{q-p} \int_0^{+\infty} \frac{t^{n+\frac{1}{p-t}} dt}{(\lambda + t \frac{p}{q-t})^{\frac{(p-1)q}{q-p}}}. \quad (2.5)$$
Observe that when \( M = \mathbb{R}^n \) and \( C = \Phi \), for each \( \lambda > 0 \), the function \( v_\lambda : \mathbb{R}^n \to \mathbb{R} \) defined by

\[
v_\lambda(x) = \left( \lambda + |x|^{\frac{p}{p-1}} \right)^{-\frac{p-1}{q-p}}
\]

is an extremal function in inequality (1.3). That is

\[
\left( \int_{\mathbb{R}^n} v_\lambda^r dx \right)^{\frac{1}{r}} = \Phi \left( \int_{\mathbb{R}^n} |\nabla v_\lambda|^p dx \right)^{\frac{\theta}{p}} \left( \int_{\mathbb{R}^n} v_\lambda^q dx \right)^{\frac{1-\theta}{q}}.
\]

(2.7)

The above equality can be rewritten as follows:

\[
(-G'(\lambda))^{\frac{p}{p-r}} = \tilde{l} \left( G(\lambda) + \frac{q-p}{(p-1)q} \lambda G'(\lambda) \right) G(\lambda)^{\frac{(1-\theta)p}{\theta q}},
\]

(2.8)

where

\[
\tilde{l} = \Phi^{\frac{p}{q-p}} \left( \frac{p}{q-p} \right)^p \left( \frac{(p-1)q}{q-p} \right)^{\frac{p}{\theta q}}.
\]

Substituting

\[
G(\lambda) = G(1)^{(p-1)}\left( \frac{n}{p} - \frac{n}{q-p} \right)
\]

into (2.8), we have

\[
\left( 1 - \frac{n(q-p)}{pq} \right)^{\frac{p}{p-r}} = \Phi^{\frac{p}{q-p}} \left( \frac{p}{q-p} \right)^p \left( \frac{(q-p)n}{pq} \right) G(1)^{\frac{p}{\theta q} + \frac{1-\theta}{q} - \frac{1}{r}}.
\]

(2.9)

Consider the constant \( A \) given by

\[
\left( 1 - \frac{n(q-p)}{pq} \right)^{\frac{p}{p-r}} = C^{\frac{p}{q-p}} \left( \frac{p}{q-p} \right)^p \left( \frac{(q-p)n}{pq} \right) A^{\frac{p}{\theta q} + \frac{1-\theta}{q} - \frac{1}{r}}.
\]

(2.10)

One can check that the function

\[
H_0(\lambda) = A^{(p-1)}\left( \frac{n}{p} - \frac{n}{q-p} \right), \quad \lambda \in (0, +\infty)
\]
satisfies the differential equation

\[
(-H_0'(\lambda))^\frac{p}{np} = l \left( H_0(\lambda) + \frac{q-p}{(p-1)q} \lambda H_0'(\lambda) \right) H_0(\lambda)^\frac{(1-\theta)p}{np}.
\] (2.11)

It follows from (2.9) and (2.10) that

\[
A = \left( \frac{\Phi}{C} \right) \left( \frac{\theta + 1 - \theta}{p} - \frac{1}{\tau} \right)^{-1} G(1),
\]

which implies that

\[
H_0(\lambda) = \left( \frac{\Phi}{C} \right) \left( \frac{\theta + 1 - \theta}{p} - \frac{1}{\tau} \right)^{-1} G(\lambda).
\] (2.12)

We claim that if \( F(\lambda_0) < H_0(\lambda_0) \), for some \( \lambda_0 > 0 \), then \( F(\lambda) < H_0(\lambda) \), \( \forall \lambda \in (0, \lambda_0] \). In order to see this, suppose that there exists some \( \tilde{\lambda} \in (0, \lambda_0) \) such that \( F(\tilde{\lambda}) \geq H_0(\tilde{\lambda}) \). Set

\[
\lambda_1 = \sup \{ \lambda < \lambda_0; F(\lambda) = H_0(\lambda) \}.
\]

Then for any \( \lambda \in [\lambda_1, \lambda_0] \), \( 0 < F(\lambda) \leq H_0(\lambda) \) and so we have from (2.4) that

\[
(-F'(\lambda))^\frac{p}{np} \leq l \left( H_0(\lambda) + \frac{\lambda(q-p)}{(p-1)q} F'(\lambda) \right) H_0(\lambda)^\frac{(1-\theta)p}{np}.
\] (2.13)

For each \( \lambda > 0 \), the function \( \phi_\lambda : [0, +\infty) \rightarrow R \) defined by

\[
\phi_\lambda(t) = t^\frac{p}{np} + \frac{l\lambda(q-p)t}{(p-1)q} H_0(\lambda)^\frac{(1-\theta)p}{np}
\]

is increasing. Hence, when \( \lambda \in [\lambda_1, \lambda_0] \), we deduce from (2.13) and (2.11) that

\[
\phi_\lambda(-F'(\lambda)) = (-F'(\lambda))^\frac{p}{np} + \frac{l\lambda(q-p)}{(p-1)q} (-F'(\lambda)) H_0(\lambda)^\frac{(1-\theta)p}{np}
\]

\[
\leq lH_0(\lambda)^{1+\frac{(1-\theta)p}{np}}
\]

\[
= \phi_\lambda(-H_0'(\lambda)),
\]
which gives $-F'(\lambda) \leq -H'_0(\lambda), \forall \lambda \in [\lambda_1, \lambda_0]$. This implies that $F - H_0$ is increasing on $[\lambda_1, \lambda_0]$. Consequently, we have

$$0 = (F - H_0)(\lambda_1) \leq (F - H_0)(\lambda_0) < 0,$$

which is a contradiction. Thus the above claim is true.

Let us come back to the expression of $F(\lambda)$. Fix a small $\varepsilon > 0$. Since

$$\lim_{u \to 0} \frac{\text{vol}[B(x_0, u)]}{V_0(u)} = 1,$$

there exists a $\delta > 0$ such that $\text{vol}[B(x_0, h)] \geq (1 - \varepsilon)V_0(h), \forall h \leq \delta$. It then follows from (2.2) that

$$F(\lambda) \geq \frac{pq}{q - p}(1 - \varepsilon) \int_0^\delta V_0(t) \frac{t^{\frac{1}{p-1}}}{(\lambda + \frac{p}{p-1})^{\frac{p(q-1)}{q-p}}} dt$$

$$= \frac{pq}{q - p}(1 - \varepsilon)\lambda^{\frac{(p-1)n}{p} + 1 - \frac{p(q-1)}{q-p}} \int_0^\delta V_0(s) \frac{s^{\frac{1}{p-1}}}{(1 + s^{\frac{p}{p-1}})^{\frac{p(q-1)}{q-p}}} ds.$$

On the other hand, we have

$$G(\lambda) = \frac{pq}{q - p} \lambda^{\frac{(p-1)n}{p} + 1 - \frac{p(q-1)}{q-p}} \int_0^{+\infty} V_0(s) \frac{s^{\frac{1}{p-1}}}{(1 + s^{\frac{p}{p-1}})^{\frac{p(q-1)}{q-p}}} ds.$$

Hence

$$\liminf_{\lambda \to 0} \frac{F(\lambda)}{G(\lambda)} \geq 1 - \varepsilon.$$

Letting $\varepsilon \to 0$, we get

$$\liminf_{\lambda \to 0} \frac{F(\lambda)}{G(\lambda)} \geq 1.$$

(2.14)

We consider now two cases.

**Case 1:** $C > \Phi$. 

In this case, it follows from (2.12) and (2.14) that

\[
\liminf_{\lambda \to 0} \frac{F(\lambda)}{H_0(\lambda)} = \left( C \Phi \left( \frac{\lambda}{p} + \frac{1 - \theta}{q} - \frac{1}{r} \right) \right)^{-1} \liminf_{\lambda \to 0} \frac{F(\lambda)}{G(\lambda)}
\]

\[
\geq \left( C \Phi \left( \frac{\lambda}{p} + \frac{1 - \theta}{q} - \frac{1}{r} \right) \right)^{-1} > 1.
\]

(2.15)

The above claim then implies that

\[
F(\lambda) \geq H_0(\lambda), \quad \forall \lambda > 0.
\]

Thus, for any \( \lambda > 0 \), it holds

\[
\int_0^{+\infty} \left( \frac{\text{vol}[B(x_0, s)] - bV_0(s)}{\lambda + t^{\frac{p}{p-1}} \left( t^{\frac{1}{p}} \right)^{\frac{p(q-1)}{q-p}}} \right) dt \geq 0,
\]

(2.16)

where

\[
b = (C^{-1} \Phi \left( \frac{\lambda}{p} + \frac{1 - \theta}{q} - \frac{1}{r} \right) \right)^{-1}.
\]

The Bishop–Gromov comparison theorem [BC,Ch,GLP] tells us that the function \( \frac{\text{vol}[B(x_0, s)]}{V_0(s)} \) is decreasing. Assume now that

\[
\lim_{s \to +\infty} \frac{\text{vol}[B(x_0, s)]}{V_0(s)} = b_0.
\]

In order to conclude the proof of Theorem 1.2 in the case that \( C > \Phi \), it suffices to show that \( b_0 \geq b \). We shall prove this fact by contradiction. Thus suppose that \( b_0 = b - \varepsilon_0 \), for some \( \varepsilon_0 > 0 \). Then there exists an \( N_0 > 0 \) such that

\[
\frac{\text{vol}[B(x_0, s)]}{V_0(s)} \leq b - \frac{\varepsilon_0}{2}, \quad \forall s \geq N_0.
\]

(2.17)

Substituting (2.17) into (2.16), we obtain for every \( \lambda > 0 \) that

\[
0 \leq \int_0^{N_0} \frac{\text{vol}[B(x_0, s)]}{V_0(s)} \cdot \frac{t^{\frac{p}{p-1}} \left( t^{\frac{1}{p}} \right)^{\frac{p(q-1)}{q-p}}} \left( \lambda + t^{\frac{p}{p-1}} \right)^{\frac{p(q-1)}{q-p}} dt
\]
\[
\begin{aligned}
&+ \int_{N_0}^\infty \left( b - \frac{\varepsilon_0}{2} \right) \frac{t^{n+\frac{1}{p-1}}}{(\lambda + t^{p-1})^{\frac{q(p-1)}{q-p}}} dt - b \int_0^\infty \frac{t^{n+\frac{1}{p-1}}}{(\lambda + t^{p-1})^{\frac{q(p-1)}{q-p}}} dt \\
\leq \int_0^{N_0} \frac{t^{n+\frac{1}{p-1}}}{(\lambda + t^{p-1})^{\frac{q(p-1)}{q-p}}} dt + \int_{N_0}^\infty \left( b - \frac{\varepsilon_0}{2} \right) \frac{t^{n+\frac{1}{p-1}}}{(\lambda + t^{p-1})^{\frac{q(p-1)}{q-p}}} dt \\
&- b \int_0^\infty \frac{t^{n+\frac{1}{p-1}}}{(\lambda + t^{p-1})^{\frac{q(p-1)}{q-p}}} dt \\
&= \int_0^{N_0} \left( 1 - b + \frac{\varepsilon_0}{2} \right) \frac{t^{n+\frac{1}{p-1}}}{(\lambda + t^{p-1})^{\frac{q(p-1)}{q-p}}} dt - \frac{\varepsilon_0}{2} \int_0^\infty \frac{t^{n+\frac{1}{p-1}}}{(\lambda + t^{p-1})^{\frac{q(p-1)}{q-p}}} dt \\
&= \int_0^{N_0} \left( 1 - b + \frac{\varepsilon_0}{2} \right) \frac{t^{n+\frac{1}{p-1}}}{(\lambda + t^{p-1})^{\frac{q(p-1)}{q-p}}} dt - \frac{\varepsilon_0}{2} \frac{q-p}{pq\omega_n} G(\lambda) \\
&\leq \left( 1 - b + \frac{\varepsilon_0}{2} \right) \lambda^{-\frac{p(p-1)}{q-p}} \frac{N_0^{n+\frac{1}{p-1}}}{(n+1+\frac{1}{p-1})} - \frac{\varepsilon_0}{2} \frac{q-p}{pq\omega_n} \lambda^{(p-1)\left(\frac{n}{p} - \frac{q}{q-p}\right)} G(1).
\end{aligned}
\]

Thus, for any $\lambda > 0$, one has

\[
\frac{\varepsilon_0}{2} \frac{q-p}{pq\omega_n} \cdot G(1) \leq \lambda^{\left(\frac{1-p}{p}\right)n-1} \cdot \left( 1 - b + \frac{\varepsilon_0}{2} \right) \frac{N_0^{n+\frac{1}{p-1}}}{(n+1+\frac{1}{p-1})}.
\]

Letting $\lambda \rightarrow +\infty$ in the above inequality and observing that $(1-p)n/p - 1 < 0$, one obtains a contradiction. This completes the proof of Theorem 1.2 in the case that $C > \Phi$.

**Case 2:** $C = \Phi$. In this case, we have for any fixed $\delta > 0$ that

\[
\|u\|_r \leq (\Phi + \delta) \|u\|_p^\theta \|u\|_q^{1-\theta}.
\]  

Thus for any $x \in M$ it follows from Case 1 that

\[
\text{vol} [B(x, r)] \geq \left( \frac{\Phi}{\Phi + \delta} \right) \left( \frac{\theta}{\theta + \frac{1-\theta}{q} + \frac{1}{r}} \right)^{-1} V_0(r), \quad \forall r > 0.
\]
Making $\delta \to 0$, one gets

$$\text{vol } [B(x,r)] \geq V_0(r), \quad \forall r > 0.$$ 

This completes the proof of Theorem 1.2 for the case that $C = \Phi$.

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References


