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The Ademollo–Gatto theorem for lattice semileptonic decays

D. Bećirević [a,*], G. Martinelli [b], G. Villadoro [b]

[a] *Laboratoire de Physique Théorique, Université Paris Sud, Centre d'Orsay, F-91405 Orsay cedex, France*
[b] *Dipartimento di Fisica, Università di Roma "La Sapienza", and INFN, Sezione di Roma, P.le A. Moro 2, I-00185 Rome, Italy*

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Abstract

We present the results of the calculation of the $K_{\ell 3}$ semileptonic form factor at zero momentum transfer, $f_+(0)$, obtained at one-loop in partially quenched chiral perturbation theory (with either $n_f = 2$, or $n_f = 3$, and with generic valence and sea quark masses). We show that for $n_f = 2$, and at the one-loop level, when the masses of the valence and sea light quarks are equal, the correction is of $\mathcal{O}[(M_K^2 - M_\pi^2)^3]$. The formulae presented here can be useful for the mass extrapolation of the results obtained in lattice simulations to the physical point.

1. Introduction

In the last two years we assisted to a renewed interest in theoretical calculations of the semileptonic form factor $f_+(q^2)$ relevant to the extraction of $|V_{us}|$ from $K \to \pi \ell \bar{\nu}_\ell$ ($K_{\ell 3}$) decays [1–5]. In particular, it has been shown that in lattice simulations the form factor at zero recoil, $f_+(q^2 = 0)$, can be extracted with the percent precision that is required for making a meaningful determination of $|V_{us}|$ [4]. Although many systematic uncertainties must still be reduced, by performing unquenched calculations at lower quark masses and on several lattice spacings, the calculation of Ref. [4] triggered a new wave of activity and the quality of the results is rapidly improving [5].

A key observation which allows to reach a good theoretical control of these transitions is the Ademollo–Gatto theorem [6], which states that the $K_{\ell 3}$ form factors $f_+(q^2)$ and $f_0(q^2)$ at zero momentum transfer ($q^2 = 0$) are renormalised only by terms of second order in the breaking of the SU(3) flavour symmetry. Besides, chiral perturbation theory (ChPT) provides an excellent tool to analyse the dependence of $f_{+,0}(0)$ on the meson (quark) masses, and a guidance for the extrapolation of the lattice form factors to the physical point. Following Leutwyler and Roos it is convenient to express the form factor in the form [7]

$$f_+(0) = 1 + f_2 + f_4 + \cdots, \tag{1}$$

where $f_n = \mathcal{O}[M_{K,\pi}^n/(4\pi f_\pi)^n]$ are the terms arising at higher orders in ChPT. Because of the Ademollo–Gatto theorem, the first non-trivial term in the chiral expansion, f_2, does not receive contributions from local operators appearing in the effective theory and can be computed unambiguously in terms of M_K, M_π and f_π [7].

Lattice calculations of the $K_{\ell 3}$ form factors have been done in quenched and partially quenched ($n_f = 2$) QCD. In the latter case simulations are performed with "valence" quark masses equal to or different from "sea" quark masses. In such a situation a number of subtleties related to the validity of the Ademollo–Gatto theorem arise. In this Letter we discuss the applicability of Ademollo–Gatto theorem in various situations (quenched, partially quenched and fully unquenched), and give the main expressions for f_2 in each case. These formulae are important for the extrapolation of the form factors to the physical point. In the following we

* Corresponding author.
E-mail address: damir.becirevic@th.u-psud.fr (D. Bećirević).

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will always work in the isospin symmetric limit, with the mass of the strange quark (m_s) different from the mass of the light quarks ($m_d = m_u$).

2. Quenched and unquenched formulae

In this section we give a brief summary of the known results for f_2, namely, in full QCD and its quenched approximation.

2.1. Full QCD

In the isospin-symmetric limit, within full QCD, the expression of the leading chiral correction f_2 is [7]

$$f_2 = \frac{3}{2} H_{\pi K} + \frac{3}{2} H_{\eta K}, \tag{2}$$

where

$$H_{PQ} = -\frac{1}{64\pi^2 f_\pi^2}\left[M_P^2 + M_Q^2 + \frac{2M_P^2 M_Q^2}{M_P^2 - M_Q^2} \log \frac{M_Q^2}{M_P^2} \right]. \tag{3}$$

Note that f_2 is completely specified in terms of pseudoscalar meson masses and decay constants ($f_\pi \approx 132$ MeV), it is negative ($f_2 \approx -0.023$ for physical masses), as implied by unitarity [7,8], and vanishes as $(M_K^2 - M_\pi^2)^2/(f_\pi^2 M_K^2)$ in the SU(3) limit, following the combined constraints of chiral symmetry and the Ademollo–Gatto theorem.

2.2. Quenched QCD

The structure of chiral logarithms appearing in Eqs. (2), (3), is valid only in the full theory. In the quenched theory, instead, the leading (unphysical) logarithms are those entering the one-loop functional of qChPT [9–11]. f_2 in the quenched case was first computed in Ref. [4].

Normalising the lowest-order qChPT Lagrangian as in Ref. [9], with a quadratic term for the singlet field $\Phi_0 = \mathrm{str}(\Phi)$ chosen as

$$\mathcal{L}_2^q\big|_{\Phi_0^2} = \frac{\alpha}{6} D_\mu \Phi_0 D^\mu \Phi_0 - \frac{M_0^2}{6}\Phi_0^2, \tag{4}$$

the result is

$$f_2^q = H_{\pi K}^q + H_{(s\bar{s})K}^q, \tag{5}$$

where

$$H_{PK}^q = \frac{M_K^2}{96\pi^2 f_\pi^2}\left[\frac{M_0^2(M_K^2 + M_P^2) - 2\alpha M_K^2 M_P^2}{(M_K^2 - M_P^2)^2} \log\left(\frac{M_K^2}{M_P^2}\right) - \alpha \right], \tag{6}$$

with $M_{(s\bar{s})}^2 = 2M_K^2 - M_\pi^2$. As anticipated, the one-loop result in Eq. (5) is finite because of the Ademollo–Gatto theorem, which is still valid in the quenched approximation [9], and thus the absence of contributions from local operators in f_2^q. A proof that the Ademollo–Gatto theorem (and more generally the Sirlin's relation [12]) holds within qChPT beyond the one-loop level can easily be obtained by applying the functional formalism to the demonstration in Ref. [12]. The latter needs only flavour symmetries for valence quarks which hold on the lattice also in the quenched case.

It is worth emphasising that the nature of the SU(3) breaking corrections in the quenched theory is completely different from that of full QCD: only contributions coming from the mixing with the flavour singlet state are present and one finds $f_2^q > 0$, which is a signal of the non-unitarity of the theory. For typical values of the singlet parameters ($M_0 \approx 0.6$ GeV and $\alpha \approx 0$ [13]) and for the physical values of pion and kaon masses, one finds $f_2^q \approx +0.022$.

3. Partially quenched results

In this section we give the new results for various set-ups relevant to partially quenched QCD. We have used the partially quenched ChPT Lagrangian defined in Refs. [14,15], and work with two sea quark masses ($m_s^{(S)}$, $m_d^{(S)}$), and two valence ones ($m_s^{(V)}$, $m_d^{(V)}$). We stress again that we always work in the exact isospin limit, i.e., $m_u = m_d$. The meson masses, at leading order in ChPT, read

$$M_\pi^2 = 2B_0 m_d^{(V)}, \qquad M_K^2 = B_0\big(m_s^{(V)} + m_d^{(V)}\big), \qquad M_{dd}^2 = 2B_0 m_d^{(S)}, \qquad M_{ss}^2 = 2B_0 m_s^{(S)}, \tag{7}$$

where B_0 is the chiral condensate (more precisely, $B_0 = -2\langle\bar{q}q\rangle/f_\pi^2$). In Appendix A we give the complete formula for f_2^{pq}, as obtained with 3 dynamical flavours and four quark masses enumerated above. Here we focus onto the limits that are particularly interesting to the situations encountered in the partially quenched QCD simulations on the lattice.

Like in the cases of full and quenched QCD, also in the partially quenched theory the Ademollo–Gatto theorem holds non-perturbatively to all orders in the chiral expansion. However, the generic structure of the lowest order correction in ChPT, expanded in the mass difference of the valence quark masses, reads

$$f_2^{pq} = \left[\frac{g_1}{m_s^{(S)}} + g_2\big(m_d^{(S)} - m_d^{(V)}\big)\right]\big(m_s^{(V)} - m_d^{(V)}\big)^2 + \mathcal{O}\big[\big(m_s^{(V)} - m_d^{(V)}\big)^3\big], \tag{8}$$

where g_1 and g_2 are functions of the valence and sea quark (meson) masses. Thus we find that in the partially quenched theory with $n_f = 2$, which is obtained by sending $m_s^{(S)} \to \infty$, the correction is at least of the third order in $m_s^{(V)} - m_d^{(V)}$ if the valence and sea light quark masses are the same. This is only an accident however: at the next order in the chiral expansion, the corrections of $\mathcal{O}[(m_s^{(V)} - m_d^{(V)})^2]$, due to the effect of the higher-dimensional local operators, will appear. This implies that a numerical analysis of the mass dependence of the form factor $f_{+,0}(0)$ in the $n_f = 2$ case and with $m_d^{(S)} = m_d^{(V)}$, could determine the constants quite precisely since the leading non-analytic corrections from f_2^{pq} are suppressed by this enhanced AG effect.

In the following we give the resulting expressions in two important cases:

Case 1. $n_f = 2$ non-degenerate valence and sea light quarks. In this case we have

$$\begin{aligned} f_2^{pq} = &-\frac{2M_K^2 + M_{dd}^2}{32\pi^2 f_\pi^2} + \frac{M_K^2[M_\pi^2 M_{dd}^2 + M_K^2(M_{dd}^2 - 2M_\pi^2)]\log\big(\frac{M_K^2}{M_\pi^2}\big)}{64\pi^2 f_\pi^2 (M_K^2 - M_\pi^2)^2} \\ &+ \frac{M_K^2[(2M_K^2 - M_{dd}^2)(2M_K^2 - M_\pi^2) - M_K^2 M_{dd}^2]\log\big(2 - \frac{M_\pi^2}{M_K^2}\big)}{64\pi^2 f_\pi^2 (M_K^2 - M_\pi^2)^2} \\ &+ \frac{(2M_K^2 - M_\pi^2 + M_{dd}^2)(M_\pi^2 + M_{dd}^2)\log\big(\frac{2M_K^2 - M_\pi^2 + M_{dd}^2}{M_\pi^2 + M_{dd}^2}\big)}{64\pi^2 f_\pi^2 (M_K^2 - M_\pi^2)}, \end{aligned} \tag{9}$$

which in the small $M_K^2 - M_\pi^2$ limit becomes

$$f_2^{pq} = -\frac{(M_K^2 - M_\pi^2)^2 (M_K^2 - M_{dd}^2)(3M_K^2 + M_{dd}^2)}{192\pi^2 f_\pi^2 M_K^4 (M_K^2 + M_{dd}^2)} + \mathcal{O}\big[\big(M_K^2 - M_\pi^2\big)^4\big]. \tag{10}$$

Note that only even powers of $M_K^2 - M_\pi^2$ appear in the expansion, as in the quenched case. The odd powers are hidden in the factor $M_K^2 - M_{dd}^2$.

Case 2. $n_f = 2$ degenerate valence and sea light quarks. By taking $M_{dd} = M_\pi$ we get the case with $m_d^{(S)} = m_d^{(V)}$. The correction is now cubic in the SU(3) breaking, namely,

$$f_2^{pq} = -\frac{2M_K^2 + M_\pi^2}{32\pi^2 f_\pi^2} - \frac{3M_K^2 M_\pi^2 \log\big(\frac{M_\pi^2}{M_K^2}\big)}{64\pi^2 f_\pi^2 (M_K^2 - M_\pi^2)} + \frac{M_K^2(4M_K^2 - M_\pi^2)\log\big(2 - \frac{M_\pi^2}{M_K^2}\big)}{64\pi^2 f_\pi^2 (M_K^2 - M_\pi^2)}, \tag{11}$$

which after expanding in $M_K^2 - M_\pi^2$ leads to

$$f_2^{pq} = -\frac{(M_K^2 - M_\pi^2)^3}{96\pi^2 f_\pi^2 M_K^4} + \mathcal{O}\big[\big(M_K^2 - M_\pi^2\big)^4\big], \tag{12}$$

thus showing the suppression of the SU(3)-breaking corrections at this order. In particular, in this case, the AG quadratic correction extracted from lattice simulation of the $K \to \pi$ vector form factor, will be free from the f_2 contributions and will start with f_4 where analytic contributions are present.

4. Conclusion

In this Letter we discussed the leading chiral corrections to the $K_{\ell 3}$ form factor $f(0)$, that are protected by the Ademollo–Gatto theorem and thus unambiguously computable in all three forms of ChPT, i.e., the ones corresponding to the full, partially quenched and quenched QCD. We provide the formulae for the partially quenched case which are needed for the mass extrapolation of

currently accessible $f(0)$ computed on the lattice to the physical kaon and pion masses. The complete formula, with generic sea and valence quark (meson) masses, is given in Appendix A, wheres the case of $n_f = 2$ is discussed in more detail in the text.[1] We show that in the latter case with the valence and sea light quarks being degenerate in mass, the form factor $f(0)$ is free from f_2^{pq} correction, thus allowing for ever cleaner determination of f_4 from the lattice.

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Appendix A

In this appendix we give the formula for f_2^{pq} for $n_f = 3$ and generic valence and sea quark masses. The case $n_f = 2$ discussed in the text is readily obtained by sending $m_s^{(S)} \to \infty$. The full case is recovered by taking $m_s^{(S)} = m_s^{(V)}$ and $m_d^{(S)} = m_d^{(V)}$, corresponding to $M_{ss}^2 = 2M_K^2 - M_\pi^2$ and $M_{dd}^2 = M_\pi^2$.

$$
\begin{aligned}
f_2^{pq} = {} & M_K^2\Big[\big(2M_K^2 - M_\pi^2\big)\Big(6M_K^2\big(2M_K^2 - M_\pi^2\big)^2 - M_{ss}^2\big[\big(2M_K^2 - M_\pi^2\big)\big(11M_K^2 - M_\pi^2\big) + 4M_K^2 M_{ss}^2\big]\Big) \\
& - 2\big[\big(5M_K^2 - M_\pi^2\big)\big(2M_K^2 - M_\pi^2\big)^2 - 3\big(2M_K^2 - M_\pi^2\big)\big(3M_K^2 - M_\pi^2\big)M_{ss}^2 + \big(3M_K^2 - M_\pi^2\big)M_{ss}^4\big]M_{dd}^2 \\
& + \big[M_\pi^2 M_{ss}^2 + M_K^2\big(4M_K^2 - 2M_\pi^2 - 3M_{ss}^2\big)\big]M_{dd}^4\Big]\frac{\log\big(\frac{2M_K^2 - M_\pi^2}{M_\pi^2}\big)}{32\pi^2 f_\pi^2\big(M_K^2 - M_\pi^2\big)^2\big(3M_\pi^2 + 2M_{ss}^2 + M_{dd}^2 - 6M_K^2\big)^2} \\
& - \frac{M_K^4\big(M_K^2 - M_{ss}^2\big)\big(M_K^2 - M_{dd}^2\big)\log\big(\frac{M_K^2}{M_\pi^2}\big)}{8\pi^2 f_\pi^2\big(M_K^2 - M_\pi^2\big)^2\big(3M_K^2 - 2M_{ss}^2 - M_{dd}^2\big)} + \frac{\big(2M_K^2 - M_\pi^2 + M_{ss}^2\big)\big(M_\pi^2 + M_{ss}^2\big)\log\big(\frac{2M_K^2 - M_\pi^2 + M_{ss}^2}{M_\pi^2 + M_{ss}^2}\big)}{128\pi^2 f_\pi^2\big(M_K^2 - M_\pi^2\big)} \\
& + \frac{\big(2M_K^2 - M_\pi^2 + M_{dd}^2\big)\big(M_\pi^2 + M_{dd}^2\big)\log\big(\frac{2M_K^2 - M_\pi^2 + M_{dd}^2}{M_\pi^2 + M_{dd}^2}\big)}{64\pi^2 f_\pi^2\big(M_K^2 - M_\pi^2\big)} \\
& - \frac{3M_K^2\big(M_K^2 - M_\pi^2\big)^2\big(M_{ss}^2 - M_{dd}^2\big)^2\big(2M_{ss}^2 + M_{dd}^2\big)\log\big(\frac{2M_{ss}^2 + M_{dd}^2}{3M_\pi^2}\big)}{4\pi^2 f_\pi^2\big(3M_K^2 - 2M_{ss}^2 - M_{dd}^2\big)\big(2M_{ss}^2 + M_{dd}^2 - 3M_\pi^2\big)^2\big(3M_\pi^2 + 2M_{ss}^2 + M_{dd}^2 - 6M_K^2\big)^2} \\
& + \frac{26M_K^4 - \big(2M_{ss}^2 + M_{dd}^2 + 3M_\pi^2\big)\big(M_{ss}^2 + 2M_{dd}^2\big)}{64\pi^2 f_\pi^2\big(3M_\pi^2 + 2M_{ss}^2 + M_{dd}^2 - 6M_K^2\big)} \\
& - \frac{M_K^2\big(39M_\pi^4 - 8M_{ss}^4 - 18M_\pi^2\big(M_{ss}^2 + 2M_{dd}^2\big) + M_{dd}^2\big(18M_{ss}^2 + 5M_{dd}^2\big)\big)}{64\pi^2 f_\pi^2\big(3M_\pi^2 - 2M_{ss}^2 - M_{dd}^2\big)\big(3M_\pi^2 + 2M_{ss}^2 + M_{dd}^2 - 6M_K^2\big)}.
\end{aligned}
$$

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